Differential and Integral Equations Volume 16, Number 7, July 2003, Pages 829–840

# ELLIPTIC PROBLEMS WITH UNBOUNDED DRIFT COEFFICIENTS AND NEUMANN BOUNDARY CONDITION

Viorel Barbu[1]
University of Iasi, 6600 Iasi, Romania

Giuseppe Da Prato[2]
Scuola Normale Superiore di Pisa, 56126, Pisa, Italy

**Abstract.** A linear elliptic equation with Neumann boundary conditions and unbounded drift coefficients is studied. The existence and uniqueness results are used to characterize the infinitesimal generator of the transition semigroup associated with a stochastic variational inequality.

## 1. Introduction and setting of the problem

We are here concerned with the following elliptic problem with Neumann boundary conditions:

$$\begin{cases} \lambda\varphi - \dfrac{1}{2}\,\Delta\varphi + \langle F(x), D\varphi(x)\rangle = f \quad \text{in } K^0 \\ \dfrac{\partial\varphi}{\partial n} = 0 \quad \text{on } \partial K, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $K$ is a closed, convex subset of $\mathbb{R}^d$ with $C^2$ boundary $\partial K$ and nonempty interior $K^0$ (containing 0 for simplicity). Moreover, the vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$(H)$

$$\begin{cases} (i) \quad \textit{There is } C_1 \in \mathbb{R},\ C_2 > 0,\ N \in \mathbb{N},\ \textit{such that} \\ \qquad \langle F(x), x\rangle \ge \omega|x|^2 + C_1, \quad |F(x)| \le C_2(|x|^N + 1) \quad \forall\, x \in \mathbb{R}^d; \\ (ii) \quad \textit{There is } \gamma \ge 0 \textit{ such that} \\ \qquad \langle F(x) - F(y), x - y\rangle \ge -\gamma|x-y|^2, \quad \forall\, x, y \in \mathbb{R}^d; \\ (iii)\ F + \gamma \textit{ is maximal monotone.} \end{cases}$$

Notice that if $F$ is continuous then (iii) follows from (ii).

---

Accepted for publication: May 2003.
AMS Subject Classifications: 35J25, 47D07, 47H06.

[1]This was done during the stay in Scuola Normale Superiore di Pisa.

[2]Partially supported by the Italian National Project MURST "Equazioni di Kolmogorov."

Moreover, $\frac{\partial\varphi}{\partial n}$ is the outward derivative to $\partial K$. We notice that, since $F(x)$ is unbounded, the usual classical results for the Neumann problem, see e.g. [8], are not available.

The first new result of this paper, Theorem 3.2 below, amounts to saying that there is a Borel probability measure $\nu$ absolutely continuous with respect to the Lebesgue measure with nonnegative density $\rho\in L^1(\mathbb{R}^d)$ vanishing on $\mathbb{R}^d\backslash K$, such that for each $f\in L^2(\mathbb{R}^d,\nu)$ equation (1.1) has a unique strong solution (in the Friedrichs sense) $\varphi\in W^{1,2}(H,\nu)$. More precisely, let $N_0$ be the linear operator defined as [3]

$$\begin{cases} N_0\varphi=\dfrac{1}{2}\,\Delta\varphi-\langle F(x),D\varphi\rangle,\\ D(N_0)=\left\{\varphi\in C^2_b(H):\ \dfrac{\partial\varphi}{\partial\nu}\ =0 \text{ on } \partial K,\ N_0\varphi\in L^2(H,\nu)\right\}. \end{cases} \tag{1.2}$$

We prove that $N_0$ is dissipative in $L^2(H,\nu)$ and that its closure is $m$–dissipative.

The second result of this paper is that $\nu$ is the unique invariant measure of the transition semigroup

$$P_t\varphi(x)=\mathbb{E}[\varphi(X(t,x))],\quad t\ge 0,\ x\in K, \tag{1.3}$$

associated with the stochastic variational inequality

$$\begin{cases} dX(t)+F(X(t))dt+\partial I_K(X(t))dt\ni dW(t)\\ X(0)=x. \end{cases} \tag{1.4}$$

Here $W(t)$ is an $n$–dimensional Brownian process defined on a stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ and $I_K$ the indicator function of $K$:

$$I_K(x)=\begin{cases} 0 \text{ if } x\in K\\ +\infty \text{ if } x\notin K. \end{cases}$$

Existence and uniqueness of a weak solution $X(t)$ for equation (1.5) were proved in [5] and [10]. In the case when $F$ is a gradient field equation (1.1) was studied in [6] where, thanks to the fact that $N_0$ is symmetric, the domain of the closure $N$ of $N_0$ was also characterized. If $F$ is not a gradient $N$ is not symmetric, and we were not able to obtain a characterization of the domain of $N$ in the general case.

The results proved in this paper are relevant in the study of the stochastic obstacle problem by Galerkin approximations as we will show in a forthcoming paper.

[3] $C_b(\mathbb{R}^d)$ is the Banach space of all uniformly continuous and bounded mappings $\varphi\colon\mathbb{R}^r\to\mathbb{R}$ endowed with the sup norm: $\|\varphi\|_0=\sup_{x\in\mathbb{R}^d}|\varphi(x)|$. For any $h\in H$ the space $C^h_b(\mathbb{R}^d)$ is defined in the usual way.

We end this section by giving some preliminaries on Yosida approximations of $F$ and $\partial I_K$. We set $F_1 = F + \gamma$ and, for any $\varepsilon > 0$,

$$J_\varepsilon(x) = (1 + \varepsilon F_1)^{-1}(x), \quad F_\varepsilon(x) = \frac{1}{\varepsilon}\,(x - J_\varepsilon(x)) - \gamma x. \tag{1.5}$$

Moreover, we introduce the Moreau–Yosida approximations of $I_K$,

$$(I_K)_\varepsilon(x) = \inf\left\{ I_K(y) + \frac{1}{2\varepsilon}\,|x-y|^2,\ y \in H \right\}. \tag{1.6}$$

Clearly we have

$$(I_K)_\varepsilon(x) = \frac{1}{2\varepsilon}\,d_K^2(x), \quad x \in H, \tag{1.7}$$

where $d_K(x)$ denotes the distance of $x$ from $K$. We recall that $(I_K)_\varepsilon$ is of class $C^{1,Lip}$ and that, setting $D(I_K)_\varepsilon = \beta_\varepsilon$, we have

$$\beta_\varepsilon(x) = (\partial I_K)_\varepsilon(x) = \frac{1}{\varepsilon}\,(x - P_K(x)), \quad x \in H, \tag{1.8}$$

where $P_K(x)$ is the projection of $x$ on $K$ so that

$$|x - P_K(x)| = d_K(x), \quad x \in H. \tag{1.9}$$

## 2. The approximating equation

Consider the approximating stochastic equation

$$\begin{cases} dX_\varepsilon(t) + F_\varepsilon(X_\varepsilon(t))dt + \beta_\varepsilon(X_\varepsilon(t))dt = dW(t) \\ X_\varepsilon(0) = x, \end{cases} \tag{2.1}$$

where $F_\varepsilon$ and $\beta_\varepsilon$ are defined in (1.5) and (1.8) respectively. Since, as is well known, $F_\varepsilon$ and $\beta_\varepsilon$ are Lipschitz continuous, problem (1.6) has a unique solution, which we denote by $X_\varepsilon(t,x)$. Moreover, we denote by $P_t^\varepsilon$ the corresponding transition semigroup:

$$P_t^\varepsilon\varphi(x) = \mathbb{E}[\varphi(X_\varepsilon(t,x))], \quad t \ge 0,\ x \in \mathbb{R}^d,\ \varphi \in C_b(\mathbb{R}^d). \tag{2.2}$$

It is well known that, under Hypothesis $(H)$, $P_t^\varepsilon$ possesses a unique invariant measure $\nu_\varepsilon$ and that $P_t^\varepsilon$ has a unique extension to a $C_0$ contraction semigroup in $L^2(H,\nu_\varepsilon)$, which we still denote by $P_t^\varepsilon$. We denote by $N_\varepsilon$ its infinitesimal generator

$$N_\varepsilon\varphi(x) = \frac{1}{2}\,\Delta\varphi(x) - \langle F_\varepsilon(x) + \beta_\varepsilon(x), D\varphi(x)\rangle.$$

It is well known, see e.g. [1], that $\nu_\varepsilon$ is absolutely continuous with respect to the Lebesgue measure $\nu_\varepsilon(dx) = \rho_\varepsilon(x)dx$ and that $\rho_\varepsilon$ is a weak solution in $L^2(\mathbb{R}^d)$ of the following equation:

$$\operatorname{div}\Big(\frac{1}{2}\,D\rho_\varepsilon + (F_\varepsilon + \beta_\varepsilon)\rho_\varepsilon\Big) = 0 \quad \text{in } \mathbb{R}^d,\ \varepsilon > 0. \tag{2.3}$$

Moreover, one has

$$\rho_\varepsilon(x) > 0, \quad \forall\, x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} \rho_\varepsilon(x)dx = 1, \quad \varepsilon > 0. \tag{2.4}$$

Before proceeding further we need some estimates.

**Lemma 2.1.** *There exists $C > 0$ such that for all $\varepsilon > 0$ we have*

$$\int_{\mathbb{R}^d} |\beta_\varepsilon(x)|\nu_\varepsilon(dx) \le C, \quad \forall\, \varepsilon > 0. \tag{2.5}$$

$$\int_{\mathbb{R}^n} |x|^{2m}\nu_\varepsilon(dx) \le C, \quad \forall\, m \in \mathbb{N}. \tag{2.6}$$

**Proof**. Let us prove (2.5). Let $u \in H$ be such that $|u| = 1$ and $\rho > 0$ be sufficiently small so that $\rho u \in K$ (recall that 0 belongs to the interior of $K$). By (1.7)–(1.8) we have

$$\beta_\varepsilon(x) = \frac{1}{2\varepsilon}\, D(d_K^2).$$

Thus, by the convexity of $d_K^2$, it follows that

$$\frac{1}{2\varepsilon}\, d_K^2(x) - \frac{1}{2\varepsilon}\, d_K^2(\rho u) \ge \langle \beta_\varepsilon(x), x - \rho u\rangle.$$

Since $d_K^2(\rho u) = 0$ we have

$$\langle \beta_\varepsilon(x), x - \rho u\rangle \ge 0. \tag{2.7}$$

Setting in (2.7) $u = \frac{\beta_\varepsilon(x)}{|\beta_\varepsilon(x)|}$, we obtain

$$\rho|\beta_\varepsilon(x)| \le \langle \beta_\varepsilon(x), x\rangle, \quad x \in \mathbb{R}^d. \tag{2.8}$$

Now let $X_\varepsilon(t)$ be the solution of (2.1). By Itô's formula, we have

$$\mathbb{E}|X_\varepsilon(t,x)|^2 = |x|^2 - \mathbb{E}\int_0^t \langle F_\varepsilon(X_\varepsilon(s,x)) + \beta_\varepsilon(X_\varepsilon(s,x)), X_\varepsilon(s,x)\rangle ds + td.$$

Taking into account Hypothesis $(H)$ and (2.8), we see that there exists a positive constant $C$ such that

$$\frac{1}{t}\, \mathbb{E}\int_0^t |\beta_\varepsilon(X_\varepsilon(s,x))|ds \le C.$$

Therefore, (2.5) follows by integrating the above inequality with respect to the invariant measure $\nu_\varepsilon$.

Let us prove (2.6). For this we consider the Liapunov function $\varphi(x) = |x|^{2m}$ (notice that $\varphi$ does not belong to the domain of $N_\varepsilon$; however, it can be easily approximated by functions of that domain). We have

$$N_\varepsilon\varphi(x) = (md + 2m(m-1))|x|^{2(m-1)} - 2m\langle F_\varepsilon(x) + \beta_\varepsilon(x), x\rangle.$$

Since $\langle \beta_\varepsilon(x), x\rangle \geq 0$ and, thanks to (H), $\langle F_\varepsilon(x), x\rangle \geq \frac{\omega}{2}\,|x|^2 + C$, we have

$$N_\varepsilon\varphi(x) \leq (md + 2m(m-1))|x|^{2(m-1)} - m\omega|x|^{2m} - 2mC.$$

Integrating this inequality with respect to $\nu_\varepsilon$ and taking into account that, by the invariance of $\nu_\varepsilon$, $\int_H N_\varepsilon\varphi d\nu_\varepsilon = 0$, we find

$$m\omega \int_H |x|^{2m}\nu_\varepsilon(dx) \leq (md + 2m(m-1)) \int_H |x|^{2(m-1)}\nu_\varepsilon(dx) + 2mC_1.$$

Now the conclusion follows by an induction argument on $m$. □

**Proposition 2.2.** *There exists an invariant measure $\nu$ for the transition semigroup $P_t$. Moreover,*

$$\int_H |x - P_K x|\; \nu(dx) = 0, \tag{2.9}$$

*so that $\nu$ is concentrated on $K$.*

**Proof**. We first note that by (2.6) it follows that $\{\nu_\varepsilon\}$ is tight. So by the Krylov–Bogoliubov theorem there exists a subsequence of $\{\nu_\varepsilon\}$, still denoted by $\{\nu_\varepsilon\}$, which is weakly convergent to a probability measure $\nu$. It is easy to check that $\nu$ is invariant for $P_t$. Moreover, by (2.5) it follows that

$$\int_H |x - P_K x|\; \nu_\varepsilon(dx) \leq C\varepsilon, \quad \varepsilon > 0,$$

which implies (2.9). □

We want now to show that the sequence $\{\rho_\varepsilon\}$ is convergent in $L^1(\mathbb{R}^d)$.

**Lemma 2.3.** *The sequence $\{\rho_\varepsilon\}$ of densities is compact in $L^1(\mathbb{R}^d)$. Hence, for $\varepsilon \to 0$,*

$$\rho_\varepsilon \to \rho \quad \text{strongly in } \; L^1(\mathbb{R}^d), \quad \nu(dx) = \rho(x)dx,\; \rho \geq 0 \text{ a.e. in } \; \mathbb{R}^d. \tag{2.10}$$

*Moreover, $\rho \in L^p_{loc}(\mathbb{R}^d)$ for $p < d/(d-1)$, and $\rho$ satisfies in the sense of distributions*

$$\begin{cases} \dfrac{1}{2}\,\Delta\rho(x) = -\text{div}\,(\rho F(x)) \quad \text{in } D \\ \rho(x) = 0 \quad \text{a.e. in } \; x \in \mathbb{R}^d \setminus D. \end{cases} \tag{2.11}$$

**Proof**. We note first that, taking into account estimate (2.6), we have

$$\int_{\{|x|\geq R\}} \rho_\varepsilon(x)dx = \int_{\{|x|\geq R\}} \nu_\varepsilon(dx) \leq \frac{1}{R^2}\, \int_{\mathbb{R}^d} |x|^2\nu_\varepsilon(dx) \leq \frac{C}{R^2} \to 0 \tag{2.12}$$

as $R \to \infty$, uniformly with respect to $\varepsilon$. Let us show now that $\{\rho_\varepsilon\}$ satisfies the conditions of the Kolmogorov compactness theorem, see e.g. [4], in each $L^1(B_R)$; i.e.,

$$\lim_{|h|\to 0} \int_{B_R} |\rho_\varepsilon(x+h) - \rho_\varepsilon(x)|dx = 0, \tag{2.13}$$

uniformly in $\varepsilon$. To this end consider a cutoff function $\chi \in C^\infty(B_R)$ such that $\chi(x) = 1$ for $|x| \le R-1$. We have, taking into account (2.3),

$$\begin{aligned}
\Delta(\rho_\varepsilon\chi) &= \rho_\varepsilon\Delta\chi + 2\langle D\rho_\varepsilon, D\chi\rangle + \chi\Delta\rho_\varepsilon \\
&= \rho_\varepsilon\Delta\chi + 2\text{div }(\rho_\varepsilon D\chi) - 2\rho_\varepsilon\Delta\chi + \chi\Delta\rho_\varepsilon \\
&= -\rho_\varepsilon\Delta\chi + 2\text{div }(\rho_\varepsilon D\chi) - 2\chi\text{div }[(F_\varepsilon + \beta_\varepsilon)\rho_\varepsilon] \qquad (2.14)\\
&= 2\text{div }[\rho_\varepsilon D\chi + \chi\rho_\varepsilon(F_\varepsilon+\beta_\varepsilon)] - \rho_\varepsilon\Delta\chi - 2\langle D\chi, (F_\varepsilon+\beta_\varepsilon)\rangle \\
&= \text{div }\Psi_1^\varepsilon + \Psi_2^\varepsilon.
\end{aligned}$$

Let $E$ be the fundamental solution of the Laplacian operator; i.e.,

$$E(x) = \begin{cases} -\dfrac{1}{(d-2)\omega_d|x|^{d-2}} & \text{if } d > 2 \\ -\dfrac{1}{2\pi\ \log|x|} & \text{if } d = 2, \end{cases}$$

where $\omega_d$ is the surface of the unit sphere of $\mathbb{R}^d$. We note that $E \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and there exists $M_p > 0$ such that

$$|DE|_{L^p(B_R)} + |E|_{L^p(B_R)} \le M_p \quad \text{for } 1 \le p < \frac{d}{d-1}\,. \tag{2.15}$$

Then by (2.14) we have

$$\begin{aligned}
\rho_\varepsilon(x)\ \chi(x) &= \int_{\mathbb{R}^d} E(x-y)\Psi_2^\varepsilon(y)dy + \int_{\mathbb{R}^d} E(x-y)\text{ div }\Psi_1^\varepsilon(y)dy \\
&= -\int_{\mathbb{R}^d} (\langle DE(x-y), \Psi_1^\varepsilon(y)\rangle - E(x-y)\ \Psi_2^\varepsilon(y))dy. \qquad (2.16)
\end{aligned}$$

Keeping in mind that by (2.5)–(2.6), there exists a constant $K > 0$ such that

$$|\Psi_2^\varepsilon|_{L^1(\mathbb{R}^d)} + |\Psi_1^\varepsilon|_{L^1(\mathbb{R}^d)} \le K \quad \forall\, \varepsilon > 0, \tag{2.17}$$

we get by (2.15) and (2.16) via Young's inequality that

$$|\rho_\varepsilon\ \chi|_{L^p(\mathbb{R}^d)} \le M_pK, \quad \forall\, \varepsilon > 0,\ p < \frac{d}{d-1}. \tag{2.18}$$

Moreover, setting $\varphi_h(x) = \varphi(x+h) - \varphi(x)$, we see by (2.16) that

$$(\rho_\varepsilon\chi)_h(x) = -\int_{\mathbb{R}^d} (\langle DE_h(x-y), \Psi_1^\varepsilon(y)\rangle - E_h(x-y)\ \Psi_2^\varepsilon(y))dy, \quad x \in \mathbb{R}^d,$$

and we obtain similarly that there exists $K_1 > 0$ such that

$$|(\rho_\varepsilon\ \chi)_h|_{L^1(\mathbb{R}^d)} \le K_1(|DE_h|_{L^1(\mathbb{R}^d)} + |E_h|_{L^1(\mathbb{R}^d)}), \tag{2.19}$$

uniformly with respect to $\varepsilon$. Taking into account estimate (2.15) we find (2.13) as desired. Hence $\{\rho_\varepsilon\}$ is compact in $L^1(B_{R-1})$, and, since $R$ is arbitrary, it follows from (2.13) that $\{\rho_\varepsilon\}$ is compact in $L^1(\mathbb{R}^d)$ and weakly compact in $L^p_{loc}(\mathbb{R}^d)$. Then (2.10) follows and clearly $\nu(dx) = \rho(x)dx$. As regards (2.11), it is an immediate consequence of the estimate (2.5) and of equation (2.3). This completes the proof. □

## 3. The existence of solution

Let us consider the operator $N_0$ defined in $L^2(K,\nu)$ by

$$\begin{cases} N_0\varphi = \dfrac{1}{2}\ \Delta\varphi - \langle F(x), D\varphi\rangle. \\ D(N_0) = \left\{\varphi \in C_b^2(\mathbb{R}^d) :\ \dfrac{\partial\varphi}{\partial n}\ = 0 \text{ on } \partial K,\ N_0\varphi \in L^2(K,\nu)\right\}. \end{cases} \tag{3.1}$$

**Lemma 3.1.** *$N_0$ is dissipative in $L^2(K,\nu)$.*

**Proof**. We note first that for each $\varphi \in C_b^2(\mathbb{R}^d)$ we have, by invariance of $\nu_\varepsilon$,

$$\int_{\mathbb{R}^d} N_\varepsilon(x)\varphi(x)\ \nu_\varepsilon(dx) = 0;$$

i.e.,

$$\int_{\mathbb{R}^d} \Big(\frac{1}{2}\ \Delta\varphi - \langle F_\varepsilon(x) + \beta_\varepsilon(x), D\varphi(x)\rangle\Big)\nu_\varepsilon(dx) = 0. \tag{3.2}$$

This yields

$$\lim_{\varepsilon\to 0} \Big[\int_{\mathbb{R}^d} N_0\varphi(x)\nu_\varepsilon(dx) + \int_{\mathbb{R}^d\setminus K} \langle\beta_\varepsilon(x), D\varphi(x)\rangle\nu_\varepsilon(dx)\Big] = 0.$$

Hence,

$$\int_{\mathbb{R}^d} N_0\varphi(x)\nu(dx) = -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\setminus K} \langle\beta_\varepsilon(x), D\varphi(x)\rangle\nu_\varepsilon(dx). \tag{3.3}$$

But the right-hand side vanishes by the following argument. Consider for each $\gamma > 0$ the set

$$E_\gamma = \{x \in \mathbb{R}^d \setminus K :\ d_K(x) \ge \gamma\}$$

and the Lyapunov function $\Phi(x) = d_K^{2m}(x)$. Since by (1.8) $Dd_K^2(x) = 2(x - P_K x)$, we have

$$D\Phi(x) = 2m\, d_K^{2(m-1)}(x)(x - P_K x),$$

so that $\Phi \in D(N_0)$. Moreover, since the boundary of $K$ is of class $C^2$

$$\begin{aligned} D^2\Phi(x) &= 4m(m-1)d_K^{2(m-2)}(x)(x - P_K x) \otimes (x - P_K x) \\ &\quad + 2m d_K^{2(m-1)}(x) D(x - P_K x), \end{aligned}$$

and we find

$$\begin{aligned} N_\varepsilon \Phi(x) &= m(m-1)d_K^{2(m-1)}(x) + m d_K^{2(m-1)}(x) \operatorname{Tr}[D(1 - P_K)(x)] \\ &\quad - 2m\, d_K^{2(m-1)}(x)\langle \beta_\varepsilon(x) + F_\varepsilon(x), x - P_K x\rangle. \end{aligned}$$

Integrating both sides of this identity with respect to $\nu_\varepsilon$ and taking into account that $\int_H N_\varepsilon \Phi\, d\nu_\varepsilon = 0$ yields

$$\begin{aligned} &(m-1)\int_H d_K^{2(m-1)}(x)\nu_\varepsilon(dx) + \int_H d_K^{2(m-1)}(x) \operatorname{Tr}[D(1 - P_K)(x)]\nu_\varepsilon(dx) \\ &= 2 \int_H d_K^{2(m-1)}(x)\langle \beta_\varepsilon(x) + F_\varepsilon(x), x - P_K x\rangle \nu_\varepsilon(dx). \end{aligned}$$

Since $1 - P_K$ is monotone and Lipschitzian, the trace $\operatorname{Tr}[D(1 - P_K)(x)]$ is bounded by a constant $\kappa > 0$. Therefore, we have

$$\begin{aligned} &\int_H d_K^{2(m-1)}(x)\langle \beta_\varepsilon(x) + F_\varepsilon(x), x - P_K x\rangle \nu_\varepsilon(dx) \\ &\le \frac{m-1+\kappa}{2} \int_H d_K^{2(m-1)}(x)\nu_\varepsilon(dx). \end{aligned}$$

Since $\langle \beta_\varepsilon(x), x - P_K x\rangle = \frac{1}{\varepsilon}\, |x - P_K x|^2 = \frac{1}{\varepsilon}\, d_K^2(x)$ and

$$|\langle F_\varepsilon(x), x - P_K x\rangle| \le C_2 |x|(|x|^N + 1),$$

we find that, for a suitable positive constant $\kappa_2$, the following estimate,

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^d \setminus K} d_K^{2m}(x)\nu_\varepsilon(dx) \le \kappa_2 \int_{\mathbb{R}^d \setminus K} (d_K^{2(m-1)}(x) + (1+|x|^{N+1}) d_K^{2m-1}(x))\nu_\varepsilon(dx),$$

holds. Since $d_K(x)| \le |x|$ for all $x \in \mathbb{R}^d$, because $0 \in K$, we have taking into account (2.6)

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^d \setminus K} d_K^{2m}(x)\nu_\varepsilon(dx) \le 2\kappa_2 C.$$

Consequently,

$$\lim_{\varepsilon \to 0} \int_H d_K^{2m}(x)\nu_\varepsilon(dx) = 0 \quad \forall\, m \in \mathbb{N}. \tag{3.4}$$

Next we represent

$$\int_{\mathbb{R}^d\setminus K}\langle\beta_\varepsilon(x),D\varphi(x)\rangle\nu_\varepsilon(dx)=I^1_\varepsilon(\gamma)+I^2_\varepsilon(\gamma),$$

where

$$I^1_\varepsilon(\gamma)=\int_{E_\gamma}\langle\beta_\varepsilon(x),D\varphi(x)\rangle\nu_\varepsilon(dx)$$

and

$$I^2_\varepsilon(\gamma)=\int_{\{d_K(x)\le\gamma\}}\langle\beta_\varepsilon(x),D\varphi(x)\rangle\nu_\varepsilon(dx).$$

By (3.4) it follows that $\lim_{\varepsilon\to 0} I^1_\varepsilon(\gamma)=0$, while, recalling that

$$x-P_K(x)=n(P_K(x))d_K(x),$$

where $n$ is the outward normal, we get

$$\int_{\{d_K(x)\le\gamma\}}\langle\beta_\varepsilon(x),D\varphi(x)\rangle\nu_\varepsilon(dx)=\int_{\{d_K(x)\le\gamma\}}\langle\beta_\varepsilon(x),D\varphi(x)-D\varphi(P_Kx)\rangle\nu_\varepsilon(dx)$$

because $\varphi\in D(N_0)$ implies that $\langle D\varphi(P_Kx),N(P_Kx)\rangle=0$. This yields, via estimate (2.5), that

$$\begin{aligned}&\Big|\int_{\{d_K(x)\le\gamma\}}\langle\beta_\varepsilon(x),D\varphi(x)\rangle\nu_\varepsilon(dx)\Big|\\&\le\frac{C}{\varepsilon}\ \sup\{|D\varphi(x)-D\varphi(P_Kx)|:\ |x-P_K(x)|\le\gamma\}\overset{\gamma\downarrow 0}{\to}\ 0.\end{aligned}$$

Then by (3.3) we infer that

$$\int_{\mathbb{R}^d}N_0\varphi\nu(dx)=0\quad\forall\ \varphi\in D(N_0).\tag{3.5}$$

The latter implies, via standard arguments, that $N_0$ is dissipative in $L^2(H,\nu)$ as claimed. □

Now we are ready to prove the main result of this paper.

**Theorem 3.2.** *The closure $N$ of $N_0$ in $L^2(H,\nu)$ is $m$–dissipative. Moreover, for each $\varphi\in D(N)$ we have*

$$\int_{\mathbb{R}^d}N\varphi\ \varphi\ d\nu=-\frac{1}{2}\ \int_{\mathbb{R}^d}|D\varphi|^2\ d\nu.\tag{3.6}$$

**Proof**. Let $f\in C^1_b(\mathbb{R}^d)$, $\lambda>0$, and $\varepsilon>0$. Consider the following Neumann problem:

$$\begin{cases}\lambda\varphi_\varepsilon-\dfrac{1}{2}\ \Delta\varphi_\varepsilon+\langle F_\varepsilon,D\varphi_\varepsilon\rangle=f\quad\text{in }K^0\\[2mm]\dfrac{\partial\varphi_\varepsilon}{\partial n}=0\quad\text{on }\partial K.\end{cases}\tag{3.7}$$

To solve problem (3.7) one can consider an approximating problem,

$$\begin{cases} \lambda\varphi_{\varepsilon,R} - \dfrac{1}{2}\,\Delta\varphi_{\varepsilon,R} + \langle F_\varepsilon, D\varphi_{\varepsilon,R}\rangle = f \quad \text{in } K_R^0 \\ \dfrac{\partial\varphi_{\varepsilon,R}}{\partial n} = 0 \quad \text{on } \partial K_R, \end{cases} \tag{3.8}$$

where $K_R = K \cap B_R$ and $B_R$ is the ball in $H$ with center 0 and radius $R$. Problem (3.8) has a unique strict solution by classical results. Then one can prove that $\varphi_{\varepsilon,R}$ converges as $R \to \infty$ to a strict solution of (3.7); see [3]. Moreover, using the classical Bernstein method, one can show that the following estimate holds (see [2]):

$$\|D\varphi_\varepsilon\|_{C_b(K)} \le C(\lambda)\, \|Df\|_{C_b(K)}. \tag{3.9}$$

Since the boundary $\partial K$ is of class $C^2$ the solution $\varphi_\varepsilon$ of (3.7) extends across the boundary to a function (again denoted $\varphi_\varepsilon$) which belongs to the domain $D(N_0)$ of $N_0$. Consequently, we can write

$$\lambda\varphi_\varepsilon - N_0\varphi_\varepsilon = f - \langle F_\varepsilon(x) - F(x), D\varphi_\varepsilon\rangle.$$

We note that, taking into account (3.9), we have

$$\langle F_\varepsilon(x) - F(x), D\varphi_\varepsilon\rangle \overset{\varepsilon\downarrow 0}{\to} 0 \;\text{ in }\; L^2(\mathbb{R}^d, \nu),$$

and so by dissipativity of $N_0$ , $\{\varphi_\varepsilon\}$ is convergent in $L^2(\mathbb{R}^d, \nu)$. Hence $\overline{N_0} = N$ is $m$–dissipative in $L^2(K, \nu)$ as claimed.

Finally, if $\varphi \in D(N_0)$ (3.5) follows by integrating with respect to $\nu$ the elementary identity

$$N_0(\varphi^2) = 2\varphi\, N_0\varphi + |D\varphi|^2.$$

Since $N$ is the closure of $N_0$, (3.5) holds for all $\varphi \in D(N)$. □

**Corollary 3.3.** *For each $\lambda > 0$ and $f \in L^2(K, \nu)$ there exists a unique strong solution in the sense of Friedrichs, $\varphi \in W^{1,2}(K, \nu)$, to the equation*

$$\begin{cases} \lambda\varphi - \dfrac{1}{2}\,\Delta\varphi + \langle F(x), D\varphi\rangle = f \quad \text{in } D \\ \dfrac{\partial\varphi}{\partial n} = 0 \quad \text{on } \partial K. \end{cases} \tag{3.10}$$

Corollary 3.3 simply means that there is a sequence $\{f_n\} \subset L^2(K, \nu)$ and $\{\varphi_n\} \subset D(N_0)$ satisfying equation (3.7) with $f = f_n$ and converging to $\varphi$ as $n \to \infty$. This is precisely the definition of a strong Friedrichs solution. This was established in [6] for $F$ of gradient type.

## 4. The case of nonregular $K$

We shall assume here that $K$ is a closed, convex subset of $\mathbb{R}^d$, with nonempty interior and piecewise–$C^2$ boundary $\partial K$. This means that $\partial K = \Gamma_0 \cup \Gamma_1$ where $\Gamma_0$ has Lebesgue measure 0 while $\Gamma_1$ is open in $\partial K$ and of class $C^2$, i.e., locally can be represented as $x_n = \psi(x'),\ x' \in \mathbb{R}^{d-1}$ where $\psi \in C^2(\mathbb{R}^{d-1})$.

It is clear that all results of Section 2 remain true in the present situation. As regards Section 3, we define the operator $\widetilde{N_0}$ in $L^2(K,\nu)$ by

$$\begin{cases} \widetilde{N_0}\varphi = \dfrac{1}{2}\,\Delta\varphi - \langle F(x), D\varphi\rangle. \\ D(\widetilde{N_0}) = \Big\{\varphi \in C_b^1(\mathbb{R}^d) \cap C^2(K^0) \cap W^{2,\infty}(\mathbb{R}^d) : \\ \qquad\qquad \dfrac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma_1,\ \widetilde{N_0}\varphi \in L^2(K,\nu)\Big\}. \end{cases} \tag{4.1}$$

We have

**Lemma 4.1.** *There exists an invariant measure $\nu$ for $P_t$ with support in $K$, and $\widetilde{N_0}$ is dissipative in $L^2(K,\nu)$.*

The proof is essentially the same as Proposition 2.2 and Lemma 3.1 and will not be reproduced here. We note only that instead of the function $\Phi$ we shall take $\Phi_\zeta(x) = (d_K(x))_\zeta^{2m}$, where $(d_K(x))_\zeta$ is a smooth approximation of $d_K$, and let $\zeta$ tend to zero. We have also

**Theorem 4.2.** *The closure $\widetilde{N}$ of $\widetilde{N_0}$ in $L^2(H,\nu)$ is $m$–dissipative. Moreover, for each $\varphi \in D(\widetilde{N})$ we have*

$$\int_{\mathbb{R}^d} \widetilde{N}\varphi\,\varphi\,d\nu = -\frac{1}{2}\int_{\mathbb{R}^d} |D\varphi|^2\,d\nu. \tag{4.2}$$

**Proof**. Let $f \in C_b^1(\mathbb{R}^d)$, $\lambda > 0$, and $\varepsilon > 0$. Consider the following Neumann problem:

$$\begin{cases} \lambda\psi_\varepsilon - \dfrac{1}{2}\,\Delta\psi_\varepsilon + \langle F_\varepsilon, D\psi_\varepsilon\rangle = f \quad \text{in } K^0 \\ \dfrac{\partial\psi_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_1. \end{cases} \tag{4.3}$$

We claim that (4.3) has a solution $\psi_\varepsilon \in C_b^1(K) \cap C^2(K^0) \cap W^{2,\infty}(K^0)$.

Here is the argument. Consider a decreasing sequence $\{K_\mu\} \supset K$ of regular convex sets such that

$$\lim_{\mu\to 0} d(K_\mu, K) = 0, \tag{4.4}$$

where $d$ denotes the Hausdorff distance; see [7]. Then the problem

$$\begin{cases} \lambda\psi_\varepsilon^\mu - \dfrac{1}{2}\,\Delta\psi_\varepsilon^\mu + \langle F_\varepsilon, D\psi_\varepsilon^\mu\rangle = f \quad \text{in } K_\mu^0 \\ \dfrac{\partial\psi_\varepsilon^\mu}{\partial n} = 0 \quad \text{on } \partial K_\mu \end{cases} \tag{4.5}$$

has as seen above a unique classical solution $\psi_\varepsilon^\mu \in C_b^2(K_\mu)$, and the following estimate holds:

$$\|\psi_\varepsilon^\mu\|_{C_b^2(K_m)} \le C(\lambda, f), \quad \forall \varepsilon, \mu > 0,$$

for a suitable constant $C(\lambda, f)$. This implies that for $\mu \to 0$ we have $\psi_\varepsilon^\mu \to \psi_\varepsilon$ in $C_b^1(K)$ and weak$^*$ in $W^{2,\infty}(K)$. Moreover, it is readily seen that $\psi_\varepsilon$ is a weak solution to (4.3); i.e.,

$$\int_{K^0} \Big(\lambda\psi_\varepsilon\varphi + \frac{1}{2}\,\langle D\psi_\varepsilon, D\varphi\rangle\Big)dx + \int_{K^0} \langle F_\varepsilon, D\psi_\varepsilon\rangle\varphi\, dx = 0 \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^d).$$

Taking into account that $\psi_\varepsilon \in C_b^1(K)$, the latter implies that $\psi_\varepsilon$ satisfies (4.3) as claimed. This means that $\psi_\varepsilon \in D(\widetilde{N_0})$ (or more precisely has an extension to $\mathbb{R}^d$ which belongs to $D(\widetilde{N_0})$), and so we may conclude the proof of Theorem 4.2 as that of Theorem 3.2. □

## References

[1] V.I. Bogachev, N.V. Krylov, and M. Röckner, *On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions*, Comm. Partial Diff. Equations, 26 (2001), 11–12.

[2] M. Bertoldi and S. Fornaro, *Gradient estimates in parabolic problems with unbounded coefficients*, preprint.

[3] M. Bertoldi and L. Lorenzi, *Estimates of the derivatives for parabolic operators with unbounded coefficients*, in preparation.

[4] H. Brézis, "Analyse Fonctionnelle," Masson, Paris, 1983.

[5] E. Cepa, *Multivalued stochastic differential equations,* C.R. Acad. Sci. Paris, Ser 1, Math. 319 (1994), 1075–1078.

[6] G. Da Prato and A. Lunardi, *Elliptic operators with unbounded drift coefficients and Neumann boundary condition*, Preprint Università di Parma n. 306, 2002.

[7] H.G. Egglestone, "Convexity," Cambridge University Press, Cambridge, 1958.

[8] O.A. Ladyzhenskaja, V.A. Solonnikov, and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, Amer. Math. Soc. (1968).

[9] G. Lumer and R.S. Phillips, *Dissipative operators in a Banach space,* Pac. J. Math., 11 (1961), 679–698.

[10] A. Rascanu, *Deterministic and stochastic differential equations in Hilbert spaces involving multivalued maximal monotone operators*, PanAmerican Math. J., 3 (1996), 83–119.