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SYMMETRY OF SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION WITH UNBOUNDED COEFFICIENTS

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Abstract. We study the equation $-\Delta u+|x|^a|u|^{q-2}u = |x|^b|u|^{p-2}u$ with Dirichlet boundary condition on $B(0,1)$ or on $\mathbb{R}^N$. We study the radial solutions of this equation on $\mathbb{R}^N$ and the symmetry breaking for ground states for $q = 2$ on $\mathbb{R}^N$. Estimates of the transition are also given when p is close to 2 or 2^* on $B(0,1)$.

1. Introduction

We consider the equation

$$\begin{cases} -\Delta u + g(x)|u|^{q-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω denotes $\mathbb{R}^N$ or the unit ball $B(0,1)$. We are interested in the existence of radial solutions of (1.1). Many papers deal with the case $\Omega = \mathbb{R}^N$, $q = 2$, g large at infinity and f superlinear, subcritical and bounded in x (see e.g. [1], [6], [5]). The case f unbounded in x was studied in [10] and [9] and the case $\Omega = B(0,1)$, $g(x) = 0$ and $f(x,u) = |x|^b u^{p-1}$ was studied in [11]; in particular, it is proved that under some hypothesis the ground states are not radial symmetric. For an introduction to symmetry-breaking results, see the survey of Brezis [3].

In this paper we consider the model problem

$$\begin{cases} -\Delta u + |x|^a|u|^{q-2}u = |x|^b|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

First, we study the general case (i.e., $q > 1$) with $\Omega = \mathbb{R}^N$. In Section 2.1, using the mountain-pass theorem, we prove the existence of positive radial

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solution of (1.2) when

$$\max(2,q) < p < 2^* + \frac{2b}{N-2}, \quad 2(b-a) < (N-1)(p-q).$$

In Section 2.2, using a symmetric mountain-pass theorem, we prove that there exist infinitely many radial solutions of (1.2). In Section 2.3, we consider a necessary condition for existence of solutions of (1.2), using a Pohozaev-type equality.

Next, we deal with the quadratic case ($q = 2$), with $\Omega = \mathbb{R}^N$ or $\Omega = B(0,1)$. In Section 3, we study

$$S_{a,b,p} := \inf_{\substack{u \in X, \\ u \neq 0.}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_{\mathbb{R}^N} |x|^b |u|^p dx\right)^{\frac{2}{p}}}$$

and $S^R_{a,b,p}$, the same infimum on the space of radially symmetric functions of X, where X is the completion of $\mathcal{D}(\Omega)$ for the norm considered. We prove that, on $\mathbb{R}^N$, we have $S_{a,b,p} < S^R_{a,b,p}$ if $ap < 2b$ and b are large enough.

Sections 4 and 5 are devoted to ground states for p close to the limit values 2 and 2^* (with $\Omega = B(0,1)$).

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2. The general problem

2.1. Existence of a radial solution. We consider the problem

$$\begin{cases} -\Delta u + |x|^a |u|^{q-2} u = |x|^b |u|^{p-2} u & \text{on } \mathbb{R}^N, \\ u > 0, \end{cases} \tag{2.1}$$

with

- $u(x)$ radially symmetric (i.e., $u(x) = u(|x|)$) such that
$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{2} + |x|^a \frac{|u|^q}{q} dx < \infty \; ; \tag{2.2}$$
- $a, b > -N$ so that $|x|^a$, $|x|^b \in L^1_{loc}$;
- $q > 1$.

The functional associated with this problem is

$$\Phi(u) := \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^2}{2} + |x|^a \frac{|u|^q}{q} - |x|^b \frac{|u|^p}{p} \Big) dx. \tag{2.3}$$

We use the following norm:

$$||u||_X := ||\nabla u||_2 + \left\| |x|^{\frac{a}{q}} u \right\|_q,$$

and we define X as the completion of $\mathcal{D}(\mathbb{R}^N)$ for this norm.

We define also the space

$$Y := \left\{ u : u \text{ is measurable on } \mathbb{R}^N \text{ and } \int_{\mathbb{R}^N} |x|^b |u|^p dx < \infty \right\}$$

with the norm $||u||_Y := \left\| |x|^{\frac{b}{p}} u \right\|_p$. We denote by X_r and Y_r the spaces of radially symmetric functions in, respectively, X and Y. We will prove that $X_r \subset Y_r$ and that the embedding is compact.

First, we recall two inequalities due, respectively, to Rother [8] and Strauss [12].

Lemma 1. *If $N \geq 3$, $1 \leq p < \infty$, $p = \frac{2N}{N-2} + \frac{2c}{N-2}$, there exists $A_{N,c} > 0$ such that, for every $u \in \mathcal{D}_r^{1,2}(\mathbb{R}^N)$,*

$$\left(\int_{\mathbb{R}^N} |x|^c |u|^p dx \right)^{\frac{2}{p}} \leq A_{N,c} \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Lemma 2. *If $N \geq 2$, there exists $B_N > 0$ such that for every $u \in H_r^1(\mathbb{R}^N)$,*

$$|x|^{\frac{N-1}{2}} |u(x)| \leq B_N |u|_2^{\frac{1}{2}} |\nabla u|_2^{\frac{1}{2}} \qquad \text{a.e. on } \mathbb{R}^N.$$

Now we can prove the compactness of the embedding $X_r \subset Y_r$.

Lemma 3. *If $2 < p < 2^* + \frac{2b}{N-2}$ and if $2(b-a) < (N-1)(p-q)$, then $X_r \subset Y_r$ and the embedding is compact.*

Proof. Assume that $u_n \rightharpoonup 0$ in X_r. We must prove that $||u_n||_Y \to 0$. First, we note that the sequence $||u_n||_X$ is bounded. We have

$$||u_n||_Y^p = \int_{|x| \leq \epsilon} |x|^b |u_n|^p dx + \int_{\epsilon \leq |x| \leq \frac{1}{\epsilon}} |x|^b |u_n|^p dx + \int_{|x| \geq \frac{1}{\epsilon}} |x|^b |u_n|^p dx. \quad (2.4)$$

We define c by $p = 2^* + \frac{2c}{N-2}$. Lemma 1 gives us, knowing that $||\nabla u_n||_2$ is bounded (see [9]), the following result:

$$\int_{|x| \leq \epsilon} |x|^b |u_n|^p dx \leq c_1 \, \epsilon^{b-c}.$$

The hypothesis $p < 2^* + \frac{2b}{N-2}$ ensures us that the first term of (2.4) tends to 0 uniformly in n as $\epsilon \to 0$.

Now we study the third term of (2.4):

$$\int_{|x|\geq\frac{1}{\epsilon}}|x|^b|u_n|^p dx=\int_{|x|\geq\frac{1}{\epsilon}}|x|^{b-a}|u_n|^{p-q}|x|^a|u_n|^q dx.$$

Using Lemma 2, we obtain

$$\int_{|x|\geq\frac{1}{\epsilon}}|x|^b|u_n|^p dx\leq c_2\int_{|x|\geq\frac{1}{\epsilon}}|x|^{b-a-\frac{N-1}{2}(p-q)}|x|^a|u_n|^q dx\leq c_3\big(\frac{1}{\epsilon}\big)^{b-a-\frac{N-1}{2}(p-q)}.$$

Using the hypothesis $2(b-a)<(N-1)(p-q)$, we see that this term tends to 0 uniformly in n as $\epsilon\to 0$.

So, it suffices to prove that $\int_\Omega |x|^b|u_n|^p dx\to 0$ as $n\to\infty$ with $\Omega:=\left\{x\in\mathbb{R}^N:\epsilon\leq|x|\leq\frac{1}{\epsilon}\right\}$. As Ω is bounded, there exists $c_4\in\mathbb{R}$ such that

$$||u_n||_{H^1(\Omega)}\leq c_4||u_n||_X.$$

As $u_n\rightharpoonup 0$ in X, we obtain $u_n\rightharpoonup 0$ in $H^1(\Omega)$. Using the fact that $H^1(\Omega)\subset\subset L^2(\Omega)$, we obtain

$$u_n\to 0 \text{ in } L^2(\Omega). \tag{2.5}$$

Now we compute, using Lemma 2,

$$\int_\Omega|x|^b|u_n|^p dx\leq c_4||u_n||_\infty^{p-2}\int_\Omega|u_n|^2dx\leq C\int_\Omega|u_n|^2dx,$$

which tends to 0 by (2.5). □

We will use the following variant of the mountain-pass theorem (see [2]).

Lemma 4. *Let E be a Banach space, $\Phi\in\mathcal{C}^1(E,\mathbb{R})$ which satisfies $\Phi(0)=0$ and the following conditions:*

- [i] *there exists $r>0$ such that $\inf_{||u||=r}\Phi(u)>0$;*
- [ii] *there exists $v\neq 0$ with $||v||>r$ such that $\Phi(v)<0$;*
- [iii] *the Palais-Smale condition:*
  if $\Phi'(u_n)\to 0$ and if $\Phi(u_n)\to c$, then (u_n) has a subsequence converging to $u\in X$;
- [iv] *there exists a continuous mapping $P:E\to E$ such that $\Phi(P(u))\leq\Phi(u)$ for every $u\in E$, $P(0)=0$ and $P(v)=v$.*

Then there exists a critical point $u^*\in\overline{P(E)}$ such that $\Phi(u^*)\geq\inf_{||u||=r}\Phi(u)$.

Now we prove the main result of this section.

Theorem 5. *If $N\geq 3$, $q>1$, $\max(2,q)<p<2^*+\frac{2b}{N-2}$ and $2(b-a)<(N-1)(p-q)$, then problem (2.1) has a radial solution.*

Proof. Consider the functional Φ restricted to X_r. We shall verify the assumptions of Lemma 4. First, we verify the geometry of the mountain-pass theorem.

[i] we must find $r > 0$ such that $\inf_{||u||=r} \Phi(u) > 0$.

We suppose $p > q \geq 2$ and we take $||u||_X \ll 1$. We have, by Lemma 3,

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{q}\left\||x|^{\frac{a}{q}}u\right\|_q^q - \frac{1}{p}\|u\|_Y^p,\\ &\geq \frac{1}{2}\|\nabla u\|_2^q + \frac{1}{q}\left\||x|^{\frac{a}{q}}u\right\|_q^q - c_5\|u\|_X^p.\end{aligned}$$

As $||\nabla u||_2 < 1$, we obtain

$$\Phi(u) \geq \frac{1}{q}\Big(\|\nabla u\|_2^q + \left\||x|^{\frac{a}{q}}u\right\|_q^q\Big) - c_5\|u\|_X^p.$$

However, $\left(\frac{a+b}{2}\right)^q \leq \max(a^q, b^q) \leq a^q + b^q$, so we have

$$\Phi(u) \geq \frac{1}{q\,2^q}\Big(\|\nabla u\|_2 + \left\||x|^{\frac{a}{q}}u\right\|_q\Big)^q - c_5||u||_X^p = c_6||u||_X^q - c_5||u||_X^p.$$

As $p > q$, we see that for $||u||_X = r$ small enough we have $\Phi(u) > 0$.

On the other hand, if we suppose $p > 2 \geq q$ a similar argument is valid. In this case we obtain

$$\Phi(u) \geq c_6||u||_X^2 - c_5||u||_X^p.$$

[ii] Let $w > 0$ be fixed; we consider $v := tw$, with $t \in \mathbb{R}$, so we obtain

$$\Phi(v) = \frac{t^2}{2}\|\nabla w\|_2^2 + \frac{t^q}{q}\left\||x|^{\frac{a}{q}}w\right\|_q^q - \frac{t^p}{p}\left\||x|^{\frac{b}{p}}w\right\|_p^p \leq c_7\,t^2 + c_8\,t^q - c_9\,t^p.$$

So, there exist t such that $\Phi(v) < 0$ and $||v|| > r$.

[iii] We verify the Palais-Smale condition.

Let $(u_n) \subset X_r$ such that $\Phi'(u_n) \to 0$ and $\Phi(u_n) \to c$; we must prove that $u_n \to u \in X_r$. First, we prove that the sequence (u_n) is bounded.

For n large enough, we have $\Phi(u_n) \leq c + 1$, $||\Phi'(u_n)|| \leq p$. So we obtain

$$\begin{aligned}c + 1 + ||u_n||_X &\geq \Phi(u_n) - \frac{1}{p}\left\langle \Phi'(u_n), u_n\right\rangle\\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\|\nabla u_n\|_2^2 + \Big(\frac{1}{q} - \frac{1}{p}\Big)\left\||x|^{\frac{a}{q}}u_n\right\|_q^q\\ &\geq m\Big(\Big(||\nabla u_n||_2 + \left\||x|^{\frac{a}{q}}u_n\right\|_q\Big)^{\min(2,q)} - 1\Big),\end{aligned}$$

where $m = \min(\frac{1}{2} - \frac{1}{p}, \frac{1}{q} - \frac{1}{p})$. Since $q > 1$, the sequence (u_n) is bounded. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in X_r.

Now we prove that $u_n \to u \in X_r$.

As $(u_n - u) \rightharpoonup 0$ in X_r, we have $\langle \Phi'(u), u_n - u\rangle \to 0$. On the other hand, as $\Phi'(u_n) \to 0$ in X_r', we have also $\langle \Phi'(u_n), u_n - u\rangle \to 0$. In consequence, we obtain

$$\Big\langle \Phi'(u_n) - \Phi'(u), u_n - u \Big\rangle \to 0.$$

It follows that

$$\Big\langle \Phi'(u_n) - \Phi'(u), u_n - u \Big\rangle = ||\nabla(u_n - u)||_2^2 + \Big\| |x|^{\frac{a}{q}}(u_n - u)\Big\|_q^q - \Big\| |x|^{\frac{b}{p}}(u_n - u)\Big\|_p^p.$$

But, as $u_n \rightharpoonup u$ in X_r, we have $u_n \to u$ in Y_r (by Lemma 3), so

$$\Big\| |x|^{\frac{b}{p}}(u_n - u)\Big\|_p^p \to 0.$$

Then we have

$$||\nabla(u_n - u)||_2^2 + \Big\| |x|^{\frac{a}{q}}(u_n - u)\Big\|_q^q \to 0,$$

which implies

$$\|\nabla(u_n - u)\|_2 \to 0, \quad \Big\| |x|^{\frac{a}{q}}(u_n - u)\Big\|_q \to 0.$$

And we obtain $u_n \to u$ in X_r.

So the conditions of the mountain-pass theorem are verified. To obtain a positive solution, it suffices to choose, in Lemma 4, the mapping $u \mapsto |u|$ as P. This mapping is continuous, $\Phi(P(u)) = \Phi(u)$ for all $u \in X_r$, $P(0) = 0$ and, if we take $v > 0$ in [ii], it is clear that $P(v) = v$.

So, using Lemma 4, we obtain a nontrivial critical point $u \geq 0$ of Φ restricted to X_r. By the maximum principle we obtain $u > 0$ on $\mathbb{R}^N$ and the principle of symmetric criticality (see e.g. [13]) ends the proof. □

2.2. **Existence of infinitely many radial solutions.** We consider the analogous problem

$$\begin{cases} -\Delta u + |x|^a|u|^{q-2}u = |x|^b|u|^{p-2}u \qquad \text{on } \mathbb{R}^N, \\ u(x) = u(|x|), \\ \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{2} + |x|^a \frac{|u|^q}{q} dx < \infty. \end{cases} \tag{2.6}$$

Here we take radial functions to have the compact embedding $X_r \subset\subset Y_r$.

We will use the same norm $||\cdot||_X$ and the same functional Φ, restricted to X_r, as in Section 2.1. We will apply the following result, a symmetric version of the mountain-pass theorem (see Rabinowitz [7]).

Lemma 6. *Let E be an infinite-dimensional Banach space and let $\Phi \in \mathcal{C}^1(E, \mathbb{R})$ be even, satisfy the (PS) condition and $\Phi(0) = 0$. If $E = V \oplus W$, where V is finite dimensional and Φ satisfies*

(A_1) *there exists $r > 0$ such that*

$$\inf_{\substack{u \in W, \\ ||u||_X = r.}} \Phi(u) > 0;$$

(A_2) *for each finite-dimensional subspace $F \subset E$, there exists ρ_F such that*

$$\max_{\substack{u \in F, \\ ||u||_X = \rho_F.}} \Phi(u) < 0;$$

then Φ possesses an unbounded sequence of critical values.

Theorem 7. *If $N \geq 3$, $q > 1$, $\max(2, q) < p < 2^* + \frac{2b}{N-2}$ and if $2(b - a) < (N - 1)(p - q)$, then problem (2.6) has a sequence of radial solutions (u_k) such that $\Phi(u_k) \to \infty$ as $k \to \infty$.*

Proof. We will verify the hypothesis of Lemma 6 with $E = X_r$, $V = 0$ and $W = X_r$. First, we note that X_r is an infinite-dimensional Banach space and that $\Phi \in \mathcal{C}^1(E, \mathbb{R})$ is even, vanishes for $u = 0$ and satisfies the Palais-Smale condition.

Condition (A_1) was already verified in the proof of Theorem 5. In order to verify condition (A_2), we will find a majoration of Φ:

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{q} \left\| |x|^{\frac{a}{q}} u \right\|_q^q - \frac{1}{p} \left\| |x|^{\frac{b}{p}} u \right\|_p^p, \\ &\leq ||u||_X^2 + ||u||_X^q - \frac{1}{p} ||u||_Y^p. \end{aligned}$$

Since on the finite-dimensional space F all norms are equivalent, relation (A_2) is satisfied for every $\rho_F > 0$ large enough.

So, using Lemma 6, we obtain that Φ possesses an unbounded sequence of critical values on X_r. The principle of symmetric criticality ends the proof. □

2.3. **Nonexistence of solution.** Now we will obtain a nonexistence result of solution for problem (2.1) by a Pohozaev-type identity. Here, the difficulty is that the domain is unbounded, so we will use a truncation argument (see e.g. [4]).

Lemma 8. *Let $u \in X$ be a solution of* (2.1); *then u satisfies*

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{N+a}{q}\int_{\mathbb{R}^N}|x|^a|u|^q\,dx-\frac{N+b}{p}\int_{\mathbb{R}^N}|x|^b|u|^p\,dx=0.$$

Proof. Let $\psi \in \mathcal{D}(\mathbb{R})$ such that $0 \le \psi \le 1$, $\psi(r)=1$ for $r \le 1$ and $\psi(r)=0$ for $r \ge 2$. We define on $\mathbb{R}^N$ the functions

$$\psi_n(x) := \psi\Big(\frac{|x|^2}{n^2}\Big).$$

There exists $c \ge 0$ such that, for every n,

$$\psi_n \le c \qquad \text{and} \qquad |x|\,|\nabla\psi_n(x)| \le c.$$

It follows from (2.1) that, for every n,

$$0=\Big(\Delta u-|x|^a|u|^{q-1}+|x|^b|u|^{p-1}\Big)\,\psi_n x\cdot\nabla u. \tag{2.7}$$

In order to simplify notation, let

$$f(x,u):=-|x|^a|u|^{q-1}+|x|^b|u|^{p-1},\quad F(x,u):=-|x|^a\frac{|u|^q}{q}+|x|^b\frac{|u|^p}{p}.$$

We have

$$\begin{aligned}
f(x,u)\psi_n x\cdot\nabla u &= \operatorname{div}\left(F(x,u)\psi_n x\right)-N\psi F(x,u)-F(x,u)x\cdot\nabla\psi_n\\
&\quad -\Big(-\frac{a}{q}|x|^a|u|^q+\frac{b}{p}|x|^b|u|^p\Big)\psi_n,\\
\psi_n\,\Delta u\,x\cdot\nabla u &= \operatorname{div}\Big(\psi_n\Big(\nabla u x\cdot\nabla u-x\frac{|\nabla u|^2}{2}\Big)\Big)+\frac{N-2}{2}\psi_n|\nabla u|^2\\
&\quad +\frac{|\nabla u|^2}{2}x\cdot\nabla\psi_n-x\cdot\nabla u\nabla\psi_n\cdot\nabla u.
\end{aligned}$$

Using these relations in (2.7) and integrating, we obtain, for every n,

$$\begin{aligned}
\int_{\mathbb{R}^N}\Big(\Big[&NF(x,u)-\frac{a}{q}|x|^a|u|^q+\frac{b}{p}|x|^b|u|^p-\frac{N-2}{2}|\nabla u|^2\Big]\psi_n\\
&+F(x,u)x\cdot\nabla\psi_n-\frac{|\nabla u|^2}{2}x\cdot\nabla\psi_n+x\cdot\nabla u\nabla\psi_n\cdot\nabla u\Big)dx=0.
\end{aligned}$$

The Lebesgue dominated convergence theorem implies that

$$\int_{\mathbb{R}^N}\Big(NF(x,u)-\frac{a}{q}|x|^a|u|^q+\frac{b}{p}|x|^b|u|^p-\frac{N-2}{2}|\nabla u|^2\Big)dx=0,$$

which concludes the proof. □

We can now prove the following nonexistence-of-solution result.

Theorem 9. *If*

$$p < 2^* + \frac{2b}{N-2} \qquad \textit{and} \qquad q(N+b) < p(N+a)$$

or if

$$p > 2^* + \frac{2b}{N-2} \qquad \textit{and} \qquad q(N+b) > p(N+a),$$

then $u = 0$ *is the unique solution of problem* (2.1).

Proof. Since u is a nontrivial solution of (2.1), we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |x|^a |u|^q dx = \int_{\mathbb{R}^N} |x|^b |u|^p dx.$$

Using Lemma 8, we obtain

$$\Big(\frac{N-2}{2} - \frac{N+b}{p}\Big) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \Big(\frac{N+a}{q} - \frac{N+b}{p}\Big) \int_{\mathbb{R}^N} |x|^a |u|^q dx = 0.$$

As the two integrals are positive, their coefficients must be of opposite sign. Since there is no order relation between these coefficients, we consider two cases.

Case 1. We suppose $\frac{N-2}{2} - \frac{N+b}{p} > 0 > \frac{N+a}{q} - \frac{N+b}{p}$. So we obtain the following contradiction:

$$p > 2^* + \frac{2b}{N-2}, \quad p(N+a) < q(N+b).$$

Case 2. We suppose $\frac{N+a}{q} - \frac{N+b}{p} > 0 > \frac{N-2}{2} - \frac{N+b}{p}$. So we obtain

$$p < 2^* + \frac{2b}{N-2}, \quad p(N+a) > q(N+b),$$

which is a contradiction. □

3. The quadratic problem on $\mathbb{R}^N$

We consider the equation

$$\begin{cases} -\Delta u + |x|^a u = |x|^b u^{p-1} \qquad \text{on } \mathbb{R}^N, \\ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |x|^a |u|^2 dx < \infty. \end{cases} \tag{3.1}$$

Let

$$R(u) := \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_{\mathbb{R}^N} |x|^b |u|^p dx\right)^{\frac{2}{p}}};$$

then we define

$$S^R_{a,b,p} := \inf_{u\in X_r,\ u\neq 0.} R(u) \qquad \text{and} \qquad S_{a,b,p} := \inf_{u\in X,\ u\neq 0.} R(u).$$

We will find a condition in order to have $S_{a,b,p} < S^R_{a,b,p}$. The main idea is to give an asymptotic estimate $S^R_{a,b,p}$ as $b \to \infty$, using a change of variables, and then to majorate $S_{a,b,p}$ considering $R(u_\alpha)$ with u_α a bump "going" to infinity. Then we will obtain

$$S_{a,b,p} \leq R(u_\alpha) < S^R_{a,b,p}$$

for some a, b and p. We assume that $N \geq 2$ and $2 < p < 2^*$ are satisfied.

3.1. **Estimation of $S^R_{a,b,p}$.** Let $v(|x|) := u(|x|^\beta)$ for β to determine. We have

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^b|u|^p dx &= \int_0^{+\infty} |u(r)|^p r^{b+N-1} dr \\ &= \beta \int_0^{+\infty} |v(\rho)|^p \rho^{\beta(b+N-1)+\beta-1} d\rho \qquad \text{with } r = \rho^\beta.\end{aligned}$$

To avoid ρ under the integral, let $\beta := \frac{N}{N+b}$. Then we obtain

$$\int_{\mathbb{R}^N} |x|^b|u|^p dx = \beta \int_0^\infty |v(\rho)|^p \rho^{N-1} d\rho = \beta||v||^p_{L^p(\mathbb{R}^N)}.$$

Using the same argument, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2 dx &= \int_0^\infty \left(\frac{|v'(\rho)|}{\beta\rho^{\beta-1}}\right)^2 \rho^{\beta(N-1)}\beta\rho^{\beta-1} d\rho \\ &= \beta^{-1} \int_{\mathbb{R}^N} |\nabla v|^2 |x|^{(N-2)(\beta-1)} dx\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^a u^2 dx &= \int_0^\infty \rho^{\beta(a+N-1)} |u(\rho^\beta)|^2 \beta\rho^{\beta-1} d\rho, \\ &= \beta \int_0^\infty |v(\rho)|^2 \rho^{\beta(N+a)-1} d\rho = \beta \int_{\mathbb{R}^N} |v(|x|)|^2 |x|^{\beta a+(\beta-1)N} dx.\end{aligned}$$

So, the Rayleigh quotient is

$$R(u) = \frac{\beta^{-1}\int_{\mathbb{R}^N} |\nabla v|^2 |x|^{(N-2)(\beta-1)} dx + \beta \int_{\mathbb{R}^N} |v|^2 |x|^a |x|^{(N+1)(\beta-1)} dx}{\beta^{\frac{2}{p}} \left(\int_{\mathbb{R}^N} |v|^p\right)^{\frac{2}{p}}}.$$

We define

$$F(v) := \frac{\int_{\mathbb{R}^N} |\nabla v|^2 |x|^{(N-2)(\beta-1)} dx}{\left(\int_{\mathbb{R}^N} |v|^p\right)^{\frac{2}{p}}},$$

$$G(v) := \frac{\int_{\mathbb{R}^N} |v|^2 |x|^a |x|^{(N+1)(\beta-1)} dx}{\left(\int_{\mathbb{R}^N} |v|^p\right)^{\frac{2}{p}}},$$

so we have

$$S^R_{a,b,p} = \inf_{v \in H^1_{r,a}(\mathbb{R}^N)} \left(\beta^{-1-\frac{2}{p}} F(v) + \beta^{1-\frac{2}{p}} G(v)\right).$$

We have the following relation:

$$\begin{aligned} \inf\left(\beta^{-1-\frac{2}{p}} F(v) + \beta^{1-\frac{2}{p}} G(v)\right) &\geq \inf\left(\beta^{-1-\frac{2}{p}} F(v)\right) + \beta^{1-\frac{2}{p}} G(v) \\ &\geq \inf\left(\beta^{-1-\frac{2}{p}} F(v)\right). \end{aligned}$$

So there exists $C > 0$ such that $S^R_{abp} \geq C\beta^{-1-\frac{2}{p}}$ when $\beta \to 0$ and $\beta = \frac{N}{b+N} \to 0$ when $b \to +\infty$.

3.2. **Estimation of $S_{a,b,p}$.** Let $u \in \mathcal{D}(B(0,1))$ be a "bump" that we translate to the infinite:

$$u_\alpha(x) := u(x - x_\alpha) \qquad \text{where } x_\alpha = (\alpha, 0, \dots, 0) \text{ with } \alpha \to \infty.$$

We want to find an upper bound for $R(u)$. To find it, we compute an upper bound of its numerator and a lower bound of its denominator. First, we note that

$$\int_{\mathbb{R}^N} |\nabla u_\alpha|^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Now we look to the denominator and the second term of the numerator:

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^b |u_\alpha(x)|^p dx &= \int_{B(x_\alpha,1)} |x|^b |u_\alpha(x)|^p dx \\ &\geq \int_{B(x_\alpha,1)} (\alpha-1)^b |u_\alpha(x)|^p dx = (\alpha-1)^b ||u||_p^p \\ \int_{\mathbb{R}^N} |x|^a |u_\alpha(x)|^2 dx &= \int_{B(x_\alpha,1)} |x|^a |u_\alpha(x)|^2 dx \leq (\alpha+1)^a ||u||_2^2. \end{aligned}$$

So we have

$$0 \leq S_{a,b,p} \leq R(u_\alpha) \leq \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + (\alpha+1)^a \int_{\mathbb{R}^N} u^2(x) dx}{(\alpha-1)^{\frac{2}{p}b} \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{2}{p}}} \leq C\alpha^{a-\frac{2}{p}b},$$

where C is a positive constant.

3.3. **Comparison between $S_{a,b,p}$ and $S^R_{a,b,p}$.** We have proved that $S_{a,b,p} \to 0$ as $\alpha \to +\infty$ when $a < b\frac{2}{p}$. On the other hand, we have $S^R_{a,b,p} \to +\infty$ as $b \to +\infty$. So we have proved the following result.

Theorem 10. *Let $N \geq 2$ and $2 < p < 2^*$, if $a < b\frac{2}{p}$; then there exists $b^* > 0$ such that for every $b > b^*$ we have $S_{a,b,p} < S^R_{a,b,p}$.*

This result is similar to Theorem 3.1 in [9].

4. The quadratic problem on $B(0,1)$: analysis for p close to 2

We consider

$$-\Delta u + |x|^a u = |x|^b u^{p-1}, \quad u \in H^1_0(B(0,1)). \tag{4.1}$$

The goal is to study b^*, a limiting exponent above which an infimum cannot be radial, when $p \to 2$.

Let $(-\Delta)^{-1}$ be the inverse of the Laplace operator with Dirichlet boundary conditions. We note that equation (4.1) can be written, with $\lambda = 1$,

$$(-\Delta)^{-1}\left(\lambda |x|^b u^{p-1} - |x|^a u\right) - u = 0. \tag{4.2}$$

We define the operator

$$\begin{aligned} P : [2,2^*) \times \mathbb{R}^+ \times \mathbb{R}^+ \times H^1_0(B(0,1)) \times \mathbb{R} \quad &\to \quad H^1_0(B(0,1)) \times \mathbb{R}, \\ (p,a,b,u,\lambda) \quad &\mapsto \quad \left((-\Delta)^{-1}\left(\lambda |x|^b u^{p-1} - |x|^a u\right) - u, ||u||^2 - 1\right), \end{aligned}$$

where

$$||u||^2 = \int_{B(0,1)} |\nabla u|^2 dx.$$

For a, b fixed and p close to 2, we will prove (using the implicit function theorem) that the (u, λ) for which P vanishes (i.e., u is a nontrivial solution of (4.1)) are functions of (a, b, p). In this section, u_0 and S_0 will denote respectively $u_{a_0,b_0,2}$ and $S_{a_0,b_0,2}$.

Lemma 11. *Let $a_0, b_0 > 0$ be fixed; then there exists $\epsilon > 0$ and functions*

$$\Lambda : [2, 2+\epsilon) \times (a_0 - \epsilon, a_0 + \epsilon) \times (b_0 - \epsilon, b_0 + \epsilon) \to (S_0 - \epsilon, S_0 + \epsilon),$$

$$U : [2, 2+\epsilon) \times (a_0 - \epsilon, a_0 + \epsilon) \times (b_0 - \epsilon, b_0 + \epsilon) \to B(u_0, \epsilon) \subset H^1_0(B(0,1)),$$

such that in $[2,2+\epsilon)\times(a_0-\epsilon,a_0+\epsilon)\times(b_0-\epsilon,b_0+\epsilon)\times B(u_0,\epsilon)\times(S_0-\epsilon,S_0+\epsilon)$ *we have*

$$P(p,a,b,u,\lambda)=0 \qquad \Leftrightarrow \qquad u=U(p,a,b) \quad \textit{and} \quad \lambda=\Lambda(p,a,b).$$

Proof. It suffices to prove that the partial derivative ∂P of P with respect to (u,λ) at $(2,a_0,b_0,u_0,S_0)$ is a homeomorphism of $H_0^1(B(0,1))\times\mathbb{R}$.

We have

$$\begin{aligned}&\langle\partial P(2,a_0,b_0,u_0,S_0),(v,t)\rangle\\&=\left((-\Delta)^{-1}\left(S_0|x|^{b_0}v-|x|^{a_0}v\right)-v+t(-\Delta)^{-1}\left(|x|^{b_0}u_0\right),2\langle u_0,v\rangle\right).\end{aligned}$$

So, if $\langle\partial P(2,a_0,b_0,u_0,S_0),(v,t)\rangle=0$, then

$$\begin{cases} t=0,\\ (-\Delta)^{-1}\left(S_0|x|^{b_0}v-|x|^{a_0}v\right)-v=0,\\ <u_0,v>=0.\end{cases}$$

But the kernel of $(-\Delta)^{-1}\left(S_0|x|^{b_0}I-|x|^{a_0}I\right)-I$ is the set of multipliers of u_0, so

$$\langle\partial P(2,a_0,b_0,u_0,S_0),(v,t)\rangle=0 \quad \Leftrightarrow \quad (v,t)=(0,0).$$

Let $(w,s)\in H_0^1(B(0,1))\times\mathbb{R}$; we must prove that in this case there exists $(v,t)\in H_0^1(B(0,1))\times\mathbb{R}$ such that

$$\langle\partial P(2,a_0,b_0,u_0,S_0),(v,t)\rangle=(w,s). \tag{4.3}$$

We define a new scalar product on $H_0^1(B(0,1))$,

$$((u,v)):=\int_{B(0,1)}\nabla u\nabla v+|x|^{a_0}uv\,dx,$$

and we consider the functional

$$H_0^1(B(0,1))\to\mathbb{R},\quad v\mapsto\int_{B(0,1)}|x|^{b_0}uv\,dx.$$

This functional is continuous, so it can be represented by a scalar product (Riesz representation theorem):

$$((Au,v))=\int_{B(0,1)}|x|^{b_0}uv\,dx.$$

So the weak form of problem (4.2) can be written in the following form:

$$((u,v))=\lambda((Au,v)).$$

But the operator A is self-adjoint and compact. So, by the Fredholm alternative, there exists $e_1, e_2, \ldots$ an orthonormal base of $X := \big(H_0^1(B(0,1)), ((.,.))\big)$ and a sequence $\lambda_1, \lambda_2, \ldots$ of eigenvalues such that $e_n - \lambda_n Ae_n = 0$ for every $n \in \mathbb{N}$. Let

$$L : X \to X, \quad u \mapsto u - \lambda_1 Au;$$

we have $KerL = e_1$. Since the operator L is self-adjoint, we have $Im(L) = (KerL)^\perp$, so we have $Im(L) = \langle e_1 \rangle^\perp$ (orthogonal with respect to the scalar product $((\cdot,\cdot))$). As u_0 is solution of (4.2) with $\lambda = S_0$ and $p = 2$, we can suppose, without loss of generality, that $e_1 = u_0$ and $\lambda_1 = S_0$, which implies that $Im(L) = \langle u_0 \rangle^\perp$.

So, we take t such that $\Delta w + t|x|^{b_0}u_0$ is orthogonal to u_0 with respect to the scalar product $((\cdot,\cdot))$. In this case, there exists $\tilde{v}$ such that $L\tilde{v} = \Delta w + T|x|^{b_0}u_0$. In order to verify relation (4.3), let $v := \tilde{v} + v_1$ with $v_1 \in KerL$. So (v,t) satisfy the first part of relation (4.3) and, taking the right v_1, also the second. □

We can now prove the following result.

Theorem 12. *Let a_0 be fixed; then for every $n \in \mathbb{N}$ there exists $\delta_n > 0$ such that the unique minimizer of $S_{a_0,b,p}$ is radial for $b \le n$ and $p \le 2 + \delta_n$.*

This means that if $p \to 2$, then there exists b^* such that for every $b < b^*$ we have $S_{a_0,b,p} = S^R_{a_0,b,p}$, and $b^* \to \infty$ as $p \to 2$.

Proof. For the sake of contradiction, assume that there exists $n \in \mathbb{N}$, a sequence $(\delta_k) \to 0_+$ and a sequence $(b_k) \subset [0,n]$ such that a minimizer of $S_{a_0,b_k,2+\delta_k}$ is nonradial for every k (i.e., such that $S_{a_0,b_k,2+\delta_k} < S^R_{a_0,b_k,2+\delta_k}$).

We begin by observing that, without loss of generality, $b_k \to b_\infty$ as $b_\infty \in [0,n]$. Moreover, if we denote by u_k the minimizer of $S_{a_0,b_k,2+\delta_k}$, we have $u_k \rightharpoonup u_\infty$ in $H_0^1(B(0,1))$.

We will now prove that $S_{a_0,b_\infty,2} = \lim_{k\to\infty} S_{a_0,b_k,2+\delta_k}$. For that, we prove the following relations:

$$S_{a_0,b_\infty,2} \le \liminf_{k\to\infty} S_{a_0,b_k,2+\delta_k}, \tag{4.4}$$

$$S_{a_0,b_\infty,2} \ge \limsup_{k\to\infty} S_{a_0,b_k,2+\delta_k}. \tag{4.5}$$

In order to prove the first relation, we begin by proving that

$$\int_{B(0,1)} |x|^{b_\infty} u_\infty^2 \ge 1.$$

With $\sigma := N(\frac{1}{2+\delta_k} - \frac{1}{2^*})$, we obtain the inequality

$$\int_{B(0,1)} |x|^{b_k} u_k^{2+\delta_k} dx \leq \Big(\int_{B(0,1)} |x|^{b_k} u_k^2 dx \Big)^{\frac{2+\delta_k}{2}\sigma} \Big(\int_{B(0,1)} u_k^{2^*} dx \Big)^{\frac{2+\delta_k}{2}(1-\sigma)}.$$

The left term is 1 for every k. So we have

$$1 \leq \Big(\int_{B(0,1)} |x|^{b_k} u_k^2 dx \Big)^{\frac{2+\delta_k}{2}\sigma} \Big(\int_{B(0,1)} u_k^{2^*} dx \Big)^{\frac{2+\delta_k}{2}(1-\sigma)}. \tag{4.6}$$

When $k \to \infty$ the second term converges to 1: the integral is uniformly bounded and the exponent tends to 0. On the other hand, we see that the first term converges to $\int_{B(0,1)} |x|^{b_\infty} u_\infty^2 dx$:

$$\begin{aligned}
&\int_{B(0,1)} \left(|x|^{b_k} u_k^2 - |x|^{b_\infty} u_\infty^2 \right) dx \\
&\leq \int_{B(0,1)} \left(|x|^{b_k} u_k^2 - |x|^{b_k} u_\infty^2 \right) dx + \int_{B(0,1)} \left| |x|^{b_k} - |x|^{b_\infty} \right| u_\infty^2 dx \\
&\leq \int_{B(0,1)} |x|^{b_k} \left(u_k^2 - u_\infty^2 \right) dx + \int_{B(0,1)} \left| |x|^{b_k} - |x|^{b_\infty} \right| u_\infty^2 dx.
\end{aligned}$$

These two terms converge to 0: the first by the Rellich theorem and the second by the Lebesgue dominated convergence theorem. So equation (4.6) gives us

$$1 \leq \int_{B(0,1)} |x|^{b_\infty} u_\infty^2 dx.$$

With this result we can prove relation (4.4):

$$S_{a_0, b_\infty, 2} \leq \frac{\int_{B(0,1)} |\nabla u_\infty|^2 + |x|^a u_\infty^2 dx}{\int_{B(0,1)} |x|^{b_\infty} u_\infty^2 dx} \leq \int_{B(0,1)} |\nabla u_\infty|^2 + |x|^a u_\infty^2 dx.$$

So, by weak lower semicontinuity, we have

$$S_{a_0, b_\infty, 2} \leq \liminf_{k \to \infty} S_{a_0, b_k, 2+\delta_k}.$$

Relation (4.5) is verified by upper semicontinuity. So we have proved that

$$S_{a_0, b_\infty, 2} = \lim_{k \to \infty} S_{a_0, b_k, 2+\delta_k}.$$

By uniform convexity, we have $u_k \to u_\infty \in H_0^1(B(0,1))$. But by Lemma 11, we know that u_k is unique for k large (i.e. for $2+\delta_k$ close to 2), so is radial, which is a contradiction. □

5. The quadratic problem for $B(0,1)$: analysis for p close to 2^*

Let us define

$$S^R_{a,b,p} := \inf_{\substack{u\in X,\\ u\neq 0.}} \frac{\int_{B(0,1)} |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_{B(0,1)} |x|^b u^p dx\right)^{\frac{2}{p}}}.$$

The following inequality is due to Ni.

Lemma 13. *If $N \geq 2$, there exists $c_{10} > 0$ such that for every $u \in \mathcal{D}^{1,2}_r(\mathbb{R}^N)$*

$$|x|^{\frac{N-2}{2}}|u(x)| \leq c_{10}|\nabla u|_2.$$

This inequality give us the following majoration:

$$\begin{aligned}\int_{B(0,1)} |x|^b|u(r)|^p dr &\leq c_{10}|\nabla u|_2^p \int_0^1 r^{b+N-1+p\frac{2-N}{2}} dr,\\ &= c_{10}|\nabla u|_2^p \left(\frac{2-N}{2}p + b + N\right)^{-1}.\end{aligned}$$

On the other hand, it is obvious that

$$|\nabla u|_2^2 \leq \int |\nabla u|^2 + |x|^a u^2 dx.$$

So we get

$$\frac{\int_{B(0,1)} |\nabla u|^2 + |x|^a u^2 dx}{\left(\int_{B(0,1)} |x|^b u^p dx\right)^{2/p}} \geq c_{11}\left(\frac{2-N}{2}p + b + N\right)^{2/p} \geq c_{11} b^{2/p}$$

because $\frac{2-N}{2}p + N > 0$. And we obtain the following lemma.

Lemma 14. *If $N \geq 3$, then there exists $c_{11} > 0$ such that for every $p \in [2, 2^*], b \geq 0$, we have $c_{11} b^{2/p} \leq S^R_{a,b,p}$.*

Now, we want to find a strict inequality between $S_{a,b,2^*}$ and $S^R_{a,b,2^*}$.

Lemma 15. *If $N \geq 3$, $a \geq 0$ and $b > 0$, then $S_{a,b,2^*} < S^R_{a,b,2^*}$.*

Proof. We begin by proving that the infimum $S_{a,b,2^*}$ equals

$$S := \inf_{u\in H^1_0(B(0,1))} \frac{\int_{B(0,1)} |\nabla u|^2 + u^2}{\left(\int_{B(0,1)} u^{2^*}\right)^{2/2^*}}.$$

We begin by proving that $S_{a,b,2^*} \le S$. Let a sequence $(u_n) \subset H_0^1(B(0,1))$ such that $\frac{\|u_n\|^2_{H_0^1(B(0,1))}}{|u_n|^2_{2^*}} \to S$, with (u_n) a concentration, i.e., based on an element $u \in \mathcal{D}(B(0,1))$. Let $y \in B(0,1)$; we have $u_n(x) := u(n(x-y_n))$, where $y_n := y\,(1-\frac{1}{n})$. Now we want to obtain the inequality $S_{a,b,2^*} \le S$ using the sequence (u_n):

$$\begin{aligned}\int_{B(0,1)} |x|^b |u_n(x)|^{2^*} dx &= \int_{B(y,\frac{1}{n})} |x|^b |u_n(x)|^{2^*} dx,\\ &\ge \int_{B(y,\frac{1}{n})} (|y| - \frac{1}{n})^b |u_n(x)|^{2^*} dx \ge (|y| - \frac{1}{n})^b |u_n(x)|^{2^*}_{2^*}.\end{aligned}$$

On the other hand, it is obvious that

$$\int_{B(0,1)} |x|^a |u_n(x)|^2 dx \le |u_n|_2^2.$$

So we obtain

$$\frac{\int_{B(0,1)} |\nabla u_n|^2 + |x|^a |u_n|^2 dx}{\left(\int_{B(0,1)} |x|^b |u_n(x)|^{2^*} dx\right)^{2/2^*}} \le (|y| - \frac{1}{n})^{-b\frac{2}{2^*}} \frac{\int_{B(0,1)} |\nabla u_n|^2 + u^2 dx}{\left(\int_{B(0,1)} u_n^{2^*} dx\right)^{2/2^*}}.$$

As $n \to +\infty$, we have $S_{a,b,2^*} \le |y|^{-b\frac{2}{2^*}} S$, which is valid for every $y \in B(0,1)$, so $S_{a,b,2^*} \le S$.

By the same argument, we obtain $|y|^a\, S \le S_{a,b,2^*}$ for every $y \in B(0,1)$. So we have $S = S_{a,b,2^*}$, with S not achieved.

On the other hand, it is easy to see that $S^R_{a,b,2^*}$ is achieved: it suffices to prove that there is no loss of compactness at the origin, as in [9].

So we have $S_{a,b,2^*} = S < S^R_{a,b,2^*}$. □

Using Lemmas 14 and 15, we can prove the main result of this section.

Theorem 16. *Let $N \ge 3$, $a \ge 0$ and $b > 0$; for every $n \in \mathbb{N}$ there exists $\delta > 0$ such that if $b \le \frac{1}{n}$ and $2^* - \delta_n < p < 2^*$, then $S_{a,b,p} < S^R_{a,b,p}$.*

Proof. For the sake of contradiction, we suppose that there exists $n \in \mathbb{N}$, $(b_k) \ge \frac{1}{n}$ and $(\delta_k) \to 0$ such that

$$S_{a,b_k,2^*-\delta_k} = S^R_{a,b_k,2^*-\delta_k}. \tag{5.1}$$

We can see that there exists $c_{12} > 0$ such that for every $p \in [2, 2^*]$ and $b \to +\infty$ we have $S_{a,b,p} \le c_{12} b^{2-N+\frac{2N}{p}}$. On the other hand, Lemma 14 implies that there exists c_{13} such that $c_{13}\, b^{\frac{2}{p}} \le S^R_{a,b,p}$ for $b \to +\infty$.

These last three relations imply that the sequence (b_k) is bounded. Without loss of generality, we can suppose that this sequence converges: $b_k \to b_\infty \geq \frac{1}{n}$. In this case, as in Section 4, we have

$$S^R_{a,b_\infty,2^*} = \lim_{k\to\infty} S^R_{a,b_k,2^*-\delta_k}. \tag{5.2}$$

On the other hand, by upper semicontinuity, we have

$$S_{a,b_\infty,2^*} \geq \limsup_{k\to\infty} S_{a,b_k,2^*-\delta_k}. \tag{5.3}$$

Using (5.3), (5.1), and (5.2) we obtain

$$S_{a,b_\infty,2^*} \geq \limsup_{k\to\infty} S_{a,b_k,2^*-\delta_k} = \limsup_{k\to\infty} S^R_{a,b_k,2^*-\delta_k} = S^R_{a,b_\infty,2^*}.$$

So we obtain the inequality $S_{a,b_\infty,2^*} \geq S^R_{a,b_\infty,2^*}$, which is a contradiction with Lemma 15. □

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