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DYNAMICS OF PARABOLIC EQUATIONS: FROM CLASSICAL SOLUTIONS TO METASOLUTIONS

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Abstract. In this paper we describe the asymptotic behavior of the positive solutions of a class of parabolic equations according to the size of a certain parameter. Within the range of values of the parameter where the model does not admit an attracting classical steady state it possesses an attracting metasolution —a very weak generalized solution. It turns out that the minimal metasolution attracts all positive solutions starting in a subsolution and that the limiting profile of any other positive solution lies in the order interval defined by the minimal and the maximal metasolution.

1. Introduction

This paper provides a complete description of the asymptotic behaviour of the positive solutions of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x) f(x,u) u & \text{in} \quad \Omega \times (0,\infty)\,, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty)\,, \\ u(\cdot,0) = u_0 \geq 0 & \text{in} \quad \Omega\,, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$ of class $\mathcal{C}^3$, $\lambda \in \mathbb{R}$, and $a \geq 0$, $a \neq 0$, is a function of class $\mathcal{C}^\mu(\bar\Omega)$, for some $\mu \in (0,1]$, satisfying the following assumptions:

(Ha1) $\Omega_+ := \{\, x \in \Omega : a(x) > 0 \,\}$ is a subdomain of Ω with $\bar\Omega_+ \subset \Omega$ whose boundary, $\partial\Omega_+$, is of class $\mathcal{C}^3$, and

$$\lim_{\substack{x\in\Omega_+ \\ \text{dist}\,(x,\partial\Omega_+)\downarrow 0}} \frac{a(x)}{\text{dist}\,(x,\partial\Omega_+)} = 0\,. \tag{1.2}$$

(Ha2) The open set $\Omega_0 := \Omega \setminus \bar\Omega_+$ consists of two components, $\Omega_{0,i}$, $i \in \{1,2\}$, such that $\bar\Omega_{0,1} \cap \bar\Omega_{0,2} = \emptyset$ and $\bar\Omega_{0,2} \subset \Omega$. Subsequently, given a regular subdomain D of Ω and $V \in C^\mu(\bar D)$, we denote by $\sigma[-\Delta +$

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$V, D]$ the principal eigenvalue of $-\Delta + V$ in D under homogeneous Dirichlet boundary conditions. Using this notation, we also assume that

$$\sigma_1 := \sigma[-\Delta, \Omega_{0,1}] < \sigma_2 := \sigma[-\Delta, \Omega_{0,2}]\,. \tag{1.3}$$

In Figure 1 we have represented a typical situation where all assumptions on these domains are satisfied. It should be noted that $\Gamma = \partial\Omega$, $\Gamma_1 = \partial\Omega_{0,1} \setminus \partial\Omega$, $\Gamma_2 := \partial\Omega_{0,2}$, $\partial\Omega_+ = \Gamma_1 \cup \Gamma_2$. As for the nonlinearity, we suppose the following:

(Hf1) $f \in C^{\mu,1+\mu}(\bar\Omega \times [0,\infty))$ satisfies $f(x,0) = 0$ and $\partial_u f(x,u) > 0$ for all $u > 0$ and $x \in \Omega$. Moreover, there exists $f_b \in \mathcal{C}^{1+\mu}([0,\infty))$ such that $f_b(0) = 0$, $f_b(u) > 0$ and $f_b'(u) > 0$ for all $u > 0$, $\lim_{u\uparrow\infty} f_b(u) = \infty$, and $f(\cdot,u) \geq f_b(u)$ if $u \geq 0$. Obviously, this implies $\lim_{u\uparrow\infty} f(x,u) = \infty$ uniformly on $\bar\Omega$.

(Hf2) For any $\beta > 0$ and compact set $K \subset \Omega_+$ consider the auxiliary function

$$h_{K,\beta}(u) := \alpha_K u f_b(u) - \beta u \tag{1.4}$$

where $\alpha_K := \inf_{x\in K} a(x) > 0$. Let $u_{K,\beta}$ denote the unique positive zero of $h_{K,\beta}$; thanks to (Hf1), it is well defined. Then, we assume that for each pair (K,β) and $u_* > u_{K,\beta}$

$$I(u_*) := \int_{u_*}^{\infty} \Big[\int_{u_*}^{u} h_{K,\beta}(z)\,dz \Big]^{-1/2} du < \infty\,. \tag{1.5}$$

Condition (Hf2) is satisfied if there exist $\eta > 0$ and $p > 0$ such that $f_b(u) \geq \eta\, u^p$ for each $u \in [0,\infty)$; it entails the existence of a regular positive solution of the singular problem

$$\begin{cases} -u'' = \beta u - \alpha_K u f_b(u) & \text{in } (-R,R) \\ u(-R) = u(R) = \infty \end{cases}$$

for each $R > 0$ (cf. [8, Remarks 3.4]).

Actually, conditions (Hf1) and (Hf2) enable us to apply the interior estimates of J. B. Keller [7] and R. Osserman [10] to get a priori bounds in $L^\infty_{loc}(\Omega_+)$ for all classical solutions, large solutions, and metasolutions (to be defined below) of

$$\begin{cases} -\Delta u = \lambda u - a f(\cdot,u)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

(cf. [6] and [8]), and, consequently, to get a priori bounds in Ω_+ for all positive solutions of the evolutionary problem (1.1). It should be noted that any nonnegative solution of (1.6) lies in the Banach space $U := \{\, u \in C^{2+\mu}(\bar\Omega) \;:\; u|_{\partial\Omega} = 0 \,\}$. Moreover, if $u \in U \setminus \{0\}$ is a nonnegative solution of

(1.6), then combining the Krein-Rutman theorem with the strong maximum principle shows that $\lambda = \sigma[-\Delta + af(\cdot,u),\Omega]$ and that u is a positive multiple of the principal eigenfunction associated to λ. Hence, $u(x) > 0$ for all $x \in \Omega$ and $\frac{\partial u}{\partial n}(x) < 0$ for all $x \in \partial\Omega$, where n is the outward unit normal to Ω at x; i.e., u lies in the interior of the cone U^+ of nonnegative functions of U. On the other hand, since for any nonnegative solution u of (1.1)

$$\frac{\partial u}{\partial t} - \Delta u = \lambda u - af(\cdot,u)u \le \lambda u\,,$$

it follows that for each $u_0 \in C(\bar\Omega)$, $u_0 \ge 0$, (1.1) has a unique solution globally defined in time: $u_{[\lambda,\Omega]}(\cdot,\cdot;u_0) \in C^{2+\mu,1+\frac{\mu}{2}}(\bar\Omega \times (0,\infty))$. By the parabolic maximum principle, $u_{[\lambda,\Omega]}(\cdot,t;u_0) \in U^+$ for each $t > 0$ if $u_0 > 0$.

Throughout this paper, given a function v it is said that $v > 0$ if $v \ge 0$ and $v \ne 0$. Moreover, we set $\sigma_0 := \sigma[-\Delta,\Omega]$, $\tilde\Omega := \Omega_+ \cup \bar\Omega_{0,2} = \Omega \setminus \bar\Omega_{0,1}$. Then, by the monotonicity of the principal eigenvalue with respect to the underlying domain, (1.3) gives $\sigma_0 < \sigma_1 < \sigma_2$. The main goal of this paper is to characterize the asymptotic behavior of all positive solutions of (1.1) according to the various ranges of the parameter $\lambda \in \mathbb{R}$. It is well known that zero is a global attractor for all positive solutions of (1.1) if $\lambda \le \sigma_0$, and that the unique positive classical solution of (1.6) is a global attractor if $\sigma_0 < \lambda < \sigma_1$ (cf., e.g., [3] and [8]). The next theorem is the main result of this paper.

Theorem 1.1. *Suppose* $u_0 \in \mathcal{C}(\bar\Omega)$, $u_0 > 0$. *Then*

1.– *In case* $\sigma_1 \le \lambda < \sigma_2$, *the following assertions are true:*
   - (*a*) $\lim_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) = \infty$ *uniformly in compact subsets of* $\bar\Omega_{0,1} \setminus \partial\Omega$.
   - (*b*) $\mathfrak{L}^{\min}_{[\lambda,\tilde\Omega]} \le \liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \le \limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \le \mathfrak{L}^{\max}_{[\lambda,\tilde\Omega]}$ *in* $\tilde\Omega$, *where* $\mathfrak{L}^{\min}_{[\lambda,\tilde\Omega]}$ *and* $\mathfrak{L}^{\max}_{[\lambda,\tilde\Omega]}$ *stand for the minimal and the maximal regular solutions—large solutions in* $\tilde\Omega$*—of the singular boundary value problem*

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & in \quad \tilde\Omega\,, \\ u = \infty & on \quad \partial\tilde\Omega (= \Gamma_1)\,. \end{cases} \tag{1.7}$$

   - (*c*) *If, in addition,* u_0 *is a subsolution of* (1.6), *then*

$$\lim_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) = \mathfrak{L}^{\min}_{[\lambda,\tilde\Omega]} \qquad in\ \tilde\Omega\,,$$

   uniformly in compact subsets.

2.– *In case* $\lambda \ge \sigma_2$, *the following assertions are true:*

(*a*) $\lim_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)=\infty$ *uniformly in compact subsets of* $\Omega\setminus\Omega_+$.

(*b*) $\mathfrak{L}^{\min}_{[\lambda,\Omega_+]} \leq \liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leq \limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leq \mathfrak{L}^{\max}_{[\lambda,\Omega_+]}$ *in* Ω_+, *where* $\mathfrak{L}^{\min}_{[\lambda,\Omega_+]}$ *and* $\mathfrak{L}^{\max}_{[\lambda,\Omega_+]}$ *stand for the minimal and the maximal regular solutions, large solutions in* Ω_+, *of the singular boundary value problem*

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in} \quad \Omega_+\,, \\ u=\infty & \text{on} \quad \partial\Omega_+\,. \end{cases} \tag{1.8}$$

(*c*) *If, in addition,* u_0 *is a subsolution of* (1.6), *then*

$$\lim_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)=\mathfrak{L}^{\min}_{[\lambda,\Omega_+]} \qquad \text{in} \;\; \Omega_+\,,$$

uniformly in compact subsets.

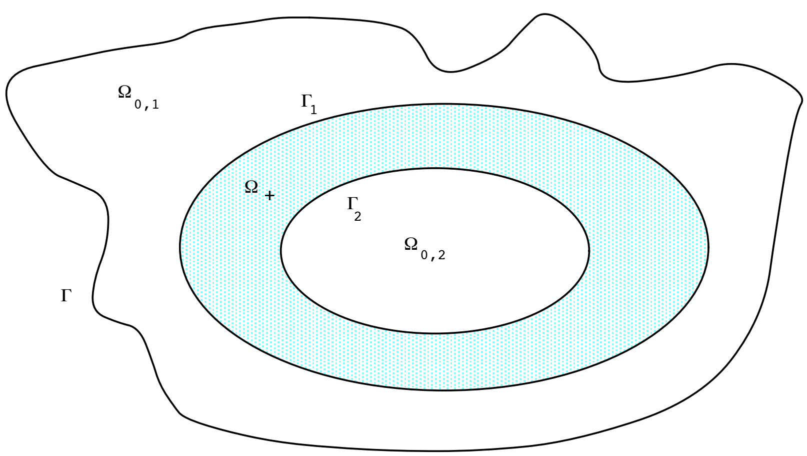

Figure 1: The nodal behavior of $a(x)$.

The solutions of (1.7) and (1.8) are referred to in the literature as *large solutions* supported in $\tilde\Omega$ and Ω_+, respectively. The extension by infinity to all of Ω of a large solution is said to be a metasolution; the concept of metasolution goes back to [5] and [6] (cf. [8] for further details), though publishing of [6] took almost four years. Theorem 1.1 establishes that within the range $\sigma_1<\lambda<\sigma_2$ the limiting behavior of the solutions of (1.1) is governed by the metasolutions of (1.6) supported in $\tilde\Omega$, while it is governed by the metasolutions of (1.6) supported in Ω_+ if $\lambda\geq\sigma_2$.

Parts 1(a) and 2(a) of Theorem 1.1 have been already shown in [8], and Parts 1(b) and 2(b) were stated in [8, Theorem 6.2], although the proof of [8, Theorem 6.2] contains a gap, since the comparison carried out in between formulas (6.4) and (6.5) is not necessarily true. Parts 1(c) and 2(c) of Theorem 1.1 are completely new, even in the simplest situation when $\Omega \setminus \bar{\Omega}_+$ is connected (cf. [2, Theorem 1.3]). It should be noted that, in general, large solutions, and, hence, metasolutions, supported in $\tilde{\Omega}$ and Ω_+ are not necessarily unique. As a consequence, the results of Parts 1(c) and 2(c) are not necessarily true when u_0 fails to be a subsolution of (1.6), except if the metasolution is unique. Some uniqueness results, when $f(x,u) = u^p$ for some $p > 0$ and $a(x)$ decays like a power of dist $(x, \partial\Omega_+)$ on $\partial\Omega_+$, can be found in [2] and [4] for the special case when $\Omega \setminus \bar{\Omega}_+$ is connected and the power is kept fixed, and in [9] for our general setting.

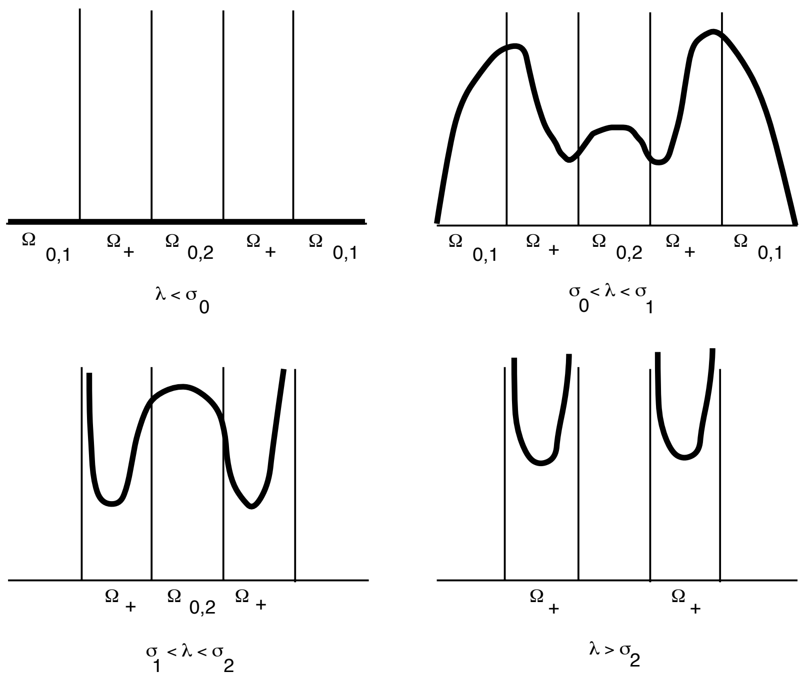

Figure 2: All possible limiting profiles as $t \nearrow \infty$.

In Figure 2 we have represented all possible limiting profiles according to each of the possible ranges of λ. In all cases we have represented a one-dimensional slice of the limiting profile. To discuss the diagram we assume that either we have uniqueness or else we are starting in a subsolution. In the last case we should refer to minimal metasolutions, instead of to metasolutions. When $\lambda < \sigma_0$, all solutions approach zero. When $\sigma_0 < \lambda < \sigma_1$ all solutions approach the unique positive steady state. As $\lambda \nearrow \sigma_1$, the steady states grow to infinity in $\Omega_{0,1}$ while they stabilize in $\tilde{\Omega} = \Omega_+ \cup \bar{\Omega}_{0,2}$;

the limiting steady state is the minimal metasolution of (1.6) supported in $\tilde{\Omega}$ for $\lambda = \sigma_1$. Quite naturally, this metasolution provides us with the limiting profile as $t \nearrow \infty$ of the solutions of (1.1) at $\lambda = \sigma_1$. Actually, the metasolutions of (1.6) supported in $\tilde{\Omega}$ provide us with the limiting profiles of the solutions of (1.1) for every $\sigma_1 < \lambda < \sigma_2$. As $\lambda \nearrow \sigma_2$ the metasolutions of (1.6) supported in $\tilde{\Omega}$ grow to infinity in $\bar{\Omega}_{0,2}$ while they stabilize to a bounded profile in Ω_+; the limiting generalized solution is the metasolution supported in Ω_+ for $\lambda = \sigma_2$. Such a metasolution is the limiting profile of all positive solutions of (1.1) as $t \nearrow \infty$. Actually, the metasolutions supported in Ω_+ provide us with the asymptotic behavior of all positive solutions of (1.1) for any $\lambda \geq \sigma_2$.

The layout of this paper is as follows. In Section 2 we collect some results from [8] and the references therein, those needed to prove Theorem 1.1. In Section 3 we prove Theorem 1.1.

It should be noted that the proof of Theorem 1.1 can be adapted to cover the general framework dealt with in [8]—the technical details will appear elsewhere. Here we have preferred to deal with simple assumptions.

2. Classical solutions, large solutions....and metasolutions

In this section we collect the results of [8] and the references therein that we shall use in the proof of Theorem 1.1.

2.1. **Classical steady-states.**

The following result comes from [3], [5], and [6].

Theorem 2.1. *The problem* (1.6) *possesses a positive solution if, and only if,* $\lambda \in (\sigma_0, \sigma_1)$*, and it is unique if it exists. It is denoted by* $\theta_{[\lambda,\Omega]}$. *Moreover,*

(*a*) *The branch* $\lambda \mapsto \theta_{[\lambda,\Omega]}$ *bifurcates from* $u = 0$ *at* $\lambda = \sigma_0$*; i.e.,*

$$\lim_{\lambda \searrow \sigma_0} \theta_{[\lambda,\Omega]} = 0 \,. \tag{2.1}$$

(*b*) 0 *is a global attractor for all positive solutions of* (1.1) *if* $\lambda \leq \sigma_0$*; i.e.,*

$$\lim_{t \nearrow \infty} u_{[\lambda,\Omega]}(\cdot, t; u_0) = 0 \qquad \text{in } \mathcal{C}^{2+\mu}(\bar{\Omega})$$

for each $u_0 > 0$.

(*c*) $\theta_{[\lambda,\Omega]}$ *is a global attractor for all positive solutions of* (1.1) *if* $\lambda \in (\sigma_0, \sigma_1)$*; i.e.,*

$$\lim_{t \nearrow \infty} u_{[\lambda,\Omega]}(\cdot, t; u_0) = \theta_{[\lambda,\Omega]} \qquad \text{in } \mathcal{C}^{2+\mu}(\bar{\Omega})$$

if $u_0 > 0$.

2.2. **Large solutions.** Given a subdomain $D \subset \Omega$, classical positive solutions of the singular problem

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in} \quad D\,, \\ u = \infty & \text{on} \quad \partial D \end{cases} \tag{2.2}$$

are called *large solutions* in D of

$$-\Delta u = \lambda u - af(\cdot,u)u\,. \tag{2.3}$$

A function $u \in \mathcal{C}^{2+\mu}(D)$ is said to be a solution of (2.2) if it solves (2.3) in D and

$$\lim_{\substack{x\in D \\ \text{dist}\,(x,\partial D)\searrow 0}} u(x) = \infty\,.$$

The following result, going back to [5], [6], and [8], characterizes the existence of large solutions of (2.3) in $D \in \{\tilde{\Omega}, \Omega_+\}$.

Theorem 2.2. *The following assertions are true:*

($\tilde{\Omega}$) *Equation* (2.3) *has a large solution in $\tilde{\Omega}$ if, and only if, $\lambda < \sigma_2$. Moreover, if this is the case, then there is a minimal and a maximal large solution, denoted by $\mathfrak{L}^{\min}_{[\lambda,\tilde{\Omega}]}$ and $\mathfrak{L}^{\max}_{[\lambda,\tilde{\Omega}]}$, respectively.*

(Ω_+) *For each $\lambda \in \mathbb{R}$ equation* (2.3) *has a large solution in Ω_+. Actually, there is a minimal and a maximal large solution, denoted by $\mathfrak{L}^{\min}_{[\lambda,\Omega_+]}$ and $\mathfrak{L}^{\max}_{[\lambda,\Omega_+]}$, respectively.*

Subsequently, for any constant $M > 0$, any subdomain $D \subset \Omega$ of class $\mathcal{C}^3$, and $u_0 \in \mathcal{C}(\bar{D})$, we denote by $u_{[\lambda,D,M]}(x,t;u_0)$ the unique solution of the evolutionary problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - af(\cdot,u)u & \text{in} \quad D \times (0,\infty) \\ u = M & \text{on} \quad \partial D \times (0,\infty) \\ u(\cdot,0) = u_0 \geq 0 & \text{in} \quad D. \end{cases} \tag{2.4}$$

Since $af(\cdot,u)u \geq 0$, $u_{[\lambda,D,M]}(x,t;u_0)$ is globally defined in time. Moreover, by parabolic regularity $u_{[\lambda,D,M]} \in C^{2+\mu,1+\frac{\mu}{2}}(\bar{D} \times (0,\infty))$. We also consider the associated elliptic problem

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in} \quad D \\ u = M & \text{on} \quad \partial D. \end{cases} \tag{2.5}$$

The next result goes back to [5] and [8].

Theorem 2.3. *The following assertions are true:*

($\tilde{\Omega}$) *Suppose $D = \tilde{\Omega}$. Then,* (2.5) *possesses a positive solution if, and only if, $\lambda < \sigma_2$, and it is unique if it exists. It is denoted by $\theta_{[\lambda,\tilde{\Omega},M]}$. Moreover, for each $\lambda < \sigma_2$, one has the following:*

(*a*) $\mathfrak{L}^{\min}_{[\lambda,\tilde\Omega]} = \lim_{M\nearrow\infty} \theta_{[\lambda,\tilde\Omega,M]}$.

(*b*) $\theta_{[\lambda,\tilde\Omega,M]}$ *is a global attractor for all positive solutions of* (2.4); *i.e., for each* $u_0 > 0$,

$$\lim_{t\nearrow\infty} u_{[\lambda,\tilde\Omega,M]}(\cdot,t;u_0) = \theta_{[\lambda,\tilde\Omega,M]} \qquad \text{in } \mathcal{C}^{2+\mu}(\bar{\tilde\Omega})\,.$$

(Ω_+) *Suppose* $D = \Omega_+$. *Then, for each* $\lambda \in \mathbb{R}$, (2.5) *possesses a unique positive solution, denoted by* $\theta_{[\lambda,\Omega_+,M]}$. *Moreover, for each* $\lambda \in \mathbb{R}$, *the following properties are satisfied:*

(a) $\mathfrak{L}^{\min}_{[\lambda,\Omega_+]} = \lim_{M\nearrow\infty} \theta_{[\lambda,\Omega_+,M]}$ *in* $\mathcal{C}^{2+\mu}(\Omega_+)$.

(b) $\theta_{[\lambda,\Omega_+,M]}$ *is a global attractor for all positive solutions of* (2.4); *i.e., for each* $u_0 > 0$,

$$\lim_{t\nearrow\infty} u_{[\lambda,\Omega_+,M]}(\cdot,t;u_0) = \theta_{[\lambda,\Omega_+,M]} \qquad \text{in } \mathcal{C}^{2+\mu}(\bar\Omega_+)\,.$$

Moreover, for each $D \in \{\tilde\Omega, \Omega_+\}$, *the mapping* $M \mapsto \theta_{[\lambda,D,M]}$ *is increasing, and, if* $\underline{u}$ *(respectively* $\overline{u}$*) is a subsolution (respectively supersolution) of* (2.5), *then* $\underline{u} \le \theta_{[\lambda,D,M]}$ *(respectively* $\overline{u} \ge \theta_{[\lambda,D,M]}$*).*

Remark 2.4. Theorem 2.3 is also true if $\tilde\Omega$ and Ω_+ are replaced by $\tilde\Omega_\delta := \{\, x \in \tilde\Omega \,:\, \operatorname{dist}(x,\partial\tilde\Omega) > \delta \,\}$ and $\Omega_+^\delta := \{\, x \in \Omega_+ \,:\, \operatorname{dist}(x,\partial\Omega_+) > \delta \,\}$, respectively, provided $\delta > 0$ is sufficiently small.

2.3. **Metasolutions.** Given $D \in \{\tilde\Omega, \Omega_+\}$, a function $\mathfrak{M}_{[\lambda,D]} \,:\, \Omega \to [0,\infty]$ is said to be a *metasolution of* (2.3)*—or* (1.6)*—supported in* D if there exists a large solution of (2.3) in D, say $\mathfrak{L}_{[\lambda,D]}$, for which

$$\mathfrak{M}_{[\lambda,D]} = \begin{cases} \infty & \text{in } \Omega \setminus \bar D \\ \mathfrak{L}_{[\lambda,D]} & \text{in } D. \end{cases}$$

In other words, metasolutions are extensions by ∞ to the whole of Ω of large solutions in D.

Theorem 2.5. *The following assertions are true:*

($\tilde\Omega$) *Equation* (2.3) *possesses a metasolution supported in* $\tilde\Omega$ *if, and only if,* $\lambda < \sigma_2$. *Moreover, if this is the case, then there is a minimal and a maximal metasolution supported in* $\tilde\Omega$, *denoted by* $\mathfrak{M}^{\min}_{[\lambda,\tilde\Omega]}$ *and* $\mathfrak{M}^{\max}_{[\lambda,\tilde\Omega]}$, *respectively.*

(Ω_+) *For each* $\lambda \in \mathbb{R}$, *equation* (2.3) *possesses a metasolution supported in* Ω_+. *Moreover, for each* $\lambda \in \mathbb{R}$, *there is a minimal and a maximal metasolution supported in* Ω_+, *denoted by* $\mathfrak{M}^{\min}_{[\lambda,\Omega_+]}$ *and* $\mathfrak{M}^{\max}_{[\lambda,\Omega_+]}$, *respectively.*

Furthermore, the following fundamental relations are satisfied:

$$\lim_{\lambda\nearrow\sigma_1}\theta_{[\lambda,\Omega]}=\mathfrak{M}^{\min}_{[\sigma_1,\tilde\Omega]}\,,\qquad \lim_{\lambda\nearrow\sigma_2}\mathfrak{M}^{\min}_{[\lambda,\tilde\Omega]}=\mathfrak{M}^{\min}_{[\sigma_2,\Omega_+]}\,. \tag{2.6}$$

The first limit is uniform on any compact subset of $\bar\Omega_{0,1}\setminus\partial\Omega$ *and* $\tilde\Omega$*, separately—not in* $\Omega=\big(\bar\Omega_{0,1}\setminus\partial\Omega\big)\cup\tilde\Omega$*. The second limit is uniform on any compact subset of* $\Omega\setminus\Omega_+$ *and* Ω_+*, separately. Therefore, these limits exist in* $\mathcal{C}^{2+\mu}_{\rm loc}(\bar\Omega_{0,1}\setminus\partial\Omega)$ *and* $\mathcal{C}^{2+\mu}_{\rm loc}(\tilde\Omega)$*, and in* $\mathcal{C}^{2+\mu}_{\rm loc}(\Omega\setminus\Omega_+)$ *and* $\mathcal{C}^{2+\mu}_{\rm loc}(\Omega_+)$*, respectively, by elliptic regularity.*

Actually, for each $D\in\{\tilde\Omega,\Omega_+\}$, we have that

$$\mathfrak{M}^{\max(\min)}_{[\lambda,D]}=\begin{cases}\infty & \text{in }\ \Omega\setminus\bar D\,,\\ \mathfrak{L}^{\max(\min)}_{[\lambda,D]} & \text{in }\ D\,.\end{cases} \tag{2.7}$$

Henceforth, the first relationship of (2.6) reads as follows:

$$\lim_{\lambda\nearrow\sigma_1}\theta_{[\lambda,\Omega]}=\begin{cases}\infty & \text{in }\ \bar\Omega_{0,1}\setminus\partial\Omega\,,\\ \mathfrak{L}^{\min}_{[\sigma_1,\tilde\Omega]} & \text{in }\ \tilde\Omega=\Omega\setminus\bar\Omega_{0,1}\,,\end{cases} \tag{2.8}$$

uniformly in $\bar\Omega_{0,1}\setminus\partial\Omega$ and $\tilde\Omega$—separately, whereas the second one can be stated under the form

$$\lim_{\lambda\nearrow\sigma_2}\mathfrak{L}^{\min}_{[\lambda,\tilde\Omega]}=\begin{cases}\infty & \text{in }\ \bar\Omega_{0,2}\,,\\ \mathfrak{L}^{\min}_{[\sigma_2,\Omega_+]} & \text{in }\ \Omega_+\,,\end{cases} \tag{2.9}$$

uniformly in $\bar\Omega_{0,2}$ and Ω_+, separately.

It should be noted that along the components Γ_1 and Γ_2 those pointwise limits blow up.

3. Proof of Theorem 1.1

Throughout this section, $u_0\in\mathcal{C}(\bar\Omega)$, $u_0>0$. Suppose

$$\sigma_1\le\lambda<\sigma_2\,. \tag{3.1}$$

Thanks to the parabolic maximum principle, for each $\varepsilon>0$ and $t\ge 0$,

$$u_{[\lambda,\Omega]}(\cdot,t;u_0)\ge u_{[\sigma_1-\varepsilon,\Omega]}(\cdot,t;u_0)\,,$$

since $\lambda>\sigma_1-\varepsilon$. Thus, thanks to Theorem 2.1(c),

$$\liminf_{t\nearrow\infty}u_{[\lambda,\Omega]}(\cdot,t;u_0)\ge\lim_{t\nearrow\infty}u_{[\sigma_1-\varepsilon,\Omega]}(\cdot,t;u_0)=\theta_{[\sigma_1-\varepsilon,\Omega]}\,. \tag{3.2}$$

As (3.2) is valid for each sufficiently small $\varepsilon>0$, using (2.6) it is apparent that

$$\liminf_{t\nearrow\infty}u_{[\lambda,\Omega]}(\cdot,t;u_0)\ge\lim_{\varepsilon\searrow 0}\theta_{[\sigma_1-\varepsilon,\Omega]}=\mathfrak{M}^{\min}_{[\sigma_1,\tilde\Omega]}\,. \tag{3.3}$$

In particular (cf. (2.8)),

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)=\infty \quad \text{unif. in compact subsets of} \quad \bar\Omega_{0,1}\setminus\partial\Omega=\Omega\setminus\tilde\Omega\,. \tag{3.4}$$

Thus, the limit is uniform on the component $\Gamma_1(=\partial\tilde\Omega)$ of $\partial\Omega_+$. So, for each $M>0$ there exists a constant $T_M>0$ such that

$$u_{[\lambda,\Omega]}(x,t;u_0)\geq M \qquad \text{for each} \quad (x,t)\in\partial\tilde\Omega\times[T_M,\infty),$$

and, hence, $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ provides us with a supersolution of the evolutionary problem

$$\begin{cases} \frac{\partial v}{\partial t}-\Delta v=\lambda v-af(\cdot,v)v & \text{in} \quad \tilde\Omega\times(T_M,\infty)\\ v=M & \text{on} \quad \partial\tilde\Omega\times(T_M,\infty)\\ v(\cdot,0)=u_{[\lambda,\Omega]}(\cdot,T_M;u_0) & \text{in} \quad \tilde\Omega. \end{cases} \tag{3.5}$$

By the parabolic maximum principle, this implies that, for each $(x,t)\in\tilde\Omega\times(T_M,\infty)$,

$$u_{[\lambda,\Omega]}(x,t;u_0)\geq u_{[\lambda,\tilde\Omega,M]}\left(x,t-T_M;u_{[\lambda,\Omega]}(\cdot,T_M;u_0)\right).$$

So, thanks to Theorem 2.3($\tilde\Omega$)(b), we have that

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)\geq\lim_{t\nearrow\infty}u_{[\lambda,\tilde\Omega,M]}\left(\cdot,t-T_M;u_{[\lambda,\Omega]}(\cdot,T_M;u_0)\right)=\theta_{[\lambda,\tilde\Omega,M]}$$

in $\tilde\Omega$. Therefore, passing to the limit as $M\nearrow\infty$, Theorem 2.3($\tilde\Omega$)(a) gives

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)\geq\mathfrak{L}^{\min}_{[\lambda,\tilde\Omega]} \qquad \text{in} \quad \tilde\Omega\,.$$

Now, combining this estimate with (3.4), it is easily realized that

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)\geq\mathfrak{M}^{\min}_{[\lambda,\tilde\Omega]} \qquad \text{in} \quad \Omega\,. \tag{3.6}$$

It should be noted that this estimate is independent of u_0, and that it is substantially sharper than (3.3) when $\lambda>\sigma_1$. Estimate (3.6) provides us with Theorem 1.1 1(a) and the lower estimate of Theorem 1.1 1(b).

Now, we assume, in addition, that u_0 is a subsolution of (1.6). Then, for each $t>0$, the function $u_t(x):=u_{[\lambda,\Omega]}(x,t;u_0)$, $x\in\bar\Omega$, is a subsolution of (1.6) (cf. [11]), because $t\mapsto u_{[\lambda,\Omega]}(x,t;u_0)$ is increasing. Now, fix $t>0$ and set $M_t:=\max_{\bar{\tilde\Omega}}u_{[\lambda,\Omega]}(\cdot,t;u_0)$. Then, due to the last assertion of Theorem 2.3, for each $M\geq M_t$ we have that

$$u_{[\lambda,\Omega]}(\cdot,t;u_0)\leq\theta_{[\lambda,\tilde\Omega,M]} \qquad \text{in} \quad \tilde\Omega,$$

and, hence, Theorem 2.3($\tilde\Omega$)(a) implies

$$u_{[\lambda,\Omega]}(\cdot,t;u_0)\leq\lim_{M\nearrow\infty}\theta_{[\lambda,\tilde\Omega,M]}=\mathfrak{L}^{\min}_{[\lambda,\tilde\Omega]} \qquad \text{in} \quad \tilde\Omega\,.$$

Therefore,

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \le \mathfrak{L}^{\min}_{[\lambda,\tilde\Omega]} \qquad \text{in} \quad \tilde\Omega,$$

and it is apparent from (3.6) that

$$\lim_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) = \mathfrak{M}^{\min}_{[\lambda,\tilde\Omega]} \qquad \text{in} \quad \Omega\,. \tag{3.7}$$

This relation provides us with Theorem 1.1 1(c).

Now, suppose $u_0 > 0$ is arbitrary—not necessarily a subsolution of (1.6)—and

$$\lambda \ge \sigma_2\,. \tag{3.8}$$

Then, thanks to the parabolic maximum principle, for each $\varepsilon > 0$ and $t > 0$, we have that

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \ge u_{[\sigma_2-\varepsilon,\Omega]}(\cdot,t;u_0) \qquad \text{in} \quad \Omega,$$

and, hence, thanks to (3.6), for each sufficiently small $\varepsilon > 0$,

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \ge \liminf_{t\nearrow\infty} u_{[\sigma_2-\varepsilon,\Omega]}(\cdot,t;u_0) \ge \mathfrak{M}^{\min}_{[\sigma_2-\varepsilon,\tilde\Omega]} \qquad \text{in} \quad \Omega \tag{3.9}$$

since $\sigma_1 < \sigma_2 - \varepsilon < \sigma_2$. Note that we cannot apply (3.7), because u_0 is not necessarily a subsolution of (1.6).

As (3.9) holds true for each sufficiently small $\varepsilon > 0$, and, thanks to the second relation in (2.6), we have

$$\lim_{\varepsilon\searrow 0} \mathfrak{M}^{\min}_{[\sigma_2-\varepsilon,\tilde\Omega]} = \mathfrak{M}^{\min}_{[\sigma_2,\Omega_+]}\,,$$

we obtain from (3.9) that

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \ge \mathfrak{M}^{\min}_{[\sigma_2,\Omega_+]} \qquad \text{in} \quad \Omega\,. \tag{3.10}$$

In particular,

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) = \infty \qquad \text{in} \quad \Omega\setminus\Omega_+\,, \tag{3.11}$$

which concludes the proof of Theorem 1.1 2(a). Moreover, (3.11) is uniform on $\partial\Omega_+$. Thus, for each $M > 0$ there exists a constant $T_M > 0$ such that

$$u_{[\lambda,\Omega]}(x,t;u_0) \ge M \qquad \text{for all} \quad (x,t) \in \partial\Omega_+ \times [T_M,\infty)\,.$$

Now, changing the role of $\tilde\Omega$ in the proof of (3.6) by Ω_+ gives

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \ge \mathfrak{L}^{\min}_{[\lambda,\Omega_+]} \qquad \text{in} \quad \Omega_+\,.$$

Thus, taking into account (3.11), we find that

$$\liminf_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \ge \mathfrak{M}^{\min}_{[\lambda,\Omega_+]} \qquad \text{in} \quad \Omega\,. \tag{3.12}$$

Relation (3.12) provides us with the lower estimate of Theorem 1.1 2(b).

Now, besides (3.8), suppose u_0 is a subsolution of (1.6). Then, for each $t > 0$, the function $u_{[\lambda,\Omega]}(\cdot, t; u_0)$ is a subsolution of (1.6), and, hence, changing the role of $\tilde{\Omega}$ in the proof of (3.7) by Ω_+ it is easy to see that

$$\limsup_{t \nearrow \infty} u_{[\lambda,\Omega]}(\cdot, t; u_0) \leq \mathfrak{L}^{\min}_{[\lambda,\Omega_+]} \qquad \text{in} \quad \Omega_+ .$$

Therefore, due to (3.12), we find that

$$\lim_{t \nearrow \infty} u_{[\lambda,\Omega]}(\cdot, t; u_0) = \mathfrak{M}^{\min}_{[\lambda,\Omega_+]} \qquad \text{in} \quad \Omega . \tag{3.13}$$

This concludes the proof of Theorem 1.1 2(c).

To complete the proof of the theorem it remains to obtain the upper estimates when $u_0 > 0$ is an arbitrary positive initial datum. The strategy adopted here to get these estimates consists in obtaining bounds in Ω_+ for any positive solution of (1.1). Such bounds can be obtained arguing as follows. Fix $\lambda \geq \sigma_1$ and freeze the solution of (1.1) at time 1 by considering the function $u_{[\lambda,\Omega]}(\cdot, 1; u_0)$. By the parabolic maximum principle, $u_{[\lambda,\Omega]}(\cdot, 1; u_0)$ lies in the interior of the cone of positive functions of $\mathcal{C}^1_0(\bar{\Omega})$. Thus, there exists a constant $\kappa = \kappa(u_0) > 0$ for which

$$u_{[\lambda,\Omega]}(\cdot, 1; u_0) < \kappa\, \varphi \tag{3.14}$$

where φ is the principal eigenfunction associated with σ_0. We claim that there exists

$$\Lambda > \max\{\lambda, \sigma_2\}$$

for which $\kappa\, \varphi$ is a subsolution of

$$\begin{cases} -\Delta u = \Lambda u - a f(\cdot, u) u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.15}$$

Indeed, since $\kappa\varphi = 0$ on $\partial\Omega$, $\kappa\varphi$ is a subsolution of (3.15) if, and only if,

$$-\Delta(\kappa\varphi) \leq \Lambda\kappa\varphi - a f(\cdot, \kappa\varphi)\kappa\varphi \qquad \text{in} \quad \Omega$$

or, equivalently,

$$a f(\cdot, \kappa\varphi) \leq \Lambda - \sigma_0 \qquad \text{in} \quad \Omega,$$

which is satisfied for any Λ sufficiently large. Now, thanks to the parabolic maximum principle, (3.14) implies that, for any $(x, t) \in \Omega \times (0, \infty)$,

$$u_{[\lambda,\Omega]}(\cdot, t+1; u_0) = u_{[\lambda,\Omega]}(\cdot, t; u_{[\lambda,\Omega]}(\cdot, 1; u_0)) \leq u_{[\lambda,\Omega]}(\cdot, t; \kappa\varphi) .$$

Similarly, $\Lambda > \lambda$ implies

$$u_{[\lambda,\Omega]}(\cdot, t; \kappa\varphi) \leq u_{[\Lambda,\Omega]}(\cdot, t; \kappa\varphi)$$

and, hence, for each $t > 0$,

$$u_{[\lambda,\Omega]}(\cdot, t+1; u_0) \le u_{[\Lambda,\Omega]}(\cdot, t; \kappa\varphi) \qquad \text{in} \quad \Omega\,. \tag{3.16}$$

Since $\kappa\varphi$ is a subsolution of (3.15), it follows from Theorem 1.1 2(c) that

$$\lim_{t\nearrow\infty} u_{[\Lambda,\Omega]}(\cdot, t; \kappa\varphi) = \mathfrak{L}^{\min}_{[\Lambda,\Omega]} \qquad \text{in} \quad \Omega_+\,.$$

Thus, thanks to (3.16),

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(\cdot, t+1; u_0) \le \mathfrak{L}^{\min}_{[\Lambda,\Omega]} \qquad \text{in} \quad \Omega_+, \tag{3.17}$$

and, therefore, $u_{[\lambda,\Omega]}(\cdot, t; u_0)$ is uniformly bounded above in any compact subset of Ω_+ for each $t > 0$. This provides us with the necessary a priori bounds to conclude the proof of the theorem.

Now, suppose (3.1), and, for each sufficiently small $\delta > 0$, consider

$$\tilde{\Omega}_\delta := \{\, x \in \tilde{\Omega} \ : \ \text{dist}\,(x, \partial\tilde{\Omega}) > \delta \,\}\,.$$

Since $\partial\tilde{\Omega}_\delta \subset \Omega_+$ and thanks to the existence of a priori bounds, there exists a constant $M_0 > 0$ such that, for each $M \ge M_0$ and $t \ge 0$,

$$u_{[\lambda,\Omega]}(\cdot, t; u_0) \le M \qquad \text{on} \quad \partial\tilde{\Omega}_\delta\,.$$

Thus, the parabolic maximum principle shows that

$$u_{[\lambda,\Omega]}(x, t; u_0) \le u_{[\lambda,\tilde{\Omega}_\delta,M]}(x, t; u_0) \qquad \text{for each} \quad (x,t) \in \tilde{\Omega}_\delta \times [0,\infty)\,,$$

where $u_{[\lambda,\tilde{\Omega}_\delta,M]}(x, t; u_0)$ stands for the solution of (2.4) for $D = \tilde{\Omega}_\delta$. Since the vanishing region of $a(x)$ in $\tilde{\Omega}_\delta$ equals $\Omega_{0,2}$ and we are assuming that $\lambda < \sigma_2$, Theorems 2.1 and 2.3 also apply in $\tilde{\Omega}_\delta$, besides in Ω. Maintaining the notation introduced there, we have that, in $\tilde{\Omega}_\delta$,

$$\limsup_{t\nearrow\infty} u_{[\lambda,\tilde{\Omega}_\delta,M]}(x, t; u_0) = \theta_{[\lambda,\tilde{\Omega}_\delta,M]}\,.$$

Hence, for each sufficiently small $\delta > 0$, we have that

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(x, t; u_0) \le \theta_{[\lambda,\tilde{\Omega}_\delta,M]} \qquad \text{in} \quad \tilde{\Omega}_\delta\,.$$

As this estimate holds for all $M \ge M_0$, passing to the limit as $M \nearrow \infty$ gives

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(x, t; u_0) \le \mathfrak{L}^{\min}_{[\lambda,\tilde{\Omega}_\delta]} \qquad \text{in} \quad \tilde{\Omega}_\delta\,. \tag{3.18}$$

Now, let $\delta_n > 0$, $n \ge 1$, be a sequence such that $\lim_{n\to\infty} \delta_n = 0$. By the existence of a priori bounds on compact subsets of Ω_+ and using a rather standard compactness argument it is easy to see that there exists a subsequence δ_{n_m}, $m \geq 1$, for which

$$\mathfrak{L}^*_{[\lambda,\tilde{\Omega}]} := \inf_{m\geq 1} \mathfrak{L}^{\min}_{[\lambda,\tilde{\Omega}_{\delta_{n_m}}]} \tag{3.19}$$

is a large solution of (2.3) in $\tilde{\Omega}$. It should be noted that condition $\lambda < \sigma_2$ provides a priori bounds in $\bar{\Omega}_{0,2}$ from bounds in Ω_+. Let $\mathfrak{L}_{[\lambda,\tilde{\Omega}]}$ be any other large solution of (2.3) in $\tilde{\Omega}$. By a rather standard comparison argument involving the strong maximum principle, the monotonicity of the nonlinearity implies that, for each sufficiently small $\delta > 0$,

$$\mathfrak{L}_{[\lambda,\tilde{\Omega}]}\Big|_{\tilde{\Omega}_\delta} \leq \mathfrak{L}^{\min}_{[\lambda,\tilde{\Omega}_\delta]} \quad \text{in} \quad \tilde{\Omega}_\delta\,.$$

In particular, there exists $m_0 \geq 1$ such that, for each $m \geq m_0$,

$$\mathfrak{L}_{[\lambda,\tilde{\Omega}]}\Big|_{\tilde{\Omega}_{\delta_{n_m}}} \leq \mathfrak{L}^{\min}_{[\lambda,\tilde{\Omega}_{\delta_{n_m}}]} \quad \text{in} \quad \tilde{\Omega}_{\delta_{n_m}}\,. \tag{3.20}$$

Thanks to (3.19), we find from (3.20) that

$$\mathfrak{L}_{[\lambda,\tilde{\Omega}]} \leq \mathfrak{L}^*_{[\lambda,\tilde{\Omega}]} \quad \text{in} \quad \tilde{\Omega},$$

and, hence,

$$\mathfrak{L}^*_{[\lambda,\tilde{\Omega}]} = \mathfrak{L}^{\max}_{[\lambda,\tilde{\Omega}]}\,,$$

by definition of maximal large solution. Consequently, (3.19) implies

$$\mathfrak{L}^{\max}_{[\lambda,\tilde{\Omega}]} := \inf_{m\geq 1} \mathfrak{L}^{\min}_{[\lambda,\tilde{\Omega}_{\delta_{n_m}}]},$$

and, therefore, since the previous argument can be repeated along any convergent sequence,

$$\mathfrak{L}^{\max}_{[\lambda,\tilde{\Omega}]} := \inf_{\delta>0} \mathfrak{L}^{\min}_{[\lambda,\tilde{\Omega}_\delta]}\,.$$

Moreover, passing to the limit as $\delta \searrow 0$ in (3.18) gives

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(x,t;u_0) \leq \mathfrak{L}^{\max}_{[\lambda,\tilde{\Omega}]} \qquad \text{in} \quad \tilde{\Omega}\,. \tag{3.21}$$

This estimate concludes the proof of Theorem 1.1 1.

Finally, suppose $\lambda \geq \sigma_2$ and, for each sufficiently small $\delta > 0$, consider the set $\Omega^\delta_+ := \{\, x \in \Omega_+ : \operatorname{dist}(x, \partial\Omega_+) > \delta \,\}$. Arguing as above, there exists a constant $M_0 > 0$ such that, for each $M \geq M_0$ and $t \geq 0$,

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leq M \qquad \text{on} \quad \partial\Omega^\delta_+ \subset \Omega_+,$$

and, hence,

$$u_{[\lambda,\Omega]}(x,t;u_0) \leq u_{[\lambda,\Omega^\delta_+,M]}(x,t;u_0) \qquad \text{for each} \quad (x,t) \in \Omega^\delta_+ \times [0,\infty)\,.$$

Moreover, in Ω_+^δ, we have that

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega_+^\delta,M]}(x,t;u_0) = \theta_{[\lambda,\Omega_+^\delta,M]}\,.$$

Henceforth, for each sufficiently small $\delta > 0$, we see that

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(x,t;u_0) \le \theta_{[\lambda,\Omega_+^\delta,M]} \qquad \text{in} \quad \Omega_+^\delta\,.$$

As this estimate holds for all $M \ge M_0$, passing to the limit as $M \nearrow \infty$ gives

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(x,t;u_0) \le \mathfrak{L}^{\min}_{[\lambda,\Omega_+^\delta]} \qquad \text{in} \quad \Omega_+^\delta\,.$$

Therefore, passing to the limit as $\delta \searrow 0$, we find that

$$\limsup_{t\nearrow\infty} u_{[\lambda,\Omega]}(x,t;u_0) \le \mathfrak{L}^{\max}_{[\lambda,\Omega_+]} \qquad \text{in} \quad \Omega_+\,.$$

The last estimate can be obtained by adapting the proof of (3.21). This concludes the proof of the theorem. □

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