Differential and Integral Equations Volume 16, Number 12, December 2003, Pages 1519–1531

MULTIPLE NONSEMITRIVIAL SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. We prove the existence of multiple nonnegative nonsemitrivial solutions for a quasilinear elliptic system, defined on an arbitrary domain (bounded or unbounded).

1. Introduction

In this paper we prove the multiplicity of solutions for the system

$$\begin{aligned} -\Delta_p u &= \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u + \frac{\mu(x)}{(\alpha+1)(\delta+1)}|u|^{\gamma-1}|v|^{\delta+1}u, \\ -\Delta_q v &= \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}\, v + \frac{\mu(x)}{(\beta+1)(\gamma+1)}|u|^{\gamma+1}|v|^{\delta-1}v, \end{aligned} \tag{1.1$_\lambda$}$$

where $x \in \Omega$ and $\Omega \subseteq \mathbb{R}^N$ is an arbitrary domain (bounded or unbounded). Throughout this work the following hypotheses are assumed:

($\mathcal{H}$) $N > p > 1$, $N > q > 1$, $\alpha \geq 0$, and $\beta \geq 0$ satisfying $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, $\gamma \geq 0$, $\delta \geq 0$, and $p < \gamma + 1$ or $q < \delta + 1$ satisfying $\frac{\gamma+1}{p^*} + \frac{\delta+1}{q^*} < 1$, where p^* and q^* denote the critical Sobolev exponents: $p^* = \frac{Np}{N-p}$ and $q^* = \frac{Nq}{N-q}$.

Accepted for publication: August 2003.
AMS Subject Classifications: 35B32, 35D05, 35D10, 35J50, 35P15.

($\mathcal{H}_1$) The exponents α, β, γ, and δ satisfy also the general condition

$$\frac{1}{(\alpha+1)(\delta+1)} + \frac{1}{(\beta+1)(\gamma+1)} < 1.$$

Let us point out that a necessary condition for ($\mathcal{H}$) to hold true is that $p+q > N$. E.g., $N = 3$, $p = q = 2$, $\alpha = \beta = 0$, and $\gamma = \delta = \frac{3}{2}$ are admissible values satisfying both ($\mathcal{H}$) and ($\mathcal{H}_1$). Furthermore, we suppose that the coefficient functions $a(x)$, $d(x)$, $b(x)$, and $\mu(x)$ satisfy the following conditions:

(Υ_1) a is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0,1)$, such that $a \in L^{N/p}(\Omega) \cap L^\infty(\Omega)$ and there exists $\Omega^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega^+| > 0$, such that $a(x) > 0$, for all $x \in \Omega^+$.

(Υ_2) d is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0,1)$, such that $d \in L^{N/q}(\Omega) \cap L^\infty(\Omega)$ and there exists $\Omega^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega^+| > 0$, such that $d(x) > 0$, for all $x \in \Omega^+$.

(Υ_3) the functions a and d satisfy one of the following hypotheses:

(G^+) $a(x) \geq 0$ and $d(x) \geq 0$, in Ω, or

(G^-) $a(x) < 0$ and $d(x) < 0$, for all $x \in \Omega^-$, on some subset $\Omega^- \subset \Omega$ with $|\Omega^-| > 0$.

(Υ_4) b is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0,1)$, $b(x) \geq 0$ in Ω, $b(x) \not\equiv 0$, and $b \in L^{\omega_1}(\Omega) \cap L^\infty(\Omega)$, where $\omega_1 = p^*q^*/[p^*q^* - (\alpha+1)q^* - (\beta+1)p^*]$.

(Υ_5) μ is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0,1)$, changing sign (i.e., $\mu^+ \not\equiv 0$, $\mu^- \not\equiv 0$) and $\mu \in L^\infty(\Omega) \cap L^{\omega_2}(\Omega)$, where $\omega_2 = p^*q^*/[p^*q^* - (\gamma+1)q^* - (\delta+1)p^*]$.

In addition the function $\mu(x)$ satisfies the following key condition:

$$(\Upsilon_6) \qquad \int_\Omega \mu(x)\, |u_1|^{\gamma+1}\, |v_1|^{\delta+1}\, dx < 0$$

where (u_1, v_1) is the positive normalized eigenfunction of the unperturbed system

$$\begin{aligned} -\Delta_p u &= \lambda\, a(x) |\, u|^{p-2} u + \lambda\, b(x)\, |u|^{\alpha-1}\, |v|^{\beta+1}\, u, \quad x \in \Omega, \\ -\Delta_q v &= \lambda\, d(x)\, |v|^{q-2} v + \lambda\, b(x)\, |u|^{\alpha+1}\, |v|^{\beta-1}\, v, \quad x \in \Omega, \end{aligned} \tag{1.2$_\lambda$}$$

corresponding to the positive principal eigenvalue λ_1 (see Section 2, Theorem 2.3).

Recently, many works have appeared about semilinear and quasilinear elliptic systems. There is a great variety of applications where such systems are involved; see [2, 6]–[9] and the references therein.

Some of the main difficulties in the study of these systems arise from the lack of homogeneity of the unperturbed problem $(1.2)_\lambda$, and the interest is not only in proving the existence of a solution, but also in investigating whether this solution is *semitrivial* (i.e., of the form $(u,0)$ or $(0,v)$) or not.

The paper is organized in four sections. In Section 2, the notion of the weak solution is defined and we recall some known results from [8] about problems $(1.1)_\lambda$ and $(1.2)_\lambda$. In Section 3, we state the main multiplicity results for the system $(1.1)_\lambda$. In Section 4, the existence of at least one nonsemitrivial solution for the problem $(1.1)_{\lambda_1}$ is proved.

This work improves the study of the quasilinear elliptic systems done in [7, 8]. It also generalizes the results for the scalar equation from [5] to the case of the systems. In fact the procedure here is based on the arguments developed in [5]. Let us note that some of the results in the present work may be considered new even for the bounded domain case. To be precise, as far as we know, the result concerning the multiplicity of nonsemitrivial solutions is new even for bounded domains.

2. Space settings—The eigenvalue problem $(1.2)_\lambda$

Consider the product space $Z := \mathcal{D}^{1,p}(\Omega) \times \mathcal{D}^{1,q}(\Omega)$ equipped with the norm $||z||_Z := ||u||_{1,p} + ||v||_{1,q},\ z = (u,v) \in Z$, where

$$||u||_{1,p} := \Big(\int_\Omega |\nabla u|^p \, dx \Big)^{1/p}.$$

Let us note that, if Ω is a bounded domain, then $||u||_{1,p}$ is a norm equivalent to the standard Sobolev norm in the space $W_0^{1,p}(\Omega)$; i.e., $W_0^{1,p}(\Omega) = \mathcal{D}^{1,p}(\Omega)$. However, in the case when Ω is an unbounded domain, $||u||_{1,p}$ is the norm of the space $\mathcal{D}^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega) \subsetneqq \mathcal{D}^{1,p}(\Omega)$. For more details we refer to the classical book [1].

We introduce the functionals J, D, B, $M : Z \to \mathbb{R}$ in the following way:

$$\begin{aligned}
J(u,v) &:= \frac{\alpha+1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{\beta+1}{q} \int_\Omega |\nabla v|^q \, dx, \\
D(u,v) &:= \frac{\alpha+1}{p} \int_\Omega a(x)|u|^p \, dx + \frac{\beta+1}{q} \int_\Omega d(x)|v|^q \, dx, \\
B(u,v) &:= \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1} \, dx, \\
M(u,v) &:= \int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1} \, dx.
\end{aligned}$$

Lemma 2.1. *The functionals J, D, B, and M are well defined. Moreover, J is continuous and D, B, and M are compact.*

Proof. The fact that the functionals J, D, B, and M are well defined may be proved by applying Hölder's inequality. The continuity of J and the compactness of D and B is proved in the works [4, Lemma 2.3] and [6, Lemma 5.1]. We prove the compactness of the operator M, with $\Omega = \mathbb{R}^N$. The other cases are similar.

Let (u_n, v_n) be a bounded sequence in Z. Hence (u_n, v_n) converges weakly (up to a subsequence) to (u_0, v_0) in Z; i.e., $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,p}$ and $v_n \rightharpoonup v_0$ in $\mathcal{D}^{1,q}$, as $n \to \infty$. It follows that

$$|M(u_n, v_n) - M(u_0, v_0)| \leq L + K,$$

where

$$\begin{aligned} L &:= \int_{\mathbb{R}^N} |\mu(x)| \left| |u_n|^{\gamma+1} - |u_0|^{\gamma+1} \right| |v_n|^{\delta+1} \, dx, \\ K &:= \int_{\mathbb{R}^N} |\mu(x)| \, |u_0|^{\gamma+1} \left| |v_n|^{\delta+1} - |v_0|^{\delta+1} \right| dx. \end{aligned}$$

For some $R > 0$ we write $L = L_1 + L_2$, where

$$L_1 := \int_{B_R} |\mu(x)| \left| |u_n|^{\gamma+1} - |u_0|^{\gamma+1} \right| |v_n|^{\delta+1} \, dx$$

$$L_2 := \int_{\mathbb{R}^N \setminus B_R} |\mu(x)| \left| |u_n|^{\gamma+1} - |u_0|^{\gamma+1} \right| |v_n|^{\delta+1} \, dx,$$

where B_R is the ball in $\mathbb{R}^N$ centered at the origin with radius $R > 0$. Applying Hölder's inequality to I_1 we obtain

$$L_1 \leq ||\mu(x)||_{L^\infty(B_R)} |||u_n|^{\gamma+1} - |u_0|^{\gamma+1}||_{L^{\frac{p'}{\gamma+1}}(B_R)} ||v_n||^{\delta+1}_{L^{q^*}(B_R)},$$

where $1 < p' < p^*$, such that $\frac{\gamma+1}{p'} + \frac{\delta+1}{q^*} = 1$.

Since $\{(u_n, v_n)\}$ is a bounded sequence in Z it is also bounded in $\mathcal{D}^{1,p}(B_R) \times \mathcal{D}^{1,q}(B_R)$. So, passing to a subsequence if necessary, we have $u_n \to u_0$ in $L^{p'}(B_R)$, as $n \to \infty$, for any $1 < p' < p^*$. Then, we have that $|u_n|^{\gamma+1} \to |u_0|^{\gamma+1}$ in $L^{\frac{p'}{\gamma+1}}(B_R)$, for any $1 < p' < p^*$. This means that that, for n large enough, we obtain $L_1 < \epsilon$. Applying Hölder's inequality to L_2 we obtain

$$L_2 \leq ||\mu(x)||_{L^{\omega_2}(\mathbb{R}^N \setminus B_R)} |||u_n|^{\gamma+1} - |u_0|^{\gamma+1}||_{L^{\frac{p^*}{\gamma+1}}(\mathbb{R}^N \setminus B_R)} ||v_n||^{\delta+1}_{L^{q^*}(\mathbb{R}^N \setminus B_R)} < \epsilon,$$

for R sufficiently large. Therefore we get that $L < 2\epsilon$. Similarly we may prove that $K < 2\epsilon$; hence, the lemma is proved. $\square$

Next, we introduce the functionals A_λ, $I_\lambda : Z \to \mathbb{R}$ in the following way.

$$\begin{aligned} A_\lambda(u,v) &:= J(u,v) - \lambda D(u,v) - \lambda B(u,v), \\ I_\lambda(u,v) &:= A_\lambda(u,v) - \frac{1}{(\gamma+1)(\delta+1)} M(u,v). \end{aligned}$$

Lemma 2.2. *The functionals A_λ and I_λ are well defined, and they are weakly lower semicontinuous.*

Proof. The proof follows from Lemma 2.1 and the convexity of J. □

We say that (u,v) is a *weak solution* of the system $(1.1)_\lambda$ if and only if (u,v) is a critical point of the functional I_λ.

Theorem 2.3. (see [2, 6, 10]) *The system $(1.2)_\lambda$ admits a positive principal eigenvalue λ_1, given by*

$$\lambda_1 = \inf_{D(u,v)+B(u,v)=1} J(u,v). \tag{2.1}$$

The associated normalized eigenfunction (u_1, v_1) belongs to Z; each component is positive and of class $\mathcal{C}^{1,\zeta}(B_r)$, for any $r > 0$, where $\zeta = \zeta(r) \in (0,1)$. In addition,

(i) the set of all eigenfunctions corresponding to the principal eigenvalue λ_1 forms a one-dimensional manifold, $E_1 \subset Z$, which is defined by $E_1 = \{(c_1 u_1, c_1^{p/q} v_1)\,;\ c_1 \in \mathbb{R}\}$.

(ii) λ_1 is the only eigenvalue of $(1.2)_\lambda$ to which there corresponds a componentwise positive eigenfunction.

(iii) λ_1 is isolated in the following sense: there exists $\eta > 0$, such that the interval $(0, \lambda_1 + \eta)$ does not contain any other eigenvalue than λ_1.

The following assertion follows from a more general result proved in [8].

Theorem 2.4. *The principal eigenvalue $\lambda_1 > 0$ of the problem $(1.2)_\lambda$ is a bifurcation point (in the sense of Rabinowitz) of $(1.1)_\lambda$; i.e., there exists a continuum C of nontrivial solutions of $(1.1)_\lambda$ such that $(\lambda_0, 0, 0) \in \bar{C}$ and C is either unbounded in $E = \mathbb{R} \times Z$, with*

$$||(\lambda, u, v)||_E = (|\lambda|^2 + ||(u,v)||_Z^2)^{1/2},\ (\lambda, u, v) \in E,$$

or there is an eigenvalue $\hat{\lambda} \neq \lambda_0$, such that $(\hat{\lambda}, 0) \in \bar{C}$.

Moreover, there exists $\eta > 0$ small enough, such that for each $(\lambda, u, v) \in C \cap B_\eta(\lambda_1, 0)$, we have $u(x) \geq 0$ and $v(x) \geq 0$, almost everywhere in Ω.

As will be clear later, it is convenient to recall the following eigenvalue problem:

$$-\Delta_p u = \lambda\, g(x)|u|^{p-2}u, \quad x \in \Omega, \tag*{$(2.2)_\lambda$}$$

where $g(x)$ satisfies condition (Υ_1). It is known that the problem $(2.2)_\lambda$ has a positive principal eigenvalue $\lambda_{p,g}$, which can be characterized variationally. This eigenvalue is simple and isolated and it is the only one having a positive eigenfunction $\phi_{p,g}$. For the details we refer to the works [2, 6].

Lemma 2.5. *Let λ be close enough to λ_1. Every nontrivial solution (u, v) of $(1.1)_\lambda$ is nonsemitrivial.*

Proof. Observe that the nonzero component of any semitrivial solution of the system $(1.1)_\lambda$ corresponds to an eigenfunction of $(2.2)_\lambda$, either with $g(x) = a(x)$ or with $g(x) = d(x)$. So it suffices to prove that $\lambda_1 < \min\{\lambda_{p,a}, \lambda_{q,d}\}$. Assume the opposite. Then the system $(1.2)_{\lambda_{p,a}}$ ($(1.2)_{\lambda_{q,d}}$, respectively) would have a solution $(\phi_{p,g}, 0)$ ($(0, \phi_{q,d})$, respectively). From the variational characterization (2.1) of the eigenvalue λ_1 this is a contradiction, and the proof is completed. □

3. MULTIPLICITY RESULTS

In this section we prove the multiplicity of solutions of the system $(1.1)_\lambda$. Before this we introduce some notation and prove some lemmas describing certain properties of the continuum C from Theorem 2.4. Let Λ_λ be the Nehari manifold associated with $(1.1)_\lambda$; i.e.,

$$\Lambda_\lambda = \big\{(u, v) \in Z : \langle I'_\lambda(u, v), (u, v)\rangle = 0\big\}.$$

Clearly, Λ_λ is closed in Z. Next, we define the following disjoint subsets of Λ_λ:

$$\begin{aligned}
\Lambda_\lambda^+ = \Big\{(u, v) \in \Lambda_\lambda : \int_\Omega \Big[&|\nabla u|^p + |\nabla v|^q - \lambda\, a(x)|u|^p - \lambda\, d(x)|v|^q \\
&-2\,\lambda\, b(x)\,|u|^{\alpha+1}\,|v|^{\beta+1}\Big]\,dx > \int_\Omega \mu(x)\,|u|^{\gamma+1}\,|v|^{\delta+1}\,dx\Big\},\\
\Lambda_\lambda^0 = \Big\{(u, v) \in \Lambda_\lambda : \int_\Omega \Big[&|\nabla u|^p + |\nabla v|^q - \lambda\, a(x)|u|^p - \lambda\, d(x)|v|^q \\
&-2\,\lambda\, b(x)\,|u|^{\alpha+1}\,|v|^{\beta+1}\Big]\,dx = \int_\Omega \mu(x)\,|u|^{\gamma+1}\,|v|^{\delta+1}\,dx\Big\},\\
\Lambda_\lambda^- = \Big\{(u, v) \in \Lambda_\lambda : \int_\Omega \Big[&|\nabla u|^p + |\nabla v|^q - \lambda\, a(x)|u|^p - \lambda\, d(x)|v|^q \\
&-2\,\lambda\, b(x)\,|u|^{\alpha+1}\,|v|^{\beta+1}\Big]\,dx < \int_\Omega \mu(x)\,|u|^{\gamma+1}\,|v|^{\delta+1}\,dx\Big\}.
\end{aligned}$$

Remark 3.1. (i) Condition $(\mathcal{H}_1)$ implies that the sets Λ_λ^+, Λ_λ^0, and Λ_λ^- may be expressed as

$$\Lambda_\lambda^+(\Lambda_\lambda^0, \Lambda_\lambda^-, resp.) = \Big\{(u,v) \in \Lambda_\lambda : \int_\Omega \mu(x)|u|^{\gamma+1}|v|^{\delta+1}dx < (=, >, resp.) 0\}.$$

(ii) Condition $\mu^+ \not\equiv 0$ implies that $\Lambda_\lambda^- \neq \varnothing$.

(iii) Any critical point of I_λ restricted on Λ_λ is a critical point of I_λ with respect to the whole space Z.

Remark 3.2. Note that the condition (Υ_6) implies that $(u_1, v_1) \notin \Lambda_\lambda^-$.

Lemma 3.3. *The solution branch C bends to the right of λ_1 at $(\lambda_1, 0, 0)$; i.e., there exists $\rho > 0$, such that $(\lambda, u, v) \in C$ and $||u||_{1,p} + ||v||_{1,q} < \rho$, implies $\lambda > \lambda_1$.*

Proof. Assume the opposite. Then, there exists a sequence $(\lambda_n, u_n, v_n) \in C$, such that $(u_n, v_n) \to 0$ in Z, $\lambda_n \leq \lambda_1$, $\lambda_n \to \lambda_1$, and

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u_n|^p - \lambda_n a(x)|u_n|^p - \lambda_n b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\Big)dx \\
&= \int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}dx, \qquad (3.1)\\
&\int_\Omega \Big(|\nabla v_n|^q - \lambda_n d(x)|v_n|^q - \lambda_n b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\Big)dx \\
&= \int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}dx. \qquad (3.2)
\end{aligned}$$

We introduce the sequences $\tilde{u}_n$ and $\tilde{v}_n$ in the following way:

$$\tilde{u}_n := \frac{u_n}{(||u_n||_{1,p}^p + ||v_n||_{1,q}^q)^{1/p}} \quad \text{and} \quad \tilde{v}_n := \frac{v_n}{(||u_n||_{1,p}^p + ||v_n||_{1,q}^q)^{1/q}}\,. \qquad (3.3)$$

The sequences $\tilde{u}_n$ and $\tilde{v}_n$ are bounded. Indeed, we have that

$$||\tilde{u}_n||_{1,p}^p + ||\tilde{v}_n||_{1,q}^q = 1, \quad \text{for every} \quad n \in \mathbb{N}.$$

Thus, we may consider that there exists some point $(\tilde{u}_0, \tilde{v}_0) \in Z$, such that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}_0, \tilde{v}_0)$ (weakly) in Z. Condition $(\mathcal{H})$ implies, also, that

$$\frac{|u_n|^{\alpha+1}\,|v_n|^{\beta+1}}{||u_n||_{1,p}^p + ||v_n||_{1,q}^q} = |\tilde{u}_n|^{\alpha+1}\,|\tilde{v}_n|^{\beta+1}, \qquad (3.4)$$

for every $n \in \mathbb{N}$. Moreover, the range of the exponents implies that

$$\frac{\int_\Omega \mu(x)|u_n|^{\gamma+1}|v_n|^{\delta+1}\,dx}{||u_n||_{1,p}^p + ||v_n||_{1,q}^q} \leq \frac{||\mu||_{\omega_2}||u_n||_{p^*}^{\gamma+1}||v_n||_{q^*}^{\delta+1}}{||u_n||_{1,p}^p + ||v_n||_{1,q}^q} \to 0, \qquad (3.5)$$

as $(u_n, v_n) \to 0$ in Z. Using now relations (3.4) and (3.5), equations (3.1) and (3.2) imply that

$$\begin{aligned} \int_\Omega \Big(|\nabla \tilde{u}_n|^p - \lambda_n\, a(x)\, |\tilde{u}_n|^p - \lambda_n\, b(x)\, |\tilde{u}_n|^{\alpha+1} |\tilde{v}_n|^{\beta+1} \Big)\, dx &\to 0, \\ \int_\Omega \Big(|\nabla \tilde{v}_n|^q - \lambda_n\, d(x)\, |\tilde{v}_n|^q - \lambda_n\, b(x)\, |\tilde{u}_n|^{\alpha+1} |\tilde{v}_n|^{\beta+1} \Big)\, dx &\to 0, \end{aligned}$$

as $n \to \infty$. Moreover, the compactness of the operators D and B (see Lemma 2.1) implies that

$$\begin{aligned} \lambda_n \int_\Omega a(x)\, |\tilde{u}_n|^p\, dx &\to \lambda_1 \int_\Omega a(x)\, |\tilde{u}_0|^p\, dx, \\ \lambda_n \int_\Omega d(x)\, |\tilde{v}_n|^q\, dx &\to \lambda_1 \int_\Omega d(x)\, |\tilde{v}_0|^q\, dx, \\ \lambda_n \int_\Omega b(x)\, |\tilde{u}_n|^{\alpha+1} |\tilde{v}_n|^{\beta+1}\, dx &\to \lambda_1 \int_\Omega b(x)\, |\tilde{u}_0|^{\alpha+1} |\tilde{v}_0|^{\beta+1}\, dx, \end{aligned}$$

as $n \to \infty$. Hence, $(\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}_0, \tilde{v}_0) \not\equiv (0,0)$ (strongly) in Z and $(\tilde{u}_0, \tilde{v}_0)$ is a solution of $(1.2)_{\lambda_1}$. Then, the simplicity of λ_1, see Theorem 2.3(i), implies that $\tilde{u}_0 = k^p u_1$ and $\tilde{v}_0 = k^q v_1$, for some positive constant k. Multiplying equations (3.1) and (3.2) by $(\alpha+1)/p$ and $(\beta+1)/q$, respectively, adding the resulting equations, and using condition $(\mathcal{H})$, we deduce that

$$A_{\lambda_n}(u_n, v_n) = c_1 \int_\Omega \mu(x)\, |u_n|^{\gamma+1} |v_n|^{\delta+1}\, dx, \quad \text{for any } n \in \mathbb{N}, \tag{3.6}$$

where $c_1 = \frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)}$. From the variational characterization (2.1) of the eigenvalue λ_1 and from equation (3.6) we conclude that

$$0 \le \lim_{n\to\infty} c_1 \int_\Omega \mu(x)\, |\tilde{u}_n|^{\gamma+1} |\tilde{v}_n|^{\delta+1}\, dx = c_2 \int_\Omega \mu(x)\, |u_1|^{\gamma+1} |v_1|^{\delta+1}\, dx < 0,$$

for some $c_2 = c_2(c_1, k) > 0$, which is a contradiction, and the proof is completed. □

Corollary 3.4. *Suppose that $(\lambda, u, v) \in C$, such that (λ, u, v) is close enough to $(\lambda_1, 0, 0)$; then $(u, v) \in \Lambda^+_\lambda$.*

Proof. Let $(\lambda_n, u_n, v_n) \in C$, such that $(u_n, v_n) \to 0$ in Z and $\lambda_n \to \lambda_1$. Then, using the same arguments as in Lemma 3.3 we may prove that

$$\int_\Omega \mu(x)\, |u_n|^{\gamma+1} |v_n|^{\delta+1}\, dx < 0, \quad \text{for } n \text{ large enough;}$$

i.e., $(u_n, v_n) \in \Lambda^+_\lambda$, when n is large enough. □

To apply variational methods the next two lemmas are useful.

Lemma 3.5. *There exists $\lambda^0 > \lambda_1$, such that for every $\lambda \in (\lambda_1, \lambda^0)$ the set Λ_λ^- is closed in Z.*

Proof. We have to prove that for any $(u_n, v_n) \in \Lambda_\lambda^-$ such that $(u_n, v_n) \to (u, v)$ in Z we have $(u, v) \in \Lambda_\lambda^-$, when $\lambda \in (\lambda_1, \lambda^0)$. Due to the characterization of Λ_λ^- in Remark 3.1(i) this will be the case if

$$\int_\Omega \mu(x)\, |\tilde{u}_n|^{\gamma+1}\, |\tilde{v}_n|^{\delta+1}\, dx \to \int_\Omega \mu(x)\, |u|^{\gamma+1}\, |v|^{\delta+1}\, dx < 0.$$

Assume that such a λ^0 does not exist. Then, there exists a sequence (λ_n, u_n, v_n), with $(u_n, v_n) \in \Lambda_\lambda^-$, such that $\lambda_n \to \lambda_1$ and $\int_\Omega \mu(x)\, |u_n|^{\gamma+1}\, |v_n|^{\delta+1}\, dx \to 0$. Since (u_n, v_n) is a solution for the system $(1.1)_{\lambda_n}$ we have that

$$\begin{aligned} \int_\Omega |\nabla u_n|^p\, dx - \lambda_n \int_\Omega a(x)|u_n|^p\, dx - \lambda_n \int_\Omega b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\, dx &\to 0, \\ \int_\Omega |\nabla v_n|^q\, dx - \lambda_n \int_\Omega d(x)|v_n|^q\, dx - \lambda_n \int_\Omega b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\, dx &\to 0. \end{aligned}$$

We may prove, as in Lemma 3.3, that the sequences $\{\tilde{u}_n\}$ and $\{\tilde{v}_n\}$ converge strongly to some $(\tilde{u}_0, \tilde{v}_0)$, and the following relations are valid: $\tilde{u}_0 = k^p u_1$ and $\tilde{v}_0 = k^q v_1$, for some positive constant k. The compactness of the operator M (see Lemma 2.1) implies that

$$0 \le \lim_{n\to\infty} c_3 \int_\Omega \mu(x)\, |\tilde{u}_n|^{\gamma+1}\, |\tilde{v}_n|^{\delta+1}\, dx = c_4 \int_\Omega \mu(x)\, |u_1|^{\gamma+1}\, |v_1|^{\delta+1}\, dx < 0,$$

for some positive constants c_3 and c_4, which leads to a contradiction, and so Λ_λ^- is closed in Z. □

Using condition $(\mathcal{H})$, we observe that

$$\begin{aligned} A_\lambda(u, v) \;=\; & \frac{\alpha+1}{p} \int_\Omega \Big(|\nabla u|^p - \lambda\, a(x)|u|^p - \lambda\, b(x)\, |u|^{\alpha+1}\, |v|^{\beta+1} \Big)\, dx \\ & + \frac{\beta+1}{q} \int_\Omega \Big(|\nabla v|^q - \lambda\, d(x)|v|^q - \lambda\, b(x)\, |u|^{\alpha+1}\, |v|^{\beta+1} \Big)\, dx. \end{aligned}$$

Then

$$I_\lambda(u, v) = \Big[\frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)} - \frac{1}{(\gamma+1)(\delta+1)} \Big] \int_\Omega \mu(x)\, |u|^{\gamma+1}\, |v|^{\delta+1}\, dx, \tag{3.7}$$

for every $(u, v) \in \Lambda_\lambda^-$. Since $p < \gamma + 1$ or $q < \delta + 1$, we deduce that

$$I_\lambda(u, v) > 0, \quad \text{for every } (u, v) \in \Lambda_\lambda^-. \tag{3.8}$$

Lemma 3.6. *The functional I_λ satisfies the (PS) condition on Λ_λ^-, whenever λ is close enough to λ_1.*

Proof. Let the sequence $(u_n, v_n) \in \Lambda_\lambda^-$ be such that $I_\lambda(u_n, v_n) \le c$ and $I'_\lambda(u_n, v_n) \to 0$, as $n \to \infty$. We first prove that (u_n, v_n) is a bounded sequence. From the following equality,

$$
\begin{aligned}
I_\lambda(u_n, v_n) &- \langle I'_\lambda(u_n, v_n), (\frac{u_n}{p}, \frac{v_n}{q})\rangle \\
&= \Big[\frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)} - \frac{1}{(\gamma+1)(\delta+1)}\Big] M(u_n, v_n),
\end{aligned}
$$

we deduce that the quantity $M(u_n, v_n)$ is bounded, for all $n \in \mathbb{N}$. The boundedness of $I_\lambda(u_n, v_n)$ and $M(u_n, v_n)$ imply that $A_\lambda(u_n, v_n)$ must be bounded, too. Next, we claim that there exists a positive constant σ, such that

$$
\frac{A_\lambda(u_n, v_n)}{||u_n||_{1,p}^p + ||v_n||_{1,q}^q} \ge \sigma > 0, \quad \text{for every } \ n \in \mathbb{N},
$$

which would imply the boundedness of (u_n, v_n) in Z. Suppose the opposite. Then, there exists a sequence (λ_n, u_n, v_n), with $(u_n, v_n) \in \Lambda_\lambda^-$, such that $\lambda_n \to \lambda_1$ and

$$
\frac{A_{\lambda_n}(u_n, v_n)}{||u_n||_{1,p}^p + ||v_n||_{1,q}^q} = A_{\lambda_n}(\tilde{u}_n, \tilde{v}_n) \to 0,
$$

where $(\tilde{u}_n, \tilde{v}_n)$ are the sequences introduced by (3.3). The boundedness of $(\tilde{u}_n, \tilde{v}_n)$ implies that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}_0, \tilde{v}_0)$ (weakly) in Z, for some $(\tilde{u}_0, \tilde{v}_0) \in Z$. From the variational characterization (2.1) of λ_1 and Lemma 2.2 we derive that

$$
0 \le A_{\lambda_1}(\tilde{u}_0, \tilde{v}_0) \le \liminf_{n\to\infty} A_{\lambda_n}(\tilde{u}_n, \tilde{v}_n) = 0. \tag{3.9}
$$

We claim that $(\tilde{u}_0, \tilde{v}_0) \not\equiv 0$. Assume the opposite. Then, from the compactness of the functionals D and B we obtain that

$$
\lim_{n\to\infty} D(\tilde{u}_n, \tilde{v}_n) = \lim_{n\to\infty} B(\tilde{u}_n, \tilde{v}_n) = 0.
$$

Hence, from (3.9) we deduce that $(\tilde{u}_n, \tilde{v}_n) \to 0$ (strongly) in Z, which contradicts the fact that $||(\tilde{u}_n, \tilde{v}_n)||_Z = 1$, for every $n \in \mathbb{N}$.

In this case, from (3.9) we must have that $\tilde{u}_0 = k^p u_1$ and $\tilde{v}_0 = k^q v_1$, for some positive constant k. Then from hypothesis (Υ_6) we get the following contradiction:

$$
0 < \int_\Omega \mu(x)\, |\tilde{u}_n|^{\gamma+1}\, |\tilde{v}_n|^{\delta+1}\, dx \to c_5 \int_\Omega \mu(x)\, |u_1|^{\gamma+1}\, |u_1|^{\gamma+1}\, dx < 0.
$$

Hence (u_n, v_n) is a bounded sequence. Using the compactness of the functionals D, B, and M and following the procedure from [7, Lemma 2.3] we obtain that $(\tilde{u}_n, \tilde{v}_n)$ has a convergent subsequence, and the proof is completed. □

The main result of this work is the following theorem.

Theorem 3.7. *Let hypotheses* $(\mathcal{H})$ *and* $(\mathcal{H}_1)$ *and* (Υ_1)–(Υ_6) *be satisfied. Then there exists* $\lambda^* > \lambda_1$*, such that the system* $(1.1)_\lambda$ *has two nonnegative nonsemitrivial solutions, for every* $\lambda \in (\lambda_1, \lambda^*)$.

Proof. The existence of a nonsemitrivial solution, which belongs in Λ_λ^+, follows directly from Theorem 2.4 and Corollary 3.4. We prove the existence of a solution for the system $(1.1)_\lambda$, which belongs in Λ_λ^-. Consider the set Λ_λ^- equipped with the metric $d(\tilde{z}_1, \tilde{z}_2) = ||\tilde{z}_1 - \tilde{z}_2||_Z$, for every $\tilde{z}_1$ and $\tilde{z}_2$ in Λ_λ^-. Then, it is clear from Lemma 3.5, that for λ^* close to λ_1, Λ_λ^- becomes a complete metric space. On the other hand, from (3.8) we have that the functional I_λ is bounded below in Λ_λ^-. Since I_λ satisfies the (PS) condition in Λ_λ^- (see Lemma 3.6), Ekeland's variational principle implies the existence of a solution for the system $(1.1)_\lambda$. This solution is nonnegative due to the fact that $I_\lambda(|u|, |v|) = I_\lambda(u, v)$, and Lemma 2.5 implies that it is also nonsemitrivial. □

4. The system $(1.1)_{\lambda_1}$

In this section we prove the existence of a nonnegative solution for the system $(1.1)_{\lambda_1}$. Recall that the solution set $\Lambda_{\lambda_1} \subset Z$ is characterized as

$$\Lambda_{\lambda_1} = \big\{(u, v) \in Z : \langle I'_{\lambda_1}(u, v), (u, v)\rangle = 0\big\}.$$

Lemma 4.1. *The value of* $I_{\lambda_1}(u, v)$ *is nonnegative, for every* $(u, v) \in \Lambda_{\lambda_1}$.

Proof. Since $(u, v) \in \Lambda_{\lambda_1}$, we have that

$$\langle \frac{1}{p}(I_{\lambda_1})'_u(u, v), (u, v)\rangle + \langle \frac{1}{q}(I_{\lambda_1})'_v(u, v), (u, v)\rangle = 0,$$

which implies that

$$A_{\lambda_1}(u, v) - \Big[\frac{1}{p(\delta + 1)} + \frac{1}{q(\gamma + 1)}\Big] M(u, v) = 0.$$

From the variational characterization of λ_1 we deduce that $A_{\lambda_1}(u, v) \geq 0$. Hence

$$M(u, v) \geq 0, \tag{4.1}$$

for every $(u,v) \in \Lambda_{\lambda_1}$. On the other hand, as in (3.7), we obtain that

$$I_{\lambda_1}(u,v) = \Big[\frac{1}{p(\delta+1)} + \frac{1}{q(\gamma+1)} - \frac{1}{(\gamma+1)(\delta+1)}\Big]\, M(u,v). \tag{4.2}$$

So, from (4.1) and (4.2) we get the conclusion. □

Theorem 4.2. *Let the hypotheses* $(\mathcal{H})$ *and* $(\mathcal{H}_1)$ *and* (Υ_1)–(Υ_6) *be satisfied. Then the system* $(1.1)_{\lambda_1}$ *has a nonnegative nonsemitrivial solution.*

Proof. It follows from the definition of Λ_{λ_1} that it is a closed set in Z. We prove that 0 is an isolated point of Λ_{λ_1}. Indeed, assume that $(u_n, v_n) \in \Lambda_{\lambda_1}$, $(u_n, v_n) \not\equiv 0$, and $(u_n, v_n) \to 0$ in Z. Then, (u_n, v_n) satisfies (3.1) and (3.2) with $\lambda_n \equiv \lambda_1$. Exactly as in the proof of Lemma 3.3 (but writing λ_1 instead of λ_n) we arrive at a contradiction. Now, repeating the same arguments as those from the proof of Lemma 3.6, it is possible to prove that I_{λ_1} satisfies the (PS) condition on $\Lambda_{\lambda_1}\backslash\{0\}$.

Due to Lemma 4.1 and Ekeland's variational principle there exists a critical point of I_{λ_1} on $\Lambda_{\lambda_1}\backslash\{0\}$ and hence a nontrivial nonnegative solution of $(1.1)_{\lambda_1}$. This solution is nonsemitrivial due to Lemma 2.5. □

Remark 4.3. In particular, the results obtained in this work are valid in the case where $a(x) \equiv 0$ and $d(x) \equiv 0$, i.e., in the case of the following system:

$$-\Delta_p u = \lambda\, b(x)\, |u|^{\alpha-1}\, |v|^{\beta+1}\, u + \frac{1}{(\alpha+1)(\delta+1)}\, \mu(x)\, |u|^{\gamma-1} |v|^{\delta+1} u, \;\; x \in \Omega,$$

$$-\Delta_q v = \lambda\, b(x)\, |u|^{\alpha+1}\, |v|^{\beta-1}\, v + \frac{1}{(\beta+1)(\gamma+1)}\, \mu(x)\, |u|^{\gamma+1} |v|^{\delta-1} v, \;\; x \in \Omega.$$

For more details about such systems we refer to the works [7, 8, 9].

Note added in Proofs. After this work was submitted for publication, similar results to Theorem 4.2 were reported in [3].

Acknowledgments. The first author was supported by the Ministry of Education, Research Plan # MSM 235200001. The second and the third author were partially supported by the program "Thales" of Research Commitee of National Technical University of Athens. The third author was also supported by a grant # 296 from State Scholarship's Foundation, Hellenic Republic.

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