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NONLINEAR INTEGRO-DIFFERENTIAL EVOLUTION PROBLEMS ARISING IN OPTION PRICING: A VISCOSITY SOLUTIONS APPROACH

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Abstract. A class of nonlinear integro-differential Cauchy problems is studied by means of the viscosity solutions approach. In view of financial applications, we are interested in continuous initial data with exponential growth at infinity. Existence and uniqueness of solution is obtained through Perron's method, via a comparison principle; besides, a first order regularity result is given. This extension of the standard theory of viscosity solutions allows to price derivatives in jump–diffusion markets with correlated assets, even in the presence of a large investor, by means of the PDE's approach. In particular, derivatives may be perfectly hedged in a completed market.

1. Introduction

This paper is devoted to the study of the Cauchy problem related to integro–differential equations of the following type

$$\partial_t u + F(x,t,u,\mathcal{I}u,Du,D^2u) = 0, \qquad (x,t) \in \mathbb{R}^{\mathrm{N}} \times (0,T], \tag{1.1}$$

$$u(x,0) = u_o(x), \qquad x \in \mathbb{R}^{\mathrm{N}}. \tag{1.2}$$

The integral term $\mathcal{I}u$ is given by

$$\mathcal{I}u(x,t) = \int_{\mathbb{R}^{\mathrm{N}}} M\big(u(x+z,t),u(x,t)\big)d\mu_{x,t}(z), \tag{1.3}$$

where $\mu_{x,t}$ are bounded positive Radon measures fulfilling

$$\lim_{(y,s)\to(x,t)} \int_{\mathbb{R}^{\mathrm{N}}} \varphi(z)d\mu_{y,s}(z) = \int_{\mathbb{R}^{\mathrm{N}}} \varphi(z)d\mu_{x,t}(z), \tag{1.4}$$

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for any continuous function with bounded support φ, and M is a Lipschitz continuous function, nondecreasing with respect to its first argument:

$$\begin{aligned} &|M(u,v) - M(u',v')| \le \mathrm{M}\left(|u-u'| + |v-v'|\right), \\ &M(u,u) = 0, \quad M(u,v) \le M(u',v) \ \text{ if } u \le u'. \end{aligned} \tag{1.5}$$

All results may be trivially extended to any vector valued nonlocal term $\mathcal{I}u = \left(\mathcal{I}^1 u, \cdots, \mathcal{I}^{\mathrm{P}} u\right)$.

We make the following general assumptions

F.0) $u_o \in \mathcal{C}(\mathbb{R}^{\mathrm{N}})$,

F.1) $F \in \mathcal{C}\left([0,T] \times \mathbb{R}^{\mathrm{N}} \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}^{\mathrm{N}}\right)$ is degenerate elliptic, namely

$$F(x,t,u,p,X) \ge F(x,t,u,p,Y) \qquad \text{if } X \le Y,$$

F.2) F is quasi–monotone as a function of u, namely there exists $g \in \mathcal{C}([0,T])$ such that $g(0) = 0$, $\int_0^1 \frac{dr}{g(r)} = +\infty$ and

$$F(x,t,u,p,X) \ge F(x,t,v,p,X) - g(u-v) \qquad \text{if } u \ge v,$$

F.3) F is monotone non-increasing with respect to the nonlocal term $\mathcal{I}u$:

$$F(x,t,u,I,p,X) \ge F(x,t,u,J,p,X) \qquad \text{if } I \le J.$$

Here $\mathcal{S}^{\mathrm{N}}$ stands for the set of symmetrical $\mathrm{N} \times \mathrm{N}$ matrices.

Integro–differential parabolic operators of type

$$\partial_t u - \operatorname{div}\left(\mathcal{A}(x,t,u,Du)Du\right) + b(x,t,u,Du) = \mathcal{J}u, \tag{1.6}$$

where $\mathcal{A}$ is a positive definite matrix and

$$\mathcal{J}u = \int_{\mathbb{R}^{\mathrm{N}}} \left[u(x+z,t) - u(x,t) - z \cdot Du(x,t)\right] \mu_{x,t}(dz), \tag{1.7}$$

have been extensively studied, see for instance Garroni and Menaldi [10] and the references therein. Even though the nonlocal terms arising in derivatives pricing may always be written in the form (1.7) (see later on Remark 2.6) their technique may not be applied to many equations of financial interest, for two reasons. The first one, which does not seem to be overcome, is that they deal at most with weak degeneracy, namely $\mathcal{A} \ge \omega(x) I_{\mathrm{N}}$, with $\omega > 0$ almost everywhere. The second one is that it is not clear how to handle a nonlinear dependence on the jump term $\mathcal{J}$.

Therefore, it is natural to look to the theory of viscosity solutions, which allows to deal with strongly degenerate equations. For instance, using this approach Alvarez and Tourin [1] studied the Cauchy problem concerning an integro-differential equation of type

$$\partial_t u + F(x,t,u,Du,D^2u) = \mathcal{I}u, \tag{1.8}$$

where the nonlocal term is given by (1.3)–(1.5), and F is a pure differential second order operator, possibly degenerate elliptic. In particular, they proved the well posedness in the class of continuous functions with linear growth for large values of $|x|$. In this paper we follow this approach and we extend the results of [1] in order to handle the wider class (1.1).

The motivation for such improvement of the known theory relies in the study of derivatives pricing in jump–diffusion markets with large investor, i.e., in that, cases where the agent policy effects the assets prices, since in this framework the pricing equation has a nonlinear behavior with respect to the integral term $\mathcal{I}u$. In Section 2, we shall give more details about the relationship between the option pricing formula and integro–partial differential problems of type (1.1)–(1.2). For the time being we just point out that, in view of these financial applications, we are interested into well posedness results and local Lipschitz regularity of the solutions to the Cauchy problem, when the initial datum has exponential growth at infinity

$$\textbf{(H.0)} \qquad e^{-n_o|x|}u_o(x) \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}) \cap L^\infty(\mathbb{R}^{\mathrm{N}}),$$

or, respectively, when the first order derivatives of the initial datum grow up exponentially at infinity

$$\textbf{(H}'\textbf{.0)} \qquad e^{-n|x|}u_o(x) \in W^{1,\infty}(\mathbb{R}^{\mathrm{N}}).$$

This is the case, for instance, when pricing any call option, since u_o is given by $u_o(x) = \max\{e^x - e^K, 0\}$, where K is the strike price of the option.

To this aim, we shall focus our interest into the semilinear case

$$\partial_t u + \mathcal{L}_{\mathcal{I}} u + H(x, t, u, \mathcal{I}u, Du) = 0, \tag{1.9}$$

where

$$\mathcal{L}_{\mathcal{I}} u = -\frac{1}{2}\mathrm{tr}\big[\sigma\sigma^T D^2 u\big] + b \cdot Du + c\, u - \mathcal{I}u$$

is a linear integro–differential operator, possibly strongly degenerate.

This paper is organized as follows. We begin by illustrating jump–diffusion models of market and by relating the price of derivatives to a Cauchy problem of type (1.9)–(1.2), in Section 2.

Section 3 concerns the well posedness of integro–differential Cauchy problems: the notion of viscosity solutions and the Perron's method established in [1] are extended to the problem (1.1)–(1.2); next, by means of the comparison principle stated by Theorem 3.4, the following result is attained.

Theorem 1.1 (Well posedness)**.** *We assume that the initial datum satisfies H.0 and that the semilinear integro–differential equation* (1.9) *has the following regularity properties*

H.1) *σ_{ij}, b_j, c are bounded and continuous functions of x, t, and σ_{ij}, b_j are locally Lipschitz continuous w.r.t. x (uniformly w.r.t. t).*

H.2) *The function H is continuous with respect to all the variables and monotone nonincreasing with respect to the forth one. For all $R > 0$, there exists a modulus of continuity ω_R such that*

$$\big|H(x,t,u,I,p) - H(y,t,u,I,p)\big| \le \omega_R\left((1+|u|+|I|+|p|)|x-y|\right) \quad (1.10)$$

if $|x|, |y| \le R$. Moreover, there exists $\mathrm{L}' \ge 0$ such that

$$H(x,t,u,I,p) - H(x,t,v,J,q) \le \mathrm{L}'\left[|u-v| + |I-J| + |p-q|\right], \quad (1.11)$$

for all $(x,t) \in \mathbb{R}^{\mathrm{N}} \times [0,T]$, $u, v, I, J \in \mathbb{R}$, and $p, q \in \mathbb{R}^{\mathrm{N}}$. Lastly, there exists $n_1 \ge 0$ such that

$$e^{-n_1|x|} H(x,t,0,0,0) \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T]) \cap L^\infty\big(0,T;L^\infty(\mathbb{R}^{\mathrm{N}})\big). \quad (1.12)$$

H.3) *There exists $n \ge \max(n_o, n_1)$ such that $1+e^{n|\cdot|}$ belongs to $L^1(\mathbb{R}^{\mathrm{N}}; \mu_{x,t})$ for all x, t and that*

$$\int \Big(1 + e^{n|z|}\Big)\, d\mu_{x,t}(z) \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0,T]) \cap L^\infty\big(0,T;L^\infty(\mathbb{R}^{\mathrm{N}})\big). \quad (1.13)$$

Then the problem (1.9)–(1.2) admits a continuous viscosity solution u fulfilling

$$e^{-\max(n_o,n_1)|x|} u(x,t) \in L^\infty\big(0,T;L^\infty(\mathbb{R}^{\mathrm{N}})\big),$$

which is unique in the class

$$\Big\{ v \in \mathcal{C}\big(\mathbb{R}^{\mathrm{N}} \times (0,T)\big) \,:\, e^{-n|x|} v(x,t) \in L^\infty\big(0,T;L^\infty(\mathbb{R}^{\mathrm{N}})\big) \Big\}.$$

Finally, Section 4 is devoted to the study of the regularity properties of the viscosity solutions. In view of the financial applications, we suppose that the initial datum is locally Lipschitz continuous and that the measures $\mu_{x,t}$ are of type

$$d\mu_{x,t}(z) = \sum_{k=1}^{\mathrm{M}} \nu^k(x,t)\, \delta(\zeta^k(x,t) - z) + \tilde{\nu}(x,t;z)\, dz,$$

where ν^k, $\tilde{\nu}$ are continuous and nonnegative real functions, ζ^k are continuous functions with values in $\mathbb{R}^{\mathrm{N}}$, and δ stands for the Dirac measure. Eventually, the following regularity preserving property is established.

Theorem 1.2 (Lipschitz continuity)**.** *We assume that the initial datum u_o satisfies H'.0 and that the hypotheses H.1–H.3 about the semilinear integro–differential equation (1.9) are strengthened by*

H'.1) $\sigma_{ij}, b_j, c \in \mathcal{C}\big(\mathbb{R}^{\rm N}\times[0,T]\big)\cap L^\infty\big(0,T;W^{1,\infty}(\mathbb{R}^{\rm N})\big)$;
H'.2) (1.10) *is strengthened by*

$$\big|H(x,t,u,I,p)-H(y,t,u,I,p)\big| \le \mathrm{L}\big(1+|u|+|I|+|p|\big)|x-y|, \tag{1.10$'$}$$

for all $x,y,p\in\mathbb{R}^{\rm N}$, $t\in[0,T]$,$u,I\in\mathbb{R}$, *and* (1.11), (1.12) *still hold;*
H'.3) $\nu^k,\zeta^k\in\mathcal{C}\big(\mathbb{R}^{\rm N}\times[0,T]\big)\cap L^\infty\big(0,T;W^{1,\infty}(\mathbb{R}^{\rm N})\big)$, *and there exists* $\mathrm{D}'>0$ *such that*

$$\int_{\mathbb{R}^{\rm N}}\Big(1+e^{n|z|}\Big)\,|\tilde\nu(x,t;z)-\tilde\nu(y,t;z)|\,dz \le \mathrm{D}'|x-y|.$$

Then, the viscosity solution u *to the problem* (1.9)–(1.2) *is locally Lipschitz continuous with respect to* x, *namely* $e^{-n|x|}u(x,t)\in L^\infty\big(0,T;W^{1,\infty}(\mathbb{R}^{\rm N})\big)$.

2. The financial model

The motivation for our study relies in the problem of derivatives pricing in jump–diffusion markets. This contest, which was already considered by Merton [20] in 1976, allows to account for abrupt price movements caused by exogenous events on information, and so to reduce the discrepancies between the standard Black & Scholes model and empirical evidence. In addition to the small variation of the trend which may be modeled by a continuous process based on Brownian motion, sudden and rare breaks are taken into account: they are described by means of some point process that counts the occurrences of rare and random events.

In this section, we describe a market model where a finite number of risky assets are traded in continuous time, evolving according to jump–diffusion processes that possibly are correlated. The presence of a large investor is allowed, actually it is assumed that the interest rate of bank deposits depends on the invested wealth. The price of derivatives is characterized by means of an integro–differential parabolic equation, which is semilinear (of type (1.9)) as a consequence of the large investor assumption. Besides it is strongly degenerate if the diffusion matrix driving the underlying assets is degenerate: it is always the case, for instance, if the market is complete (see later on Remark 2.3).

Let $(\Omega,\mathcal{F},\mathcal{P})$ be a given probability space, endowed with a filtration $(\mathcal{F}_t)_t$. Take then a D–dimensional Brownian motion $W_t=(W_t^1,\cdots,W_t^{\rm D})^T$ and an M–dimensional Poisson process $N_t=(N_t^1,\cdots,N_t^{\rm M})^T$, with non-homogeneous intensities $\lambda^1(t),\cdots,\lambda^{\rm M}(t)$ fulfilling

$$\lambda^k\in\mathcal{C}[0,T],\qquad \lambda^k>0. \tag{2.1}$$

We assume that the processes W_t and N_t are uncorrelated.

Next we consider in this probability setting a market composed by N risky assets, whose prices will be denoted by $S = (S^1, \cdots, S^{\mathrm{N}})^T$, and a money market account, whose value will be denoted by B.

We assume that the vector of risky assets prices evolves according to a jump–diffusion process in $\mathbb{R}^{\mathrm{N}}_+$, described by the stochastic differential equation

$$dS_t = \mathbf{S}_{t^-}\left[\alpha dt + \sigma dW_t + \gamma dN_t\right], \tag{2.2}$$

where $\mathbf{S} = \operatorname{diag}(S^1, \cdots, S^{\mathrm{N}})$, $\alpha = (\alpha_1, \cdots, \alpha_{\mathrm{N}})^T$ is the drift vector, $\sigma = (\sigma_{ij})_{\substack{i=1,\cdots,\mathrm{N}\\ j=1,\cdots,\mathrm{D}}}$ is the diffusion matrix, and $\gamma = (\gamma_{ik})_{\substack{i=1,\cdots,\mathrm{N}\\ k=1,\cdots,\mathrm{M}}}$ is the jump matrix. The coefficients $\alpha_j, \sigma_{ij}, \gamma_{ik}$ are assumed to be deterministic and to satisfy

A.1) $\alpha_j, \sigma_{ij}, \gamma_{ik}$ are continuous functions of S and t on $\mathbb{R}^{\mathrm{N}}_+ \times [0,T]$, with the regularity property

$$f \in L^\infty\big(0,T;W^{1,\infty}(\mathbb{R}^{\mathrm{N}}_+)\big), \qquad S\cdot Df \in L^\infty(\mathbb{R}^{\mathrm{N}}_+ \times [0,T]), \tag{2.3}$$

for f equal to α_j, σ_{ij}, and γ_{ik}. Moreover, $\inf \gamma_{ik} > -1$.

Assumption A.1 leads a wide extension to previously studied jump–diffusion models like the one in [16], where all coefficients are assumed to be constant. Notice that the first part of property (2.3) is just needed to guarantee the well–posedness of (2.2) (see, for instance, [12]). With regard to the second part, it is a technical request needed for Theorem 2.8. It is worth pointing out that (2.3) does not bring any restriction to the application to “real” problems. Actually in practice all the coefficients are calibrated on a discrete grid of statistical data, so that one may obtain all the needed regularity by suitable extending them to the continuum.

The value of the bank deposit evolves according to the differential equation

$$dB(t) = rB(t)dt, \tag{2.4}$$

where r stands for the interest rate. We assume that r depends, beneath on the stocks price S and on the time t, on the wealth invested in bonds, which will be denoted by ξ. Usually r is a non-increasing function of ξ; nevertheless we shall suppose that $r(S,t,\xi)\xi$ is non-decreasing with respect to ξ (as S,t fixed). If not, the agents where boosted to draw back their deposits from the bank. In addition, we do not impose that r is continuous with respect to ξ at zero, in order to include interesting examples as well as 2.1. Therefore we assume that

A.2) r is a non–negative continuous function of S, t, ξ on $\mathbb{R}^{\mathrm{N}}_{+} \times [0,T] \times \mathbb{R}\backslash\{0\}$. For all $\xi \in \mathbb{R} \setminus \{0\}$ the function $(S,t) \mapsto r(S,t,\xi)$ satisfies (2.3), uniformly with respect to ξ, and for all $(S,t) \in \mathbb{R}^{\mathrm{N}}_{+} \times [0,T]$, the function $\xi \mapsto r(S,t,\xi)\xi$ is non–decreasing and Lipschitz continuous on $\mathbb{R}$, uniformly with respect to (S,t).

This large investor model describes situations not included in the standard theory.

Example 2.1 (Different interest rates for borrowing or lending)**.**

$$r(S,t,\xi) = \begin{cases} R(S,t) & \text{if } \xi \leq 0, \qquad R \text{ borrowing rate,} \\ \rho(S,t) & \text{if } \xi > 0, \qquad \rho \text{ lending rate;} \end{cases}$$

where R, ρ are bounded smooth functions and $R(S,t) > \rho(S,t)$.

Example 2.2 (Large institutional investor)**.** The interest rate decreases when too many wealth is invested in bonds, according to the law $r(S,t,\xi) = R(S,t)\, f(\xi)$, where R is bounded and smooth, and f is a positive Lipschitz continuous function on $\mathbb{R}$, such that $f(\xi) = 1$ if $\xi \leq \xi_o$, $-f(\xi)\xi \leq f'(\xi) \leq 0$ almost everywhere in (ξ_o, ∞).

Large investor models of this kind have been studied by Cvitanič and Ma [7], assuming a pure diffusion process for assets' price and rank $\sigma = \mathrm{N} = \mathrm{D}$ (equivalently, that the market is complete), in the framework of forward–backward stochastic differential equations. Existence of solution to the stochastic equation characterizing the price of derivatives is attained through existence and regularity of the solution to an associate partial differential equation. In the non–linear setting caused by the large investor assumption, the PDE's approach is nowadays the only way to price claims, to the author's knowledge.

We recall (see, for instance, [17, Théorèmes 2.6, 3.4]) that the market composed by S_t and B_t is without arbitrage opportunities (respectively, complete) if and only if there exists a probability $\mathcal{P}^*$ (respectively, a unique probability $\mathcal{P}^*$), equivalent to the given one, such that the processes S^i_t/B_t are martingales under $\mathcal{P}^*$. In force of the characterization [3, VIII T10] (see also [21]), the market is without arbitrage opportunities if and only if

$$\begin{gathered} \text{there exist } \theta \in \mathbb{R}^{\mathrm{D}} \text{ and } \phi \in \mathbb{R}^{\mathrm{M}}_{+} \text{ such that} \\ \alpha - r\mathbf{1} = (\sigma, \gamma) \begin{pmatrix} \theta \\ -\lambda\phi \end{pmatrix}, \end{gathered} \tag{2.5}$$

where $\mathbf{1}$ stands for the vector of $\mathbb{R}^{\mathrm{N}}$ $(1, \cdots, 1)^T$ and $\lambda = \operatorname{diag}(\lambda^1, \cdots, \lambda^{\mathrm{M}})$.

Moreover, the couple θ, ϕ satisfying (2.5) (usually named market price for risks) is in one–to–one correspondence with an equivalent martingale measure $\mathcal{P}^*$. Therefore, the market is complete if and only the market price for risks is unique.

Remark 2.3. Notice that, in the jump–diffusion setting, if $\operatorname{rank}\sigma = \mathrm{N} = \mathrm{D}$, the market is not complete, because for any choice of $\phi \in \mathbb{R}^{\mathrm{M}}_+$, there exists $\theta = \sigma^{-1}(\alpha - r\mathbf{1} + \gamma\lambda\phi)$ satisfying (2.5).

Instead a sufficient condition for completeness is

$$\operatorname{rank}(\sigma, \gamma) = \mathrm{N} = \mathrm{D} + \mathrm{M}, \tag{2.6}$$

which has already been used in [16]. We emphasize that, in this case, $\operatorname{rank}\sigma \le \mathrm{D} < \mathrm{N}$.

Furthermore, if $\theta^1, \theta^2 \in \mathbb{R}^{\mathrm{D}}$ and $\phi^1, \phi^2 \in \mathbb{R}^{\mathrm{M}}$ (only depending by S and t) are defined by

$$\begin{pmatrix} \theta^1 \\ -\lambda\phi^1 \end{pmatrix} = (\sigma, \gamma)^{-1}\alpha, \qquad \begin{pmatrix} \theta^2 \\ -\lambda\phi^2 \end{pmatrix} = -(\sigma, \gamma)^{-1}\mathbf{1}, \tag{2.7}$$

the non–arbitrage condition (2.5) becomes $\phi^1 + r\phi^2 \in \mathbb{R}^{\mathrm{M}}_+$. Since in the large investor case r also depends by the quantity ξ (that is decided by the investor), we are forced to strengthen it by asking that

$$\phi^1, \phi^2 \in \mathbb{R}^{\mathrm{M}}_+. \tag{2.8}$$

In Mathematical Finance completeness is a crucial point, because it allows to perfectly hedge the risk endowed in any contingent claim and, as a consequence, to price them unambiguously. It is worth mentioning that, in the jump–diffusion setting, one may reduce to a market which satisfies condition (2.6), and therefore which is complete, even if dealing with a claim on some underlying assets $S^1, \cdots, S^{\tilde{\mathrm{N}}}$ which do not compose a complete market. This practice, often referred to as "completion of the market", stands in adding to the natural market composed by $S^1, \cdots, S^{\tilde{\mathrm{N}}}$, some new assets $S^{\tilde{\mathrm{N}}+1}, \cdots, S^{\mathrm{D}+\mathrm{M}}$ until condition (2.6) is fulfilled. It is not a mere theoretical trick, since in most applications the role of $S^{\tilde{\mathrm{N}}+1}, \cdots, S^{\mathrm{D}+\mathrm{M}}$ is played by some call options on the former assets $S^1, \cdots, S^{\tilde{\mathrm{N}}}$ which are really marketed; see [23], [18] for more details about this topic.

By now we suppose that the market composed by S and B fulfills (2.6)–(2.7) and we take an European contingent claim with maturity T and payoff $G(S_T)$. We recall that a hedging strategy is a dynamic portfolio which invests the wealth $\Delta_t = (\Delta^1_t, \cdots, \Delta^{\mathrm{N}}_t)$ in the risky stokes and ξ_t in the bank, in such a way that

$$\begin{aligned} d\,(\Delta_t \mathbf{1} + \xi_t) &= \Delta_t \mathbf{S}_t^{-1} dS_t + \xi_t B_t^{-1} dB_t, \qquad \text{for all } t \in (0,T), \\ \Delta_T \mathbf{1} + \xi_T &= G(S_T). \end{aligned}$$

By making use of the generalized Ito Lemma (see, for instance, [8, Theorem VIII.27]), the well known Δ–hedging technique (illustrated in [9, Paragraph 3.5]) characterizes the replicating strategy in the given probability $\mathcal{P}$ as

$$\begin{aligned} \Delta_t &= \big(DU\mathbf{S}\sigma, \tilde{\mathcal{I}}U\big)(\sigma,\gamma)^{-1} \\ \xi_t &= U - \mathbf{S}\sigma\theta^2 \cdot DU - \lambda\phi^2 \cdot \tilde{\mathcal{I}}U, \end{aligned} \tag{2.9}$$

where $U(S_t, t)$ is the arbitrage price of the claim at time $t < T$. Moreover, the deterministic function U solves the modified "Black & Scholes equation"

$$\begin{aligned} -\partial_t U = \frac{1}{2}\mathrm{tr}\big[(\mathbf{S}\sigma)(\mathbf{S}\sigma)^T D^2 U\big] + \mathbf{S}(\alpha - \sigma\theta^1)\cdot DU + \lambda\phi^1\cdot\tilde{\mathcal{I}}U \\ -r\Big(S, t, U - \mathbf{S}\sigma\theta^2\cdot DU - \lambda\phi^2\cdot\tilde{\mathcal{I}}U\Big)\times\Big(U - \mathbf{S}\sigma\theta^2\cdot DU - \lambda\phi^2\cdot\tilde{\mathcal{I}}U\Big) \end{aligned} \tag{2.10}$$

on $(0,\infty)^{\mathrm{N}} \times [0,T)$ and satisfies the final condition

$$U(S,T) = G(S). \tag{2.11}$$

Here $DU = (\partial_{S^1}U, \cdots, \partial_{S^{\mathrm{N}}}U)^T$ is the gradient of U, $D^2U = \big(\partial^2_{S^iS^j}U\big)$ is the Hessian matrix of U, and the nonlocal term $\tilde{\mathcal{I}}U$ takes into account the increments of U due to the jumps of the underlying assets, being

$$\tilde{\mathcal{I}}_k U = U(S + \mathbf{S}\gamma_k, t) - U(S,t)$$

(γ_k standing for the k^{th} row of γ) for $k = 1, \cdots \mathrm{M}$.

Remark 2.4. It is worth noticing that the dependence of equation (2.10) by the drift vector α is merely seeming, because by making use of the relation (2.7) one gets that $\alpha - \sigma\theta^1 = -\gamma\lambda\phi^1$.

Remark 2.5. Notice that r gives rise to a nonlinear, nonlocal term, because of the large investor assumption. Actually r depends on the wealth invested in bonds, i.e., on the process ξ_t characterized by (2.9).

The equation (2.10) has the same boundary degeneracy of the classical Black & Scholes equation, that can be removed by the change of variables $x_i = \log S^i$. Now U solves (2.10)–(2.11) if and only if

$$u(x,t) = U\left(e^{x_1}, \cdots, e^{x_{\mathrm{N}}}, T - t\right)$$

solves a semilinear integro–differential Cauchy problem of the type (1.9)–(1.2). Still writing $\alpha, \sigma, \gamma, \theta, \phi$ for the economical parameters composed with the change of variables, we have

$$\mathcal{L}_{\mathcal{I}} u = -\frac{1}{2}\mathrm{tr}\big[\sigma\sigma^T D^2 u\big] + b \cdot Du - \mathcal{I}^1 u$$

with $b_i = \frac{1}{2}\sigma_{ii}^2 - (\alpha - \sigma\theta^1)_i$, and

$$H(x,t,u,\mathcal{I}^2 u, Du) = r\left(e^{x_1}, \cdots, e^{x_{\mathrm{N}}}, T-t, u - \sigma\theta^2 \cdot Du - \mathcal{I}^2 u\right) \times \big(u - \sigma\theta^2 \cdot Du - \mathcal{I}^2 u\big).$$

As $\ell = 1,2$ $\mathcal{I}^\ell$ is a nonlocal term of type (1.3) where $\mu_{x,t}^\ell$ is the discrete bounded measure

$$\mu_{x,t}^\ell(z) = \sum_{k=1}^{\mathrm{M}} \lambda^k \phi_k^\ell\, \delta(\zeta_k(x,t) - z).$$

Here ζ_k stands for the vector of $\mathbb{R}^{\mathrm{N}}$ $(\log(1+\gamma_{1k}), \cdots, \log(1+\gamma_{\mathrm{N}k}))$, and δ stands for the Dirac measure.

Remark 2.6. By making use of the relations (2.7), it is possible to rewrite the equation with integral terms of type (1.7) (according to the same measures $\mu_{x,t}^\ell$), by setting

$$\mathcal{L}_{\mathcal{J}} u = -\tfrac{1}{2}\mathrm{tr}\big[\sigma\sigma^T D^2 u\big] + \hat{b} \cdot Du - \mathcal{J}^1 u,$$

$$H(x,t,u,\mathcal{J}^2 u, Du) = r\left(e^{x_1}, \cdots, e^{x_{\mathrm{N}}}, T-t, u - \mathrm{div}\, u - \mathcal{J}^2 u\right) \times \big(u - \mathrm{div}\, u - \mathcal{J}^2 u\big),$$

where $\hat{b}_1 = \frac{1}{2}\sigma_{ii}^2$.

Since in the new variables $u_o(x) = G(e^{x_1}, \cdots, e^{x_{\mathrm{N}}})$, the study of the former problem (2.10)–(2.11) with payoff G growing polynomially at infinity is equivalent to the study of the Cauchy problem (1.9)–(1.2) with initial datum u_o growing exponentially at infinity.

In principle, one may directly study the problem (2.10)–(2.11), for which polynomial growth is sufficient; but in this case one has to face the difficulties related to the boundary degeneracy in $\partial\mathbb{R}_+^{\mathrm{N}}$ (notice that this boundary is not of class $W^{3,\infty}$, which is a standard assumption for the viscosity solutions approach) and of the rate of growth of the coefficients of the linear part (which may play a role when checking Lipschitz continuity). Since the two problems are equivalent, we have chosen here to study the one without boundary, with bounded coefficient, but with faster increasing datum.

We emphasize that the obtained equation of type (1.9) is always of strongly degenerate type (i.e., $\operatorname{rank}\sigma < \mathrm{N}$ everywhere) in force of the completeness condition (2.6).

In the linear case corresponding to a small investor, integro–differential equations arising in finance have been studied by Pham (see [21] and the references therein) for $\mathrm{N} = \mathrm{D} = 1$, and by Mastroeni and Matzeu (see [19] and the references therein) for $\mathrm{N} = \mathrm{D}$ arbitrary, under the non-degeneracy assumption $\operatorname{rank}\sigma = \mathrm{N}$. Mastroeni and Matzeu studied also a semilinear case, allowing nonlinearities with respect to DU (but not with respect to $\tilde{\mathcal{I}}U$). Their variational technique deals with weak degeneracy: namely, $\sigma\sigma^T$ is positive, defined at all (S, t), except a set of zero measure in which it vanishes. Hence it may not be applied to problems of type ($\tilde{1}$.sl)–($\tilde{2}$), when the completeness condition (2.6) holds true.

Let us state the main results that may be obtained as easy consequences of Theorems 1.1 and 1.2, respectively.

Theorem 2.7 (Well posedness)**.** *We assume that the market parameters satisfy A.1, A.2, and that* (2.6)–(2.8) *hold true with* $\inf\det(\sigma, \gamma) > 0$. *Then the final value problem* (2.10)–(2.11) *is well posed in the class of continuous functions with polynomial growth for large* $|S|$. *In particular, for all payoffs* $G \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}_+)$ *with* $0 \le G(S) \le \mathrm{B}_o(1 + |S|^{n_o})$, *the viscosity solution satisfies* $0 \le U(S, t) \le \mathrm{B}(1 + |S|^{n_o})$.

Theorem 2.8 (First order estimate)**.** *Under the same assumptions of Theorem* 2.7, *and for all payoffs* $G \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}}_+)$ *with*

$$|G(S)| + |DG(S)| \le \mathrm{B}_o(1 + |S|^n),$$

the viscosity solution of (2.10)–(2.11) *is locally Lipschitz continuous with respect to* S, *namely*

$$|U(S, t)| + |DU(S, t)| \le \mathrm{B}(1 + |S|^n)$$

for all $t \in [0, T]$.

The regularity property stated by Theorem 2.8 is quite relevant. Indeed, the trader expects that, in order to hedge from the risk of his contingent claim, he need to place a quantity of money which grows up exactly in the same way of the expected payoff, as the prices S increase. Since the gradient of U is connected with the hedging portfolio by the relation (2.9), such property is just established by Theorem 2.8.

3. Existence and uniqueness of viscosity solution

In this section, we show that the Perron's method, introduced by Ishii [13] for viscosity solutions, can be used also for the integro–differential problem (1.1)–(1.2); then, by a comparison principle, we obtain existence and uniqueness of a continuous solution. We refer to [4] for a general presentation of the theory of viscosity solutions in the pure differential case and to Alvarez and Tourin [1] for a simpler nonlocal situation.

We begin by defining viscosity solutions for integro–differential problems with non local term $\mathcal{I}u$ of type (1.3), under the structural assumptions (1.4), (1.5).

At this step we may deal with the wide class of integro–differential problems (1.1), (1.2) under the assumptions F.0–F.3.

In order to define viscosity solutions to nonlocal equations, it is needed to face two main difficulties: the first one is that the meaning of "viscosity inequality" changes by using the local notion of semijets, or the global one; the second one is that, in general, the integral term $\mathcal{I}$ does not preserves semicontinuity. In force of assumption F.3, they may be overcame by following [1].

Definition 3.1. Given a function u and $(x,t) \in \mathbb{R}^N \times [0,T)$, we shall say that $\partial_t u + F(x,t,u,\mathcal{I}u,Du,D^2u) \leq 0$ (respectively, ≥ 0) in **viscosity sense** at (x,t) if one of the following equivalent items holds:

(1) $\tau + F(x,t,u(x,t),\mathcal{I}u(x,t),p,X) \leq 0$ (resp., ≥ 0) for all (τ,p,X) in $\mathcal{P}^+u(x,t)$ (respectively, $\mathcal{P}^-u(x,t)$);

(2) $\partial_t\varphi(x,t) + F\big(x,t,u(x,t),\mathcal{I}u(x,t),D\varphi(x,t),D^2\varphi(x,t)\big) \leq 0$ (resp., ≥ 0) in classical sense, for each smooth function φ on $\mathbb{R}^N \times [0,T)$ such that $u - \varphi$ has a local maximum (resp., a minimum) at (x,t);

(3) $\partial_t\varphi + F\big(x,t,\varphi,\mathcal{I}\varphi,D\varphi,D^2\varphi\big) \leq 0$ (resp., ≥ 0) in classical sense, for all smooth functions φ on $\mathbb{R}^N \times [0,T)$ such that $u-\varphi$ has a global strict maximum (resp., a minimum) at (x,t).

We recall that the parabolic semijets $\mathcal{P}^{\pm}$ are defined as

$$\mathcal{P}^{\pm}u(x,t) = \Big\{ \ (\tau,p,X) \in \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N : \ u(y,s) \leq \ (\text{ resp. } \geq) \\ u(x,t) + \tau(t-s) + p\cdot(x-y) + \frac{1}{2}X(x-y)\cdot(x-y) \\ +o\left(|t-s| + |x-y|^2\right) \ \text{ as } (y,s) \to (x,t) \Big\},$$

Next we define sub/supersolutions to the problem (1.1)–(1.2) as follows.

Definition 3.2. An upper (resp., lower) semicontinuous, locally bounded function u on $\mathbb{R}^N \times [0,T)$ is a **subsolution** (resp., a **supersolution**) to the problem (1.1)–(1.2) if $\partial_t u + F(x,t,u,\mathcal{I}u,Du,D^2u) \le 0$ (resp., $\ge$) in viscosity sense at all $(x,t) \in \mathbb{R}^N \times (0,T]$, and $u(x,0) \le u_o(x)$ (resp., $\ge$) at all $x \in \mathbb{R}^N$.

A **solution** is any function which is is both a sub/supersolution.

It is worst mentioning that, in general, $\mathcal{I}u$ is not upper (respectively, lower) semicontinuous for any subsolution (respectively, supersolution). On the other hand [1, Lemma 1] establishes that $\mathcal{I}u$ is upper (resp., lower) semicontinuous at a point $(x,t) \in \mathbb{R}^N \times [0,T]$ if u satisfies the additional property

$$\begin{array}{c}\text{there exists } \Phi \in \mathcal{B}_{x,t} \text{ such that for all } y,s \text{ near at } x,t \text{ and } z \in \mathbb{R}^N \\ M(u(y+z,s),u(y,s)) \le \Phi(z) \qquad (\text{resp., } \ge -\Phi(z)).\end{array} \tag{3.1}$$

Here the set $\mathcal{B}_{x,t}$ is defined by

$$\begin{aligned}\mathcal{B}_{x,t} \;=\; \Big\{ & \Phi \in \mathcal{C}\big(\mathbb{R}^N\big) \cap L^1\big(\mathbb{R}^N;\mu_{x,t}\big) : \Phi \ge 0, \\ & \lim_{(y,s)\to(x,t)} \int_{\mathbb{R}^N} \Phi(z)\, d\mu_{y,s}(z) = \int_{\mathbb{R}^N} \Phi(z)\, d\mu_{x,t}(z) \Big\}.\end{aligned}$$

Notice that, since $\mu_{x,t}$ are finite measures, such property is implied by a suitable rate of growth at infinity. Actually, the proof of next result is immediate.

Lemma 3.1. *Assume that there exists a subadditive, nonnegative* $g \in \mathcal{C}[0,\infty)$ *such that the function* $(x,t) \to \int [1+g(|z|)]\, d\mu_{x,t}(z)$ *is continuous on* $\mathbb{R}^N \times [0,T)$. *Then all upper* (*resp., lower*) *semicontinuous functions such that* $u(x,t)/g(|x|)$ *is bounded satisfy* (3.1).

In particular, if $\mu_{x,t}$ has compact support for any fixed x,t, all semicontinuous and locally bounded functions satisfy (3.1).

It is useful to observe that the initial condition can be relaxed, in the same spirit of [2, Proposition 5] for the purely differential case. In the integro-differential case, the relaxation of the initial condition has to be balanced by imposing (3.1) at $t = 0$.

Definition 3.3. u is a subsolution (respectively, a supersolution) of (1.1)–(1.2) with **generalized initial condition**, GIC, if the initial condition $u(x,0) \le u_o(x)$ (respectively, $\ge$) is replaced by

$$\begin{array}{c}\min\big\{\partial_t u + F(x,0,u,\mathcal{I}u,Du,D^2u), u-u_o\big\} \le 0 \\ (\text{resp., } \max\big\{\partial_t u + F(x,0,u,\mathcal{I}u,Du,D^2u), u-u_o\big\} \ge 0\ )\end{array}$$

in viscosity sense and property (3.1) is satisfied at $t = 0$.

In the next Lemma we show that a sub/supersolution with generalized initial condition is a sub/supersolution in the sense of Definition 3.2, indeed.

Lemma 3.2. *If u is a subsolution (resp., a supersolution) to (1.1)–(1.2) with GIC, then $u(x,0) \le u_o(x)$ (resp. $\ge$) for all $x \in \mathbb{R}^N$.*

Proof. Assume by contradiction that $u(x_o,0) > u_o(x_o)$; by continuity, $u(x_o,0) > u_o(x)$ for x near x_o. Let V be the neighborhood of $(x_o,0)$ in which property (3.1) is fulfilled and $u(x_o,0) > u_o(x)$, in addition. We consider the family of test functions

$$\varphi_n(x,t) = \frac{n}{2}\,|x-x_o|^2 + \mathrm{c}_n t,$$

where $\mathrm{c}_n > \max\Big(n, -\inf_{((x,t),v)\in(V,u(V))} F\Big(x,t,v,\int \Phi\mu_{x,t}, n(x-x_o), nI\Big)\Big)$.

By upper semicontinuity, $u - \varphi_n$ has a maximum point (x_n,t_n) in the compact set $\overline{V}$ and, up to an extracted sequence, we may assume that (x_n,t_n) converges to $(\hat{x},\hat{t}) \in \overline{V}$. Because

$$u(x_o,0) \le \lim_{n\to\infty}[u(x_n,t_n) - \varphi_n(x_n,t_n)] \le u(\hat{x},\hat{t}) - \lim_{n\to\infty} n\Big(\frac{1}{2}\,|x_n-x_o|^2 + t_n\Big),$$

we have that $(\hat{x},\hat{t}) = (x_o,0)$. Therefore, for large n, $(x_n,t_n) \in V$ is a local maximum point for $u - \varphi_n$ on $\mathbb{R}^N \times [0,T]$ and Definition 3.1.2 gives

$$\mathrm{c}_n \le -F(x_n,t_n,u(x_n,t_n),\mathcal{I}u(x_n,t_n),n(x_n-x_o),nI).$$

Next, since $\mathcal{I}u(x_n,t_n) \le \int \Phi d\mu_{x_nt_n}$ by (3.1), hypothesis F.3 implies the contradiction

$$\mathrm{c}_n \le -F\Big(x_n,t_n,u(x_n,t_n),\int \Phi d\mu_{x_nt_n}, n(x_n-x_o), nI\Big).$$

3.1. **Perron's method.** In [1], Alvarez and Tourin showed that Perron's method can be used for integro–differential Cauchy problem of the kind (1.8)–(1.2), under assumptions F.0–F.2. Here we show that the monotonicity assumption F.3 is sufficient to apply Perron's method to the wider class (1.1)–(1.2).

Theorem 3.3 (Perron Method)**.** *Assume that $h, k \in \mathcal{C}(\mathbb{R}^N \times [0,T])$ are respectively a viscosity sub/supersolution of (1.1)–(1.2) such that:*

(i) $h \le k$,

(ii) *for every* $(x,t) \in \mathbb{R}^N \times [0,T]$*, there exist* $\Phi \in \mathcal{B}_{x,t}$ *and a neighborhood* V *of* (x,t) *such that*

$$-\Phi(z) \le M\left(h(y+z,s),w\right) \le M\left(k(y+z,s),w\right) \le \Phi(z)$$

for all $(y,s) \in V$*,* $z \in \mathbb{R}^N$*, and* w *in the segment line* $[h(y,s),k(y,s)]$*.*

We denote by $\mathcal{S}$ *the set of subsolutions* v *of* (1.1)–(1.2) *such that* $h \le v \le k$ *on* $\mathbb{R}^N \times [0,T]$*, by*

$$u(x,t) = \sup\left\{v(x,t) \ : \ v \in \mathcal{S}\right\}, \tag{3.2}$$

and by u^**,* u_* *the upper and lower semicontinuous envelope of* u*, respectively. Then* u^* *and* u_* *are respectively a sub/supersolution to* (1.1)–(1.2).

Proof. We first observe that, since h and k are continuous and M is non–decreasing with respect to its first argument, hypothesis (ii) guarantees that u^* and u_* satisfy the property (3.1) at all points of $\mathbb{R}^N \times [0,T]$.

Therefore, thanks to Lemma 3.2, we may prove that u^* is a subsolution to (1.1)–(1.2) by checking that it is a subsolution with GIC. Take $(x,t) \in \mathbb{R}^N \times [0,T]$, with $u^*(x,0) > u_o(x)$ if $t=0$ (otherwise there is nothing to prove): our goal is to show that $\partial_t\varphi + F\big(x,t,u^*,\mathcal{I}u^*,D\varphi,D^2\varphi\big) \le 0$ for every smooth function φ that $u^*-\varphi$ has a strict maximum at (x,t). By standard arguments, one may find two sequences $v_n \in \mathcal{S}$ and $(x_n,t_n) \in \mathbb{R}^N \times [0,T]$ such that $(x_n,t_n,v(x_n,t_n))$ converges to $(x,t,u^*(x,t))$, and (x_n,t_n) are points of local maximum for $v_n - \varphi$. Up to an extracted sequence, we may suppose that $t_n > 0$ for all n: otherwise, we should have $u^*(x,0) = \lim v_n(x_n,0) \le \lim u_o(x_n) = u_o(x)$ because of the continuity of u_o. Thus $\partial_t\varphi + F\big(x_n,t_n,v_n,\mathcal{I}v_n,D\varphi,D^2\varphi\big) \le 0$. By passing to the limit as n goes to infinity and by taking advantage of assumption F.3, we have

$$\begin{aligned} 0 \ &\ge \liminf_{n\to\infty}\left[\partial_t\varphi(x_n,t_n) + F\big(x_n,t_n,v_n,\mathcal{I}v_n,D\varphi,D^2\varphi\big)\right] \\ &= \partial_t\varphi(x,t) + F\big(x,t,u^*,\limsup_{n\to\infty}\mathcal{I}v_n(x_n,t_n),D\varphi,D^2\varphi\big). \end{aligned}$$

Hence, the proof may be completed by showing that

$$\limsup_{n\to\infty}\mathcal{I}v_n(x_n,t_n) \le \mathcal{I}u^*(x,t),$$

but this last inequality is an easy consequence of [1, Lemma 1] and hypothesis (ii).

Next, we shall prove that u_* is a supersolution to (1.1)–(1.2). By Lemma 3.2, it is sufficient to show that it is a supersolution with GIC. Suppose by contradiction that u_* is not a supersolution with GIC; then there exist

$(x_o,t_o)\in\mathbb{R}^{\mathrm{N}}\times[0,T]$ (with $u_*(x_o,0)<u_o(x_o)$ if $t_o=0$) and a smooth function φ such that $(u_*-\varphi)(x_o,t_o)=0$ is a global strict minimum and

$$\partial_t\varphi+F\big(x_o,t_o,\varphi,\mathcal{I}\varphi,D\varphi,D^2\varphi\big)<0.$$

We carry out the proof by constructing a subsolution in $\mathcal{S}$ strictly bigger than u^* at one point. We begin by setting

$$B_r=\left\{(x,t)\in\mathbb{R}^{\mathrm{N}}\times[0,T)\,:\,|x-x_o|^2+(t-t_o)^2<r\right\},$$
$$\varphi^\delta(x,t)=\max\left(\varphi(x,t)+\delta,h(x,t)\right).$$

Notice that, by construction, $\varphi^\delta\equiv\varphi(x,t)+\delta$ near at (x_o,t_o). So, the same arguments of [1, Proposition 1] yield that, for suitable δ and $r>0$, $\varphi^\delta\le k$ and $\mathcal{I}\varphi^\delta$ is lower semicontinuous near at (x_o,t_o). Therefore F.3 implies that

$$\partial_t\varphi+F\big(x,t,\varphi^\delta,\mathcal{I}\varphi^\delta,D\varphi,D^2\varphi\big)\le 0,\quad \text{on } B_r,$$

after possibly choosing a smaller r. Moreover, as $u_*-\varphi$ has a strict minimum at (x_o,t_o) and $u^*\ge h$, we may suppose without loss of generality that $u^*\ge\varphi^\delta$ on $\mathbb{R}^{\mathrm{N}}\times[0,T]\setminus B_{\frac{r}{2}}$. Next we set

$$w(x,t)=\begin{cases} u^*(x,t), & (x,t)\in\mathbb{R}^{\mathrm{N}}\times[0,T]\setminus B_r,\\ \max\big(u^*(x,t)\ ,\ \varphi^\delta(x,t)\big), & (x,t)\in B_r.\end{cases}$$

Hypothesis (ii) guarantees that w satisfies property (3.1), so that it easily follows by construction that w is a subsolution to (1.1)–(1.2) with generalized initial condition. On the other hand, standard semicontinuity arguments show that there is at least one point near (x_o,t_o) where $w>u^*$. □

In general, the function u produced in (3.2) does not have any regularity property, but, whenever its continuity is achieved, then u is a viscosity solution. Actually, the only availability of a comparison principle among sub/supersolutions of (1.1)–(1.2) is sufficient to get the existence of one solution, via the Perron's method.

3.2. **The Comparison Principle.** It is well understood that, in order to establish a comparison principle for the Cauchy problem, it is necessary to select a rate of growth at infinity for sub/supersolutions. In view of the financial applications, we are interested in dealing with exponential growth, namely we shall assume that the initial datum satisfies, for a suitable n_o,

$$(H.0)\qquad e^{-n_o|x|}u_o(x)\in\mathcal{C}(\mathbb{R}^{\mathrm{N}})\cap L^\infty(\mathbb{R}^{\mathrm{N}}).$$

As a consequence, a mere extension to the integro–differential setting of the classical results is not sufficient, because they only allow linear growth (see, for instance [5], [11], or [14]). However, the equation of financial interest has some nice properties: first of all, the coefficients of the linear part respect the growth conditions of the classical Phragmèn-Lindelof principle (see for instance [22, Theorem 10]); the nonlinear term is globally Lipschitz–continuous, and finally the nonlocal term $\mathcal{I}$ has a natural growth restriction by virtue of the boundedness of the economical parameters. Therefore, we focus here our interest into the semilinear integro–differential equation

$$\partial_t u + \mathcal{L}_{\mathcal{I}} u + H(x,t,u,\mathcal{I}u,Du) = 0, \quad (x,t) \in \mathbb{R}^{\mathrm{N}} \times (0,T), \tag{1.9}$$

assuming that H.1–H.3 hold.

Let us mention that assumptions H.1 and H.2-(1.10) are standard (see, for instance, [15, 4]). In addition, the Lipschitz–continuity property H.2-(1.11) and the condition on the rate of growth of the source term **H.2**-(1.12) are imposed in order to deal with faster increasing solutions, in the spirit of [6].

With respect to hypothesis H.3, we stress that some compatibility among the space $L^1(\mathbb{R}^{\mathrm{N}};\mu_{x,t})$ and the rate of growth at infinity of the initial datum and of the source term is necessary, in view of uniqueness of solutions, also in the linear case. We refer to [1, Remark 4], which shows lack of uniqueness in the class of bounded functions, when the constant function 1 does not belong to $L^1(\mathbb{R}^{\mathrm{N}};\mu_{x,t})$.

The main result of this subsection is the following.

Theorem 3.4 (Comparison Principle)**.** *Let* $\underline{u}$ *and* $\overline{u}$*, satisfying*

$$e^{-n|x|}\underline{u},\ e^{-n|x|}\overline{u} \in L^\infty\big(0,T;L^\infty(\mathbb{R}^{\mathrm{N}})\big), \tag{3.3}$$

be respectively sub/supersolution for the Cauchy problem (1.9)–(1.2)*.*

Then $\underline{u} \le \overline{u}$ *in* $\mathbb{R}^{\mathrm{N}} \times [0,T]$*.*

Notice that the double sided estimate (3.3) is imposed to the rate of growth of sub/supersolutions, while a one sided estimate is sufficient in the purely differential setting. Indeed, the additional estimate is related to the request (3.1), which is needed in order to estimate the integral term $\mathcal{I}u$. Actually hypothesis H.3 and the growth condition (3.3) imply that $\underline{u}$ and $\overline{u}$ satisfy (3.1), via Lemma 3.1.

The idea for handling faster increasing sub/supersolutions is the same os the classical Phragmèn–Lindelof principle, namely to multiply $\underline{u}$ and $\overline{u}$ for a suitable weight function in such a way that their difference is bounded from above and take maximum in $\mathbb{R}^{\mathrm{N}} \times [0,T]$. As we shall show during the proof, this technique may be extended to nonlinear equations under assumption H.2-(1.11)

Before entering into details of the proof, we recall for the sake of completeness a Lemma previously used by Ishii and Kobayasi [14], which establishes the well known Osgood condition in the framework of viscosity solutions.

Lemma 3.5 ([14, Theorem 3, page 918]). *Let $g \in \mathcal{C}([0,T])$ be such that*

$$g(0) = 0, \qquad \int_0^1 \frac{dr}{g(r)} = +\infty.$$

Let then f be an upper semicontinuous function on $[0,T)$, nonnegative, bounded from above and suppose that

$$\min\big(f'(t) - g(f(t))\,,\, f(t)\big) \le 0, \qquad \text{for all } t \in [0,T), \tag{3.4}$$

in viscosity sense, then $f \equiv 0$.

Proof of Theorem 3.4. Without loss of generality, we may replace the norm $|x|$ with $\ell(x) = \sqrt{1+|x|^2}$, with the advantage that ℓ is of class $\mathcal{C}^2$. By hypothesis $\underline{v}(x,t) = e^{-n\ell(x)}\underline{u}(x,t)$ and $\overline{v}(y,t) = e^{-n\ell(y)}\overline{u}(y,t)$ are bounded. Besides, $\underline{v}$ and $\overline{v}$ are respectively a sub/supersolution for the modified integro–differential Cauchy problem

$$(1.9^\sharp) \qquad \partial_t v + \mathcal{L}_{\mathcal{I}}^\sharp v + H^\sharp\Big(x,t,v,\mathcal{I}^\sharp v, Dv\Big) = 0, \qquad (x,t) \in \mathbb{R}^{\mathrm{N}} \times (0,T],$$

$$(1.2^\sharp) \qquad v(x,0) = e^{-n\ell(x)}u_o(x) \in \mathcal{C}(\mathbb{R}^{\mathrm{N}}) \cap L^\infty(\mathbb{R}^{\mathrm{N}}), \qquad x \in \mathbb{R}^{\mathrm{N}}.$$

Here, we have used the notations

$$\mathcal{L}_{\mathcal{I}}^\sharp v = -\frac{1}{2}\mathrm{tr}\big[\sigma\sigma^T D^2 v\big] + b^\sharp \cdot Dv + c^\sharp v - \mathcal{I}^\sharp v,$$

where $b^\sharp = b - \frac{1}{2}n\, D\ell\sigma\sigma^T$ and $c^\sharp = c + n\,\mathbf{b}\cdot D\ell - \frac{1}{2}n\,\mathrm{tr}\big[\sigma\sigma^T D^2\ell\big] - \frac{1}{2}\overline{n}^2\,|D\ell\sigma|^2$,

$$\begin{aligned} H^\sharp\Big(x,t,v,\mathcal{I}^\sharp v, Dv\Big) &= e^{-n\ell(x)} H\left(x,t,e^{n\ell(x)}\Big(v,\mathcal{I}^\sharp v, nvD\ell(x) + Dv\Big)\right), \\ \mathcal{I}^\sharp v(x,t) &= e^{-n\ell(x)} \int_{\mathbb{R}^{\mathrm{N}}} \mathcal{M}^\sharp v(x,t,z)\,\mu_{xt}(dz), \\ \mathcal{M}^\sharp v(x,t,z) &= e^{-n\ell(x)} M\left(e^{n\ell(x+z)}v(x+z,t), e^{n\ell(x)}v(x,t)\right). \end{aligned}$$

Notice that the coefficients of $\mathcal{L}_{\mathcal{I}}^\sharp$ still satisfies H.1, and that the new Hamiltonian $H^\sharp$ still fulfills (1.10) (for a new modulus of continuity $\omega^\sharp$) and (1.11); in addition the source term $H^\sharp(\cdot,0,0,0)$ is bounded. Theorem 3.4 is equivalent to the comparison principle among bounded sub/supersolutions to $(1.9^\sharp)$–$(1.2^\sharp)$. We set $\theta(t) = \sup\big\{\,(\underline{v}(x,t) - \overline{v}(x,t))^+ \,:\, x \in \mathbb{R}^{\mathrm{N}}\big\}$, and we prove the thesis by checking that $\theta \equiv 0$. By invoking Lemma 3.5, it suffices to show that its upper semicontinuous envelope θ^* fulfills (3.4) for

$$g(\theta^*) = \left(-\inf c^\sharp + \mathrm{MD} + (1+\mathrm{MD})\mathrm{L}'\right)^+ \theta^*,$$

where $\mathrm{D} = \left\| \int \left(1 + e^{n|z|}\right) d\mu_{x,t}(z) \right\|_\infty$.

To this aim we fix $t_o \in [0,T]$, with $\theta^*(t_o) > 0$ (if not, there is nothing to prove) and we take a smooth function $\varphi(t)$ such that $\theta^*(t_o) - \varphi(t_o) = 0$ is a strict global maximum.

For any fixed $\delta > 0$, the function

$$\Psi_\delta(x,t) = \underline{v}(x,t) - \overline{v}(x,t) - \frac{\delta}{2}|x|^2 - \varphi(t)$$

has a global maximum point (x_δ, t_δ) on $\mathbb{R}^{\mathrm{N}} \times [0,T]$. We claim that it is possible to choose an infinitesimal sequence of parameters (that we still denote by δ) in such a way that

$$\frac{\delta}{2}|x_\delta|^2 \to 0, \quad t_\delta \to t_o, \quad \underline{v}(x_\delta, t_\delta) - \overline{v}(x_\delta, t_\delta) \to \theta^*(t_o), \tag{3.5}$$

as $\delta \to 0$. As a consequence

$$\theta(t_\delta) \to \theta^*(t_o) \qquad \text{as } \delta \to 0. \tag{3.6}$$

Moreover, we may suppose that $t_\delta > 0$ for all δ, even if $t_o = 0$: otherwise we should have $0 < \theta^*(t_o) = \lim_{\delta\to 0} [\underline{v}(x_\delta, 0) - \overline{v}(x_\delta, 0)] \le 0$ in force of ($1.2^\sharp$).

Afterward we double variables by considering $(x_{\alpha\delta}, y_{\alpha\delta}, t_{\alpha\delta})$ a maximum point for the function

$$\Psi_{\alpha\delta}(x,y,t) = \underline{v}(x,t) - \overline{v}(y,t) - \frac{\alpha}{2}|x-y|^2 - \frac{\delta}{2}|x|^2 - \varphi(t)$$

on $\mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}} \times [0,T]$. It is an easy exercise to check that, for any fixed δ,

$$\begin{gathered} \frac{\alpha}{2}|x_{\alpha\delta} - y_{\alpha\delta}|^2 \to 0, \qquad t_{\alpha\delta} \to t_\delta, \\ \underline{v}(x_{\alpha\delta}, t_\delta) - \overline{v}(y_{\alpha\delta}, t_{\alpha\delta}) \to \underline{v}(x_\delta, t_\delta) - \overline{v}(x_\delta, t_\delta), \end{gathered} \tag{3.7}$$

as $\alpha \to \infty$. Hence, in view of (3.5), we are allowed to assume that $|x_{\alpha\delta}|$, $|y_{\alpha\delta}| \le R(\delta)$, $t_{\alpha\delta} > 0$, $\underline{v}(x_{\alpha\delta}, t_\delta) - \overline{v}(y_{\alpha\delta}, t_{\alpha\delta}) > 0$ for large α.

We defer to check (3.5) and (3.7) later on. For the time being, we want to take advantage from the fact that $\underline{v}$ and $\overline{v}$ are sub/supersolutions of the integro–differential equation ($1.9^\sharp$) to achieve an estimate from above of $\varphi'(t_o)$. To this end, we apply [4, Theorem 8.3] with $\epsilon = 1/\alpha$ to $v_1(x,t) = \underline{v}(x,t) - \frac{\delta}{2}|x|^2$, $v_2(y,t) = -\overline{v}(y,t)$, $\phi(x,y,t) = \frac{\alpha}{2}|x-y|^2 + \varphi(t)$. It yields that there are $\tau_1, \tau_2 \in \mathbb{R}$ and two symmetrical matrices X, Y (depending on α, δ) such that

$$\tau_1 - \tau_2 = \varphi'(t_{\alpha\delta}), \quad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

and

$$\tau_1 + F^\sharp\Big(x, t, \underline{v}, \mathcal{I}^\sharp\underline{v}, \alpha(x-y)+\delta x, X+\delta I\Big) \leq 0,$$
$$\tau_2 + F^\sharp\big(y, t, \overline{v}, \mathcal{I}^\sharp\overline{v}, \alpha(x-y), Y\big) \geq 0,$$

where we have omitted the dependence on α, δ and we have used the abbreviated notation

$$F^\sharp(x,t,v,I,p,X) = -\frac{1}{2}\mathrm{tr}\big[\sigma\sigma^T X\big] + b^\sharp \cdot p + c^\sharp v - I + H^\sharp(x,t,v,I,p)\,.$$

Subtracting this two inequalities yields

$$\varphi'(t) \leq F^\sharp\Big(y, t, \overline{v}, \mathcal{I}^\sharp\overline{v}, \alpha(x-y), Y\Big) - F^\sharp\Big(x, t, \underline{v}, \mathcal{I}^\sharp\underline{v}, \alpha(x-y)+\delta x, X+\delta I\Big)\,.$$

By making use of (3.7) and repeating the computations of [4, Example 3.6], [15, Theorem II.1], passing to the limit as $\alpha \to \infty$ yields

$$\varphi'(t_\delta) \leq -\inf c^\sharp\theta(t_\delta) + \limsup_{\alpha\to\infty}\Big|\mathcal{I}^\sharp\underline{v}(x_{\alpha\delta}, t_{\alpha\delta}) - \mathcal{I}^\sharp\overline{v}(x_{\alpha\delta}, t_{\alpha\delta})\Big|$$
$$+ \limsup_{\alpha\to\infty}\omega^\sharp_{R(\delta)}\Big(\Big(1 + \|\overline{v}\|_\infty + \Big\|\mathcal{I}^\sharp\overline{v}\Big\|_\infty + \alpha|x_{\alpha\delta}-y_{\alpha\delta}|\Big)|x_{\alpha\delta}-y_{\alpha\delta}|\Big)$$
$$+\mathrm{L}'\Big(\theta(t_\delta) + \limsup_{\alpha\to\infty}\Big|\mathcal{I}^\sharp\underline{v}(x_{\alpha\delta}, t_{\alpha\delta}) - \mathcal{I}^\sharp\overline{v}(x_{\alpha\delta}, t_{\alpha\delta})\Big|\Big).$$

Since $\overline{v}$ and $\mathcal{I}^\sharp\overline{v}$ are bounded by construction (recalling H.3), the third term on the right–hand side is zero, via (3.7). Moreover, thanks to hypothesis H.3, $\mathcal{I}^\sharp\underline{v}$, $\mathcal{I}^\sharp\overline{v}$ are continuous, via Lemma 3.1 and [1, Lemma 1]. In particular, for all fixed δ we have

$$\limsup_{\alpha\to\infty}\Big|\mathcal{I}^\sharp\underline{v}(x_{\alpha\delta}, t_{\alpha\delta}) - \mathcal{I}^\sharp\overline{v}(x_{\alpha\delta}, t_{\alpha\delta})\Big|$$
$$\leq \int_{\mathbb{R}^{\mathrm{N}}}\Big|\mathcal{M}^\sharp\underline{v}(x_\delta, t_\delta; z) - \mathcal{M}^\sharp\overline{v}(x_\delta, t_\delta; z)\Big|\, d\mu_{x_\delta,t_\delta}(dz) \leq \mathrm{MD}\theta(t_\delta).$$

Summing up, we have obtained $\varphi'(t_\delta) \leq g(\theta(t_\delta))$.

Eventually passing to the limit as $\delta \to 0$ and taking advantage of (3.6) yields the desired inequality $\varphi'(t_o) \leq g(\theta^*(t_o))$, because φ' and g are continuous.

Lastly we sketch how (3.5) and (3.7) can be checked, for the sake of completeness.

With respect to (3.5), we set $M_\delta = \max \psi_\delta = \psi_\delta(x_\delta, t_\delta)$. By construction $M_\delta \leq 0$. Besides, by definition there is a sequence $(\hat{x}_n, \hat{t}_n)$ such that $\hat{t}_n \to t_0$ and $\underline{v}(\hat{x}_n, \hat{t}_n) - \overline{v}(\hat{x}_n, \hat{t}_n) \to \theta^*(t_0)$. Hence by choosing δ_n in such a way that $\frac{\delta_n}{2}|\hat{x}_n|^2 \to 0$ we obtain $M_{\delta_n} \to 0$ and therefore $\frac{\delta_n}{2}|x_{\delta_n}|^2 \to 0$ and

$$\lim_{n\to 0} [\underline{v}(x_{\delta_n}, t_{\delta_n}) - \overline{v}(x_{\delta_n}, t_{\delta_n})] = \lim_{n\to 0} \varphi(t_{\delta_n}). \tag{3.8}$$

Up to a subsequence, we may suppose that t_{δ_n} converges to some $\hat{t}_o \in [0, T]$. So (3.8) implies $\theta^*(\hat{t}_o) \geq \varphi(\hat{t}_o)$ and therefore $\hat{t}_o = t_o$, because t_o is a strict maximum point for $\theta^* - \varphi$, . Lastly, recalling that φ is continuous, the same (3.8) yields that $\underline{v}(x_{\delta_n}, t_{\delta_n}) - \overline{v}(x_{\delta_n}, t_{\delta_n}) \to \varphi(t_o) = \theta^*(t_o)$, so that (3.5) is established.

In order to check (3.7) we set $N_{\alpha\delta} = \max \Psi_{\alpha\delta}(x, y, t) = \Psi_{\alpha\delta}(x_{\alpha\delta}, y_{\alpha\delta}, t_{\alpha\delta})$. By construction $N_{\alpha\delta} \geq M_\delta$ for all α, so $\frac{\alpha}{2}|x_{\alpha\delta} - y_{\alpha\delta}|^2$ is equibounded with respect to α and then $|x_{\alpha\delta} - y_{\alpha\delta}| \to 0$ as $\alpha \to \infty$. If by contradiction $|x_{\alpha\delta}|$ where not bounded with respect to α, then $N_{\alpha\delta} \to -\infty$. Therefore, up to a subsequence, we may suppose that $(x_{\alpha\delta}, y_{\alpha\delta}, t_{\alpha\delta})$ converges to some point $(\hat{x}_\delta, \hat{x}_\delta, \hat{t}_\delta)$ as $\alpha \to \infty$. So standard semicontinuity arguments yield that $\frac{\alpha}{2}|x_{\alpha\delta} - y_{\alpha\delta}|^2 \to 0$ and $\Psi_{\alpha,\delta}(x_{\alpha\delta}, y_{\alpha\delta}, t_{\alpha\delta}) \to \Psi_\delta(\hat{x}_\delta, \hat{t}_\delta) = M_\delta$, up to an extracted sequence. In particular $(\hat{x}_\delta, \hat{t}_\delta)$ is a maximum point for Ψ_δ, so that we may suppose without loss of generality that it is equal to (x_δ, t_δ) considered before. Eventually, recalling that φ is continuous, we obtain the second and third items of (3.7). □

Now we are ready to prove, as an immediate corollary of Theorems 3.3 and 3.4, the well posedness result stated by Theorem 1.1.

Proof of Theorem 1.1. It is an easy exercise to show that there are three positive constants λ, A, B such that $h(x,t) = -e^{\lambda t}\left[\mathrm{A}t + \mathrm{B}e^{\max(n_o,n_1)\ell(x)}\right]$ and $k(x,t) = e^{\lambda t}\left[\mathrm{A}t + \mathrm{B}e^{\max(n_o,n_1)\ell(x)}\right]$ are respectively classical sub/super-solution. So, in view of Lemma 3.1, we may apply Theorem 3.3 and we obtain that u, defined in (3.2), is such that u^* and u_* are respectively a sub/super-solution. Next Theorem 3.4 yields that $u^* \leq u_*$ on $\mathbb{R}^{\mathrm{N}} \times [0, T]$. Therefore, since the inverse inequality holds by construction, $u \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times [0, T))$ is a viscosity solution, indeed.

Uniqueness follows immediately from comparison principle 3.4. □

We conclude this section by applying our results to the pricing of derivatives.

Proof of Theorem 2.7. The assumptions of Theorem 2.7 grant that the related Cauchy problem (1.9)–(1.2) fulfills H.0–H.3; in particular, condition

H.3 holds for all n and the source term $H(x,t,0,0,0)$ is zero. Hence, Theorem 1.1 establishes the existence of a continuous viscosity solution u, with $e^{-n_o|x|}u(x,t)$ bounded, which is unique in the class

$$\left\{v \in \mathcal{C}\big(\mathbb{R}^{\mathrm{N}} \times (0,T)\big) \,:\, |v(x,t)| = O(\exp n|x|) \text{ as } |x| \to \infty, \ \text{ for any } n\right\}.$$

In terms of the economical variable S, we have established the well posedness of final value problem (2.10)–(2.11) in the class of continuous functions with polynomial growth, that where stated in the first part of Theorem 2.7.

Moreover, if $0 \le G(S) \le \mathrm{B}_o(1 + |S|^{n_o})$, one may easily check that 0 and $e^{\lambda t}[\mathrm{A}t + \mathrm{B}e^{n_o\ell(x_1^+,\cdots,x_{\mathrm{N}}^+)}]$ are respectively sub/supersolution for the related Cauchy problem (1.9)–(1.2). Hence, the comparison principle stated in Theorem 3.4 guarantees that $0 \le U(S,t) \le \mathrm{B}_1(1 + |S|^{n_o})$.

4. Regularity result

Lastly, we investigate the regularity of the solution with respect to x: we expect that the solution of an integro–differential Cauchy problem is as smooth as its initial datum. In the pure differential setting, a regularity result with respect to the x–variable may be stated by refining the proof of comparison principle; actually, the solution of the Cauchy problem $u(\cdot,t)$ has the same kind of modulus of continuity of its initial datum (see [4] and the references therein).

We now prove analogous result for the integro–differential problem (1.9)–(1.2), under the assumptions H'.0–H'.3.

Proof of Theorem 1.2. We shall prove directly that

$$v = e^{-n\ell}u \in L^\infty\big(0,T;W^{1,\infty}(\mathbb{R}^{\mathrm{N}})\big),$$

by taking advantage from the fact that v is the viscosity solution of the Cauchy problem (1.9$^\sharp$)–(1.2$^\sharp$) assigned in the proof of Theorem 3.4. Notice that the coefficient of the linear operator $\mathcal{L}^\sharp$ still satisfies H'.1 and that $H^\sharp$ satisfies, beneath (1.11) (with $|u-v|$ replaced by $(1+n)|u-v|$), an estimate of type (1.10') for a new constant L (after replacing $|u|$ by $(1+n)|u|$). In addition v is continuous and bounded by virtue of Theorem 1.1, and $v(\cdot,0) = e^{-n\ell}u_o \in W^{1,\infty}(\mathbb{R}^{\mathrm{N}})$ by hypothesis H'.0. Thus, the thesis may be achieved by finding a suitable λ such that the function

$$\vartheta(t) = \sup\left\{v(x,t) - v(y,t) - \mathrm{B}_o e^{\lambda t}|x-y| \,:\, x,y \in \mathbb{R}^{\mathrm{N}}\right\}$$

is equal to 0. Here $\mathrm{B}_o = \|D\left(e^{-n\ell}u_o\right)\|_\infty$. By invoking Lemma 3.5, it is sufficient to establish that its upper semicontinuous envelope ϑ^* fulfills (3.4) for

$$g(\vartheta^*) = \Big((1+n)\text{L}' - \inf c^\sharp + (\text{L}'-1)\text{MD}\Big)^+ \vartheta^*,$$

where

$$\text{D} = \sum_{k=1}^{\text{M}} \Big\| \nu^k \left(1 + e^{n\ell\zeta^k}\right) \Big\|_\infty + \Big\| \int \left(1 + e^{n|z|}\right) \tilde{\nu}(\cdot; z) dz \Big\|_\infty.$$

To this purpose, we fix $t_o \in [0,T]$ such that $\vartheta^*(t_o) > 0$ and a smooth function φ such that $(\vartheta^* - \varphi)(t_o) = 0$ is a strict maximum. It is straightforward to check that there exists a sequence $(x_\delta, y_\delta, t_\delta) \in \mathbb{R}^{\text{N}} \times \mathbb{R}^{\text{N}} \times [0,T]$ of points of local maximum for the function

$$\mathbb{R}^{\text{N}} \times \mathbb{R}^{\text{N}} \times [0,T] \ni (x,y,t) \mapsto v(x,t) - v(y,t) - \frac{\delta}{2}|x|^2 - \text{B}_o e^{\lambda t}|x-y| - \varphi(t),$$

such that

$$\begin{gathered} \frac{\delta}{2}|x_\delta|^2 \to 0, \qquad t_\delta \to t_o, \\ v(x_\delta, t_\delta) - v(y_\delta, t_\delta) - \text{B}_o e^{\lambda t_\delta}|x_\delta - y_\delta| \to \vartheta^*(t_o) > 0, \end{gathered} \tag{4.1}$$

as $\delta \to 0$. Therefore we may suppose without loss of generality that $x_\delta \neq y_\delta$ and that $t_\delta > 0$ for all $\delta > 0$, because $v(x_\delta, 0) - v(y_\delta, 0) - \text{B}_o|x_\delta - y_\delta| \le 0$ by construction; moreover

$$\vartheta(t_\delta) \to \vartheta^*(t_o), \qquad \text{as } \delta \to 0. \tag{4.2}$$

On the other hand, by applying [4, Theorem 8.3] with $\varepsilon = \text{B}_o e^{\lambda t_\delta}|x_\delta - y_\delta| > 0$ to $v_1(x,t) = v(x,t) + \frac{\delta}{2}|x|^2$, $v_2(y,t) = -v(y,t)$, $\phi(x,y,t) = \text{B}_o e^{\lambda t}|x-y| + \varphi(t)$, and proceeding as in the proof of Theorem 3.4, we have

$$\begin{aligned} \varphi'(t) + \lambda \text{B}_o e^{\lambda t}|x-y| \le\; & F^\sharp\Big(y, t, v(y,t), \mathcal{I}^\sharp v(y,t), \text{B}_o e^{\lambda t}\frac{x-y}{|x-y|}, Y\Big) \\ & - F^\sharp\Big(x, t, v(x,t), \mathcal{I}^\sharp v(x,t), \text{B}_o e^{\lambda t}\frac{x-y}{|x-y|} + \delta x, X + \delta I\Big), \end{aligned} \tag{4.3}$$

where we have omitted the dependence from δ for simplicity of notations.

With respect to the nonlocal term $\mathcal{I}^\sharp v$, we notice that, since $|v(\xi,t) - v(\eta,t)| \le \vartheta(t) + \text{B}_o e^{\lambda t}|\xi - \eta|$ for all ξ, η, hypothesis H'.3 provides the following estimate:

$$\Big| \mathcal{I}^\sharp v(x,t) - \mathcal{I}^\sharp v(y,t) \Big| \le \text{MD}\,\vartheta(t) + \text{C}_1 \text{B}_o e^{\lambda t}|x-y|,$$

where the parameter C_1 depends, besides on M and on $\|v\|_\infty$, on the $W^{1,\infty}$–norms of ν^k, ζ^k, and on $\tilde{\text{D}}'$. Hence, by making use of the assumptions H'.1, H'.2, after computations one gets

$$F^\sharp\Big(y,t,v(y,t),\mathcal{I}^\sharp v(y,t),\mathrm{B}_o e^{\lambda t}\frac{x-y}{|x-y|},Y\Big)$$
$$-F^\sharp\Big(x,t,v(x,t),\mathcal{I}^\sharp v(x,t),\mathrm{B}_o e^{\lambda t}\frac{x-y}{|x-y|}+\delta x,X+\delta I\Big)\le$$
$$\le g(\vartheta(t))+\big[\mathrm{C}_1+\mathrm{L}'(\mathrm{C}_1+\mathrm{C}_2)\big]\,\mathrm{B}_o e^{\lambda t}|x-y|+o(1)$$

as δ goes to zero. Here, the parameter C_2 only depends, besides on $\|v\|_\infty$, on the $W^{1,\infty}$–norms of $\sigma_{ij}, b_i^\sharp, c^\sharp$, $H^\sharp(x,t,0,0,0)$, and on L.

Plugging this estimate in (4.3) and choosing $\lambda=\mathrm{C}_1+\mathrm{L}'(\mathrm{C}_1+\mathrm{C}_2)$ yields $\varphi'(t_\delta)\le g(\vartheta(t_\delta))+o(1)$. By taking advantage of the continuity of φ', g and of (4.2), passing to the limit with respect to δ gives $\varphi'(t_o)\le g(\vartheta^*(t_o))$, as wanted. □

We lastly illustrate, for the sake of completeness, how the Lipschitz regularity of the derivatives' price function follows by Theorem 1.2.

Proof of Theorem 2.8. The same assumptions of Theorem 2.7 grant that the related Cauchy problem (1.9)–(1.2) fulfills H'.0–H'.3. Hence, Theorem 1.2 may be applied and the belonging of $e^{-n\ell}u$ to $L^\infty\big(0,T;W^{1,\infty}(\mathbb{R}^{\mathrm{N}})\big)$ is attained. After coming back to the economical variable S, one exactly get the statement of Theorem 2.8. □

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