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WELL-POSEDNESS FOR THE FOURTH-ORDER NONLINEAR SCHRÖDINGER-TYPE EQUATION RELATED TO THE VORTEX FILAMENT

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Abstract. We consider the time-local well-posedness for the initial-value problem of the fourth-order nonlinear Schrödinger-type equation in one space dimension which describes the motion of the vortex filament. By using the method of Fourier restriction norm introduced by Bourgain [3] and Kenig-Ponce-Vega [17]–[19], we show the time-local well-posedness in the Sobolev space $H^s(\mathbb{R})$ with $s \geq 1/2$ under certain coefficient conditions.

1. Introduction

In this paper, we are concerned with the initial-value problem for the fourth-order nonlinear Schrödinger-type equation

$$\begin{cases} i\partial_t u + \partial_x^2 u + \nu\partial_x^4 u = F(u, \overline{u}, \partial_x u, \partial_x \overline{u}, \partial_x^2 u, \partial_x^2 \overline{u}), & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbf{C}$ is an unknown function and the nonlinear term F is given by

$$F(u, \overline{u}, \partial_x u, \partial_x \overline{u}, \partial_x^2 u, \partial_x^2 \overline{u}) = -\frac{1}{2}|u|^2 u + \lambda_1 |u|^4 u + \lambda_2 (\partial_x u)^2 \overline{u} + \lambda_3 |\partial_x u|^2 u + \lambda_4 u^2 \partial_x^2 \overline{u} + \lambda_5 |u|^2 \partial_x^2 u$$

with $\lambda_1 = -3\mu/4$, $\lambda_2 = -2\mu + \nu/2$, $\lambda_3 = -4\mu - \nu$, $\lambda_4 = -\mu$, $\lambda_5 = -2\mu + \nu$, and real constants ν and μ.

We consider the three-dimensional motion of an isolated vortex filament embedded in inviscid incompressible fluid fulfilled in an infinite region. In [7], Da Rios introduced the following model for the motion of the vortex filament by using "localized induction approximation": We denote the centerline of the vortex filament by $\mathbf{X} = \mathbf{X}(x,t)$, represented as functions of arclength x and time t. Let (κ, τ) be the curvature and torsion, and let $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ be

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the Frenet-Serret frame of the centerline of the vortex filament, respectively. Then $\mathbf{X}$ satisfies following equation:

$$\partial_t \mathbf{X} = \lambda \kappa \mathbf{b}, \tag{1.2}$$

where λ is a real constant.

By introducing "the Hasimoto transform" (see [11]) defined by

$$u(x,t) = \kappa(x,t) e^{i \int_0^x \tau(y,t) dy}, \tag{1.3}$$

equation (1.2) is transformed to the well-known cubic nonlinear Schrödinger equation

$$i\partial_t u + \lambda \big\{ \partial_x^2 u + \frac{1}{2}|u|^2 u \big\} + A(t)u = 0. \tag{1.4}$$

Here $A(t)$ is an arbitrary function of t. Hence, the localized induction equation (1.2) is a completely integrable equation equivalent to the cubic nonlinear Schrödinger equation.

To describe the motion of an actual vortex filament precisely, some detailed models taking into account the effect from higher-order corrections of the equation have been introduced by several authors.

Fukumoto-Miyazaki [9] proposed the following equation for the motion of a vortex filament with axial flow:

$$\partial_t \mathbf{X} = \lambda \big\{ \kappa \mathbf{b} + \nu \big[\frac{1}{2}\kappa^2 \mathbf{t} + \partial_x \kappa \mathbf{n} + \kappa \tau \mathbf{b} \big] \big\}, \tag{1.5}$$

where λ and ν are real constants.

Similar to (1.2), the above equation (1.5) is transformed by the Hasimoto map (1.3) to the "Hirota equation"; that is, the third-order nonlinear Schrödinger-type equation

$$i\partial_t u + \lambda \big\{ \partial_x^2 u + \frac{1}{2}|u|^2 u \big\} + A(t)u - i\lambda\nu \big\{ \partial_x^3 u + \frac{3}{2}|u|^2 \partial_x u \big\} = 0. \tag{1.6}$$

If further, the local self-induced flow around the core comprises not only a uniform flow but also a straining field which deforms the core into an ellipse, Fukumoto-Moffatt [10] proposed the following model:

$$\partial_t \mathbf{X} = \lambda \{ \kappa \mathbf{b} + \nu [(2(\partial_x \kappa)\tau + \kappa(\partial_x \tau))\mathbf{n} + (\kappa\tau^2 - \partial_x^2 \kappa)\mathbf{b} + \kappa^2 \tau \mathbf{t}] + \mu \kappa^3 \mathbf{b} \}, \tag{1.7}$$

where λ, ν, and μ are real constants. Again by (1.3), the model (1.7) is transformed to the fourth-order nonlinear Schrödinger-type equation

$$\begin{aligned} i\partial_t u &+ \lambda \big\{ \partial_x^2 u + \frac{1}{2}|u|^2 u \big\} + A(t)u - \lambda\nu \big\{ \partial_x^4 u + \frac{3}{2}(|u|^2 \partial_x^2 u + (\partial_x u)^2 \overline{u}) \\ &+ \big(\frac{3}{8}|u|^4 + \frac{1}{2}\partial_x^2 |u|^2 \big) u \big\} + \lambda \big(\mu + \frac{\nu}{2} \big) \big\{ \partial_x^2 (|u|^2 u) + \frac{3}{4}|u|^4 u \big\} = 0. \end{aligned} \tag{1.8}$$

A brief summary of the above theory is given by Fukumoto [8].

We are interested in the solvability and well-posedness of those equations. Here, the well-posedness stands for the existence and uniqueness of the solutions and continuous dependence upon the initial data. The solvability and well-posedness of those equations have been studied by several authors. Nishiyama-Tani [26] and Tani-Nishiyama [31] constructed the weak solution for initial- and initial-boundary-value problems for the localized induction equation (1.2) and (1.5). Tsutsumi [32] showed the global well-posedness for the initial-value problem of the cubic nonlinear Schrödinger equation (1.4) in L^2 (see also Cazenave-Weissler [6] and the references therein). For the initial-value problem for the third-order nonlinear Schrödinger-type equation (1.6), Staffilani [27] proved local well-posedness in the Sobolev space $H^{1/4}(\mathbb{R})$ (see also Laurey [23] and Takaoka [30]). Here, the Sobolev space is defined by $H^s(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}); \langle\xi\rangle^s \mathcal{F}_x f \in L^2(\mathbb{R})\}$, where $\langle x\rangle = (1+|x|^2)^{1/2}$ and $\mathcal{F}_x f$ is the Fourier transform of f with respect to x. The purpose of this paper is to show time-local well-posedness for (1.1) for a larger class of initial data in the Sobolev space $H^s(\mathbb{R})$ when $\nu < 0$ and $\mu - \nu/2 = 0$. Here the restriction $\nu < 0$ with $\mu - \nu/2 = 0$ is required only because of the mathematical point of view. So far, it is not clear whether this restriction is meaningful in physical phenomena. However, as is mentioned in Remark 2.3 below, the case $\mu - \nu/2 = 0$ corresponds to the noncompletely integrable case of the equation, which is covered in our theorem.

To prove the well-posedness, we combine the Fourier restriction norm with the general Strichartz estimates for the group generated by the linear part of (1.1).

The method of Fourier restriction norm is first used by Bourgain [3] for the well-posedness problems. He applied it to the cubic nonlinear Schrödinger equation and the Korteweg-de Vries (K-dV) equation. After that, Kenig-Ponce-Vega [17, 18, 19] refined this method in the case of one spatial dimension and applied it to the semilinear Schrödinger equation with quadratic form and the K-dV equation. The merit of the method of Fourier restriction norm is to be able to prove the solvability for data belonging to a weaker class than general energy method. Nowadays, the method of Fourier restriction norm has been applied to various nonlinear equations by many authors (see, e.g., [1, 2, 21, 29, 30]).

We should note that the method of Fourier restriction norm is not always applicable to any nonlinear dispersive equation. In fact, Molinet-Saut-Tzvetkov [24] showed that it is not possible to prove the solvability in $H^s(\mathbb{R})$ with $s \in \mathbb{R}$ for a dispersive equation like the Benjamin-Ono (B-O) equation and the intermediate long wave equation by the method of Fourier restriction norm. Concerning the fourth-order nonlinear Schrödinger equation (1.1), however, it is possible to prove local well-posedness even if it has stronger nonlinear terms with second-order derivative. This equation has a stronger smoothing effect which stems from the fourth-order space derivative. This effect enables us to handle the nonlinear terms which are stronger than that of the K-dV- or the B-O-type equations.

We further consider the optimality of our method to show the well-posedness. To see this, we show that the crucial nonlinear estimate essentially given by a trilinear estimate fails in $H^s(\mathbb{R})$ where $s < 1/2$. As a consequence, we show that the data-solution flow-map is not C^5 as is shown by Bourgain [4] and Tzvetkov [33].

In what follows we introduce several notations and function spaces. For a function $u(x,t)$, we denote by $\mathcal{F}_x u(\xi,t)$ and $\mathcal{F}_t u(x,\tau)$ the Fourier transforms in the variables x and t, and by $\mathcal{F}_x^{-1} u(\xi,t)$ and $\mathcal{F}_t^{-1} u(x,\tau)$ their inverse transforms in the variables x and t respectively. We denote by $\widehat{\widehat{u}}(\xi,\tau) = \mathcal{F}_x \mathcal{F}_t u(\xi,\tau)$ the transform in both x and t. Let $\langle x \rangle = (1+|x|^2)^{1/2}$. The operator $\langle D_x \rangle$ is given by $\langle D_x \rangle = \mathcal{F}_x^{-1} \langle \xi \rangle \mathcal{F}_x$. For a subset A of $\mathbb{R}$, $\chi_A(\cdot)$ denotes a characteristic function on A. For $\nu \in \mathbb{R}$, we denote by $W_\nu(t)$ the group generated by the linear part of (1.1); i.e.,

$$W_\nu(t)v(x) = C \int_{-\infty}^{\infty} e^{ix\xi - it(\xi^2 - \nu\xi^4)} \mathcal{F}_x v(\xi) d\xi.$$

For $1 \le p,q \le \infty$, Let $L_t^p L_x^q = L_t^p(\mathbb{R}; L_x^q(\mathbb{R}))$. We define the Sobolev space $H_t^b H_x^s = H_t^b(\mathbb{R}; H_x^s(\mathbb{R})) = \{f \in \mathcal{S}'(\mathbb{R}^2); \langle D_t \rangle^b f \in H_x^s(\mathbb{R})\}$ for $b, s \in \mathbb{R}$ and denote its norm by $\|\cdot\|_{H_t^b H_x^s}$. Various constants are simply denoted by C or $C_i (i = 1, 2, \dots)$.

2. Main result

Here, we state the main result. Let $\psi(t)$ be a smooth cut-off function to the interval $[-1,1]$; i.e, $\psi \in C_0^\infty(\mathbb{R})$ and

$$\psi(t) = \begin{cases} 1 & \text{for } |t| \le 1, \\ 0 & \text{for } |t| \ge 2. \end{cases}$$

For $\delta > 0$, we set $\psi_\delta(t) = \psi(t/\delta)$.

We define the solution of (1.1) as the solution of following integral equation:

$$u(t) = W_\nu(t)u_0 - i \int_0^t W_\nu(t-t')F(t')dt', \tag{2.1}$$

for $t \in [-T, T]$.

Our well-posedness theorem is as follows:

Theorem 2.1. *Let $\nu < 0$ and $\mu - \nu/2 = 0$, i.e., $\lambda_5 = 0$ in* (1.1)*, and let $s \geq 1/2$ and $b \in (1/2, 5/8)$. Then for $u_0 \in H^s(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u(t)$ of the initial-value problem* (1.1) *in $t \in [-T, T]$ satisfying*

$$u \in C([-T,T]; H^s(\mathbb{R})), \quad \psi_T W_\nu(-t)u \in H_t^b(\mathbb{R}; H_x^s(\mathbb{R})),$$
$$\psi_T W_\nu(-t)F \in H_t^{b-1}(\mathbb{R}; H_x^s(\mathbb{R})).$$

Moreover, for a given $T' \in (0,T)$, two maps, $u_0 \mapsto u$ from $H^s(\mathbb{R})$ to $C([-T',T'];\ H^s(\mathbb{R}))$ and $u_0 \mapsto \psi_{T'} W_\nu(-t)u$ from $H^s(\mathbb{R})$ to $H_t^b(\mathbb{R}; H_x^s(\mathbb{R}))$ are Lipschitz continuous, respectively.

Remark 2.2. Because of a technical reason the cases $\nu \geq 0$ and $\mu - \nu/2 \neq 0$ are not covered by our theorem (see Remark 3.3 and Remark 4.8 for the case $\nu < 0$, and Remark 4.9 for $\mu - \nu/2 = 0$ later). If we allow higher regularity for the initial data, then the general theory of semilinear evolution equations (see, e.g., [5] and [12]) shows the well-posedness.

Remark 2.3. When $\mu + \nu/2 = 0$, it is known that (1.8) is the completely integrable equation (see Fukumoto [8]) and has infinitely many conserved quantities (see Langer-Perline [22]); namely,

$$\Phi_1(u) = \frac{1}{2}\int_{-\infty}^{\infty} |u|^2 dx, \qquad \Phi_2(u) = -\frac{i}{2}\int_{-\infty}^{\infty} (\partial_x u)\overline{u} dx,$$
$$\Phi_3(u) = -\frac{1}{2}\int_{-\infty}^{\infty} (\partial_x^2 u)\overline{u} dx - \frac{1}{8}\int_{-\infty}^{\infty} |u|^4 dx, \ldots.$$

In Theorem 2.1, we assume $\mu - \nu/2 = 0$ in (1.1) and (1.8). Therefore, we are not able to extend the local well-posedness to a global result via the above conservation laws.

As stated in the introduction, to prove Theorem 2.1 we use the method of Fourier restriction norm introduced in [3]. Now, we define the Fourier restriction space $X_{s,b}^\nu$ with $b, s, \nu \in \mathbb{R}$ concerning the equation (1.1) as follows:

$$X_{s,b}^\nu = \{u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{X_{s,b}^\nu} < \infty\},$$
$$\|u\|_{X_{s,b}^\nu} = \|\langle \tau + \xi^2 - \nu\xi^4\rangle^b \langle \xi \rangle^s \widehat{\widehat{u}}(\xi,\tau)\|_{L_\xi^2 L_\tau^2} = \|W_\nu(-t)u(t)\|_{H_t^b H_x^s}.$$

The remainder of this paper is organized as follows. In Sections 3 and 4, we prove the preliminary estimates to Theorem 2.1. In particular, in Section 3 we prove the linear estimates, and in Section 4 we prove the nonlinear estimates. In Section 5 we reduce Theorem 2.1. In Section 6, we give the remark concerning the well-posedness for the initial value problem (1.1).

3. Linear estimates

In this section we prove the linear estimates needed for the proof of Theorem 2.1. In what follows, we consider only the case $\nu = -1$. The general case $\nu < 0$ can be obtained in a similar manner. For simplicity, we denote $W_{-1}(t) = W(t) and X_{s,b}^{-1} = X_{s,b}$.

First, we state some known linear estimates that are used in the proof.

Proposition 3.1. (Kenig-Ponce-Vega [17, 18]) *Let $s \in \mathbb{R},\ 1/2 < b < b' < 1$, and $\delta \in (0,1)$. Then we have*

$$\|\psi_\delta W(t)u_0\|_{X_{s,b}} \le C\delta^{(1-2b)/2}\|u_0\|_{H^s}, \tag{3.1}$$

$$\|\psi_\delta F\|_{X_{s,b-1}} \le C\delta^{b'-b}\|F\|_{X_{s,b'-1}}, \tag{3.2}$$

$$\left\|\psi_\delta \int_0^t W(t-t')F(t')dt'\right\|_{X_{s,b}} \le C\delta^{(1-2b)/2}\|F\|_{X_{s,b-1}}, \tag{3.3}$$

$$\left\|\psi_\delta \int_0^t W(t-t')F(t')dt'\right\|_{L_t^\infty((0,T);H_x^s)} \le C\delta^{(1-2b)/2}\|F\|_{X_{s,b-1}}. \tag{3.4}$$

Next, we give the estimates for the group.

Proposition 3.2. *Let $u_0 \in L_x^2(\mathbb{R})$. Then*

$$\|D_x^{\zeta\eta\theta/2}W(t)u_0\|_{L_t^qL_x^p} \le C\|u_0\|_{L_x^2}, \tag{3.5}$$

$$\|W(t)u_0\|_{L_x^4L_t^\infty} \le C\|D_x^{1/4}u_0\|_{L_x^2}, \tag{3.6}$$

$$\|W(t)u_0\|_{L_x^\infty L_t^2} \le C\|D_x^{-3/2}u_0\|_{L_x^2}, \tag{3.7}$$

where $(\zeta,\eta,\theta) \in [0,1]\times[0,1]\times[0,1]$ and $(q,p) = (8/\zeta(\eta+1), 2/(1-\zeta))$.

Remark 3.3. The inequalities (3.5), (3.6), and (3.7) are called the Strichartz [28] estimate, Kenig-Ruiz [20] estimate and Kato [13] estimate, respectively. In Theorem 2.1, we stated the results in the case $\nu < 0$ for the initial-value problem (1.1). In this case the essential inequalities are as follows:

$$1 \le C|1-6\nu\xi^2|, \tag{3.8}$$

$$|\xi|^2 \le C|1-6\nu\xi^2|, \tag{3.9}$$

$$|1-\nu\xi^2| \le C|1-6\nu\xi^2|, \tag{3.10}$$

$$|1-2\nu\xi^2| \le C|1-6\nu\xi^2|, \tag{3.11}$$

$$|\xi|^2 \le C|1-2\nu\xi^2|, \tag{3.12}$$

for any $\xi \in \mathbb{R}$. The inequalities (3.8)–(3.12) are required in the proof of the estimates for the group, but in the case $\nu > 0$ inequalities (3.8)–(3.12) fail. However, Takaoka [30] proved the local well-posedness of (1.6) by using the

Fourier restriction norm. In that proof, regarding the third-order derivative term as the main linear term, he rewrote (1.6) as follows:

$$i\partial_t u + i\partial_x^3 u + \partial_x^2 u = i\partial_t u + i\partial_x^3 u + \partial_x^2 Qu + \partial_x^2 (I-Q)u,$$

where $\widehat{\widehat{Qu}}(\xi,\tau) = q(\xi)\widehat{\widehat{u}}(\xi,\tau)$ and $q \in C(\mathbb{R})$ is given by $q(\xi) = q(|\xi|)$,

$$q(\xi) = \begin{cases} 1 - \frac{(\xi-5)^4}{5^4}, & 0 \le \xi \le 5, \\ 1, & 5 < \xi. \end{cases}$$

Proof of Proposition 3.2. First, we prove the Strichartz estimate (3.5). Let $\zeta \in [0,1]$ and $(q,p) = (4/\zeta, 2/(1-\zeta))$; then by the result of Kenig-Ponce-Vega [15, Theorem 2.1], we have

$$\Big\| \int_{-\infty}^{\infty} e^{ix\xi - it(\xi^2+\xi^4)} |2 + 12\xi^2|^{\zeta/4} \mathcal{F}_x u_0(\xi) d\xi \Big\|_{L_t^q L_x^p} \le C \|u_0\|_{L_x^2}. \tag{3.13}$$

From the inequalities (3.8) and (3.9),

$$\|W(t)u_0\|_{L_t^q L_x^p} \le C\|u_0\|_{L_x^2}, \tag{3.14}$$

$$\|D_x^{\zeta/2} W(t)u_0\|_{L_t^q L_x^p} \le C\|u_0\|_{L_x^2}. \tag{3.15}$$

Interpolating between (3.14) and (3.15), we obtain for any $\theta \in [0,1]$,

$$\|D_x^{\zeta\theta/2} W(t)u_0\|_{L_t^q L_x^p} \le C\|u_0\|_{L_x^2}. \tag{3.16}$$

By the Sobolev embedding $W^{\zeta/8,4/\zeta}(\mathbb{R}) \hookrightarrow L^{8/\zeta}(\mathbb{R})$, (3.10), and (3.13),

$$\begin{aligned} \|W(t)u_0\|_{L_t^{8/\zeta} L_x^p} &\le C\|D_t^{\zeta/8} \mathcal{F}_x u_0\|_{L_t^{4/\zeta} L_x^p} \qquad (3.17) \\ &= C\Big\| \int_{-\infty}^{\infty} e^{ix\xi - it(\xi^2+\xi^4)} |\xi|^{\zeta/4} |1+\xi^2|^{\zeta/8} \mathcal{F}_x u_0(\xi) d\xi \Big\|_{L_t^{4/\zeta} L_t^p} \le C\|u_0\|_{L_x^2}. \end{aligned}$$

Again by interpolation between (3.16) and (3.17), we have (3.5) for any $\eta \in [0,1]$.

Next, we prove the Kenig-Ruiz estimate (3.6). From [15, Theorem 2.5] and inequality (3.11),

$$\begin{aligned} \|W(t)u_0\|_{L_x^4 L_t^\infty} &\le C\Big(\int_{-\infty}^{\infty} |\mathcal{F}_x u_0(\xi)|^2 \Big| \frac{2+4\xi^2}{2+12\xi^2} \Big|^{1/2} |\xi|^{1/2} d\xi \Big)^{1/2} \\ &\le C\Big(\int_{-\infty}^{\infty} |\mathcal{F}_x u_0(\xi)|^2 |\xi|^{1/2} d\xi \Big)^{1/2}. \end{aligned}$$

Therefore, we have (3.6).

Finally, we prove the Kato estimate (3.7). From [15, Theorem 4.1] and inequality (3.12),

$$\|W(t)u_0\|_{L_x^\infty L_t^2} \le C\Big(\int_{-\infty}^{\infty} \frac{|\mathcal{F}_x u_0(\xi)|^2}{|2\xi+4\xi^3|}d\xi\Big)^{1/2} \le C\Big(\int_{-\infty}^{\infty} \frac{|\mathcal{F}_x u_0(\xi)|^2}{|\xi|^3}d\xi\Big)^{1/2}.$$

Hence we have (3.7). □

For $b \in \mathbb{R}$, define F_b, F_b^*, and G_b via the Fourier transform

$$\widehat{\widehat{F_b}}(\xi,\tau) = \frac{f(\xi,\tau)}{\langle \tau+\xi^2+\xi^4\rangle^b}, \qquad \widehat{\widehat{F_b^*}}(\xi,\tau) = \frac{\overline{f}(\xi,\tau)}{\langle \tau-\xi^2-\xi^4\rangle^b},$$
$$\widehat{\widehat{G_b}}(\xi,\tau) = \frac{g(\xi,\tau)}{\langle \tau+\xi^2+\xi^4\rangle^b}. \tag{3.18}$$

By Proposition 3.2., we have following lemma.

Lemma 3.4. *Let $b > 1/2$ and $f \in L_\xi^2 L_\tau^2$. Then*

$$\|F_b\|_{L_x^6 L_t^6} \le C\|f\|_{L_\xi^2 L_\tau^2}, \tag{3.19}$$
$$\|F_b\|_{L_x^{10} L_t^{10}} \le C\|f\|_{L_\xi^2 L_\tau^2}, \tag{3.20}$$
$$\|D_x^{-1/4}F_b\|_{L_x^4 L_t^\infty} \le C\|f\|_{L_\xi^2 L_\tau^2}, \tag{3.21}$$
$$\|D_x^{3/2}F_b\|_{L_x^\infty L_t^2} \le C\|f\|_{L_\xi^2 L_\tau^2}. \tag{3.22}$$

Similar estimates hold for F_b^* and G_b replacing F_b and $\overline{f}$, and g replacing f, respectively.

Proof of Lemma 3.4. We prove only (3.19). The other estimates are shown in the same manner.

Using the change of variables $\omega = \tau + \xi^2 + \xi^4$ ($\tau \to \omega$),

$$\begin{aligned}
\|F_b\|_{L_x^6 L_t^6} &= C\Big\| \iint_{\mathbb{R}^2} e^{ix\xi+it\tau}\frac{f(\xi,\tau)}{\langle \tau+\xi^2+\xi^4\rangle^b}d\xi\, d\tau\Big\|_{L_x^6 L_t^6} \\
&= C\Big\| \int_{-\infty}^{\infty}\Big\{e^{it\omega}\langle\omega\rangle^{-b}\Big(\int_{-\infty}^{\infty} e^{ix\xi-it(\xi^2+\xi^4)}f(\xi,\omega-\xi^2-\xi^4)d\xi\Big)\Big\}d\omega\Big\|_{L_x^6 L_t^6} \\
&= C\Big\|\int_{-\infty}^{\infty} e^{it\omega}\langle\omega\rangle^{-b}\{W(t)g_\omega\}d\omega\Big\|_{L_x^6 L_t^6},
\end{aligned}$$

where $\mathcal{F}_x g_\omega(\xi) = f(\xi,\omega-\xi^2-\xi^4)$.

By Minkowski's inequality, Proposition 3.2, (3.5), and Plancherel's theorem,

$$\|F_b\|_{L_x^6 L_t^6} \le C\int_{-\infty}^{\infty} \|e^{it\omega}\langle\omega\rangle^{-b}W(t)g_\omega\|_{L_x^6 L_t^6}d\omega$$

$$\leq C\int_{-\infty}^{\infty}\langle\omega\rangle^{-b}\|W(t)g_\omega\|_{L_x^6L_t^6}d\omega \leq C\int_{-\infty}^{\infty}\langle\omega\rangle^{-b}\|g_\omega\|_{L_x^2}d\omega$$
$$\leq C\|g_\omega\|_{L_x^2L_\omega^2} = C\|f(\xi,\omega-\xi^2-\xi^4)\|_{L_\xi^2L_\omega^2} = C\|f\|_{L_\xi^2L_\tau^2}.$$

4. Nonlinear estimates

In this section, by using the linear estimates in Lemma 3.4, we derive the crucial nonlinear estimates (see Proposition 4.7 below). First, we give some preliminary lemmas to the crucial nonlinear estimates. We set

$$\begin{aligned}&\sigma=\tau+\xi^2+\xi^4, \qquad \sigma_1=\tau-\tau_1+(\xi-\xi_1)^2+(\xi-\xi_1)^4,\\ &\sigma_2=\tau_1-\tau_2+(\xi_1-\xi_2)^2+(\xi_1-\xi_2)^4, \qquad \overline{\sigma}_3=\tau_2-\xi_2^2-\xi_2^4,\end{aligned} \tag{4.1}$$

and

$$\begin{cases}\rho=\tau+\xi^2+\xi^4, \qquad \rho_1=\tau-\tau_1+(\xi-\xi_1)^2+(\xi-\xi_1)^4,\\ \overline{\rho}_2=\tau_1-\tau_2-(\xi_1-\xi_2)^2-(\xi_1-\xi_2)^4,\\ \rho_3=\tau_2-\tau_3+(\xi_2-\xi_3)^2+(\xi_2-\xi_3)^4,\\ \overline{\rho}_4=\tau_3-\tau_4-(\xi_3-\xi_4)^2-(\xi_3-\xi_4)^4, \qquad \rho_5=\tau_4+\xi_4^2+\xi_4^4.\end{cases} \tag{4.2}$$

For $b\in\mathbb{R}$, we put

$$\begin{aligned}
h_1(\xi,\xi_1,\xi_2)&\equiv\frac{\max\{\langle\xi-\xi_1\rangle,\langle\xi_1-\xi_2\rangle,\langle\xi_2\rangle\}^{7/4}}{\langle\xi-\xi_1\rangle^{1/4}\langle\xi_1-\xi_2\rangle^{1/4}\langle\xi_2\rangle^{1/4}}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b},\\
h_2(\xi,\xi_1,\xi_2)&\equiv\frac{\max\{\langle\xi\rangle,\langle\xi_1-\xi_2\rangle,\langle\xi_2\rangle\}^{7/4}}{\langle\xi\rangle^{1/4}\langle\xi_1-\xi_2\rangle^{1/4}\langle\xi_2\rangle^{1/4}}\times\frac{1}{\langle\sigma\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b},\\
h_3(\xi,\xi_1,\xi_2)&\equiv\frac{\max\{\langle\xi\rangle,\langle\xi-\xi_1\rangle,\langle\xi_2\rangle\}^{7/4}}{\langle\xi\rangle^{1/4}\langle\xi-\xi_1\rangle^{1/4}\langle\xi_2\rangle^{1/4}}\times\frac{1}{\langle\sigma\rangle^b\langle\sigma_1\rangle^b\langle\overline{\sigma}_3\rangle^b},\\
h_4(\xi,\xi_1,\xi_2)&\equiv\frac{\max\{\langle\xi\rangle,\langle\xi-\xi_1\rangle,\langle\xi_1-\xi_2\rangle\}^{7/4}}{\langle\xi\rangle^{1/4}\langle\xi-\xi_1\rangle^{1/4}\langle\xi_1-\xi_2\rangle^{1/4}}\times\frac{1}{\langle\sigma\rangle^b\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b},\\
h_5(\xi,\xi_1,\xi_2,\xi_3,\xi_4)&\equiv\frac{1}{\langle\rho_1\rangle^b\langle\overline{\rho}_2\rangle^b\langle\rho_3\rangle^b\langle\overline{\rho}_4\rangle^b\langle\rho_5\rangle^b}.
\end{aligned} \tag{4.3}$$

Lemma 4.1. Let $g,f\in L_\xi^2L_\tau^2$. Let us define

$$I_j\equiv\begin{cases}\int_{\mathbb{R}^6}h_j(\xi,\xi_1,\xi_2)g(\xi,\tau)f(\xi-\xi_1,\tau-\tau_1)f(\xi_1-\xi_2,\tau_1-\tau_2)\overline{f}(\xi_2,\tau_2)\\ \qquad\times d\xi d\tau d\xi_1 d\tau_1 d\xi_2 d\tau_2, \qquad\qquad \text{for } j=1,2,\dots,4,\\ \int_{\mathbb{R}^{10}}h_j(\xi,\xi_1,\xi_2,\xi_3,\xi_4)g(\xi,\tau)f(\xi-\xi_1,\tau-\tau_1)\overline{f}(\xi_1-\xi_2,\tau_1-\tau_2)\\ \qquad\times f(\xi_2-\xi_3,\tau_2-\tau_3)\overline{f}(\xi_3-\xi_4,\tau_3-\tau_4)f(\xi_4,\tau_4)d\xi\\ \qquad\qquad\times d\tau d\xi_1 d\tau_1\cdots d\xi_4 d\tau_4, \qquad\qquad \text{for } j=5.\end{cases}$$

Then for $b > 1/2$, we have

$$I_j \le C \begin{cases} \|g\|_{L^2_\xi L^2_\tau} \|f\|^2_{L^2_\xi L^2_\tau} \|\overline{f}\|_{L^2_\xi L^2_\tau}, & j = 1, 2, \dots, 4, \\ \|g\|_{L^2_\xi L^2_\tau} \|f\|^3_{L^2_\xi L^2_\tau} \|\overline{f}\|^2_{L^2_\xi L^2_\tau}, & j = 5. \end{cases} \tag{4.4}$$

Proof of Lemma 4.1. For the integrals I_1–I_4, we may assume $\max\{|\xi - \xi_1|, |\xi_1 - \xi_2|, |\xi_2|\} = |\xi_2|$ by symmetry. First, we estimate the integral I_1. We split the integral domain into two sets: $|\xi_2| \ge 1$ and $|\xi_2| \le 1$.

Case $|\xi_2| \ge 1$. By regarding the definition (3.18), we estimate by Plancherel's identity, Lemma 3.4, (3.21)–(3.22),

$$\begin{aligned} I_1 \le & C \int_{\mathbb{R}^6} g(\xi,\tau) \frac{f(\xi-\xi_1, \tau-\tau_1)}{|\xi-\xi_1|^{1/4} \langle\sigma_1\rangle^b} \frac{f(\xi_1-\xi_2, \tau_1-\tau_2)}{|\xi_1-\xi_2|^{1/4}\langle\sigma_2\rangle^b} \frac{|\xi_2|^{3/2}\overline{f}(\xi_2,\tau_2)}{\langle\overline{\sigma}_3\rangle^b} \\ & \quad \times d\xi\, d\tau\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2 \\ \le & C \int_{\mathbb{R}^2} |G_0(x,t)| |D_x^{-1/4} F_b(x,t)|^2 |D_x^{3/2} F_b^*(x,t)| dx\, dt \\ \le & C \|G_0\|_{L^2_x L^2_t} \|D_x^{-1/4} F_b\|^2_{L^4_x L^\infty_t} \|D_x^{3/2} F_b^*\|_{L^\infty_x L^2_t} \le C \|g\|_{L^2_\xi L^2_\tau} \|f\|^2_{L^2_\xi L^2_\tau} \|\overline{f}\|_{L^2_\xi L^2_\tau}. \end{aligned}$$

Case $|\xi_2| \le 1$. In similar way, by using Lemma 3.4, (3.19),

$$\begin{aligned} I_1 \le & C \int_{\mathbb{R}^6} g(\xi,\tau) \frac{f(\xi-\xi_1, \tau-\tau_1)}{\langle\sigma_1\rangle^b} \frac{f(\xi_1-\xi_2, \tau_1-\tau_2)}{\langle\sigma_2\rangle^b} \frac{\overline{f}(\xi_2,\tau_2)}{\langle\overline{\sigma}_3\rangle^b} \\ & \quad \times d\xi\, d\tau\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2 \\ \le & C \int_{\mathbb{R}^2} |G_0(x,t)| |F_b(x,t)|^2 |F_b^*(x,t)| dx\, dt \\ \le & C \|G_0\|_{L^2_x L^2_t} \|F_b\|^2_{L^6_x L^6_t} \|F_b^*\|_{L^6_x L^6_t} \le C \|g\|_{L^2_\xi L^2_\tau} \|f\|^2_{L^2_\xi L^2_\tau} \|\overline{f}\|_{L^2_\xi L^2_\tau}. \end{aligned}$$

For the case I_2, change variables $\xi - \xi_1$ and σ_1 into ξ and σ, respectively, for I_3 and I_4; the proof is similar to changing variables.

Finally, we estimate the integral I_5. By regarding the definition (3.18), we estimate by the Plancherel's identity, Lemma 3.4, (3.20),

$$\begin{aligned} I_5 \le & C \int_{\mathbb{R}^{10}} g(\xi,\tau) \frac{f(\xi-\xi_1, \tau-\tau_1)}{\langle\rho_1\rangle^b} \frac{\overline{f}(\xi_1-\xi_2, \tau_1-\tau_2)}{\langle\overline{\rho}_2\rangle^b} \frac{f(\xi_2-\xi_3, \tau_2-\tau_3)}{\langle\rho_3\rangle^b} \\ & \quad \times \frac{\overline{f}(\xi_3-\xi_4, \tau_3-\tau_4)}{\langle\overline{\rho}_4\rangle^b} \frac{f(\xi_4,\tau_4)}{\langle\rho_5\rangle^b} d\xi\, d\tau\, d\xi_1\, d\tau_1 \cdots d\xi_4\, d\tau_4 \\ \le & C \int_{\mathbb{R}^2} |G_0(x,t)| |F_b(x,t)|^3 |F_b^*(x,t)|^2 dx\, dt \\ \le & C \|G_0\|_{L^2_x L^2_t} \|F_b\|^3_{L^{10}_x L^{10}_t} \|F_b^*\|^2_{L^{10}_x L^{10}_t} \le C \|g\|_{L^2_\xi L^2_\tau} \|f\|^3_{L^2_\xi L^2_\tau} \|\overline{f}\|^2_{L^2_\xi L^2_\tau}. \qquad \square \end{aligned}$$

For the next step, we separate $\mathbb{R}^3$ into the following regions.

$$\begin{aligned} A &\equiv \Big\{(\xi,\xi_1,\xi_2)\in\mathbb{R}^3 : |\xi-\xi_1+\xi_2|\le 1\Big\},\\ B &\equiv \Big\{(\xi,\xi_1,\xi_2)\in\mathbb{R}^3 : |\xi_1|\le 1\Big\},\\ C &\equiv \Big\{(\xi,\xi_1,\xi_2)\in\mathbb{R}^3 : |\xi_1|\ge 1 \text{ and } |\xi-\xi_1+\xi_2|\ge 1\Big\}. \end{aligned} \tag{4.5}$$

We further split the region C into the following regions.

$$\begin{aligned} R_1 &= \Big\{(\xi,\xi_1,\xi_2)\in C : \max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\overline{\sigma}_3|\}=|\sigma|\Big\},\\ R_2 &= \Big\{(\xi,\xi_1,\xi_2)\in C : \max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\overline{\sigma}_3|\}=|\sigma_1|\Big\},\\ R_3 &= \Big\{(\xi,\xi_1,\xi_2)\in C : \max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\overline{\sigma}_3|\}=|\sigma_2|\Big\},\\ R_4 &= \Big\{(\xi,\xi_1,\xi_2)\in C : \max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\overline{\sigma}_3|\}=|\overline{\sigma}_3|\Big\}. \end{aligned} \tag{4.6}$$

Lemma 4.2. *Let $a\in(-1/2,-1/4]$ and $b>1/2$. Then*

$$\begin{aligned} H_1(\xi,\xi_1,\xi_2) &\equiv \frac{\langle\xi-\xi_1\rangle^{1/2}\langle\xi_1-\xi_2\rangle^{1/2}\langle\xi_2\rangle^{1/2}}{\langle\xi\rangle^{1/2}}\frac{1}{\langle\sigma\rangle^{|a|}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}\langle\overline{\sigma}_3\rangle^{b}}\\ &\le Ch_1(\xi,\xi_1,\xi_2)\chi_{A\cup B}(\xi,\xi_1,\xi_2)+C\sum_{j=1}^{4}h_j(\xi,\xi_1,\xi_2)\chi_{R_j}(\xi,\xi_1,\xi_2), \end{aligned} \tag{4.7}$$

where σ, σ_1, σ_2, and $\overline{\sigma}_3$ are given by (4.1), *h_j are given by* (4.3) *and A, B, and R_j are defined by* (4.5)–(4.6).

Proof of Lemma 4.2. We note that

$$\begin{aligned} &|\sigma|+|\sigma_1|+|\sigma_2|+|\overline{\sigma}_3|\ge|\sigma-\sigma_1-\sigma_2-\overline{\sigma}_3|\\ &=2|\xi_1||\xi-\xi_1+\xi_2||2\xi^2+\xi_1^2+2\xi_2^2-\xi\xi_1-\xi_1\xi_2-2\xi_2\xi+1|\\ &=2|\xi_1||\xi-\xi_1+\xi_2|\\ &\times\left\{\frac{1}{2}\xi^2+\frac{1}{2}(\xi-\xi_1)^2+\frac{1}{2}(\xi_1-\xi_2)^2+\frac{1}{2}\xi_2^2+(\xi_2-\xi)^2+1\right\}. \end{aligned} \tag{4.8}$$

Therefore, the left-hand side of (4.8) is bounded below as follows:

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\overline{\sigma}_3|\}\ge\frac{1}{4}|\xi_1||\xi-\xi_1+\xi_2|\max\{|\xi|,|\xi-\xi_1|,|\xi_1-\xi_2|,|\xi_2|\}^2 \tag{4.9}$$

Estimate on A. Since $|\xi-\xi_1+\xi_2|\le 1$, we note the following inequalities:

$$|\xi_1-\xi_2|^{1/2}\le|\xi-\xi_1+\xi_2|^{1/2}+|\xi|^{1/2}\le\langle\xi\rangle^{1/2}. \tag{4.10}$$

From (4.10),

$$\begin{aligned} H_1(\xi,\xi_1,\xi_2) &\le C\langle \xi-\xi_1\rangle^{1/2}\langle\xi_2\rangle^{1/2}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\\ &= C\frac{\langle\xi-\xi_1\rangle^{3/4}}{\langle\xi-\xi_1\rangle^{1/4}}\frac{\langle\xi_1-\xi_2\rangle^{1/4}}{\langle\xi_1-\xi_2\rangle^{1/4}}\frac{\langle\xi_2\rangle^{3/4}}{\langle\xi_2\rangle^{1/4}}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\le Ch_1(\xi,\xi_1,\xi_2). \end{aligned}$$

Estimate on B. Since $|\xi_1|\le 1$, we note the following inequalities:

$$|\xi-\xi_1|^{1/2}\le|\xi_1|^{1/2}+|\xi|^{1/2}\le\langle\xi\rangle^{1/2}. \tag{4.11}$$

Similarly to the estimate on B, we have from (4.11) that

$$\begin{aligned} H_1(\xi,\xi_1,\xi_2) &\le C\langle \xi_1-\xi_2\rangle^{1/2}\langle\xi_2\rangle^{1/2}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\\ &= C\frac{\langle\xi-\xi_1\rangle^{1/4}}{\langle\xi-\xi_1\rangle^{1/4}}\frac{\langle\xi_1-\xi_2\rangle^{3/4}}{\langle\xi_1-\xi_2\rangle^{1/4}}\frac{\langle\xi_2\rangle^{3/4}}{\langle\xi_2\rangle^{1/4}}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\le Ch_1(\xi,\xi_1,\xi_2). \end{aligned}$$

Estimate on C. We further split above four regions R_1–R_4.
Estimate on $\mathbf{R}_1$. I.e., $\max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\overline{\sigma}_3|\}=|\sigma|$. From (4.9) and the definition of R_1, $|\sigma|\ge\max\{|\xi-\xi_1|,|\xi_1-\xi_2|,|\xi_2|\}/4$. Combining this inequality with $\max\{|\xi-\xi_1|,|\xi_1-\xi_2|,|\xi_2|\}\ge|\xi|/3$,

$$\begin{aligned} \frac{1}{\langle\sigma\rangle^{|a|}} &\le \frac{C}{\max\{\langle\xi-\xi_1\rangle,\langle\xi_1-\xi_2\rangle,\langle\xi_2\rangle\}^{2|a|}}\\ &\le\frac{C}{\langle\xi-\xi_1\rangle^{2|a|/3}\langle\xi_1-\xi_2\rangle^{2|a|/3}\langle\xi_2\rangle^{2|a|/3}}. \end{aligned} \tag{4.12}$$

Since we have the assumption $1/2+2a/3\le1/3$ and (4.12) it follows that

$$\begin{aligned} H_1(\xi,\xi_1,\xi_2) &\le C\langle\xi-\xi_1\rangle^{\frac12+2a/3}\langle\xi_1-\xi_2\rangle^{\frac12+2a/3}\langle\xi_2\rangle^{\frac12+2a/3}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\\ &\le C\langle\xi-\xi_1\rangle^{1/3}\langle\xi_1-\xi_2\rangle^{1/3}\langle\xi_2\rangle^{1/3}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\\ &= C\frac{\langle\xi-\xi_1\rangle^{7/12}}{\langle\xi-\xi_1\rangle^{1/4}}\frac{\langle\xi_1-\xi_2\rangle^{7/12}}{\langle\xi_1-\xi_2\rangle^{1/4}}\frac{\langle\xi_2\rangle^{7/12}}{\langle\xi_2\rangle^{1/4}}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\le Ch_1(\xi,\xi_1,\xi_2). \end{aligned}$$

Estimate on $\mathbf{R}_2$. I.e., $\max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\overline{\sigma}_3|\}=|\sigma_1|$. We note $\max\{|\xi|,|\xi_1-\xi_2|,|\xi_2|\}\ge|\xi-\xi_1|/3$. From (4.9),

$$\frac{1}{\langle\sigma_1\rangle^{|a|}}\le\frac{C}{\max\{\langle\xi\rangle,\langle\xi_1-\xi_2\rangle,\langle\xi_2\rangle\}^{2|a|}}\le\frac{C}{\langle\xi_1-\xi_2\rangle^{|a|}\langle\xi_2\rangle^{|a|}}.$$

Therefore, combining the above inequality with (4.9), and $\frac{9}{4} + 2a \leq \frac{7}{4}$,

$$\begin{aligned}
&H_1(\xi, \xi_1, \xi_2) \leq C\langle \xi - \xi_1\rangle^{1/2}\langle \xi_1 - \xi_2\rangle^{1/2}\langle \xi_2\rangle^{1/2} \times \frac{1}{\langle\sigma\rangle^b\langle\sigma_1\rangle^{|a|}\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b} \\
&\leq C\langle \xi - \xi_1\rangle^{1/2}\langle \xi_1 - \xi_2\rangle^{1/2+a}\langle \xi_2\rangle^{1/2+a} \times \frac{1}{< \sigma\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b} \\
&= C\frac{\langle\xi\rangle^{1/4}}{\langle\xi\rangle^{1/4}}\langle \xi - \xi_1\rangle^{1/2}\frac{\langle \xi_1 - \xi_2\rangle^{3/4+a}}{\langle \xi_1 - \xi_2\rangle^{1/4}}\frac{\langle\xi_2\rangle^{3/4+a}}{\langle\xi_2\rangle^{1/4}} \times \frac{1}{\langle\sigma\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b} \\
&\leq Ch_2(\xi, \xi_1, \xi_2).
\end{aligned}$$

Estimate on $\mathbf{R}_3$ and $\mathbf{R}_4$. By using a similar way as in the estimate on R_2 and the estimates

$$\frac{1}{\langle\sigma_2\rangle^{|a|}} \leq \frac{1}{\langle \xi - \xi_1\rangle^{|a|}\langle\xi_2\rangle^{|a|}} \text{ and } \frac{1}{\langle\sigma_3\rangle^{|a|}} \leq \frac{1}{\langle \xi - \xi_1\rangle^{|a|}\langle\xi_1 - \xi_2\rangle^{|a|}},$$

on R_2 and R_3, respectively, we are able to make an estimate in those cases. □

Lemma 4.3. *Let $a \leq 0$ and $b > 1/2$. Then*

$$H_2(\xi, \xi_1, \xi_2, \xi_3, \xi_4) \equiv \frac{1}{\langle\rho\rangle^{|a|}\langle\rho_1\rangle^b\langle\overline{\rho}_2\rangle^b\langle\rho_3\rangle^b\langle\overline{\rho}_4\rangle^b\langle\rho_5\rangle^b} \leq h_5(\xi, \xi_1, \xi_2, \xi_3, \xi_4), \tag{4.13}$$

where $\rho, \rho_1, \overline{\rho}_2, \ldots, \rho_5$ are given by (4.2) *and h_5 is given by* (4.3).

Lemma 4.3 follows directly from the definition of h_5 (4.3).

Lemma 4.4. *Let $a \in (-1/2, -1/4]$ and $b > 1/2$. Then*

$$\begin{aligned}
&H_3(\xi, \xi_1, \xi_2) \equiv \langle\xi\rangle^{1/2}\frac{|\xi - \xi_1|}{\langle\xi - \xi_1\rangle^{1/2}}\frac{|\xi_1 - \xi_2|}{\langle\xi_1 - \xi_2\rangle^{1/2}}\frac{1}{\langle\xi_2\rangle^{1/2}} \times \frac{1}{\langle\sigma\rangle^{|a|}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b} \\
&\leq Ch_1(\xi, \xi_1, \xi_2)\chi_{A\cup B}(\xi, \xi_1, \xi_2) + C\sum_{j=1}^{4} h_j(\xi, \xi_1, \xi_2)\chi_{R_j}(\xi, \xi_1, \xi_2),
\end{aligned} \tag{4.14}$$

where σ, σ_1, σ_2, and $\overline{\sigma}_3$ are given by (4.1), *h_j are given by* (4.3), *and A, B, and R_j are defined by* (4.5)–(4.6).

Proof of Lemma 4.4. From the inequality (4.8),

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|, |\overline{\sigma}_3|\} \geq \frac{1}{2}|\xi_1||\xi - \xi_1 + \xi_2||\xi - \xi_1||\xi_1 - \xi_2|. \tag{4.15}$$

Estimate on A. Since $|\xi - \xi_1 + \xi_2| \leq 1$ in this case, we note the following inequalities:

$$|\xi - \xi_1|^{1/4} \leq |\xi - \xi_1 + \xi_2|^{1/4} + |\xi_2|^{1/4} \leq \langle\xi_2\rangle^{1/4}. \tag{4.16}$$

Hence we have from (4.16) and $\max\{|\xi-\xi_1|,|\xi_1-\xi_2|,|\xi_2|\}\geq|\xi|/3$ that

$$H_3(\xi,\xi_1,\xi_2)\leq C\langle\xi\rangle^{1/2}\frac{|\xi-\xi_1|^{3/4}}{\langle\xi-\xi_1\rangle^{1/2}}\frac{|\xi_1-\xi_2|}{\langle\xi_1-\xi_2\rangle^{1/2}}\frac{\langle\xi_2\rangle^{1/4}}{\langle\xi_2\rangle^{1/2}}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}$$

$$\leq C\langle\xi\rangle^{1/2}\frac{\langle\xi-\xi_1\rangle^{1/2}}{\langle\xi-\xi_1\rangle^{1/4}}\frac{\langle\xi_1-\xi_2\rangle^{3/4}}{\langle\xi_1-\xi_2\rangle^{1/4}}\frac{1}{\langle\xi_2\rangle^{1/4}}\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\leq Ch_1(\xi,\xi_1,\xi_2).$$

Estimate on B. Since $|\xi_1|\leq 1$ in this case, we note the following inequalities:

$$|\xi_1-\xi_2|^{1/4}\leq|\xi_1|^{1/4}+|\xi_2|^{1/4}\leq\langle\xi_2\rangle^{1/4}. \tag{4.17}$$

Hence we have from (4.17) that

$$H_3(\xi,\xi_1,\xi_2)\leq C\langle\xi\rangle^{1/2}\frac{|\xi-\xi_1|}{\langle\xi-\xi_1\rangle^{1/2}}\frac{|\xi_1-\xi_2|^{3/4}}{\langle\xi_1-\xi_2\rangle^{1/2}}\frac{\langle\xi_2\rangle^{1/4}}{\langle\xi_2\rangle^{1/2}}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}$$

$$\leq C\langle\xi\rangle^{1/2}\frac{\langle\xi-\xi_1\rangle^{3/4}}{\langle\xi-\xi_1\rangle^{1/4}}\frac{\langle\xi_1-\xi_2\rangle^{1/2}}{\langle\xi_1-\xi_2\rangle^{1/4}}\frac{1}{\langle\xi_2\rangle^{1/4}}\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\leq Ch_1(\xi,\xi_1,\xi_2).$$

Estimate on C. We handle this estimate in region R_1.

From definition of R_1 in (4.6) and (4.15),

$$\frac{1}{\langle\sigma\rangle^{|a|}}\leq\frac{1}{|\xi-\xi_1|^{|a|}|\xi_1-\xi_2|^{|a|}}. \tag{4.18}$$

From $\max\{\langle\xi-\xi_1\rangle,\langle\xi_1-\xi_2\rangle,\langle\xi_2\rangle\}^{1/2}\geq\langle\xi\rangle^{1/2}/3$, (4.18), and the assumption that $2+2a\leq 7/4$,

$$H_3(\xi,\xi_1,\xi_2)\leq C\max\{\langle\xi-\xi_1\rangle,\langle\xi_1-\xi_2\rangle,\langle\xi_2\rangle\}^{1/2}$$
$$\times\frac{|\xi-\xi_1|^{1+a}}{\langle\xi-\xi_1\rangle^{1/2}}\frac{|\xi_1-\xi_2|^{1+a}}{\langle\xi_1-\xi_2\rangle^{1/2}}\frac{1}{\langle\xi_2\rangle^{1/2}}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}$$
$$\leq C\max\{\langle\xi-\xi_1\rangle,\langle\xi_1-\xi_2\rangle,\langle\xi_2\rangle\}^{1/2}$$
$$\times\frac{\langle\xi-\xi_1\rangle^{3/4+a}}{\langle\xi-\xi_1\rangle^{1/4}}\frac{\langle\xi_1-\xi_2\rangle^{3/4+a}}{\langle\xi_1-\xi_2\rangle^{1/4}}\frac{1}{\langle\xi_2\rangle^{1/4}}\times\frac{1}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}\leq Ch_1(\xi,\xi_1,\xi_2).$$

For the estimate on R_2–R_4, the proof is similar to that of the estimate on R_2 in the proof Lemma 4.2 and the estimate of C in the proof of Lemma 4.4. Here, the estimate on R_4 requires that $9/4+2a\leq 7/4$. □

Lemma 4.5. *Let $a\in(-1/2,-1/4]$ and $b>1/2$. Then*

$$H_4(\xi,\xi_1,\xi_2)\equiv\langle\xi\rangle^{\frac{1}{2}}\frac{1}{\langle\xi-\xi_1\rangle^{1/2}}\frac{|\xi_1-\xi_2|}{\langle\xi_1-\xi_2\rangle^{1/2}}\frac{|\xi_2|}{\langle\xi_2\rangle^{1/2}}\times\frac{1}{\langle\sigma\rangle^{|a|}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b\langle\overline{\sigma}_3\rangle^b}$$

$$\leq Ch_1(\xi,\xi_1,\xi_2)\chi_{A\cup B}(\xi,\xi_1,\xi_2)+C\sum_{j=1}^{4}h_j(\xi,\xi_1,\xi_2)\chi_{R_j}(\xi,\xi_1,\xi_2), \tag{4.19}$$

where σ, σ_1, σ_2, *and* $\overline{\sigma}_3$ *are given by* (4.1), h_j *are given by* (4.3), *and* A, B, R, *and* R_j *are defined by* (4.5)–(4.6).

Proof of Lemma 4.5. From the inequality (4.8),

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|, |\overline{\sigma}_3|\} \geq \frac{1}{4}|\xi_1||\xi - \xi_1 + \xi_2||\xi_1 - \xi_2||\xi_2|. \tag{4.20}$$

Estimate on A. Since $|\xi - \xi_1 + \xi_2| \leq 1$ in this case, we note the following inequalities:

$$|\xi_2|^{1/4} \leq |\xi - \xi_1 + \xi_2|^{1/4} + |\xi - \xi_1|^{1/4} \leq \langle \xi - \xi_1 \rangle^{1/4}. \tag{4.21}$$

Hence we have from (4.21) and $\max\{|\xi - \xi_1|, |\xi_1 - \xi_2|, |\xi_2|\}^{1/2} \geq |\xi|^{1/2}/3$ that

$$H_4(\xi, \xi_1, \xi_2) \leq C\langle \xi \rangle^{1/2} \frac{|\xi - \xi_1|^{1/4}}{\langle \xi - \xi_1 \rangle^{1/2}} \frac{|\xi_1 - \xi_2|}{\langle \xi_1 - \xi_2 \rangle^{1/2}} \frac{|\xi_2|^{3/4}}{\langle \xi_2 \rangle^{1/2}} \times \frac{1}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b}$$
$$\leq C\langle \xi \rangle^{\frac{1}{2}} \frac{1}{\langle \xi - \xi_1 \rangle^{1/4}} \frac{\langle \xi_1 - \xi_2 \rangle^{3/4}}{\langle \xi_1 - \xi_2 \rangle^{1/4}} \frac{\langle \xi_2 \rangle^{1/2}}{\langle \xi_2 \rangle^{1/4}} \times \frac{1}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b} \leq Ch_1(\xi, \xi_1, \xi_2).$$

Estimate on B. In this case, we note the following inequalities:

$$|\xi|^{1/4} \leq |\xi_1|^{1/4} + |\xi - \xi_1|^{1/4} \leq \langle \xi - \xi_1 \rangle^{1/4}. \tag{4.22}$$

Hence we have from (4.22) and $\max\{|\xi - \xi_1|, |\xi_1 - \xi_2|, |\xi_2|\}^{1/2} \geq |\xi|^{1/2}/3$ that

$$H_4(\xi, \xi_1, \xi_2) \leq C\langle \xi \rangle^{1/4} \frac{\langle \xi - \xi_1 \rangle^{1/4}}{\langle \xi - \xi_1 \rangle^{1/2}} \frac{|\xi_1 - \xi_2|}{\langle \xi_1 - \xi_2 \rangle^{1/2}} \frac{|\xi_2|}{\langle \xi_2 \rangle^{1/2}} \times \frac{1}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b}$$
$$\leq C\langle \xi \rangle^{1/4} \frac{1}{\langle \xi - \xi_1 \rangle^{1/4}} \frac{\langle \xi_1 - \xi_2 \rangle^{3/4}}{\langle \xi_1 - \xi_2 \rangle^{1/4}} \frac{\langle \xi_2 \rangle^{3/4}}{\langle \xi_2 \rangle^{1/4}} \times \frac{1}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b} \leq Ch_1(\xi, \xi_1, \xi_2).$$

Estimate on C. This estimate is similar to the estimate of C as in the proof of Lemma 4.4. □

Lemma 4.6. *Let* $a \in (-1/2, -3/8]$ *and* $b > 1/2$. *Then*

$$\begin{aligned} H_5(\xi, \xi_1, \xi_2) &\equiv \langle \xi \rangle^{1/2} \frac{1}{\langle \xi - \xi_1 \rangle^{1/2}} \frac{1}{\langle \xi_1 - \xi_2 \rangle^{1/2}} \frac{|\xi_2|^2}{\langle \xi_2 \rangle^{1/2}} \\ &\quad \times \frac{1}{\langle \sigma \rangle^{|a|} \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b} \\ &\leq Ch_1(\xi, \xi_1, \xi_2)\chi_{A \cup B}(\xi, \xi_1, \xi_2) + C\sum_{j=1}^{4} h_j(\xi, \xi_1, \xi_2)\chi_{R_j}(\xi, \xi_1, \xi_2), \end{aligned} \tag{4.23}$$

where σ, σ_1, σ_2, *and* $\overline{\sigma}_3$ *are given by* (4.1), h_j *are given by* (4.3), *and* A, B, *and* R_j *are defined by* (4.5)–(4.6).

Proof of Lemma 4.6. From the inequality (4.8)

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|, |\overline{\sigma}_3|\} \geq \frac{1}{4}|\xi_1||\xi - \xi_1 + \xi_2||\xi_2|^2. \tag{4.24}$$

Estimate on A. Since $|\xi - \xi_1 + \xi_2| \leq 1$ in this domain, we note the following inequalities:

$$\begin{aligned} |\xi|^{1/4} &\leq |\xi - \xi_1 + \xi_2|^{1/4} + |\xi_1 - \xi_2|^{1/4} \leq \langle \xi_1 - \xi_2 \rangle^{1/4}, \\ |\xi_2|^{1/4} &\leq |\xi - \xi_1 + \xi_2|^{1/4} + |\xi - \xi_1|^{1/4} \leq \langle \xi - \xi_1 \rangle^{1/4}. \end{aligned} \tag{4.25}$$

Hence we have from (4.25) and $\max\{|\xi - \xi_1|, |\xi_1 - \xi_2|, |\xi_2|\} \geq |\xi|/3$ that

$$\begin{aligned} &\langle \xi \rangle^{1/2} \langle \xi_2 \rangle^{1/2} \leq C \langle \xi \rangle^{1/4} \langle \xi - \xi_1 \rangle^{1/4} \langle \xi_1 - \xi_2 \rangle^{1/4} \langle \xi_2 \rangle^{1/4} \\ &\leq C \max\{\langle \xi - \xi_1 \rangle, \langle \xi_1 - \xi_2 \rangle, \langle \xi_2 \rangle\}^{1/4} \langle \xi - \xi_1 \rangle^{1/4} \langle \xi_1 - \xi_2 \rangle^{1/4} \langle \xi_2 \rangle^{1/4}. \end{aligned}$$

Therefore, $H_5(\xi, \xi_1, \xi_2) \leq C h_1(\xi, \xi_1, \xi_2)$.

Estimate on B. Since $|\xi_1| \leq 1$ we note the following inequalities:

$$\begin{aligned} |\xi|^{1/4} &\leq |\xi_1|^{1/4} + |\xi - \xi_1|^{1/4} \leq \langle \xi - \xi_1 \rangle^{1/4}, \\ |\xi_2|^{1/4} &\leq |\xi_1|^{1/4} + |\xi_1 - \xi_2|^{1/4} \leq \langle \xi_1 - \xi_2 \rangle^{1/4}. \end{aligned} \tag{4.26}$$

Hence, we have from the (4.26) and $\max\{|\xi - \xi_1|, |\xi_1 - \xi_2|, |\xi_2|\} \geq |\xi|/3$ that

$$\begin{aligned} &\langle \xi \rangle^{1/2} \langle \xi_2 \rangle^{1/2} \leq C \langle \xi \rangle^{1/4} \langle \xi - \xi_1 \rangle^{1/4} \langle \xi_1 - \xi_2 \rangle^{1/4} \langle \xi_2 \rangle^{1/4} \\ &\leq C \max\{\langle \xi - \xi_1 \rangle, \langle \xi_1 - \xi_2 \rangle, \langle \xi_2 \rangle\}^{1/4} \langle \xi - \xi_1 \rangle^{1/4} \langle \xi_1 - \xi_2 \rangle^{1/4} \langle \xi_2 \rangle^{1/4}. \end{aligned}$$

Therefore, $H_5(\xi, \xi_1, \xi_2) \leq C h_1(\xi, \xi_1, \xi_2)$.

Estimate on C. We consider only two sets, R_1 and R_2.

Estimate on R$_1$. From the definition of R_1, (4.24), and $2 + 2a \leq 3/2$,

$$\begin{aligned} &H_5(\xi, \xi_1, \xi_2) \leq \langle \xi \rangle^{1/2} \frac{1}{\langle \xi - \xi_1 \rangle^{1/2}} \frac{1}{\langle \xi_1 - \xi_2 \rangle^{1/2}} \frac{|\xi_2|^{2+2a}}{\langle \xi_2 \rangle^{1/2}} \times \frac{1}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b} \\ &\leq \frac{\max\{\langle \xi - \xi_1 \rangle, \langle \xi_1 - \xi_2 \rangle, \langle \xi_2 \rangle\}^{9/4+2a}}{\langle \xi - \xi_1 \rangle^{1/4} \langle \xi_1 - \xi_2 \rangle^{1/4} \langle \xi_2 \rangle^{1/4}} \times \frac{1}{\langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b} \leq C h_1(\xi, \xi_1, \xi_2). \end{aligned}$$

Estimate on R$_2$. From the definition of R_2, (4.24), and $5/2 + 2a \leq 7/4$,

$$\begin{aligned} &H_5(\xi, \xi_1, \xi_2) \leq \langle \xi \rangle^{1/2} \frac{1}{\langle \xi - \xi_1 \rangle^{1/2}} \frac{1}{\langle \xi_1 - \xi_2 \rangle^{1/2}} \frac{|\xi_2|^{2+2a}}{\langle \xi_2 \rangle^{1/2}} \times \frac{1}{\langle \sigma \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b} \\ &\leq \frac{\max\{\langle \xi \rangle, \langle \xi_1 - \xi_2 \rangle, \langle \xi_2 \rangle\}^{9/4+2a}}{\langle \xi_1 - \xi_2 \rangle^{1/2} \langle \xi_2 \rangle^{1/4}} \times \frac{1}{\langle \sigma \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b} \\ &\leq \frac{\max\{\langle \xi \rangle, \langle \xi_1 - \xi_2 \rangle, \langle \xi_2 \rangle\}^{5/2+2a}}{\langle \xi \rangle^{1/4} \langle \xi_1 - \xi_2 \rangle^{1/4} \langle \xi_2 \rangle^{1/4}} \times \frac{1}{\langle \sigma \rangle^b \langle \sigma_2 \rangle^b \langle \overline{\sigma}_3 \rangle^b} \leq C h_2(\xi, \xi_1, \xi_2). \quad \square \end{aligned}$$

From the above lemmas, we have the following "multilinear estimates."

Proposition 4.7. *Let $s \geq -1/2$, $a \in (-1/2, -1/4]$, and $b > 1/2$. Then for any $u \in X_{s,b}$, we have*

$$\||u|^2 u\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}. \tag{4.27}$$

Let $s \geq 0$, $a \leq 0$, and $b > 1/2$. Then for any $u \in X_{s,b}$,

$$\||u|^4 u\|_{X_{s,a}} \leq C\|u\|^5_{X_{s,b}}. \tag{4.28}$$

Let $s \geq 1/2$, $a \in (-1/2, -1/4]$, and $b > 1/2$. Then for any $u \in X_{s,b}$, we have

$$\|(\partial_x u)^2 \overline{u}\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}, \tag{4.29}$$

$$\||\partial_x u|^2 u\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}. \tag{4.30}$$

Let $s \geq 1/2$, $a \in (-1/2, -3/8]$, and $b > 1/2$. Then for any $u \in X_{s,b}$, we have

$$\|u^2 \partial_x^2 \overline{u}\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}. \tag{4.31}$$

Remark 4.8. If $\nu < 0$ then Proposition 4.7 holds for $X^\nu_{s,a}$ replacing $X_{s,a}$ and $X^\nu_{s,b}$ replacing $X^\nu_{s,b}$, respectively. From above lemmas, in this case, essential inequalities are as follows:

$$\begin{aligned} &|\sigma| + |\sigma_1| + |\sigma_2| + |\overline{\sigma}_3| \geq 2|\xi_1||\xi - \xi_1 + \xi_2| \\ &\quad \times \Big| -\nu\Big\{\frac{1}{2}\xi^2 + \frac{1}{2}(\xi - \xi_1)^2 + \frac{1}{2}(\xi_1 - \xi_2)^2 + \frac{1}{2}\xi_2^2 + (\xi - \xi_2)^2\Big\} + 1\Big| \\ &\geq 2|\nu||\xi_1||\xi - \xi_1 + \xi_2|\Big\{\frac{1}{2}\xi^2 + \frac{1}{2}(\xi - \xi_1)^2 + \frac{1}{2}(\xi_1 - \xi_2)^2 + \frac{1}{2}\xi_2^2 + (\xi - \xi_2)^2\Big\}, \end{aligned}$$

where σ, σ_1, σ_2, and $\overline{\sigma}_3$ are given by (4.1). In the case $\nu > 0$, the above inequality fails.

Remark 4.9. In Theorem 2.1, we assume $\mu - \nu/2 = 0$, because we are not able to show the"trilinear estimate"

$$\||u|^2 \partial_x^2 u\|_{X_{s,a}} \leq C\|u\|^3_{X_{s,b}}. \tag{4.32}$$

If one can estimate an inequality like (4.32), then the well-posedness with $\mu - \nu/2 \neq 0$ follows by our argument.

Proof of Proposition 4.7. By density, we have only to prove (4.27)–(4.31) for $u \in \mathcal{S}(\mathbb{R}^2)$. Let

$$f(\xi, \tau) = \langle \sigma \rangle^b \langle \xi \rangle^s |\widehat{u}(\xi, \tau)|, \qquad \overline{f}(\xi, \tau) = \langle \overline{\sigma} \rangle^b \langle \xi \rangle^s |\widehat{\overline{u}}(\xi, \tau)|,$$

where $\sigma = \tau + \xi^2 + \xi^4$ and $\overline{\sigma} = \tau - \xi^2 - \xi^4$. By a duality argument, (4.27) is reduced to showing that for any $0 \le g \in L^2_\xi L^2_\tau$

$$\begin{aligned}\int_{\mathbb{R}^6} &\frac{\langle \xi \rangle^s g(\xi,\tau)}{\langle \sigma \rangle^{|a|}} \frac{f(\xi-\xi_1,\tau-\tau_1)}{\langle \sigma_1 \rangle^b \langle \xi-\xi_1 \rangle^s} \\ &\times \frac{f(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle \sigma_2 \rangle^b \langle \xi_1-\xi_2 \rangle^s} \frac{\overline{f}(\xi_2,\tau_2)}{\langle \overline{\sigma}_3 \rangle^b \langle \xi_2 \rangle^s} d\xi\, d\tau\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2 \\ \le &C\|g\|_{L^2_\xi L^2_\tau} \|f\|^2_{L^2_\xi L^2_\tau} \|\overline{f}\|_{L^2_\xi L^2_\tau}.\end{aligned} \tag{4.33}$$

It suffices to show (4.33) only in the case $s = -1/2$ since $\langle \xi \rangle^{s+1/2} \le \langle \xi - \xi_1 \rangle^{s+1/2} \langle \xi_1 - \xi_2 \rangle^{s+1/2} \langle \xi_2 \rangle^{s+1/2}$. In this case the weight function on the left-hand side of (4.33) is equal to H_1. So, (4.33) follows from Lemma 4.1 (4.4) and Lemma 4.2 (4.7).

Similarly, (4.28)–(4.31) is reduced to showing that for any nonnegative function $g \in L^2_\xi L^2_\tau$,

$$\begin{aligned}\int_{\mathbb{R}^{10}} &\frac{g(\xi,\tau)}{\langle \rho \rangle^{|a|}} \frac{f(\xi-\xi_1,\tau-\tau_1)}{\langle \rho_1 \rangle^b} \frac{\overline{f}(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle \overline{\rho}_2 \rangle^b} \\ &\times \frac{f(\xi_2-\xi_3,\tau_2-\tau_3)}{\langle \rho_3 \rangle^b} \frac{\overline{f}(\xi_3-\xi_4,\tau_3-\tau_4)}{\langle \overline{\rho}_4 \rangle^b} \frac{f(\xi_4,\tau_4)}{\langle \rho_5 \rangle^b} d\xi\, d\tau \cdots d\xi_4\, d\tau_4 \\ &\le C\|g\|_{L^2_\xi L^2_\tau} \|f\|^3_{L^2_\xi L^2_\tau} \|\overline{f}\|^2_{L^2_\xi L^2_\tau},\end{aligned} \tag{4.34}$$

$$\begin{aligned}\int_{\mathbb{R}^6} &\frac{\langle \xi \rangle^{1/2} g(\xi,\tau)}{\langle \sigma \rangle^{|a|}} \frac{|\xi-\xi_1| f(\xi-\xi_1,\tau-\tau_1)}{\langle \sigma_1 \rangle^b \langle \xi-\xi_1 \rangle^{1/2}} \\ &\times \frac{|\xi_1-\xi_2| f(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle \sigma_2 \rangle^b \langle \xi_1-\xi_2 \rangle^{1/2}} \frac{\overline{f}(\xi_2,\tau_2)}{\langle \overline{\sigma}_3 \rangle^b \langle \xi_2 \rangle^{1/2}} d\xi\, d\tau\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2 \\ &\le C\|g\|_{L^2_\xi L^2_\tau} \|f\|^2_{L^2_\xi L^2_\tau} \|\overline{f}\|_{L^2_\xi L^2_\tau},\end{aligned} \tag{4.35}$$

$$\begin{aligned}\int_{\mathbb{R}^6} &\frac{\langle \xi \rangle^{1/2} g(\xi,\tau)}{\langle \sigma \rangle^{|a|}} \frac{f(\xi-\xi_1,\tau-\tau_1)}{\langle \sigma_1 \rangle^b \langle \xi-\xi_1 \rangle^{1/2}} \\ &\times \frac{|\xi_1-\xi_2| f(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle \sigma_2 \rangle^b \langle \xi_1-\xi_2 \rangle^{1/2}} \frac{|\xi_2| \overline{f}(\xi_2,\tau_2)}{\langle \overline{\sigma}_3 \rangle^b \langle \xi_2 \rangle^{1/2}} d\xi\, d\tau\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2 \\ &\le C\|g\|_{L^2_\xi L^2_\tau} \|f\|^2_{L^2_\xi L^2_\tau} \|\overline{f}\|_{L^2_\xi L^2_\tau},\end{aligned} \tag{4.36}$$

$$\begin{aligned}\int_{\mathbb{R}^6} &\frac{\langle \xi \rangle^{1/2} g(\xi,\tau)}{\langle \sigma \rangle^{|a|}} \frac{f(\xi-\xi_1,\tau-\tau_1)}{\langle \sigma_1 \rangle^b \langle \xi-\xi_1 \rangle^{1/2}} \\ &\times \frac{f(\xi_1-\xi_2,\tau_1-\tau_2)}{\langle \sigma_2 \rangle^b \langle \xi_1-\xi_2 \rangle^{1/2}} \frac{|\xi_2|^2 \overline{f}(\xi_2,\tau_2)}{\langle \overline{\sigma}_3 \rangle^b \langle \xi_2 \rangle^{1/2}} d\xi\, d\tau\, d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2\end{aligned}$$

$$\leq C\|g\|_{L^2_\xi L^2_\tau}\|f\|^2_{L^2_\xi L^2_\tau}\|\overline{f}\|_{L^2_\xi L^2_\tau}, \tag{4.37}$$

respectively. In the inequalities (4.34)–(4.37), the weight functions in each integral are equal to H_2–H_5, respectively. Therefore, in a way analogous to (4.33), we have (4.34) from Lemma 4.1 (4.4) and Lemma 4.3 (4.13). Similarly, (4.35) follows from Lemma 4.1 (4.4) and Lemma 4.4 (4.14), (4.36) follows from Lemma 4.1 (4.4) and Lemma 4.5 (4.19), and (4.37) follows from Lemma 4.1 (4.4) and Lemma 4.6 (4.23), respectively. □

5. Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. First, we prove the existence of a solution u to the integral equation (2.1).

Let $r = \|u_0\|_{H^s}$. For $\delta \in (0,1)$, define a complete metric space

$$\mathcal{B}(r) = \{u \in X_{s,b} : \ \|u\|_{X_{s,b}} \leq 2C_0 r\},$$

and a map

$$\Phi(u) = \psi(t)W(t)u_0 - i\psi(t)\int_0^t W(t-t')\psi_\delta(t')F(t')dt'.$$

We show that the map Φ is from $\mathcal{B}(r)$ onto $\mathcal{B}(r)$ and is a contraction.

According to Proposition 3.1 (3.1)–(3.3) and Proposition 4.7 (4.27)–(4.31), we have for b, b' with $1/2 < b < b' < 5/8$ and for $u \in \mathcal{B}(r)$,

$$\begin{aligned}\|\Phi(u)\|_{X_{s,b}} &\leq C_0 r + C_1\|\psi_\delta F\|_{X_{s,b-1}} \leq C_0 r + C_1\delta^{b'-b}\|F\|_{X_{s,b-1}}\\ &\leq C_0 r + C_1\delta^{b'-b}(\|u\|^3_{X_{s,b}} + \|u\|^5_{X_{s,b}}) \leq C_0 r + C_1\delta^{b'-b}(1+r^2)r^3.\end{aligned}$$

Therefore, we choose

$$\delta^{b'-b} \leq \frac{C_0}{(1+r^2)r^2 C_1}. \tag{5.1}$$

Then $\Phi(u) \in \mathcal{B}(r)$. Similarly, we have for $u, \tilde{u} \in \mathcal{B}(r)$,

$$\begin{aligned}&\|\Phi(u) - \Phi(\tilde{u})\|_{X_{s,b}} \leq C_2\delta^{b'-b}\|F\|_{X_{s,b-1}}\\ &\leq C_2\delta^{b'-b}\{(\|u\|^2_{X_{s,b}} + \|\tilde{u}\|^2_{X_{s,b}}) + (\|u\|^4_{X_{s,b}} + \|\tilde{u}\|^4_{X_{s,b}})\}\|u-\tilde{u}\|_{X_{s,b}}\\ &\leq C_2\delta^{b'-b}(1+r^2)r^2\|u-\tilde{u}\|_{X_{s,b}}.\end{aligned}$$

We choose δ satisfying (5.1) and

$$\delta^{b'-b} \leq \frac{1}{2(1+r^2)r^2 C_2}. \tag{5.2}$$

Then Φ is a contraction map on $\mathcal{B}(r)$.

Therefore, there exists a unique solution $u \in \mathcal{B}(r)$ satisfying

$$u(t) = \psi(t)W(t)u_0 - i\psi(t)\int_0^t W(t-t')\psi_\delta(t')F(t')dt'.$$

We choose $T < \delta$; then $u(t)$ satisfies (1.1) in $t \in [-T, T]$.

Next, we prove the uniqueness of the solution. We define the following auxiliary norms introduced by Bekiranov-Ogawa-Ponce [2].

For $T > 0$, we let

$$\|u\|_{X_T} = \inf_w \{\|w\|_{X_{s,b}}; w \in X_{s,b} \text{ such that } u(t) = w(t), t \in [-T, T] \text{ in } H^s(\mathbb{R})\}.$$

If $\|u - \tilde{u}\|_{X_T} = 0$, we have $u(t) = \tilde{u}(t)$ in $H^s(\mathbb{R})$ for $t \in [-T, T]$. By an argument similar to that in [2], we reduce the uniqueness.

The persistence property follows the Sobolev's embedding $H_t^b(\mathbb{R}; H_x^s(\mathbb{R})) \subset C(\mathbb{R}; H^s(\mathbb{R}))$ (see Proposition 3.1 (3.4)). The continuous dependence upon data also follows from a similar argument. □

6. Remark on the local well-posedness

In the previous sections, we proved the local well-posedness in $H^{1/2}$ for initial-value problem (1.1). In this section, we show that the index 1/2 is optimal in some sense. For simplicity, we consider the following initial-value problem:

$$\begin{cases} i\partial_t u + \partial_x^2 u + \nu\partial_x^4 u = \partial_x^2(|u|^2 u), & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{6.1}$$

According to the proof of Theorem 2.1 in the preceding section, the initial-value problem (1.1) with $\nu < 0$ and $2\mu - \nu = 0$ is locally well-posed in H^s with $s \geq 1/2$. In this section, we show that one can not solve the initial-value problem for (6.1) via the Duhamel's formula in H^s with $s < 1/2$. Therefore we expect that a similar result also holds for the full equation (1.1). As with a view of Theorem 2.1, the exponent 1/2 seems the critical for the well-posedness results in our method. More precisely, we have following theorem.

Theorem 6.1. *Let $s < 1/2$ and $T > 0$. Then there exists no space X_T satisfying*

$$X_T \hookrightarrow C([-T,T]; H^s(\mathbb{R})), \tag{6.2}$$

$$\|W_\nu(t)u_0\|_{X_T} \leq C\|u_0\|_{H^s}, \tag{6.3}$$

and

$$\left\|\int_0^t W_\nu(t-t')\partial_x^2(|u|^2 u)(t')dt'\right\|_{X_T} \leq C\|u\|_{X_T}^3. \tag{6.4}$$

Remark 6.2. Similar results were obtained by Bourgain [4], Molinet-Saut-Tzvetkov [24], and Tzvetkov [33] for the K-dV equation and the B-O equation (see also Takaoka [29]). To prove Theorem 6.1, we employ a method analogous to that of [24] and [33]. In the above Theorem 6.1, we do not need to assume that $\nu < 0$. Namely, the estimate fails for any $\nu \neq 0$.

As a consequence of Theorem 6.1, we have that the data-solution flow-map of (6.1) is not C^3-differentiable in H^s with $s < 1/2$. More precisely, we have following corollary.

Corollary 6.3. *Let $s < 1/2$. Then there is no $T > 0$ such that the data-solution flow-map $S_t : u_0 \longmapsto u(t)$, $t \in [-T, T]$, for (6.1) is C^3 (Fréchet) differentiable at zero from $H^s(\mathbb{R})$ to itself.*

Proof of Theorem 6.1. For simplicity, we assume $\nu = -1$. The assumption of (6.2)–(6.4) implies

$$\Big\| \int_0^t W(t-t')\partial_x^2[|W(t')u_0|^2 W(t')u_0]dt' \Big\|_{H^s} \leq C\|u_0\|_{H^s}^3. \tag{6.5}$$

Let $N \gg 1$ and $0 < \gamma \ll 1$ to be chosen later. For given $0 < T < 1$, we consider the following initial data:

$$u_0(x) = \gamma^{-1/2} N^{-s} \int_{|\xi - N| \leq \gamma} e^{ix\xi} d\xi;$$

that is,

$$\mathcal{F}_x u_0(\xi) = \gamma^{-1/2} N^{-s} \chi_{[N-\gamma, N+\gamma]}(\xi). \tag{6.6}$$

We easily see $\|u_0\|_{H^s} \leq 4$. By (6.6),

$$W(t)u_0 = C\gamma^{-1/2} N^{-s} \int_{|\xi - N| \leq \gamma} e^{-itp(\xi) + ix\xi} d\xi;$$

that is,

$$\mathcal{F}_x W(t) u_0(\xi, t) = \gamma^{-1/2} N^{-s} e^{-itp(\xi)} \chi_{[N-\gamma, N+\gamma]}(\xi), \tag{6.7}$$

where $p(\xi) = \xi^2 + \xi^4$.

By substituting (6.7) into the left-hand side of (6.5) we find

$$\begin{aligned} &\int_0^t W(t-t')\partial_x^2[|W(t')u_0|^2 W(t')u_0]dt' \\ &\quad = C\gamma^{-3/2} N^{-3s} \int_D \xi^2 e^{-itp(\xi) + ix\xi} \\ &\qquad \times \frac{e^{it(p(\xi) - p(\xi - \xi_1) - p(\xi_1 - \xi_2) + p(\xi_2))} - 1}{p(\xi) - p(\xi - \xi_1) - p(\xi_1 - \xi_2) + p(\xi_2)} d\xi \, d\xi_1 \, d\xi_2, \end{aligned} \tag{6.8}$$

where

$$D \equiv \Big\{(\xi,\xi_1,\xi_2)\in\mathbb{R}^3 : |\xi-\xi_1-N|\le\gamma,\ |\xi_1-\xi_2-N|\le\gamma \text{ and } |\xi_2+N|\le\gamma\Big\}.$$

On D, we see $|\xi_1|\le 2\gamma$, $|\xi-\xi_1+\xi_2|\le 2\gamma$, and $\max\{|\xi|,|\xi-\xi_1|,|\xi_1-\xi_2|,|\xi_2-\xi|,|\xi_2|\}\le 3N$. Since

$$\begin{aligned}&|p(\xi)-p(\xi-\xi_1)-p(\xi_1-\xi_2)+p(\xi_2)|\\ &= 2|\xi_1||\xi-\xi_1+\xi_2|\{\tfrac12\xi^2+\tfrac12(\xi-\xi_1)^2+\tfrac12(\xi_1-\xi_2)^2+\tfrac12\xi_2^2+(\xi_2-\xi)^2+1\},\end{aligned}$$

we have $|p(\xi)-p(\xi-\xi_1)-p(\xi_1-\xi_2)+p(\xi_2)|\le C_0\gamma^2N^2$. Now, we choose γ so that $N^2\gamma^2=1/2C_0$. Hence if we further assume $T/2\le t\le T$, then

$$\Big|\frac{e^{it(p(\xi)-p(\xi-\xi_1)-p(\xi_1-\xi_2)+p(\xi_2))}-1}{p(\xi)-p(\xi-\xi_1)-p(\xi_1-\xi_2)+p(\xi_2)}\Big|\ge C. \tag{6.9}$$

Since

$$\Big\|\int_D e^{ix\xi}d\xi\,d\xi_1\,d\xi_2\Big\|_{H^s}\ge C\gamma^{5/2}N^s,$$

from (6.5) and (6.8)–(6.9),

$$\begin{aligned}64C_1 &\ge C_1\|u_0\|_{H^s}^3 \ge C\Big\|\int_0^t W(t-t')\partial_x^2[|W(t')u_0|^2W(t')u_0]dt'\Big\|_{H^s}\\ &\ge C\gamma^{-3/2}N^{-3s}N^2\Big\|\int_D e^{ix\xi}d\xi\,d\xi_1\,d\xi_2\Big\|_{H^s}\ge\gamma N^{2-2s}=N^{1-2s}/\sqrt{2C_0}.\end{aligned}$$

Then if $N>(64\sqrt{2C_0}C_1)^{1/(1-2s)}$, we need the relation $1-2s\le 0$, i.e., $s\ge 1/2$. This is in contradiction to the first assumption $s<1/2$. □

Proof of Corollary 6.3. Let $u_\delta(x,t)$ be a solution to

$$\begin{cases} i\partial_t u+\partial_x^2u-\partial_x^4u=\partial_x^2(|u|^2u), & (x,t)\in\mathbb{R}\times\mathbb{R},\\ u(x,0)=\delta u_0(x), & x\in\mathbb{R},\end{cases} \tag{6.10}$$

where $\delta\in\mathbb{R}$. We rewrite (6.10) to be

$$u_\delta(x,t)=\delta W_\nu(t)u_0-i\int_0^t W_\nu(t-t')\partial_x^2(|u_\delta|^2u_\delta)(t')dt'.$$

We suppose that $u_\delta(x,t)$ is expressed by the Taylor expansion

$$u_\delta(x,t)=\sum_{k=1}^\infty \delta^k v_k(x,t).$$

Then we have

$$\frac{\partial}{\partial\delta}u_\delta(x,t)\Big|_{\delta=0}=v_1(x,t)=W(t)u_0, \tag{6.11}$$

and

$$\frac{\partial^3}{\partial\delta^3}u_\delta(x,t)\Big|_{\delta=0} = Cv_3(x,t) = C\int_0^t W(t-t')\partial_x^2(|v_1|^2v_1)(t')dt'. \qquad (6.12)$$

The assumption of C^3-differentiability of S_t and (6.11)–(6.12) imply (6.5). However, (6.5) fails. □

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