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ASYMPTOTIC PROFILES OF VARIATIONAL SOLUTIONS FOR A FITZHUGH-NAGUMO-TYPE ELLIPTIC SYSTEM

Hiroshi Matsuzawa
Department of Mathematics, Tokyo Metropolitan University
Minami-Osawa 1-1, Hachioji-shi, Tokyo 192-0397, Japan

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Abstract. In this paper we consider a semilinear elliptic system of FitzHugh-Nagumo type on a bounded domain with the same diffusion constant λ^{-1} under the Dirichlet boundary condition $-\Delta u = \lambda(f(u) - v)$, $-\Delta v = \lambda(\delta u - \gamma v)$ in Ω. In some parameter range on (δ, γ) this system has at least two nontrivial solutions when λ is sufficiently large, and these solutions are obtained by variational methods. We study the asymptotic profiles of these solutions for large λ in some parameter range on (δ, γ), especially for small δ and large γ.

1. Introduction and main results

In this paper, we consider the following FitzHugh-Nagumo-type elliptic system:

$$(\mathrm{P}_\lambda) \begin{cases} -\Delta u = \lambda(f(u) - v) & \text{in } \Omega, \\ -\Delta v = \lambda(\delta u - \gamma v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, δ, γ are positive constants, $\lambda > 0$ is a parameter, and f is given by $f(u) = u(u-a)(1-u)$ where $0 < a < 1/2$. This problem is the stationary problem for the FitzHugh-Nagumo equation:

$$(\mathrm{D}_\lambda) \begin{cases} u_t - \lambda^{-1}\Delta u = f(u) - v & \text{in } \mathbb{R}^+ \times \Omega, \\ v_t - \lambda^{-1}\Delta v = \delta u - \gamma v & \text{in } \mathbb{R}^+ \times \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x). \end{cases}$$

These equations are used as a model for nerve conduction and other chemical and biological systems. See [16] and the references therein about the case

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where the diffusion constant of u is much smaller than the diffusion constant of v.

If we set $\delta = 0$ in (P_λ), then the problem is reduced to the scalar problem,

$$(\mathrm{S}_\lambda)\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the function f is the one given in the above. It is well known that for large $\lambda > 0$ there are at least two positive solutions. One is obtained as the global minimizer of

$$I_\lambda(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \lambda F(u)dx$$

and has a boundary layer of width $O(\lambda^{-1/2})$.

The other is obtained as a mountain-pass solution and has a spiky shape if Ω is convex (see [11]). Moreover, if Ω is a ball, Ouyang and Shi [17] obtained the exact multiplicity of solutions to (S_λ) for any $\lambda > 0$.

The aim of our study is to understand the complete dynamics of solutions for (D_λ). Although the Lyapunov functional has been obtained in [7], we need to study the structure of solutions to (P_λ) in detail to understand the complete dynamics of solutions to (D_λ). In this paper we focus on the study of the asymptotic profiles of solutions to (P_λ) as a first step of this program.

Now we recall briefly two approaches to construct solutions to (P_λ). See Section 2 for the details. Since the second equation can be inverted to solve v in terms of u, the problem (P_λ) can be then written as a single equation for u including a nonlocal term. More precisely, if we define the operator $B_\lambda := (-\lambda^{-1}\Delta + \gamma)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$, then the problem (P_λ) is reduced to the following problem:

$$(\mathrm{NL}_\lambda)\begin{cases} -\Delta u + \lambda\delta B_\lambda u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Klaasen and Mitidieri [13] obtained two nontrivial solutions $(\underline{u}_\lambda, \underline{v}_\lambda)$ and $(\overline{u}_\lambda, \overline{v}_\lambda)$ in some parameter range as a critical points of the functional

$$J_\lambda(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{2}\delta(B_\lambda u)u - \lambda F(u)dx$$

on $H_0^1(\Omega)$, where $F(u) = \int_0^u f(s)ds$. Using an a priori estimate for the solution to (P_λ), the function f will be modified for large $|u|$, so that the functional J_λ is well defined on $H_0^1(\Omega)$. The pair $(\overline{u}_\lambda, \overline{v}_\lambda)$ is obtained as a global minimizer and $(\underline{u}_\lambda, \underline{v}_\lambda)$ is obtained by the well-known *mountain-pass* *theorem*. We will often call $(\underline{u}_\lambda, \underline{v}_\lambda)$ a *mountain-pass solution*. See section 2 for details.

On the other hand, recently in [20] Reinecke and Sweers used a nice transformation (P_λ) to a quasimonotone system (such an idea was first used in [15] for other systems of equations) and obtained a solution (U_λ, V_λ) by using the method of sub-supersolutions for a somewhat restricted parameter range. This solution (U_λ, V_λ) is stable and has a boundary layer of width $O(\lambda^{-1/2})$. Moreover, (U_λ, V_λ) is a unique solution in a certain order interval. Hence we will call (U_λ, V_λ) a *boundary-layer solution*. However, the relationship between the solutions obtained by these different approaches was unclear.

In this paper, we show the global minimizer $(\overline{u}_\lambda, \overline{v}_\lambda)$ coincides with the boundary-layer solution (U_λ, V_λ) for sufficiently large $\lambda > 0$. Moreover, we prove that a mountain-pass solution $(\underline{u}_\lambda, \underline{v}_\lambda)$ has a spiky asymptotic profile for large $\lambda > 0$ when Ω is ball.

To state our main results precisely, we need to assume the following three conditions on the parameters γ, δ, and a.

Conditions.

(C1) $\dfrac{\delta}{\gamma} < a < \gamma - 2\sqrt{\delta}$;

(C2) $\gamma - 2\sqrt{\delta} > M := \dfrac{(1-a)^2}{2} + \dfrac{1+a}{2}\sqrt{(1-a)^2 + 4\frac{\delta}{\gamma}} + 3\dfrac{\delta}{\gamma}$;

(C3) $\dfrac{2a^2 - 5a + 2}{9} > \beta := \dfrac{1}{2}(\gamma - M) - \dfrac{1}{2}\sqrt{(\gamma - M)^2 - 4\delta}$.

Remark 1. De Figueiredo and Mitidieri [6] showed that under the condition (C1) every nontrivial solution to the problem (NL_λ) is positive (see Proposition 2.4). Next we will use the condition (C2) to transform (P_λ) into some quasimonotone system and use the condition (C3) to construct a subsolution to the quasimonotone system. We also note that the condition (C3) implies $(2a^2 - 5a + 2)/9 > (\delta/\gamma)$ (see (2.2) in Section 2). If δ is sufficiently small and γ is sufficiently large then all conditions (C1), (C2), and (C3) are satisfied.

Remark 2. Since we compere the global minimizer $\overline{u}_\lambda$ with the boundary-layer solution U_λ obtained by the quasimonotone method as in [20], we assume slightly stronger conditions than those of [20] and use a milder modification of f.

Now we state our main results. First one is a new characterization of the boundary-layer solution (U_λ, V_λ).

Theorem 1.1. *Suppose that conditions (C2) and (C3) hold. Then there exist* $\varepsilon > 0$ *and* $\lambda^\sharp > 0$ *such that if* (u_λ, v_λ) *is a positive solution of* (P_λ) *with* $\max_\Omega u_\lambda \in (\rho^+_{\delta/\gamma} - \varepsilon, \rho^+_{\delta/\gamma})$ *and* $\lambda > \lambda^\sharp$ *then* $u_\lambda = U_\lambda$.

Using Theorem 1.1, we can show that the global minimizer $(\overline{u}_\lambda, \overline{v}_\lambda)$ coincides with the boundary-layer solution (U_λ, V_λ) for sufficiently large $\lambda > 0$.

Theorem 1.2. *Suppose that conditions (C1), (C2), and (C3) are satisfied. Then there exists* $\lambda^\flat > 0$ *such that for* $\lambda > \lambda^\flat$, $\overline{u}_\lambda = U_\lambda$ *holds.*

Lastly, we show a spiky profile of a mountain-pass solution $(\underline{u}_\lambda, \underline{v}_\lambda)$, when Ω is a ball.

Theorem 1.3. *Let* $\Omega = B_1(0)$ *be the unit ball in* $\mathbb{R}^N$ *Also, let* $(\underline{u}_\lambda, \underline{v}_\lambda)$ *be a mountain-pass solution to* (P_λ). *Then the following hold.*

(1) $\underline{u}_\lambda(0) \geq \rho^-_{\delta/\gamma}$, *where* $\rho^-_{\delta/\gamma}$ *is a positive constant independent of* λ *and will be defined in Section 2.*

(2) *If we set* $\tilde{u}_\lambda(x) = \underline{u}_\lambda(\lambda^{-1/2}x)$, $\tilde{v}_\lambda(x) = \underline{v}_\lambda(\lambda^{-1/2}x)$, *the set of functions* $\{\tilde{u}_\lambda\}$, $\{\tilde{v}_\lambda\}$ *are precompact in* $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ *and have subsequences which converge to a positive radially symmetric solution to the problem*

$$(\mathrm{P})\begin{cases} -\Delta u = f(u) - v & \text{in } \mathbb{R}^N, \\ -\Delta v = \delta u - \gamma v & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \\ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

(3) $\underline{u}_\lambda \to 0$, $\underline{v}_\lambda \to 0$ *as* $\lambda \to +\infty$ *uniformly on every compact subset of* $\overline{B_1(0)}\backslash\{0\}$.

This paper is organized as follows. In Section 2, we recall preliminary known results. In Section 3 we first establish an a priori bound for positive solutions. Next we prove Theorems 1.1 and 1.2 and we show a lower-bound estimate for the maximum of the positive solution. Finally we prove Theorem 1.3. In Section 4 we state open questions for the problem (P_λ).

2. Preliminary known results

In this section we collect some preliminary known results. First we define the operator $B_\lambda : L^2(\Omega) \to L^2(\Omega)$ as follows: for all $w \in L^2(\Omega)$, $v = B_\lambda w$ is the unique weak solution to

$$\begin{cases} -\lambda^{-1}\Delta v + \gamma v = w & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Then the second equation of (P_λ) is equivalent to $v = \delta B_\lambda u$, and by substituting into the first equation of (P_λ) we obtain the single equation including a nonlocal term

$$(\mathrm{NL}_\lambda) \begin{cases} -\Delta u + \lambda\delta B_\lambda u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The definition of B_λ implies that $\int_\Omega (B_\lambda u)u\,dx \geq 0$ and B_λ is a bounded operator in $L^2(\Omega)$ with $\|B_\lambda\|_{\mathcal{L}(L^2(\Omega))} \leq 1/\gamma$. See [13] for proofs of these results.

First we describe how to construct the variational solutions in our setting. For the construction we just impose the following condition,

$$\frac{2a^2 - 5a + 2}{9} > \frac{\delta}{\gamma} \tag{2.2}$$

which is weaker than the condition (C3). Condition (2.2) is equivalent to the following:

$g(u) := f(u) - \dfrac{\delta}{\gamma}u$ has three roots $0 < \rho^-_{\delta/\gamma} < \rho^+_{\delta/\gamma} < 1$ and satisfies

$$\int_0^{\rho^+_{\delta/\gamma}} \Big(f(u) - \frac{\delta}{\gamma}u\Big)du > 0.$$

Next we state a priori estimate for the solutions to (P_λ).

Proposition 2.1. ([14, Lemma 3]) *Suppose that there exists $m = m(\delta/\gamma) > 0$ such that*

$$\frac{f(y)}{y} < -\frac{\delta}{\gamma} \quad \textit{for } y : |y| > m,$$

and let (u, v) be a solution to (P_λ). Then $|u(x)| \leq m$ for all $x \in \Omega$.

To obtain the variational solution, we have to define the energy functional. We have to modify the function f as follows so that it is well defined and its critical points are the solution to the problem (NL_λ). Now we assume furthermore condition (C1):

$$\frac{\delta}{\gamma} < a < \gamma - 2\sqrt{\delta}.$$

We note that the direct calculation for $f(u) = u(u-a)(1-u)$ yields

$$m = m(\delta/\gamma) = \frac{a+1}{2} + \frac{1}{2}\sqrt{(a-1)^2 + 4\tfrac{\delta}{\gamma}}, \tag{2.3}$$

$$M = M(\delta/\gamma) = \max\{-f'(u) | 0 \leq u \leq m(\delta/\gamma)\} \tag{2.4}$$

$$= \frac{(1-a)^2}{2} + \frac{1+a}{2}\sqrt{(1-a)^2 + 4\frac{\delta}{\gamma}} + 3\frac{\delta}{\gamma} > 1 - a > a$$

(see Figure 1).

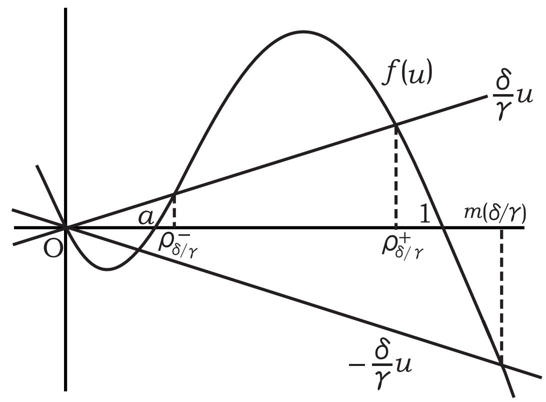

FIGURE 1

Using this estimate we modify the function f to $\tilde{f}$ satisfying the following conditions.

(1) $f(u) = \tilde{f}(u)$ for $0 < u \leq m$.
(2) $\frac{\tilde{f}(u)}{u} < -\frac{\delta}{\gamma}$ for $|u| > m$.
(3) $\tilde{f}'(u) = -a < -\frac{\delta}{\gamma}$ for large $u > m$ and for all $u < 0$.
(4) $\tilde{f}'(u) + M \geq 0$ for all $u \in \mathbb{R}$.
(5) $\tilde{f}$ is smooth.

Since we are interested in positive variational solutions, we use the modified function $\tilde{f}$ instead of f in the problem (NL_λ). And later we show that every nontrivial solution to (NL_λ), with modified function $\tilde{f}$, is positive. Hereafter we consider the problem (NL_λ) with $\tilde{f}$.

Next we define the following functional:

$$J_\lambda(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{2}\delta(B_\lambda u)u\,dx - \lambda\tilde{F}(u)dx, \tag{2.5}$$

where $\tilde{F}(u) = \int_0^u \tilde{f}(s)ds$. Then we can show that $u \in H_0^1(\Omega)$ is a critical point of J_λ if and only if u is a weak solution to the (NL_λ). Moreover by the standard bootstrap argument, $(u,v) = (u, \delta B_\lambda u)$ is a classical solution of (P_λ).

Now we state the existence result.

Proposition 2.2. ([13, Theorem 1, Theorem 2]) *Let us assume conditions* (2.2) *and (C1). Then there exists* $\lambda^\dagger > 0$ *such that for all* $\lambda > \lambda^\dagger$ *there exist two nontrivial solutions* $(\overline{u}_\lambda, \overline{v}_\lambda)$, $(\underline{u}_\lambda, \underline{v}_\lambda)$ *to* (P_λ) *satisfying* $J_\lambda(\overline{u}_\lambda) < 0, J_\lambda(\underline{u}_\lambda) > 0$.

We note that $(\overline{u}_\lambda, \overline{v}_\lambda)$ is obtained as a global minimizer of J_λ and $(\underline{u}_\lambda, \underline{v}_\lambda)$ is obtained by the *mountain-pass theorem* (see [3]).

Actually existence of these two nontrivial solutions to (P_λ) has been proved in [13] without condition (C1). We can show that the solutions obtained by the same procedure as in [13] to (P_λ) with the modified function $\tilde{f}$ are solutions to (P_λ) with the original f by Proposition 2.1 and the following argument.

Namely, we can show that the variational solutions obtained by the procedure as in [13] to (P_λ) with the modified f are positive.

Since the positivity of the solutions is invariable by the scaling,

$$\tilde{u}_\lambda(x) = u(\lambda^{-1/2}x),\ \tilde{v}_\lambda(x) = v(\lambda^{-1/2}x)$$
$$\text{for } x \in \lambda^{1/2}\Omega := \{y \in \mathbb{R}^N | \lambda^{1/2}y \in \Omega\},$$

we may assume $\lambda = 1$, and we consider the problem (NL_1). Let us define the operator

$$T := -\Delta + \delta B_1, \text{ with } D(T) := H^2(\Omega) \cap H_0^1(\Omega).$$

T is a closed and self-adjoint operator. Let us denote by $0 < \mu_1 < \mu_2 \le \mu_3 \le \cdots$ the eigenvalues of $-\Delta$ with Dirichlet boundary condition and by $\{\phi_k\}$ the corresponding eigenfunctions. It is easily seen that

$$\hat{\mu}_k = \mu_k + \frac{\delta}{\gamma + \mu_k}, \quad k = 1, 2, \ldots,$$

are the eigenvalues of the operator T. Since $\{\phi_k\}$ is a complete orthonormal system in $L^2(\Omega)$, it is readily shown that $\{\hat{\mu}_k\}$ are the only eigenvalues of T.

The following proposition follows from the positivity of the resolvent operator of T (see [6, Corollary 1.3]).

Proposition 2.3. ([6, Remark 1.3]) *Let* $\gamma + \mu_1 > \sqrt{\delta}$ *and* $2\sqrt{\delta} - \gamma \le \mu < \hat{\mu}_1$. *If* $z \in L^2(\Omega)$, $z \ge 0$ almost everywhere, *and* w *is a weak solution to*

$$\begin{cases} -\Delta w + \delta B_1 w - \mu w = z & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

then $w \geq 0$ almost everywhere. Moreover, if $z \in C(\overline{\Omega})$, $z \geq 0$ in Ω, then $w > 0$ in Ω and the outward-normal derivative satisfies $(\partial w/\partial\nu) < 0$ on $\partial\Omega$.

Now we show the positivity of solutions to problem (NL_λ) with the modified function $\tilde{f}$. We note that our modification implies that $\tilde{f}(u) \geq -au$ for all $u \in \mathbb{R}$. And we can easily check that all conditions of Proposition 2.3 with $\mu = -a$ are satisfied. Therefore every nontrivial solution u to

$$\begin{cases} -\Delta u + \delta B_1 u - (-a)u = \tilde{f}(u) + au & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is positive. Hence the following proposition holds (see [6, Remark 2.8]).

Proposition 2.4. ([6]) *Let us assume the condition (C1). Then every nontrivial solution to* (NL_λ) *with the modified function* $\tilde{f}$ *is positive.*

Next we recall the other construction of a solution to (P_λ) due to Reinecke and Sweers [20]. Since our assumption and the modification of f are slightly different from those in [20], we present them in detail, although the strategy is the same one as in [20]. Problem (P_λ) can be transformed to a quasimonotone system in some parameter range. First we state the definition and properties of a quasimonotone system.

Definiton 2.5. Let $F_1, F_2 \in C^1(\mathbb{R} \times \mathbb{R})$. An elliptic system

$$\begin{cases} -\Delta u = F_1(u, w) & \text{in } \Omega, \\ -\Delta w = F_2(u, w) & \text{in } \Omega \end{cases} \tag{2.6}$$

is called *quasimonotone* if

$$\left|\frac{\partial F_1}{\partial u}\right|, \left|\frac{\partial F_2}{\partial w}\right| \leq K,$$

for some $K > 0$ and

$$\frac{\partial F_1}{\partial w}(u, w) \geq 0 \quad \text{and} \quad \frac{\partial F_2}{\partial u}(u, w) \geq 0, \quad \text{for all } (u, w) \in \mathbb{R} \times \mathbb{R}.$$

Definiton 2.6. $(u, w) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ is called a *subsolution* (*supersolution*) to the elliptic problem

$$\begin{cases} -\Delta u = F_1(u, w) & \text{in } \Omega, \\ -\Delta w = F_2(u, w) & \text{in } \Omega, \\ u = w = 0 & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

if it satisfies

(1)

$$\begin{aligned} -\Delta u &\le (\ge) F_1(u,w) \quad \text{in } \mathcal{D}'(\Omega), \\ -\Delta w &\le (\ge) F_2(u,w) \quad \text{in } \mathcal{D}'(\Omega) \end{aligned}$$

and

(2) $(u,w) \le (\ge)(0,0)$ *on* $\partial\Omega$.

$(u,w) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ is called a *C-solution* to the problem (2.7) if it is a subsolution and a supersolution.

Proposition 2.7. ([20]) *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with smooth boundary and assume* (2.7) *is a quasimonotone system.*

If $(\underline{u}, \underline{w})$ *and* $(\overline{u}, \overline{w})$ *are a supersolution and a subsolution to* (2.7)*, respectively, with* $(\underline{u}, \underline{v}) \le (\overline{u}, \overline{v})$ *in* Ω*, then there exists a C-solution* (u,w) *to* (2.7) *with*

$$(\underline{u}, \underline{w}) \le (u,w) \le (\overline{u}, \overline{w}).$$

We note that since Ω is a bounded domain with smooth boundary $\partial\Omega$ and F_1, F_2 are C^1, any C-solution (u,w) is actually in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$.

The next proposition is an extension of the result of Gidas, Ni, and Nirenberg [9], due to Troy [21], to the quasimonotone system.

Proposition 2.8. ([21, Theorem 1]) *Suppose that* $\Omega = B_R(0)$ *and* (2.7) *is quasimonotone. If* $u > 0$, $w > 0$ *is a solution to this system with* $u, w \in C^2(\overline{B_R(0)})$*, then* u, w *is radially symmetric and* $\partial u/\partial r, \partial w/\partial r < 0$ *on* $(0,R)$.

Next we explain how to transform (P_λ) to some quasimonotone system.

Under the condition (C2),

$$\gamma - 2\sqrt{\delta} > M,$$

we can define β and α by

$$\beta := \frac{1}{2}(\gamma - M) - \frac{1}{2}\sqrt{(\gamma - M)^2 - 4\delta} > 0,$$
$$\alpha = \gamma - \beta > 0.$$

Note that $-\beta(\beta + M) = \delta - \gamma\beta$ and that

$$\theta := 1 - \frac{\delta}{\gamma\beta} > 0.$$

One may verify that (u,w) is a positive solution to

$$(\mathrm{Q}_\lambda) \begin{cases} -\Delta u = \lambda(\tilde{f}(u) - \beta u + \beta w) & \text{in } \Omega, \\ -\Delta w = \lambda(\tilde{f}(u) + Mu - \alpha w) & \text{in } \Omega, \\ u = w = 0 & \text{on } \partial\Omega \end{cases}$$

if and only if $(u, \beta u - \beta w)$ is a positive solution to (P_λ). We note that from our modification of f, we have $\tilde{f}'(s) + M \geq 0$ on $\mathbb{R}$, and hence $\tilde{f}(s) + Ms$ is monotone increasing on $\mathbb{R}$. Moreover, $\tilde{f}'$ is bounded on $\mathbb{R}$. Therefore the system (Q_λ) is quasimonotone.

Next we construct a solution for (Q_λ). We assume the condition (C3):

$$\frac{2a^2 - 5a + 2}{9} > \beta.$$

It is easy to see that the condition (C3) implies the condition (2.2).

Next, to construct the subsolutions to (Q_λ) we also need the following proposition. The following proposition corresponds to Proposition 3.1 of [20]. Although our modification of f is different from that of [20], we can show Proposition 2.9 in a way similar to that of [20]. For the reader's convenience, we give the proof of the proposition in detail.

Proposition 2.9. *Suppose that conditions (C2) and (C3) are satisfied and let $B = B_1(0) := \{x \in \mathbb{R}^N : |x| < 1\}$. Then there exists $\lambda_B > 0$ such that*

$$\begin{cases} -\Delta u = \lambda_B(\tilde{f}(u) - \beta u + \beta w) & \text{in } B, \\ -\Delta w = \lambda_B(\tilde{f}(u) + Mu - \alpha w) & \text{in } B, \\ u = w = 0 & \text{on } \partial B \end{cases} \tag{2.8}$$

has a solution (U_B, W_B) with the following properties:

(1) $0 \leq (U_B, W_B) < (\rho^+_{\delta/\gamma}, \theta\rho^+_{\delta/\gamma})$ *with* $\theta = 1 - \delta/(\gamma\beta)$.

(2) U_B, W_B *is radially symmetric with*

$$U_B'(0) = W_B'(0) = 0 \quad \text{and} \quad U_B'(r), W_B'(r) < 0 \text{ on } (0, 1].$$

(3) $(U_B(0), W_B(0)) > (\rho^-_{\delta/\gamma}, \theta\rho^-_{\delta/\gamma})$ *and* $W_B(0) \geq \theta U_B(0)$.

Proof. Since the condition (C3) holds, for fixed large $\lambda = \lambda_B$, there exists a positive solution $\underline{u}$ to

$$\begin{cases} -\Delta u = \lambda(\tilde{f}(u) - \beta u) & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

with $\max \underline{u} \in (\rho^-_\beta, \rho^+_\beta)$ (see [5]), where ρ^-_β, ρ^+_β are the positive roots of $\tilde{f}(u) - \beta u$. Since $(\underline{u}, 0)$ is a subsolution to (2.9), and $(\rho^+_{\delta/\gamma}, \theta\rho^+_{\delta/\gamma})$ is a supersolution with $(\underline{u}, 0) < (\rho^+_{\delta/\gamma}, \theta\rho^+_{\delta/\gamma})$, there exists a solution (U_B, W_B), with $\underline{u} \leq U_B < \rho^+_{\delta/\gamma}$ and $0 \leq W_B < \theta\rho^+_{\delta/\gamma}$, to (2.9); see [20, Proposition A.3]. By Proposition 2.8 we have that U_B and W_B are radially symmetric with $U_B'(0) = W_B'(0) = 0$ and $U_B'(r), W_B'(0) < 0$ on the interval $(0, 1)$. Also $(-\Delta + \lambda_B\alpha)W_B =$

$\lambda_B(\tilde{f}(U_B) + MU_B) \geq 0$, and by the strong maximum principle $W_B'(1) < 0$. Let $\tau := U_B(0)$ and $V_B := \beta(U_B - W_B)$; it also follows from the maximum principle that

$$\max V_B < \frac{\delta}{\gamma}\tau. \tag{2.9}$$

Indeed, $(-\Delta + \lambda_B\gamma)(V_B - \delta\tau/\gamma) = \lambda_B(U_B - \tau) \leq 0$ in B with $V_B = 0$ on ∂B. Since by (2.9)

$$V_B(0) = \beta(\tau - W_B(0)) < \frac{\delta}{\gamma}\tau,$$

we have

$$W_B(0) > \Big(1 - \frac{\delta}{\gamma\beta}\Big)\tau = \theta\tau > \theta\rho^-_{\delta/\gamma}.$$

Since $(-\Delta + \lambda_B\gamma)V_B = \lambda_B\delta U_B \geq 0$, $V_B'(1) = \beta(U_B'(1) - W_B'(1)) < 0$, and hence $U_B'(1) < W_B'(1) < 0$. □

Using the solution obtained above, we construct subsolutions to (Q_λ). First we fix $z^* \in \Omega$ and set

$$\lambda(z^*) := \lambda_B \mathrm{dist}(z^*, \partial\Omega)^{-2}.$$

Next for all $\lambda > \lambda(z^*)$, we set

$$Z_\lambda(x) := \begin{cases} (U_B, W_B)\left((\lambda/\lambda_B)^{1/2}(x - z^*)\right) & \text{for } |x - z^*| \leq (\lambda_B/\lambda)^{1/2}, \\ 0 & \text{for } |x - z^*| > (\lambda_B/\lambda)^{1/2} \end{cases}$$

with (U_B, W_B) as in Proposition 2.9. Next we set

$$Z^y_\lambda(x) := Z_\lambda(x + z^* - y)$$

for $y \in \Omega$ satisfying $\mathrm{dist}(y, \partial\Omega) > (\lambda_B/\lambda)^{1/2}$ and define the following family of functions:

$$\mathcal{S}_\lambda = \{Z^y_\lambda : y \in \Omega \text{ such that } \mathrm{dist}(y, \partial\Omega) > (\lambda_B/\lambda)^{1/2}\}.$$

We recall that since $\partial\Omega$ is smooth, Ω satisfies the following *uniform interior sphere condition*:

there exists $\varepsilon_\Omega > 0$ such that

$$\Omega = \bigcup\{B(y, \varepsilon) : y \in \Omega \text{ and } \mathrm{dist}(y, \partial\Omega) > \varepsilon_\Omega\}.$$

We may suppose that $\Omega_\nu := \{y \in \Omega : \mathrm{dist}(y, \partial\Omega) > \nu\}$ is connected for all $\varepsilon \leq \varepsilon_\Omega$ (see [5]).

The following statements, especially the part (2), are included implicitly in [20].

Proposition 2.10. ([20, Lemma 3.2]) *Suppose that conditions (C2) and (C3) are satisfied. Then*

(1) *For all* $\lambda > \lambda(z^*)$, Z_λ *is a subsolution to* (Q_λ) *and*

$$Y := (\rho^+_{\delta/\gamma}, \theta\rho^+_{\delta/\gamma})$$

is a supersolution to (Q_λ) *with* $Z_\lambda < Y$. *Hence there exists a solution* (U_λ, W_λ) *to* (Q_λ) *in the order interval* $[Z_\lambda, Y]$.

(2) *There exist* $\lambda^\times > \lambda(z^*)$ *such that for all* $\lambda > \lambda^\times$ *every element in* $\mathcal{S}_\lambda$ *is a subsolution to* (Q_λ). *Moreover, if* (u, w) *is a solution to* (Q_λ) *in* $[Z_\lambda, Y]$ *then for every* $Z^y_\lambda \in \mathcal{S}_\lambda$, (u, w) *is a solution to* (Q_λ) *in* $[Z^y_\lambda, Y]$.

Proof. (1) It follows directly that Y is a supersolution. Next denote $Z_\lambda = (Z^1_\lambda, Z^2_\lambda), Y = (Y^1, Y^2)$ and take $\varphi \in C^\infty_0(\Omega)$ with $\varphi \geq 0$. Then if we set $B = B_{(\lambda_B/\lambda)^{1/2}}(z^*)$, we obtain by the Green's identity

$$\begin{aligned}
&\int_\Omega Z^1_\lambda(-\Delta\varphi)dx = \int_B Z^1_\lambda(-\Delta\varphi)dx \\
= \;& -\int_B \Delta Z^1_\lambda \varphi\, dx - \int_{\partial B}\Big(Z^1_\lambda \frac{\partial\varphi}{\partial\nu} - \frac{\partial Z^1_\lambda}{\partial\nu}\varphi\Big)d\sigma \\
\leq \;& \int_\Omega (\tilde f(Z^1_\lambda) - \beta Z^1_\lambda + \beta^2_\lambda)\varphi\, dx.
\end{aligned}$$

A similar result holds for Z^2_λ.

Finally, $\max Z^1_\lambda = Z^1_\lambda(z^*) < \rho^+_{\delta/\gamma} = Y^1$, $\max Z^2_\lambda = Z^2_\lambda(z^*) < \theta\rho^-_{\delta/\gamma} = Y^2$. Hence $Z_\lambda < Y$.

(2) We can show that Z^y_λ is a subsolution in a similar way as in (1). Next we show that for large $\lambda > 0$ if (u, w) is a solution to (Q_λ) in $[Z_\lambda, Y]$ then for every $y \in \Omega$ satisfying $\mathrm{dist}(y, \partial\Omega) > (\lambda_B/\lambda)^{1/2}$, (u, w) is a solution to (Q_λ) in $[Z^y_\lambda, Y]$. Let $\lambda^\times := \{\lambda(z^*), \lambda_B\varepsilon^{-2}_\Omega\}$. Suppose that $(u, w) \in [Z_\lambda, Y]$ is a solution to (Q_λ) with $\lambda > \lambda^\times$. As in [5] there exists, for every $y \in \Omega_{(\lambda_B/\lambda)^{1/2}}$, a curve in $\Omega_{(\lambda_B/\lambda)^{1/2}}$ connecting y with z^*. Using the sweeping principle (see [20, Proposition A.6]), it follows that $(u, w) > Z^y_\lambda$ for all $y \in \Omega_{(\lambda/\lambda_B)^{1/2}}$. □

Using the earlier notation, we arrive at the important results in [20].

Proposition 2.11. ([20, Theorem 2.1, Lemma 4.2]) *Suppose conditions (C2) and (C3) are satisfied. Then there exists* $\lambda^\star > 0$ *and a function*

$$\Lambda \in C^1([\lambda^\star, +\infty), C^2(\overline{\Omega}) \times C^2(\overline{\Omega}))$$

such that $(U_\lambda, V_\lambda) := \Lambda(\lambda)$ *is a positive solution to* (P_λ) *for all* $\lambda \geq \lambda^\star$. *Furthermore,*

(1) $(U_\lambda, W_\lambda) = (U_\lambda, \beta(U_\lambda - V_\lambda))$ *is unique solution to* (Q_λ) *in the order interval* $[Z_\lambda, Y]$.

(2) $\max U_\lambda \in (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$ *and* $\max V_\lambda \in \dfrac{\delta}{\gamma}(\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$,

(3) $\lim_{\lambda\to\infty} \Lambda(\lambda) = (\rho^+_{\delta/\gamma}, \frac{\delta}{\gamma}\rho^+_{\delta/\gamma})$ *uniformly on compact subsets of* Ω.

Using the results of Propositions 2.10 and 2.11, we can obtain the following proposition.

Proposition 2.12. *Suppose that conditions (C2) and (C3) and* $\lambda > \lambda^\star$ *are satisfied. Let* $y_1, y_2 \in \Omega$ *be such that*

$$\mathrm{dist}(y_1, \partial\Omega),\ \mathrm{dist}(y_2, \partial\Omega) > (\lambda_B/\lambda)^{1/2}.$$

Then (u, w) *is a solution to* (Q_λ) *in* $[Z^{y_1}_\lambda, Y]$ *if and only if* (u, w) *is a solution to* (Q_λ) *in* $[Z^{y_2}_\lambda, Y]$.

It is shown that the solution U_λ obtained by Proposition 2.11 has a boundary layer of width $O(\lambda^{-1/2})$ (see [20] for details). Hence we often call this solution a *boundary-layer solution.*

3. Proof of main results

In this section we prove the main results. We need some lemmas and propositions. Hereafter we also use the same notation f and F for the modified functions $\tilde{f}$ and $\tilde{F}$.

Lemma 3.1. *Suppose that conditions (C2) and (C3) hold. Then for every positive solution* (u, w) *to* (Q_λ) *we have*

$$u(x) \leq \rho^+_{\delta/\gamma}, \quad w(x) \leq \theta\rho^+_{\delta/\gamma} = \big(1 - \frac{\delta}{\gamma\beta}\big)\rho^+_{\delta/\gamma}.$$

Proof. Let us assume that $u_0 := \max_\Omega u > \rho^+_{\delta/\gamma}$.

Step 1. First we show that $w(x) \leq \theta u_0$. From the second equation of (Q_λ) we have

$$-\Delta(w - \theta u_0) + \lambda\alpha(w - \theta u_0) = \lambda(f(u) + Mu - \alpha\theta u_0).$$

Next we have

$$\alpha\theta u_0 - (f(u_0) + Mu_0) = (\gamma - \beta)\Big(1 - \frac{\delta}{\gamma\beta}\Big)u_0 - (f(u_0) + Mu_0)$$

$$= \Big(\frac{\beta\gamma-\delta}{\beta} - \beta + \frac{\delta}{\gamma}\Big)u_0 - (f(u_0) + Mu_0)$$
$$= \Big(\beta + M - \beta + \frac{\delta}{\gamma}\Big)u_0 - (f(u_0) + Mu_0) = \frac{\delta}{\gamma}u_0 - f(u_0) > 0.$$

Here we use the relation $-\beta(\beta+M) = \delta - \beta\gamma$. Hence by the monotonicity of $f(s) + Ms$ we have

$$-\Delta(w - \theta u_0) + \lambda\alpha(w - \theta u_0) \le 0.$$

By the maximum principle $w(x) \le \theta u_0$ follows.

Step 2. Next we show that at a maximum point x_0 of u, $-\Delta u(x_0) < 0$. In fact, from the first equation of (Q_λ)

$$\begin{aligned}
-\Delta u(x_0) &= \lambda(f(u(x_0)) - \beta u(x_0) + \beta w(x_0)) \\
&\le \lambda(f(u(x_0)) - \beta u(x_0) + \beta\theta u(x_0)) \\
&= \lambda\Big(f(u(x_0)) - \frac{\delta}{\gamma}u(x_0) + \frac{\delta}{\gamma}u(x_0) - \beta u(x_0) + \beta\theta u(x_0)\Big) \\
&= \lambda\Big(f(u(x_0)) - \frac{\delta}{\gamma}u(x_0)\Big) < 0.
\end{aligned}$$

On the other hand, $-\Delta u(x_0) \ge 0$, since x_0 is maximum point. This is a contradiction. Hence we can conclude $u(x) \le \rho^+_{\delta/\gamma}$.

Step 3. Finally we show that $w(x) \le \theta\rho^+_{\delta/\gamma}$. At first, from the second equation of (Q_λ), we have

$$-\Delta w + \lambda\alpha w = \lambda(f(u) + Mu).$$

Next we note that

$$\lambda\alpha\theta\rho^+_{\delta/\gamma} = \lambda(f(\rho^+_{\delta/\gamma}) + M\rho^+_{\delta/\gamma}).$$

Subtracting and using the monotonicity of $f(s) + Ms$ it follows that

$$-\Delta(w - \theta\rho^+_{\delta/\gamma}) + \lambda\alpha(w - \theta\rho^+_{\delta/\gamma}) = \lambda(f(u) + Mu - (f(\rho^+_{\delta/\gamma}) + M\rho^+_{\delta/\gamma})) \le 0.$$

Hence by the maximum principle $w \le \theta\rho^+_{\delta/\gamma}$ follows. □

By the strong maximum principle we obtain the following result.

Proposition 3.2. *Suppose that conditions (C2) and (C3) hold. Let Ω be any domain and the pair (u, w) be the positive solution to*

$$\begin{cases} -\Delta u = \mu(f(u) - \beta u + \beta w) & \text{in } \Omega \\ -\Delta w = \mu(f(u) + Mu - \alpha w) & \text{in } \Omega \end{cases}$$

with $u(x) \le \rho^+_{\delta/\gamma}$, $w(x) \le \theta\rho^+_{\delta/\gamma}$ *in* Ω, *and* $\mu > 0$. *And if* $u(x_0) = \rho^+_{\delta/\gamma}$ (*respectively* $w(x_0) = \theta\rho^+_{\delta/\gamma}$) *at some point* $x_0 \in \Omega$, *then* $u(x) \equiv \rho^+_{\delta/\gamma}$ (*respectively* $w(x) \equiv \theta\rho^+_{\delta/\gamma}$) *on* Ω *hold.*

To prove Theorem 1.1 we also need the following lemma.

Lemma 3.3. *Suppose that conditions (C2) and (C3) hold. Also, let* Z^1_λ, Z^2_λ *be the first and second components of* Z_λ, *respectively, and* Y^1, Y^2 *be the first and second components of* Y, *respectively. Let* (u, w) *be the solution to* (Q_λ) *such that* $Z^1_\lambda \le u \le Y^1$ *in* Ω. *Then* $Z^2_\lambda \le w \le Y^2$ *in* Ω.

Proof. First, since the condition implies that u is a positive solution, from the second equation of (Q_λ) we have

$$-\Delta w + \lambda\alpha w = \lambda(f(u) + Mu) \ge 0 \quad \text{in } \Omega.$$

Since $w = 0$ on $\partial\Omega$ by the maximum principle we obtain that $w \ge 0$ in Ω.

Next we show that $Z^2_\lambda \le w$ in Ω. Since on $\Omega\backslash B_{(\lambda_B/\lambda)^{1/2}}(z^*)$, $Z^2_\lambda = 0$ (see Proposition 2.10), we have only to show it on $B_{(\lambda_B/\lambda)^{1/2}}(z^*)$ (note that Z_λ is smooth on $B_{(\lambda_B/\lambda)^{1/2}}(z^*)$). Indeed, Z_λ is a subsolution to (Q_λ) and w is a solution to (Q_λ); we have

$$\begin{aligned} -\Delta Z^2_\lambda + \lambda\alpha Z^2_\lambda \le \quad & \lambda(f(Z^1_\lambda) + MZ^1_\lambda) \quad \text{in } B_{(\lambda_B/\lambda)^{1/2}}(z^*) \\ -\Delta w + \lambda\alpha w = \quad & \lambda(f(u) + Mu) \quad \text{in } B_{(\lambda_B/\lambda)^{1/2}}(z^*). \end{aligned}$$

Subtracting, we have

$$-\Delta(Z^2_\lambda - w) + \lambda\alpha(Z^2_\lambda - w) \le \lambda(f(Z^1_\lambda) + MZ^1_\lambda - (f(u) + Mu)) \le 0,$$

since $Z^1_\lambda \le u$ and $f(s) + Ms$ is an increasing function. Also, we have $Z^2_\lambda - w \le 0$ on $\partial B_{(\lambda_B/\lambda)^{1/2}}(z^*)$. By the maximum principle we can conclude that $Z^2_\lambda \le w$ in Ω. We can show that $w \le Y^2$ in a similar way as in the proof of $Z^2_\lambda \le w$. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. If the result is false, there exists $\{\lambda_n\} \subset \mathbb{R}_+$ such that

$$\lambda_n \nearrow \infty \quad \text{and} \quad u_{\lambda_n} \ne U_{\lambda_n} \quad \text{and} \quad \max_\Omega u_{\lambda_n} \to \rho^+_{\delta/\gamma}.$$

Let $u_{\lambda_n}(x_n) = \max_\Omega u_{\lambda_n}$.

Case 1. $\{x_n\}$ is bounded away from $\partial\Omega$; that is, there exists $C > 0$ such that

$$\mathrm{dist}(x_n, \partial\Omega) > C > 0, \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Let us set

$$\tilde{u}_{\lambda_n}(x) = u_{\lambda_n}(\lambda_n^{-1/2}x + x_n), \quad \tilde{v}_\lambda(x) = v_{\lambda_n}(\lambda_n^{-1/2}x + x_n) \ \text{ in } B_{R_n}(0),$$

where $R_n = \lambda_n^{1/2}\text{dist}(x_n, \partial\Omega)$. Fix $R > 0$; since $R_n \to \infty$ as $n \to \infty$, $\tilde{u}_{\lambda_n}, \tilde{v}_{\lambda_n}$ is well defined in $B_R(0)$ if n is sufficiently large. By Lemma 3.2 and the positivity of u_{λ_n}

$$0 < \tilde{u}_{\lambda_n} < \rho^+_{\delta/\gamma} \ \text{ and } \ \tilde{u}_{\lambda_n}(0) = \max_\Omega u_{\lambda_n} \to \rho^+_{\delta/\gamma} \ \text{ as } n \to \infty.$$

For fixed $R > R' > 0$, $(\tilde{u}_{\lambda_n}, \tilde{v}_{\lambda_n})$ satisfies

$$\begin{aligned} -\Delta\tilde{u}_{\lambda_n} &= f(\tilde{u}_{\lambda_n}) - \tilde{v}_{\lambda_n} && \text{in } B_R(0), \\ -\Delta\tilde{v}_{\lambda_n} &= \delta\tilde{u}_{\lambda_n} - \gamma\tilde{v}_{\lambda_n} && \text{in } B_R(0) \end{aligned}$$

and $(\tilde{u}_{\lambda_n}, \tilde{w}_{\lambda_n}) := (\tilde{u}_{\lambda_n}, \tilde{u}_{\lambda_n} - (1/\beta)\tilde{v}_{\lambda_n})$ satisfies

$$\begin{aligned} -\Delta\tilde{u}_{\lambda_n} &= f(\tilde{u}_{\lambda_n}) - \beta\tilde{u}_{\lambda_n} + \beta\tilde{w}_{\lambda_n} && \text{in } B_R(0), \\ -\Delta\tilde{w}_{\lambda_n} &= f(\tilde{u}_{\lambda_n}) + M\tilde{u}_{\lambda_n} - \alpha\tilde{w}_{\lambda_n} && \text{in } B_R(0) \end{aligned}$$

for sufficiently large n. Note that $\{f(\tilde{u}_{\lambda_n})\}$ is uniformly bounded in L^∞-norm; thus $\{\tilde{u}_{\lambda_n}\}, \{\tilde{w}_{\lambda_n}\}$ is uniformly bounded in $C^\alpha(\overline{B_R(0)})$-norm for some $0 < \alpha < 1$, by elliptic L^p estimates. Thus, by Schauder's estimates, $\{\tilde{u}_{\lambda_n}\}$, $\{\tilde{w}_{\lambda_n}\}$ is uniformly bounded in $C^{2,\alpha}(\overline{B_{R'}(0)})$, and is relatively compact in $C^2(\overline{B_{R'}(0)})$. Hence, there exist $U, W \in C^2(\overline{B_{R'}(0)})$ with $0 \le U \le \rho^+_{\delta/\gamma}$ satisfying

$$\begin{aligned} &-\Delta U = f(U) - \beta U + \beta W && \text{in } B_{R'}(0), \\ &-\Delta W = f(U) + MU - \alpha W && \text{in } B_{R'}(0), \\ &U(0) = \rho^+_{\delta/\gamma}. \end{aligned}$$

Then, by Proposition 3.2, $U \equiv \rho^+_{\delta/\gamma}$ on $\overline{B_{R'}(0)}$.

On the other hand, by (3.1), if n is sufficiently large, z^* and $x_n \in \Omega$ satisfy

$$\text{dist}(z^*, \partial\Omega), \text{dist}(x_n, \partial\Omega) > (\lambda_B/\lambda_n)^{1/2}.$$

Hence, by Proposition 2.12, U_{λ_n} is the first component of the *unique* solution to (Q_λ) in the order interval $[Z_\lambda^{x_n}, Y]$. Then by Lemma 3.3 and the assumption $u_{\lambda_n} \neq U_{\lambda_n}$ we have

$$u_{\lambda_n}(x) < Z_\lambda^{x_n,1}(x) = U_B((\lambda_n/\lambda_B)^{1/2}(x - x_n)) < U_B(0) < \rho^+_{\delta/\gamma}$$

at some $x \in B_{(\lambda_B/\lambda_n)^{1/2}}(x_n)$, where the function $Z_\lambda^{x_n,1}$ is the first component of $Z_\lambda^{x_n}$ and the functions U_B and constant λ_B are as in Proposition 2.9. Thus

$$\tilde{u}_{\lambda_n}(x) < U_B(0) < \rho^+_{\delta/\gamma}$$

for some $x \in B_{\lambda_B^{1/2}}(0)$, and therefore $\tilde{u}_{\lambda_n}$ cannot possess a subsequence which converges to $\rho^+_{\delta/\gamma}$ uniformly on $\overline{B_{\lambda_B^{1/2}}(0)}$. This leads to a contradiction and completes the proof for the Case 1.

Case 2. $x_n \to \overline{x} \in \partial\Omega$ as $n \to \infty$.

To make use of the same argument as in Case 1, it suffices to show that $\lambda_n^{1/2}\mathrm{dist}(x_n, \partial\Omega) \to \infty$.

First, since $\partial\Omega$ is smooth, there exists an open neighborhood Π of $\overline{x}$ and a function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that

$$\Pi \cap \Omega = \{x \in \Pi : x^1 > \phi(x') = \phi(x^2, \dots, x^N)\}, \quad \nabla_{x'}\phi(\overline{x}) = 0.$$

Let $\tilde{x}_n \in \partial\Omega$ be such that $\mathrm{dist}(x_n, \partial\Omega) = |x_n - \tilde{x_n}|$, and we denote $x_n = (x_n^1, \dots, x_n^N) = (x_n^1, x_n')$, $\tilde{x}_n = (\tilde{x}_n^1, \dots, \tilde{x}_n^N) = (\tilde{x}_n^1, \tilde{x}_n')$, and $\overline{x} = (\overline{x}^1, \dots, \overline{x}^N) = (\overline{x}^1, \overline{x}')$. Next we define the diffeomorphism $\Phi_0 : \Pi \to \mathbb{R}^N_\xi$ and $\Phi_n : \Pi \to \mathbb{R}^N_\xi$ for every n as follows:

$$\begin{aligned} \xi &= {}^t(\xi^1, \dots, \xi^N) = \Phi_0(x) = {}^t(\Phi_0^1(x), \dots, \Phi_0^N(x)) \\ &:= {}^t(x^1 - \phi(x'), x^2 - \overline{x}^2, \dots, x^N - \overline{x}^N) \end{aligned}$$

and

$$\begin{aligned} \xi &= {}^t(\xi^1, \dots, \xi^N) = \Phi_n(x) = {}^t(\Phi_n^1(x), \dots, \Phi_n^N(x)) \\ &:= {}^t(x^1 - \phi(x'), x^2 - \tilde{x}^2, \dots, x^N - \tilde{x}^N). \end{aligned}$$

In these coordinates the Laplacian is given by

$$\Delta u = \sum_{i,j=1}^N a_{ij}^n(\xi) \frac{\partial^2}{\partial\xi^i \partial\xi^j}\tilde{u} + \sum_{j=1}^N b_j^n(\xi) \frac{\partial}{\partial\xi^j}\tilde{u},$$

where

$$a_{ij}^n(\xi) = \sum_{l=1}^N \frac{\partial\Phi_n^l}{\partial x^i}\frac{\partial\Phi_n^l}{\partial x^j}\bigg|_{x=\Phi_n^{-1}(\xi)}, a_{ij}^n \in C^2,\ b_j^n \in C^1,$$
$$u(x) = \tilde{u}(\Phi_n(x)), \tilde{u}(\xi) = u(\Phi_n^{-1}(\xi)).$$

We can take $r > 0$ such that $\Xi := \Phi_0(B_r(\overline{x}) \cap \overline{\Omega}) \subset \Phi_n(\overline{\Omega} \cap \Pi)$ for all $n \in \mathbb{N} \cup \{0\}$. We can also easily check that for fixed $\xi \in D$, $a_{ij}^n(\lambda_n^{-1/2}\xi) \to a_{ij}^0(0) = \delta_{ij}$ as $n \to \infty$. Next we define the functions

$$\begin{aligned} \tilde{U}_n(\xi) &= \tilde{u}_{\lambda_n}(\lambda_n^{-1/2}\xi) = u_{\lambda_n}(\Phi_n^{-1}(\lambda_n^{-1/2}\xi)) \\ \tilde{W}_n(\xi) &= \tilde{w}_{\lambda_n}(\lambda_n^{-1/2}\xi) = w_{\lambda_n}(\Phi_n^{-1}(\lambda_n^{-1/2}\xi)), \end{aligned}$$

where $w_{\lambda_n} = u_{\lambda_n} - (1/\beta)v_{\lambda_n}$. It follows from a simple computation that $(\tilde{U}_n, \tilde{W}_n)$ satisfies the equation

$$(\tilde{\mathrm{Q}}_n)\begin{cases} -\displaystyle\sum_{i,j=1}^{N} a_{ij}^n(\lambda_n^{-1/2}\xi)\frac{\partial^2}{\partial\xi^i\partial\xi^j}U + \sum_{j=1}^{N}\lambda_n^{-1/2}b_j^n(\lambda_n^{-1/2}\xi)\frac{\partial}{\partial\xi^j}U \\ = f(U) - \beta U + \beta W \\ -\displaystyle\sum_{i,j=1}^{N} a_{ij}^n(\lambda_n^{-1/2}\xi)\frac{\partial^2}{\partial\xi^i\partial\xi^j}U + \sum_{j=1}^{N}\lambda_n^{-1/2}b_j^n(\lambda_n^{-1/2}\xi)\frac{\partial}{\partial\xi^j}U \\ = f(U) + MU - \alpha W \end{cases}$$

in $\lambda_n^{1/2}\Xi$. For any compact set $K' \subset\subset K \subset\subset D := \{\xi \in \mathbb{R}^N : \xi_1 > 0\}$, $\{f(\tilde{U}_n)\}$ is uniformly bounded in $C^\alpha(K)$ norm and $\{(\tilde{\mathrm{Q}}_n)\}$ is a family of elliptic equations with ellipticity constants uniformly bounded away 0. This implies that $\{\tilde{U}_n\}$, $\{\tilde{W}_n\}$ are relatively compact sets in $C^2(K')$.

Now we take a sequence of compact subsets $\{K_n\}$ such that $K_n \subset K_{n+1}$ and $\bigcup_{n=1}^\infty K_n = D$ and select, by means of the usual diagonal process, subsequences of $\{\tilde{U}_n\}$, $\{\tilde{W}_n\}$ which converge to functions U, $W \in C^2(D) \cap C(\overline{D})$ satisfying

$$\begin{cases} -\Delta U = f(U) - \beta U + \beta W & \text{in } D = \{\xi \in \mathbb{R}^N : \xi_1 > 0\} \\ -\Delta W = f(U) + MU - \alpha W & \text{in } D \\ U = W = 0 & \text{on } \partial D = \{\xi \in \mathbb{R}^N : \xi_1 = 0\}. \end{cases}$$

Applying Proposition 3.2, we see that $0 < U < \rho^+_{\delta/\gamma}$ in D. But

$$\tilde{U}_n(\lambda_n^{1/2}\Phi_n(x_n)) = \tilde{u}_{\lambda_n}(\Phi_n(x_n)) = u_{\lambda_n}(x_n) \to \rho^+_{\delta/\gamma} \text{ as } n \to \infty,$$

and this implies that $\lambda_n^{1/2}|\Phi_n(x_n)| \to \infty$ as $n \to \infty$. We remark that the Φ_n's are uniformly Lipschitz continuous. Hence,

$$\begin{aligned} \lambda_n^{1/2}|\Phi_n(x_n)| &= \lambda_n^{1/2}|\Phi_n(x_n) - \Phi_n(\tilde{x}_n)| \\ &\le L\lambda_n^{1/2}|x_n - \tilde{x}_n| = L\lambda_n^{1/2}\mathrm{dist}(x_n, \partial\Omega) \end{aligned}$$

hold for some $L > 0$, and $\lambda_n^{1/2}\mathrm{dist}(x_n, \partial\Omega) \to \infty$ as desired. □

Next we prove Theorem 1.2.

Proof of Theorem 1.2. First if u is the first component of the solution to (P_λ), then

$$-\Delta u + \lambda\delta B_\lambda u = \lambda f(u).$$

Multiplying u and using Green's formula, we have

$$\int_\Omega |\nabla u|^2 + \lambda\delta(B_\lambda u)u - \lambda f(u)u\,dx = 0.$$

Substituting this into the energy functional

$$J_\lambda(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{2}\delta(B_\lambda u)u - \lambda F(u)dx,$$

we have

$$J_\lambda(u) = \lambda\int_\Omega \frac{1}{2}f(u)u - F(u)dx.$$

We set $H(u) := (1/2)f(u)u - F(u)$ and let u^* be such that

$$\frac{f(u^*)}{u^*} = f'(u^*).$$

Then we note that the assumption on f implies that H is decreasing on $(u^*, +\infty)$ and $\rho^+_{\delta/\gamma} > u^*$. Next we set

$$G(u) = \int_0^u g(v)dv = \int_0^u \Big(f(v) - \frac{\delta}{\gamma}v\Big)dv.$$

Claim 1. $H(\rho^+_{\delta/\gamma}) < 0$. In fact our condition implies that

$$g(\rho^+_{\delta/\gamma}) = 0 \quad\text{and}\quad G(\rho^+_{\delta/\gamma}) = \int_0^{\rho^+_{\delta/\gamma}} g(v)dv > 0.$$

Then we have

$$\begin{aligned} H(\rho^+_{\delta/\gamma}) &= \tfrac{1}{2}f(\rho^+_{\delta/\gamma})\rho^+_{\delta/\gamma} - F(\rho^+_{\delta/\gamma}) \\ &= \tfrac{1}{2}g(\rho^+_{\delta/\gamma})\rho^+_{\delta/\gamma} + \frac{\delta}{2\gamma}(\rho^+_{\delta/\gamma})^2 - G(\rho^+_{\delta/\gamma}) - \frac{\delta}{2\gamma}(\rho^+_{\delta/\gamma})^2 \\ &= \tfrac{1}{2}g(\rho^+_{\delta/\gamma})\rho^+_{\delta/\gamma} - G(\rho^+_{\delta/\gamma}) = -G(\rho^+_{\delta/\gamma}) < 0. \end{aligned}$$

Claim 2. There exists $\lambda^\flat > 0$ such that for $\lambda > \lambda^\flat$, $\overline{u}_\lambda = U_\lambda$. If not, there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \nearrow \infty$ and $\overline{u}_{\lambda_n} \neq U_{\lambda_n}$. From Theorem 1.1, there exists $\varepsilon > 0$ and $\lambda^\sharp > 0$ such that if (u, v) is a positive solution to (P_λ) with $\max_\Omega u \in (\rho^+_{\delta/\gamma} - \varepsilon, \rho^+_{\delta/\gamma})$ and $\lambda > \lambda^\sharp$, then $u = U_\lambda$. Since by Proposition 2.4, $\overline{u}_{\lambda_n}$ is positive, for sufficiently large n, $\max_\Omega \overline{u}_{\lambda_n} \notin (\rho^+_{\delta/\gamma} - \varepsilon, \rho^+_{\delta/\gamma})$.

Next we choose $\varepsilon_1, \varepsilon_2 > 0$ and $\Omega' \subset\subset \Omega$ in the following way. First we choose $\varepsilon_2 > 0$ such that

(1) $0 > H(\rho^+_{\delta/\gamma} - \varepsilon) > H(\rho^+_{\delta/\gamma} - \varepsilon_2), \quad \varepsilon < \varepsilon_2.$

We note that by taking $\varepsilon > 0$ small, if necessary we may assume that $H(\rho^+_{\delta/\gamma} - \varepsilon) < 0$, and we also note that $H(u)$ is decreasing near $\rho^+_{\delta/\gamma}$. Next we choose $\varepsilon_1 > 0$ so small that

(2) $(\sup_{u\geq 0} H(u) - H(\rho^+_{\delta/\gamma}))\varepsilon_1 < (H(\rho^+_{\delta/\gamma} - \varepsilon) - H(\rho^+_{\delta/\gamma} - \varepsilon_2))|\Omega|$,

where $|\Omega|$ denotes the measure of Ω. Finally we choose $\Omega' \subset\subset \Omega$ so that

(3) $|\Omega \backslash \Omega'| < \varepsilon_1$.

Then by Proposition 2.11 there exist $\lambda^\natural > 0$ such that for all $\lambda > \lambda^\natural$ and for all $x \in \Omega'$

$$\rho^+_{\delta/\gamma} - \varepsilon_2 < U_\lambda(x) < \rho^+_{\delta/\gamma}.$$

Then we have

$$J_{\lambda_n}(\overline{u}_{\lambda_n}) = \lambda_n \int_\Omega H(\overline{u}_{\lambda_n})dx \geq \lambda_n|\Omega|H(\rho^+_{\delta/\gamma} - \varepsilon)$$

and

$$\begin{aligned} J_{\lambda_n}(U_{\lambda_n}) &= \lambda_n \int_\Omega H(U_{\lambda_n})dx = \lambda_n \int_{\Omega'} H(U_{\lambda_n})dx + \lambda_n \int_{\Omega\backslash\Omega'} H(U_{\lambda_n})dx \\ &\leq \lambda_n(H(\rho^+_{\delta/\gamma} - \varepsilon_2)|\Omega'| + \sup_{u\geq 0} H(u)\varepsilon_1) \\ &\leq \lambda_n(H(\rho^+_{\delta/\gamma} - \varepsilon_2)(|\Omega| - \varepsilon_1) + \sup_{u\geq 0} H(u)\varepsilon_1) \\ &= \lambda_n(H(\rho^+_{\delta/\gamma} - \varepsilon_2)|\Omega| + (\sup_{u\geq 0} H(u) - H(\rho^+_{\delta/\gamma} - \varepsilon_2))\varepsilon_1) \\ &\leq \lambda_n(H(\rho^+_{\delta/\gamma} - \varepsilon_2)|\Omega| + (\sup_{u\geq 0} H(u) - H(\rho^+_{\delta/\gamma}))\varepsilon_1). \end{aligned}$$

Here we used that $|\Omega'| \geq |\Omega| - \varepsilon_1$ and $H(\rho^+_{\delta/\gamma} - \varepsilon_2)|\Omega'| \leq H(\rho^+_{\delta/\gamma} - \varepsilon_2)(|\Omega| - \varepsilon_1)$. Therefore,

$$\begin{aligned} &\lambda_n^{-1}(J_{\lambda_n}(U_{\lambda_n}) - J_{\lambda_n}(\overline{u}_{\lambda_n})) \\ &\leq (H(\rho^+_{\delta/\gamma} - \varepsilon_2)|\Omega| + (\sup_{u\geq 0} H(u) - H(\rho^+_{\delta/\gamma}))\varepsilon_1 - |\Omega|H(\rho^+_{\delta/\gamma} - \varepsilon) \\ &= (\sup_{u\geq 0} H(u) - H(\rho^+_{\delta/\gamma}))\varepsilon_1 - (H(\rho^+_{\delta/\gamma} - \varepsilon) - H(\rho^+_{\delta/\gamma} - \varepsilon_2))|\Omega| < 0. \end{aligned}$$

This contradicts the fact that $\overline{u}_{\lambda_n}$ is the global minimizer of J_{λ_n}. □

To show Theorem 1.3, we prepare two lemmas. The following lemma shows that the maximum of any positive solution is bounded away from 0 uniformly in λ.

Lemma 3.4. *Suppose that conditions (C2) and (C3) hold. Then every positive solution* (u,v) *of* (P_λ) *satisfies* $\max_\Omega u \geq \rho^-_{\delta/\gamma}$.

Proof. If we set $w = u - (1/\beta)v$, then

$$\begin{array}{ll} -\Delta u = \lambda(f(u) - \beta u + \beta w) & \text{in } \Omega, \\ -\Delta w = \lambda(f(u) + Mu - \alpha w) & \text{in } \Omega, \\ u = w = 0 & \text{on } \partial\Omega. \end{array}$$

Now we assume that $\max_\Omega u < \rho^-_{\delta/\gamma}$ and set $u_0 := \max_\Omega u > 0$.
Step 1. We show that $w(x) \leq \theta \max_\Omega u = \theta u_0$. In fact we have

$$\begin{aligned} &(-\Delta + \lambda\alpha)(w - \theta u_0) = -\Delta w + \lambda\alpha w - \lambda\alpha\theta u_0 \\ &= \lambda(f(u) + Mu) - (\gamma - \beta)\Big(1 - \frac{\delta}{\gamma\beta}\Big)u_0 \\ &= \lambda\Big(f(u) - \frac{\delta}{\gamma}u_0 + Mu - \Big(\frac{\gamma\beta - \delta}{\beta} - \beta\Big)u_0\Big) \\ &= \lambda\Big(f(u) - \frac{\delta}{\gamma}u_0 + M(u - u_0)\Big) < 0. \end{aligned}$$

Then by the maximum principle $w(x) \leq \theta u_0$ follows.
Step 2. If $u(x_0) = \max_\Omega u = u_0$, then $-\Delta u(x_0) < 0$. In fact we have

$$\begin{aligned} &-\Delta u(x_0) = \lambda(f(u(x_0)) - \beta u(x_0) + \beta w(x_0)) \\ &\leq \lambda\Big(f(u(x_0)) - \frac{\delta}{\gamma}u(x_0) + \frac{\delta}{\gamma}u(x_0) - \beta u(x_0) + \beta\theta u(x_0)\Big) \\ &= \lambda\Big(f(u(x_0)) - \frac{\delta}{\gamma}u(x_0)\Big) < 0. \end{aligned}$$

On the other hand, since $x_0 \in \Omega$ is a maximum point of u, then we have $-\Delta u(x_0) \geq 0$. This is a contradiction. □

Next, by using Proposition 2.8, we obtain the following proposition.

Proposition 3.5. *Let* $\Omega = B_R(0)$ *and suppose* (u,v) *is a positive solution to* (P_λ). *Then* u, v *are radially symmetric,* $u'(r), v'(r) < 0$, *on* $(0,R]$, *and* $u'(0) = v'(0) = 0$, *where* $'$ *is the derivative in* $r = |x|$.

Proof. Let us set $w = u - (1/\beta)v$. (u,w) satisfies the quasimonotone system (Q_λ), and we note that w is positive in $B_R(0)$ since u is positive. Then by Proposition 2.8 u and w are radially symmetric and decreasing in $r = |x|$.

We also have that v is radially symmetric. Next we note that v is the solution to the problem

$$\begin{aligned} &-\lambda^{-1}\Delta v + \gamma v = \delta u && \text{in } B_R(0), \\ &v = 0 && \text{on } \partial B_R(0). \end{aligned}$$

By the regularity of solutions, we differentiate the above equation in r; then we have

$$\begin{aligned} &-\lambda^{-1}\Delta v' + \Big(\frac{N-1}{\lambda|x|^2} + \gamma\Big)v' = \delta u' < 0 && \text{in } B_R(0)\backslash\{0\}, \\ &v' = \frac{\partial v}{\partial \nu} < 0 && \text{on } \partial B_R(0), \end{aligned} \tag{3.2}$$

since u is decreasing in r, where ν is an outward unit-normal vector of $\partial B_R(0)$. Then we can conclude $v' < 0$ on $(0, R]$. Indeed, if $\max_{r\in(0,R]} v'(r) \geq 0$ then we have $\max_{r\in(0,R]} v'(r) = v'(r_0)$ for some $r_0 \in (0, R)$. Then we have

$$-\lambda^{-1}\Delta v'(r_0) + \Big(\frac{N-1}{\lambda r_0^2} + \gamma\Big)v'(r_0) \geq 0.$$

This contradicts (3.2). The proof is completed. □

We can obtain Theorem 1.3 by using a similar argument as in [19]. For the reader's convenience, we give the proof of Theorem 1.3 in detail.

Proof of Theorem 1.3. First we note that from Proposition 2.8 and Lemma 3.4, we have $\underline{u}_\lambda(0) \geq \rho^-_{\delta/\gamma}$, which is (1) of Theorem 1.3. Also, from Theorem 1.1 we have that $\max \underline{u}_\lambda$ is bounded from above by $\rho^+_{\delta/\gamma}$ uniformly for sufficiently large λ. We also note that from Proposition 2.8 $\underline{u}_\lambda$ and $\underline{v}_\lambda$ are radially symmetric, decreasing in $r = |x|$, and satisfy $u'(0) = v'(0) = 0$, where $'$ represents a differentiation with respect to $r = |x|$.
Part 1. Proof of (2).
Step 1.1. Let $\lambda_1 > 0$ be sufficiently large. The functions $\{\tilde{u}_\lambda : \lambda > 2\lambda_1\}$ and $\{\tilde{v}_\lambda : \lambda > 2\lambda_1\}$ satisfy

$$\begin{cases} -\Delta\tilde{u}_\lambda = f(\tilde{u}_\lambda) - \tilde{v}_\lambda & \text{in } B_{\sqrt{2\lambda_1}}(0), \\ -\Delta\tilde{v}_\lambda = \delta\tilde{u}_\lambda - \gamma\tilde{v}_\lambda & \text{in } B_{\sqrt{2\lambda_1}}(0), \end{cases}$$

and from Lemma 3.1, we have

$$\|\tilde{u}_\lambda\|_{L^\infty(B_{\sqrt{2\lambda_1}}(0))} \leq \rho^+_{\delta/\gamma}, \quad \|\tilde{v}_\lambda\|_{L^\infty(B_{\sqrt{2\lambda_1}}(0))} \leq \frac{\delta}{\gamma}\rho^+_{\delta/\gamma},$$

and also

$$\|f(\tilde{u}_\lambda)\|_{L^\infty(B_{\sqrt{2\lambda_1}}(0))} \leq K_f := \sup_{0\leq x\leq 1} |f(x)|.$$

Using interior elliptic estimates, Schauder's interior estimates, and the fact that f is locally Lipschitz, we find that $\{\tilde{u}_\lambda : \lambda > 2\lambda_1\}$ and $\{\tilde{v}_\lambda : \lambda > 2\lambda_1\}$ are bounded in $C^{2,\alpha}(\overline{B_{\sqrt{\lambda_1}}(0)})$ for some $0 < \alpha < 1$ and hence precompact in $C^2(\overline{B_{\sqrt{\lambda_1}}(0)})$. Then there exists a sequence $\{\lambda_{1,n}\}$ such that $\lambda_1 < \lambda_{1,n} \nearrow \infty$ as $n \to \infty$ and $\{\tilde{u}_{\lambda_{1,n}}\}$, $\{\tilde{v}_{\lambda_{1,n}}\}$ converge in $C^2(\overline{B_{\sqrt{\lambda_1}}(0)})$.

We set for $x \in \overline{B_{\sqrt{\lambda_1}}(0)}$

$$u_1(x) := \lim_{n\to\infty} \tilde{u}_{\lambda_{1,n}}(x), \quad v_1(x) := \lim_{n\to\infty} \tilde{v}_{\lambda_{1,n}}(x).$$

On $\overline{B_{\sqrt{\lambda_1}}(0)}$ the functions u_1, v_1 are solutions of the equation

$$-\Delta u_1 = f(u_1) - v_1$$
$$-\Delta v_1 = \delta u_1 - \gamma v_1$$

Let $\lambda_2 := \lambda_{1,1}$ and repeat the argument in Step 1.1 to obtain that $\{\tilde{u}_{\lambda_{1,n}}\}$ and $\{\tilde{v}_{\lambda_{1,n}}\}$ are bounded sequences in $C^{2,\alpha}(\overline{B_{\sqrt{\lambda_2}}(0)})$ and precompact in $C^2(\overline{B_{\sqrt{\lambda_2}}(0)})$. Again we extract subsequences $\{\lambda_{2,n}\}$ from $\{\lambda_{1,n}\}$ such that $\{\tilde{u}_{\lambda_{2,n}}\}$ and $\{\tilde{v}_{\lambda_{2,n}}\}$ converge in $C^2(\overline{B_{\sqrt{\lambda_2}}(0)})$. We extend the functions u_1 and v_1 to $\overline{B_{\sqrt{\lambda_2}}(0)}$ by defining for $x \in \overline{B_{\sqrt{\lambda_2}}(0)}$

$$u_2(x) := \lim_{n\to\infty} \tilde{u}_{\lambda_{2,n}}(x), \; v_2(x) := \lim_{n\to\infty} \tilde{v}_{\lambda_{2,n}}(x).$$

These functions satisfy the equations on $B_{\sqrt{\lambda_2}}(0)$.

By repeating this process we obtain for every $k \in \mathbb{N}$ subsequences $\{\lambda_{k,n}\}$ from $\{\lambda_{k-1,n}\}$ such that $\{\tilde{u}_{\lambda_{k,n}}\}$ and $\{\tilde{v}_{\lambda_{k,n}}\}$ converge in $C^2(\overline{B_{\sqrt{\lambda_k}}(0)})$. Also, we obtain the functions u_k and v_k such that for $\overline{B_{\sqrt{\lambda_k}}(0)}$

$$u_k(x) := \lim_{n\to\infty} \tilde{u}_{\lambda_{k,n}}(x), \; v_k(x) := \lim_{n\to\infty} \tilde{v}_{\lambda_{k,n}}(x)$$

satisfy the equation on $\overline{B_{\sqrt{\lambda_k}}(0)}$. In addition, we can choose λ_k so that $\lambda_k \nearrow \infty$ as $k \to \infty$.

Step 1.2. We define the function U, V defined on $\mathbb{R}^N$ as follows. For $x \in \mathbb{R}^N$ there exists $k \in \mathbb{N}$ such that $x \in B_{\sqrt{\lambda_k}}(0)$. Then we define $U(x) = u_k(x)$ and $V(x) = v_k(x)$. Therefore U, V satisfies

$$\begin{cases} -\Delta U = f(U) - V & \text{on } \mathbb{R}^N, \\ -\Delta V = \delta U - \gamma V & \text{on } \mathbb{R}^N. \end{cases}$$

By Lemma 3.4, we have $\max_{B_{\sqrt{\lambda}}(0)} \tilde{u}_\lambda \geq \rho^-_{\delta/\gamma}$, and hence $\max_{\mathbb{R}^N} U \geq \rho^-_{\delta/\gamma} > 0$. Consequently, $U, V \neq 0$.

Step 1.3. It remains to show that $u(x), v(x) \to 0$ as $|x| \to \infty$. By Proposition 3.5 all the functions $\tilde{u}_\lambda$ and $\tilde{v}_\lambda$ are radially symmetric. We will consider $\tilde{u}_\lambda, \tilde{v}_\lambda, U, V$ as functions of one variable $r = |x|$; in particular we have that $U'(r) \le 0, V'(r) \le 0$ for $r > 0$ and $U'(0) = V'(0) = 0$. Let

$$l_u := \lim_{r\to\infty} U(r) = \inf_{r>0} U(r), \quad l_v := \lim_{r\to\infty} V(r) = \inf_{r>0} V(r). \tag{3.3}$$

In Step 1.4 we show that

$$l_u \in \{0, \rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma}\} \text{ and } l_v = \frac{\delta}{\gamma} l_u. \tag{3.4}$$

Then by Lemma 3.1 and Theorem 1.1, there exists $\varepsilon > 0$ for sufficiently large λ

$$\tilde{u}_\lambda(x) \le \rho^+_{\delta/\gamma} - \varepsilon < \rho^+_{\delta/\gamma}.$$

Hence, we have $l_u \le \rho^+_{\delta/\gamma} - \varepsilon < \rho^+_{\delta/\gamma}$ and $l_u \ne \rho^+_{\delta/\gamma}$. To exclude the possibility $l_u = \rho^-_{\delta/\gamma}$ we show in Step 1.5 that

$$\int_0^{l_u} \Big(f(s) - \frac{\delta}{\gamma} s \Big) ds = F(l_u) - \frac{\delta}{2\gamma} l_u^2 \ge 0. \tag{3.5}$$

Then it cannot be that $l_u = \rho^-_{\delta/\gamma}$. Then the only remaining possibility is that $l_u = l_v = 0$.

Step 1.4. We prove (3.4). Because of the radial symmetry we have that

$$\begin{cases} -U'' - \dfrac{N-1}{r} U' = f(U) - V & r > 0, \\ -V'' - \dfrac{N-1}{r} V' = \delta U - \gamma V & r > 0, \\ U'(0) = V'(0) = 0. \end{cases} \tag{3.6}$$

Multiplying the first equation by U' and the second equation by V' and integrating on $(0, R)$ one finds that for all $R > 0$

$$\frac{1}{2} U'(R)^2 + (N-1) \int_0^R \frac{(U')^2}{r} dr = F(U(0)) - F(U(R)) + \int_0^R U' V dr$$

and

$$\begin{aligned} &\frac{1}{2} V'(R)^2 + (N-1) \int_0^R \frac{(V')^2}{r} dr \\ = \quad & -\delta(U(R)V(R) - U(0)V(0)) + \delta \int_0^R U' V dr + \frac{\gamma}{2}(V(R)^2 - V(0)^2). \end{aligned}$$

Adding the above identities we find that

$$\frac{U'(R)^2+\delta^{-1}V'(R)^2}{2}+(N-1)\int_0^R\frac{(U')^2+\delta^{-1}(V')^2}{r}dr-2\int_0^R U'V\,dr$$
$$=F(U(0))-F(U(R))-(U(R)V(R)-U(0)V(0))+\frac{\gamma}{2\delta}(V(R)^2-V(0)^2) \tag{3.7}$$

and subtracting that

$$\frac{1}{2}(U'(R)^2-\delta^{-1}V'(R)^2)+(N-1)\int_0^R\frac{(U')^2-\delta^{-1}(V')^2}{r}dr \tag{3.8}$$
$$=F(U(0))-F(U(R))-U(0)V(0)+U(R)V(R)-\frac{\gamma}{2\delta}(V(R)^2-V(0)^2).$$

Because $U'(R),V'(R)\le 0$ and $U(R),V(R)$ stay bounded as $R\to\infty$ we have from (3.7) that $U'(R)\to 0$ and $V'(R)\to 0$ as $R\to\infty$. Also we see from (3.6) that

$$-U''(R)\to f(l_u)-l_v \quad\text{and}\quad -V''(R)\to \delta l_u-\gamma l_v \quad\text{as } R\to\infty$$

so that $f(l_u)-l_v=0$ and $\delta l_u-\gamma l_v=0$, and hence (3.4) follows.

Step 1.5. Next we prove (3.5). We first note that $(\sqrt{\delta}/\beta)-1\ge 0$. In fact

$$\frac{\beta}{\sqrt{\delta}}=\frac{\gamma-M}{2\sqrt{\delta}}+\sqrt{\Big(\frac{\gamma-M}{2\sqrt{\delta}}\Big)^2-1}\le 1.$$

Next we set $\tilde{w}_\lambda=\tilde{u}_\lambda-(1/\beta)\tilde{v}_\lambda$. Then we have

$$\begin{aligned}\tilde{u}'_\lambda-\delta^{-1/2}\tilde{v}'_\lambda &= \tilde{u}'_\lambda-(\sqrt{\delta}/\beta)^{-1}\tilde{u}'_\lambda+(\sqrt{\delta}/\beta)^{-1}\tilde{w}'_\lambda\\ &= (\sqrt{\delta}/\beta)^{-1}(\sqrt{\delta}/\beta-1)\tilde{u}'_\lambda+(\sqrt{\delta}/\beta)^{-1}\tilde{w}'_\lambda\le 0,\end{aligned}$$

and hence we have

$$\tilde{u}'_\lambda(r)^2-\delta^{-1}\tilde{v}'_\lambda(r)^2=(\tilde{u}'_\lambda(r)-\delta^{-1/2}\tilde{v}'_\lambda(r))(\tilde{u}'_\lambda(r)+\delta^{-1/2}\tilde{v}'_\lambda(r))\ge 0. \tag{3.9}$$

From (3.8) we see by letting $R\to\infty$ that

$$\begin{aligned}&(N-1)\int_0^\infty\frac{U'(r)^2-\delta^{-1}V'(r)^2}{r}dr\\ =\;&F(U(0))-F(l_u)-U(0)V(0)+\frac{\delta}{2\gamma}l_u^2+\frac{\gamma}{2\gamma}V(0)^2.\end{aligned} \tag{3.10}$$

On the other hand, for every solution $(\tilde{u}_\lambda,\tilde{v}_\lambda)$ it holds that

$$\frac{1}{2}(\tilde{u}'_\lambda(\sqrt{\lambda})^2-\delta^{-1}\tilde{v}'_\lambda(\sqrt{\lambda})^2)+(N-1)\int_0^{\sqrt{\lambda}}\frac{\tilde{u}'_\lambda(r)^2-\delta^{-1}\tilde{v}'_\lambda(r)^2}{r}dr$$

$$= \quad F(\tilde{u}_\lambda(0)) - \tilde{u}_\lambda(0)\tilde{v}_\lambda(0) + \frac{\gamma}{2\delta}\tilde{v}_\lambda(0)^2.$$

Hence, from (3.9), for all $K > 0$ and all $\lambda > K^2$ it holds that

$$(N-1)\int_0^K \frac{\tilde{u}'_\lambda(r)^2 - \delta^{-1}\tilde{v}'_\lambda(r)^2}{r}dr \le F(\tilde{u}_\lambda(0)) - \tilde{u}_\lambda(0)\tilde{v}_\lambda(0) + \frac{\gamma}{2\delta}\tilde{v}_\lambda(0)^2$$

so that

$$(N-1)\int_0^K \frac{U'(r)^2 - \delta^{-1}V'(r)^2}{r}dr \le F(U(0)) - U(0)V(0) + \frac{\gamma}{2\delta}V(0)^2.$$

Letting $K \to \infty$ we find that

$$(N-1)\int_0^\infty \frac{U'(r)^2 - \delta^{-1}V'(r)^2}{r}dr \le F(U(0)) - U(0)V(0) + \frac{\gamma}{2\delta}V(0)^2. \quad (3.11)$$

From (3.10) and (3.11) we have

$$\begin{aligned} &F(U(0)) - F(l_u) - U(0)V(0) + \frac{\delta}{2\gamma}l_u^2 + \frac{\gamma}{2\delta}V(0)^2 \\ &\le \quad F(U(0)) - U(0)V(0) + \frac{\gamma}{2\delta}V(0)^2, \end{aligned}$$

which is precisely (3.5).

Part 2. Finally we prove the (3); i.e., $\underline{u}_\lambda \to 0$ and $\underline{v}_\lambda \to 0$ as $\lambda \to +\infty$ on every compact subset of $\overline{B_1(0)}\backslash\{0\}$. We provide a proof only for $\underline{u}_\lambda$. If the result is false, there exist $\Omega' \subset\subset \overline{B_1(0)}\backslash\{0\}$, $\varepsilon > 0$, and a sequence $\{\lambda_n\} \subset \mathbb{R}^+$ such that $\lambda_n \nearrow \infty$ as $n \to \infty$ and

$$\sup_{\overline{\Omega'}} |\underline{u}_{\lambda_n}(x)| \ge \varepsilon. \quad (3.12)$$

Since $\overline{\Omega'}$ is compact in $\overline{B_1(0)}\backslash\{0\}$, there exists $r_0 > 0$ such that

$$r_0^{-1} \le |x| \le r_0 \quad \text{for all } x \in \overline{\Omega'}.$$

Then since $\underline{u}_{\lambda_n}$ is decreasing in $r = |x|$, we have

$$0 \le \underline{u}_{\lambda_n}(r_0) \le \underline{u}_{\lambda_n}(x) \le \underline{u}_{\lambda_n}(r_0^{-1}) \quad \text{for all } x \in \overline{\Omega'},$$

where $\underline{u}_{\lambda_n}(r_0)$ and $\underline{u}_{\lambda_n}(r_0^{-1})$ are the values of the function $\underline{u}_{\lambda_n}$ considered as a function of one variable $r = |x|$ at $r = r_0$ and r_0^{-1}. Hence,

$$0 \le \tilde{u}_{\lambda_n}(\lambda_n^{1/2}r_0) \le \underline{u}_{\lambda_n}(x) \le \tilde{u}_{\lambda_n}(\lambda_n^{1/2}r_0^{-1}) \quad \text{for all } x \in \overline{\Omega'}$$

and

$$\sup_{\overline{\Omega'}} |\underline{u}_{\lambda_n}(x)| \le \tilde{u}_{\lambda_n}(\lambda_n^{1/2}r_0^{-1}). \quad (3.13)$$

On the other hand since $\tilde{u}_{\lambda_n}$ is decreasing in r, for fixed $r > 0$ and sufficiently large n we have

$$\tilde{u}_{\lambda_n}(\lambda_n^{1/2} r_0^{-1}) \le \tilde{u}_{\lambda_n}(r). \tag{3.14}$$

Letting $n \to \infty$ in (3.13) and (3.14), if necessary taking a subsequence, we have

$$\overline{\lim_{n\to\infty}} \sup_{\overline{\Omega'}} |\underline{u}_{\lambda_n}(x)| \le \overline{\lim_{n\to\infty}} \tilde{u}_{\lambda_n}(\lambda_n^{1/2} r_0^{-1}) \le U(r).$$

Letting $r \to \infty$ we obtain

$$0 \le \overline{\lim_{n\to\infty}} \sup_{\overline{\Omega'}} |\underline{u}_{\lambda_n}(x)| \le 0.$$

This contradicts (3.12). The proofs of (3) and Theorem 1.3 are completed. □

From the proof of Theorem 1.3, we can obtain the following corollary.

Corollary 3.6. *Suppose that the all conditions of Theorem* 1.3 *hold, and let* (u_λ, v_λ) *be a solutions to* (P_λ) *such that* $u_\lambda \ne U_\lambda$ *for all sufficiently large* $\lambda > 0$*. Then the same results of Theorem* 1.3 *hold.*

4. Open questions

By Theorems 1.2 and 1.3, we obtained the asymptotic profiles of variational solutions at least for the case where $\Omega = B_R(0)$ is a ball. However, in order to understand the complete dynamics of solutions for (D_λ), the following problems still remain:

(Q1) Linearized stability of solutions.
(Q2) Exact multiplicity of solutions.
(Q3) Asymptotic profile of the mountain-pass solution when Ω is not ball.

First we discuss Problem (Q1). In Reinecke and Sweers [20], linearized stability is considered in the space $X := C(\overline{\Omega}) \times C(\overline{\Omega})$. First we define the linearized operator $A_\lambda(U,V) : D(A_\lambda(U,V)) \subset X \to X$ around the solution (U,V) to (P_λ), which is given by

$$\begin{cases} A_\lambda(U,V)\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \lambda \begin{pmatrix} f'(U) & -1 \\ \delta & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \\ D(A_\lambda) := \{(u,v) \in X : u = v = 0 \text{ on } \partial\Omega,\ (\Delta u, \Delta v) \in X\}, \end{cases}$$

where in the definition of $D(A_\lambda)$, Δu and Δv are to be understood in the distributional sense. If the spectrum $\sigma(A_\lambda(U,V))$ is contained in $\{\nu \in \mathbb{C} : \mathrm{Re}\ \nu \ge 0\}$ the solution (U,V) to (P_λ) is called linearly stable; in particular, if there exists $\nu_0 > 0$ such that $\sigma(A_\lambda(U,V))$ is contained in $\{\nu \in \mathbb{C} : \mathrm{Re}\ \nu \geq \nu_0\}$ the solution (U,V) to (P_λ) is called exponentially stable. Also, if $\sigma(A_\lambda(U,V)) \cap \{\nu \in \mathbb{C} : \mathrm{Re}\ \nu < 0\} \neq \emptyset$, then (U,V) is called linearly unstable. In Reinecke and Sweers [20] it is shown that the boundary-layer solution (U_λ, V_λ) is exponentially stable; that is, the following results hold.

Proposition 4.1. ([20, Theorem 2.2]) *Assume that the all conditions (C1), (C2), and (C3) hold, and let $\lambda^\star$ and Λ be as in Proposition* 2.11. *For every $\lambda \geq \lambda^\star$ the solution $\Lambda(\lambda) = (U_\lambda, V_\lambda)$ to* (P_λ) *is an exponentially stable stationary solution to the initial-value problem* (D_λ); *i.e., for every $\lambda \geq \lambda^\star$ there exists $\nu_\lambda > 0$ such that the spectrum $\sigma(A_\lambda(U_\lambda, V_\lambda))$ is contained in $\{\nu \in \mathbb{C} : \mathrm{Re}\ \nu > \nu_\lambda\}$.*

Hence, by Theorem 1.2, the global minimizer is linearly stable for sufficiently large $\lambda > 0$. However, the linearized stability of the mountain-pass solution is not yet known, although we believe that a mountain-pass solution is linearly unstable.

Next, with regard to Problem (Q2), in the scalar case (S_λ), if Ω is a ball it is shown that there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, the problem (S_λ) has exactly two positive solutions, exactly one nontrivial solution for $\lambda = \lambda_0$, and no solution for $\lambda < \lambda_0$ (see [17]). Taking into account that the quasimonotone system would have similar properties as in the scalar equation, we can expect that problem (P_λ) has exactly two nontrivial solutions in our parameter range. In particular, Gardner and Peletier [8] have shown that the problem (S_λ) has exactly two solutions for sufficiently large $\lambda > 0$. In [8], the exact multiplicity of solutions was investigated based on the uniqueness of positive radially symmetric solutions of the problem

$$(\mathrm{S}) \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

(see Peletier and Serrin [18]). Hence, when considering Problem (Q2), it would be necessary to consider the uniqueness of positive radially symmetric solutions for the problem

$$(\mathrm{P}) \begin{cases} -\Delta u = f(u) - v & \text{in } \mathbb{R}^N, \\ -\Delta v = \delta u - \gamma v & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \\ v(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

simultaneously. We believe that the solution to (P) is unique at least for small $\delta > 0$. However, it seems no result for the uniqueness of positive radially symmetric solution to (P) exists as far as we know.

Finally, about Problem (Q3), when Ω is a general domain, the asymptotic profile of the mountain-pass solution is not yet known. We believe that a mountain-pass solution has a spiky profile when Ω is convex as in the result about the scalar case in [11].

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