Differential and Integral Equations Volume 16, Number 12, December 2003, Pages 1505–1518

SYMMETRIC PALAIS-SMALE CONDITIONS WITH APPLICATIONS TO THREE SOLUTIONS IN TWO-BUMP DOMAINS

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(Submitted by: Takashi Suzuki)

Abstract. In this article, we prove a necessary and sufficient condition for symmetric Palais-Smale conditions, then apply it to assert the existence of three positive solutions of the equation (1.1) in an axially symmetric domain D_R in which one is axially symmetric and the other two are nonaxially symmetric.

1. Introduction

Let $N \geq 2$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$. Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u \text{ in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$ and $H_0^1(\Omega)$ is the Sobolev space in Ω with dual space $H^{-1}(\Omega)$. Associated with equation (1.1), we consider the energy functionals a, b, and J, for each $u \in H_0^1(\Omega)$:

$$a(u) = \int_\Omega (|\nabla u|^2 + u^2), \quad b(u) = \int_\Omega |u|^p, \quad J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u).$$

By Rabinowitz [19, Proposition B. 10], a, b, and J are of class $C^{1,1}$. It is well-known that the solutions of equation (1.1) are the critical points of the energy functional J.

That the existence of solutions of equation (1.1) is affected by the shape of the domain Ω has been the focus of a great deal of research in recent years. By the Rellich compactness theorem, it is easy to obtain a solution of equation (1.1) in a bounded domain. For general unbounded domains Ω,

Accepted for publication: August 2003.
AMS Subject Classifications: 35j20, 35J25, 35J60.

because of the lack of compactness, the existence of solutions of equation (1.1) is a big open question. Recently, there has been some progress in the existence and multiplicity of solutions as follows: Bahri-Lions [1], Coti Zelati [9], Chabrowski [3], Chen-Lee-Wang [6], Chen-Wang[7], Chen-Lin-Wang [8], Lien-Tzeng-Wang [17], del Pino-Felmer [11], [12], and Wang [21] used the (PS) theory to treat the existence of solutions of equation (1.1). Byeon [2], Chen-Ni-Zhou [4], Dancer [10], and Wang-Wu [22] asserted the existence of three positive solutions of semilinear elliptic equations in a dumbbell domain. Jimbo [15] and [16] asserted the existence of solutions depending on the width of the corridor of the dumbbell.

In this article we recall several known results of indexes of domains in Section 2. We then present new analyses in Section 3. The analyses are interesting on their own and will be also used in Section 4. Actually, we can use the analyses in Section 3 to assert the following: Let

$$\Omega_R = \Big[\mathbf{P}^+ + \big(0, \frac{R}{2}\big)\Big] \cup B^N(0;R) \cup \Big[\mathbf{P}^- - \big(0, \frac{R}{2}\big)\Big],$$

where $\mathbf{P}^+$ and $\mathbf{P}^-$ are defined at the beginning of Section 3. Then Ω_R is a y-symmetric, large domain Ω_R in $\mathbb{R}^N$ separated by a y-symmetric, bounded domain; there exists $R_0 > 0$, such that for $R \geq R_0$, there is a y-symmetric, positive solution of equation (1.1) in Ω_R (see Theorem 22). In Section 4, we assert the following: for $R > 0$, let Ω_R^1 and Ω_R^2 be two disjoint, bounded domains in $\mathbb{R}^N$ such that Ω_R^1 contains a ball of radius R and $\Omega_R^2 = \big\{(x,y) : (x,-y) \in \Omega_R^1\big\}$. Let Θ be a proper, achieved y-symmetric domain in $\mathbb{R}^N$ bounded in the x-direction such that $\Theta \cap \Omega_R^1 \neq \emptyset$. Let $D_R = \Omega_R^1 \cup \Theta \cup \Omega_R^2$. Then D_R is called a two-bump domain. Then there is an $R_0 > 0$ such that for $R > R_0$ the equation (1.1) on D_R has three positive solutions in which one is y-symmetric and other two are nonaxially symmetric (see Theorem 28). Since a finite dumbbell is a two-bump domain, the results of Byeon [2], Chen-Ni-Zhou [4], and Dancer [10] are the consequences of our Theorem 28.

Acknowledgment. The authors are grateful for the referee's valuable suggestions.

2. Indexes of domains

In this article, we focus on the problems in two Hilbert spaces: the whole Sobolev space $H_0^1(\Omega)$ and its closed, linear subspace $H_s(\Omega)$ defined as follows: Let $z = (x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and Ω be a domain in $\mathbb{R}^N$.

Throughout this article, we let Ω be a y-symmetric domain in $\mathbb{R}^N$ and $H_s(\Omega)$ the H^1 closure of the space $\{u \in C_0^\infty(\Omega) : u \text{ is } y\text{-symmetric}\}$ and let

$X(\Omega)$ be either the whole space $H_0^1(\Omega)$ or the y-symmetric Sobolev space $H_s(\Omega)$. Then $H_s(\Omega)$ is a closed, linear subspace of $H_0^1(\Omega)$. Let $H_s^{-1}(\Omega)$ be the dual space of $H_s(\Omega)$.

In this section, we recall several known results which will be used for later sections.

We define the Palais-Smale (simply, (PS)) sequences, (PS) values, and (PS) conditions in $X(\Omega)$ for J as follows:

Definition 1. We define

(i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\text{PS})_\beta$ sequence in $X(\Omega)$ for J if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $X^{-1}(\Omega)$ as $n \to \infty$;

(ii) $\beta \in \mathbb{R}$ is a (PS) value in $X(\Omega)$ for J if there is a $(\text{PS})_\beta$ sequence in $X(\Omega)$ for J;

(iii) J satisfies the $(\text{PS})_\beta$ condition in $X(\Omega)$ if every $(\text{PS})_\beta$ sequence in $X(\Omega)$ for J contains a convergent subsequence;

(iv) J satisfies the (PS) condition in $X(\Omega)$ if for every $\beta \in \mathbb{R}$, J satisfies the $(\text{PS})_\beta$ condition in $X(\Omega)$.

By the principle of symmetric criticality, we have a $(\text{PS})_\beta$ sequence in $X(\Omega)$ for J is a $(\text{PS})_\beta$ sequence in $H_0^1(\Omega)$ for J.

Lemma 2. *(i) For a $\mu \in X^{-1}(\Omega)$, we can extend it to be $\mu \in H^{-1}(\Omega)$ such that $||\mu||_{X^{-1}} = ||\mu||_{H^{-1}}$;*

(ii) Let $\{u_n\}$ be in $X(\Omega)$ satisfying $J'(u_n) = \text{o}(1)$ in $X^{-1}(\Omega)$; then $J'(u_n) = \text{o}(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$;

(iii) If $J'(u) = 0$ in $X^{-1}(\Omega)$, then $J'(u) = 0$ in $H^{-1}(\Omega)$.

Proof. By Palais [18, Theorem]. □

For any $\beta \in \mathbb{R}$, a $(\text{PS})_\beta$ sequence in $X(\Omega)$ for J is bounded. Moreover, a (PS) value β should be nonnegative.

Lemma 3. *Let $\beta \in \mathbb{R}$ and $\{u_n\}$ be a $(PS)_\beta$ sequence in $X(\Omega)$ for J; then there exists a positive sequence $\{c_n(\beta)\}$ such that $\|u_n\|_{H^1} \le c_n(\beta) \le c$ for each n and $c_n(\beta) = o(1)$ as $n \to \infty$ and $\beta \to 0$. Furthermore,*

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1)$$

and $\beta \ge 0$.

Proof. See Willem [20]. □

(A) Consider the constrained maximizing problem

$$\alpha_\gamma(\Omega) = (\frac{1}{2} - \frac{1}{p})\gamma(\Omega)^{\frac{2p}{2-p}},$$

where $\gamma(\Omega) = \sup \{b(u) \mid u \in X(\Omega) \setminus \{0\} \mid a(u) = 1\}$.

(B) Consider the Nehari minimization problem

$$\alpha_{\mathbf{M}}(\Omega) = \inf_{v \in \mathbf{M}(\Omega)} J(v),$$

where $\mathbf{M}(\Omega) = \{u \in X(\Omega) \backslash \{0\} : a(u) = b(u)\}$. Note that $\mathbf{M}(\Omega)$ contains every nonzero solution of equation (1.1).

We have the following useful lemmas.

Lemma 4. *Let $\beta > 0$ and $\{u_n\}$ in $X(\Omega) \backslash \{0\}$ be a sequence for J such that $J(u_n) = \beta + \mathrm{o}(1)$ and $a(u_n) = b(u_n) + \mathrm{o}(1)$. Then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $s_n = 1 + \mathrm{o}(1)$, $\{s_n u_n\}$ is in $\mathbf{M}(\Omega)$ and $J(s_n u_n) = \beta + \mathrm{o}(1)$. In particular, if $\{u_n\}$ is a (PS)$_\beta$ sequence in $X(\Omega)$ for J, then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\}$ is in $\mathbf{M}(\Omega)$ and $\{s_n u_n\}$ is a (PS)$_\beta$ sequence in $X(\Omega)$ for J.*

Proof. See Wang-Wu [22, Lemma 7]. □

Every minimizing sequence $\{u_n\}$ in $\mathbf{M}(\Omega)$ of $\alpha_{\mathbf{M}}(\Omega)$ is a $(\mathrm{PS})_{\alpha_{\mathbf{M}}(\Omega)}$ sequence in $X(\Omega)$ for J .

Lemma 5. *Let $\{u_n\}$ be in $X(\Omega)$. Then $\{u_n\}$ is a (PS)$_{\alpha_{\mathbf{M}}(\Omega)}$ sequence in $X(\Omega)$ for J if and only if $J(u_n) = \alpha_X(\Omega) + \mathrm{o}(1)$ and $a(u_n) = b(u_n) + \mathrm{o}(1)$. In particular, every minimizing sequence $\{u_n\}$ in $\mathbf{M}(\Omega)$ of $\alpha_{\mathbf{M}}(\Omega)$ is a (PS)$_{\alpha_{\mathbf{M}}(\Omega)}$ sequence in $X(\Omega)$ for J.*

Proof. By Wang-Wu [22, Lemmas 8 and 9]. □

If u achieves $\alpha_{\mathbf{M}}(\Omega)$, then u is a nonzero solution of equation (1.1).

Lemma 6. *Let $u \in \mathbf{M}(\Omega)$ such that $J(u) = \min_{v \in \mathbf{M}(\Omega)} J(v)$. Then u is a nonzero solution of equation* (1.1) *in $X(\Omega)$.*

Proof. By Wang-Wu [22, Lemma 10]. □

(C) Consider the minimax problem

$$\alpha_\Gamma(\Omega) = \inf_{g \in \Gamma(\Omega)} \max_{t \in [0,1]} J(g(t)),$$

where $e \neq 0$, $J(e) = 0$, and $\Gamma(\Omega) = \{g \in C([0,1], X(\Omega)) : g(0) = 0, g(1) = e\}$.

(D) Consider the infimum of positive (PS) values in $X(\Omega)$ for J:

$$\alpha_{\mathbf{P}}(\Omega) = \inf_{\beta \in \mathbf{P}(\Omega)} \beta,$$

where $\mathbf{P}(\Omega)$ is the set of all positive (PS) values in $X(\Omega)$ for J.

We have

Theorem 7. *Let β be a positive (PS) value in $X(\Omega)$ for J. Then*
(i) $\beta \geq \alpha_\gamma(\Omega)$; *(ii)* $\beta \geq \alpha_{\mathbf{M}}(\Omega)$; *(iii)* $\beta \geq \alpha_\Gamma(\Omega)$; *and (iv)* $\beta \geq \alpha_{\mathbf{P}}(\Omega)$.

Proof. By Wang [21]. □

Theorem 8. $\alpha_\gamma(\Omega) = \alpha_{\mathbf{M}}(\Omega) = \alpha_\Gamma(\Omega) = \alpha_{\mathbf{P}}(\Omega)$.

Proof. By Wang [21]. □

Definition 9. By Theorem 8, we conclude that the positive (PS) values $\alpha_\gamma(\Omega)$, $\alpha_\Gamma(\Omega)$, $\alpha_{\mathbf{M}}(\Omega)$, and $\alpha_{\mathbf{P}}(\Omega)$ in $X(\Omega)$ for J are the same. Any one of them is called the index of J in $X(\Omega)$ and denoted by $\alpha_X(\Omega)$. By the definition of $\alpha_{\mathbf{M}}(\Omega)$, if u is a nonzero solution of equation (1.1), then $u \in \mathbf{M}(\Omega)$. Thus, $J(u) \geq \alpha_{\mathbf{M}}(\Omega) = \alpha_X(\Omega)$. We say that a nonzero solution u of equation (1.1) in $X(\Omega)$ is a ground-state solution if $J(u) = \alpha_X(\Omega)$, and is a higher-energy solution if $J(u) > \alpha_X(\Omega)$.

Remark 1. We denote $\alpha_X(\Omega)$ by $\alpha(\Omega)$ for $X(\Omega) = H_0^1(\Omega)$ and $\alpha_X(\Omega)$ by $\alpha_s(\Omega)$ for $X(\Omega) = H_s(\Omega)$.

As a consequence of Lemma 3, for each $(\mathrm{PS})_\beta$ sequence $\{u_n\}$ in $X(\Omega)$ for J, there is a subsequence $\{u_n\}$ and u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then u is a solution in $X(\Omega)$ of equation (1.1).

Theorem 10. *We have*

(i) Let $\{u_n\}$ be a $(PS)_{\alpha_X(\Omega)}$ sequence in $X(\Omega)$ for J satisfying $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then u is a solution in $X(\Omega)$ of equation (1.1);

(ii) Let $\{u_n\}$ be a $(PS)_{\alpha_X(\Omega)}$ sequence in $X(\Omega)$ for J such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$ and u is nonzero. Then u is a positive ground-state solution in $X(\Omega)$ of equation (1.1) *and $u_n \to u$ strongly in $X(\Omega)$;*

(iii) The $(PS)_{\alpha_X(\Omega)}$ condition holds in $X(\Omega)$ for J if and only if for each $(PS)_{\alpha_X(\Omega)}$ sequence $\{u_n\}$ in $X(\Omega)$ for J, there is a subsequence $\{u_n\}$and a nonzero u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$.

Proof. By Wang-Wu [22, Lemma 21]. □

Let $\Omega^1 \subsetneqq \Omega^2$ and $\alpha_X^i = \alpha_X(\Omega^i)$ for $i = 1, 2$; then clearly $\alpha_X^2 \leq \alpha_X^1$. If $\alpha_X^2 = \alpha_X^1$, then we have the following useful results.

Theorem 11. *Let $\Omega^1 \subsetneqq \Omega^2$ and $J : X(\Omega^2) \to \mathbb{R}$ be the energy functional. Suppose that $\alpha_X^2 = \alpha_X^1$. Then*

(i) J does not satisfy the $(PS)_{\alpha_X^1}$ condition;

(ii) α_X^1 does not admit any ground-state solution;

(iii) J does not satisfy the $(PS)_{\alpha_X^2}$ condition.

Proof. By Wang-Wu [22, Lemma 22]. □

J satisfies the $(\mathrm{PS})_{\alpha_X(\Omega)}$ condition in $X(\Omega)$ if Ω is a bounded domain.

Theorem 12. *Let Ω be a bounded domain in $\mathbb{R}^N$. Then the $(PS)_{\alpha_X(\Omega)}$ condition holds in $X(\Omega)$ for J. In particular, a bounded domain is achieved (see Definition 2.4).*

Proof. By Wang-Wu [22, Lemma 25]. □

Let Ω be any unbounded domain, and let $\xi \in C^\infty([0,\infty))$ be such that $0 \le \xi \le 1$ and

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [0,1]; \\ 1 & \text{for } t \in [2,\infty). \end{cases}$$

Let

$$\xi_n(z) = \xi(\frac{2|z|}{n}). \tag{2.1}$$

Then we have the following results.

Theorem 13. *Let $Q_n = \Omega \cap B^N(0;n)$; then the following properties are equivalent:*

(i) J does not satisfy the $(PS)_{\alpha_X(\Omega)}$ condition in $X(\Omega)$ for J;

(ii) There is a $(PS)_{\alpha_X(\Omega)}$ sequence $\{u_n\}$ in $X(\Omega)$ for J such that

$$\int_{Q_n} |u_n|^p = \mathrm{o}(1);$$

(iii) There is a $(PS)_{\alpha_X(\Omega)}$ sequence $\{u_n\}$ in $X(\Omega)$ for J such that $\{\xi_n u_n\}$ is also a $(PS)_{\alpha_X(\Omega)}$ sequence $\{u_n\}$ in $X(\Omega)$ for J.

Proof. By Wang-Wu [22, Lemma 24]. □

For $k \ge 1$, $i = 1, 2, \ldots, k$, let Ω be an unbounded domain and Ω_i a proper domain in Ω such that $\Omega = \cup_{i=1}^k \Omega_i$, $\Omega_i \cap \Omega_j$ is bounded, and at least one of Ω_i is unbounded. Let $\alpha_X = \alpha_X(\Omega)$, $\alpha_X^i = \alpha_X(\Omega_i)$,

$$\mathbf{M} = \{u \in X(\Omega)\backslash\{0\} : a(u) = b(u)\},$$
$$\mathbf{M}_i = \left\{u \in H_0^1(\Omega_i)\backslash\{0\} : a(u) = b(u)\right\} \text{ for } i = 1, 2, \ldots, k.$$

Since $X(\Omega_i) \subset X(\Omega)$ and $\mathbf{M}_i \subset \mathbf{M}$, for $i = 1, 2, \ldots, k$, we have $\alpha_X \le \min\{\alpha_X^1, \alpha_X^2, \ldots, \alpha_X^k\}$.

Theorem 14. *The following properties are equivalent:*

(i) J satisfies the $(PS)_{\alpha_X}$ condition;

(ii) For every $(PS)_{\alpha_X}$ sequence $\{u_n\}$ in $X(\Omega)$ for J, there are a subsequence $\{u_n\}$ and $u \ne 0$ in $X(\Omega)$ such that $u_n \to u$ strongly in $X(\Omega)$;

(iii) For every $(PS)_{\alpha_X}$ sequence $\{u_n\}$ in $X(\Omega)$ for J, there are $c > 0$, a subsequence $\{u_n\}$, and positive integers K and n_0 such that for each $n \geq n_0$, we have

$$\int_{\Omega \cap \{|z|<K\}} |u_n|^p \geq c;$$

(iv) For every $(PS)_{\alpha_X}$ sequence $\{u_n\} \subset X(\Omega)$ for J, there is a subsequence $\{u_n\}$ such that for $\varepsilon > 0$, there is a measurable set E such that $|E| < \infty$ and $\int_{E^c} |u_n|^p dz < \varepsilon$ for each $n \in \mathbb{N}$;

(v) $\alpha_X < \min\{\alpha_X^1, \alpha_X^2, \ldots, \alpha_X^k\}$.

Proof. See Chen-Lin-Wang [8]. □

3. Symmetric Palais-Smale conditions

First, denote the ball $B^N(z_0; s)$, the infinite plate $\mathbb{R}^N_{-\rho,\rho}$, the infinite strip $\mathbf{A}^r$, a section strip $\mathbf{A}^r_s$, upper paraboloid solid $\mathbf{P}^+$ and lower paraboloid solid $\mathbf{P}^-$ as follows:

$$B^N(z_0; s) = \{z \in \mathbb{R}^N : |z - z_0| < s\}, \quad \mathbb{R}^N_{-\rho,\rho} = \left\{(x, y) \in \mathbb{R}^N : |y| < \rho\right\},$$
$$\mathbf{A}^r = \{(x, y) \in \mathbb{R}^N : |x| < r\}, \quad \mathbf{A}^r_{-s,s} = \{(x, y) \in \mathbf{A}^r : |y| < s\},$$
$$\mathbf{P}^+ = \{(x, y) \in \mathbb{R}^N : y > |x|^2\}, \quad \mathbf{P}^- = \{(x, -y) : (x, y) \in \mathbf{P}^+\}.$$

In this section, we focus on the problems on two Hilbert spaces: the whole Sobolev space $H^1_0(\Omega)$ and its closed linear subspace $H_s(\Omega)$. Let $z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and Ω a domain in $\mathbb{R}^N$.

Definition 15. *(i)* A domain Ω in $\mathbb{R}^N$ is large if for any $r > 0$ there exists $z \in \Omega$ such that $B^N(z; r) \subset \Omega$;

(ii) Suppose that $(x, y) \in \Omega$ if and only if $(x, -y) \in \Omega$; then we call Ω a y-symmetric domain;

(iii) Let Ω be a y-symmetric domain and Θ be a y-symmetric, bounded domain in $\mathbb{R}^N$. If there exist two disjoint subdomains Ω_1 and Ω_2 of Ω such that

$$(x, y) \in \Omega_2 \text{ if and only if } (x, -y) \in \Omega_1, \qquad \Omega \setminus \bar{\Theta} = \Omega_1 \cup \Omega_2,$$

then we say that Ω is separated by Θ;

(iv) Let Ω be a y-symmetric domain in $\mathbb{R}^N$. If a function $u : \Omega \to \mathbb{R}$ satisfies $u(x, y) = u(x, -y)$ for $(x, y) \in \Omega$, then we call u a y-symmetric (axially symmetric) function.

Example 16. *(i)* Let Ω be a y-symmetric domain in $\mathbf{A}^r$, and $B^N(0; r+1)$ be the N-ball. Then Ω is separated by $B^N(0; r+1)$;

(ii) For each $\rho > 0$, let $\Omega = (\mathbb{R}^N \setminus \overline{\mathbb{R}^N_{-\rho,\rho}}) \cup \mathbf{A}^r$. Then Ω is a y-symmetric, large domain in $\mathbb{R}^N$ separated by a bounded domain $\mathbf{A}^r_\rho$;

(iii) Let $\Omega = \left[\mathbf{P}^{+} + \left(0, \frac{R}{2}\right)\right] \cup B^N(0;R) \cup \left[\mathbf{P}^{-} - \left(0, \frac{R}{2}\right)\right]$; then Ω is a y-symmetric, large domain in $\mathbb{R}^N$ separated by the bounded domain $B^N(0;R)$.

Let Ω be a y-symmetric domain in $\mathbb{R}^N$, and denote the space $H_s(\Omega)$ by the H^1 closure of the space $\{u \in C_0^\infty(\Omega) : u \text{ is } y\text{-symmetric}\}$.

Theorem 17. *If Ω is a large domain in $\mathbb{R}^N$, then $\alpha(\Omega) = \alpha(\mathbb{R}^N)$.*

Proof. See Lien-Tzeng-Wang [17, Lemma 2.5]. □

Theorem 18. *We have that*
(i) $\alpha_s(B^N(0;R)) = \alpha(B^N(0;R))$;
(ii) $\alpha_s(\mathbb{R}^N) = \alpha(\mathbb{R}^N)$;
(iii) $\alpha_s(\mathbf{A}^r_{-t,t}) = \alpha(\mathbf{A}^r_{-t,t})$;
(iv) $\alpha_s(\mathbf{A}^r) = \alpha(\mathbf{A}^r)$.

Proof. By Lien-Tzeng-Wang [17] and Theorem 12, there is a ground-state solution of equation (1.1) in $B^N(0;R)$, $\mathbb{R}^N$, $\mathbf{A}^r_{-t,t}$, and $\mathbf{A}^r$. By Gidas-Ni-Nirenberg [13] and [14] and Chen-Chen-Wang [5], every positive solution of equation (1.1) in $B^N(0;R)$, $\mathbb{R}^N$, $\mathbf{A}^r_{-t,t}$, and $\mathbf{A}^r$ is y-symmetric. □

We need the following symmetric results to assert our main result.

Theorem 19. *Suppose that Ω is a y-symmetric, large domain in $\mathbb{R}^N$ separated by a y-symmetric, bounded domain; then $\alpha_s(\Omega) \leq 2\alpha(\Omega)$.*

Proof. First, by Lien-Tzeng-Wang [17] and Gidas-Ni-Nirenberg [14], for $\Omega = \mathbb{R}^N$, we have there is a positive solution u_0 of the equation (1.1) with radial symmetry such that $J(u_0) = \alpha\left(\mathbb{R}^N\right)$. Consider the cut-off function $\eta \in C_c^\infty([0,\infty))$ such that

$$0 \leq \eta \leq 1, \quad \eta(t) = \begin{cases} 1 & \text{for } t \in [0,1] \\ 0 & \text{for } t \in [2,\infty). \end{cases}$$

Since Ω is a y-symmetric, proper, large domain in $\mathbb{R}^N$, for $n = 1, 2, \ldots$, we have that there exist sequences $\{z_n\}$ and $\{r_n\}$ such that $B^N(z_n; r_n) \subset \Omega$ and $r_n \to \infty$ as $n \to \infty$. Let $\eta_n(z) = \eta(\frac{2|z-z_n|}{r_n})$, and $u_n(z) = \eta_n(z)\, u_0(z - z_n)$. Then $u_n(z) \in H_0^1(\Omega)$, and

$$J(u_n) = J(u_0) + o(1) = \alpha\left(\mathbb{R}^N\right) + o(1), \quad a(u_n) = b(u_n) + o(1).$$

By Lemma 5 and Theorem 17, u_n is a $(\mathrm{PS})_{\alpha(\Omega)}$ sequence in $H_0^1(\Omega)$ for J. Moreover, we let $w_n = u_n(x, -y)$; then w_n is also a $(\mathrm{PS})_{\alpha(\Omega)}$ sequence in $H_0^1(\Omega)$ for J such that $\mathrm{supp} w_n \cap \mathrm{supp} u_n = \emptyset$ and $\{u_n + w_n\} \subset H_s(\Omega)$. Since $\mathrm{supp} w_n \cap \mathrm{supp} u_n = \emptyset$, we have

$$a(u_n + w_n) = \int_\Omega |\nabla(u_n + w_n)|^2 + (u_n + w_n)^2$$

$$
\begin{aligned}
&= \int_\Omega |\nabla u_n|^2 + u_n^2 + \int_\Omega |\nabla w_n|^2 + w_n^2 + 2\int_\Omega \nabla u_n \nabla w_n + 2\int_\Omega u_n w_n \\
&= a(u_n) + a(w_n),
\end{aligned}
$$

$$
b(u_n + w_n) = \int_\Omega |u_n + w_n|^p = \int_\Omega |u_n|^p + \int_\Omega |w_n|^p = b(u_n) + b(w_n).
$$

Hence,

$$
\begin{aligned}
J(u_n + w_n) &= \frac{1}{2} a(u_n + w_n) - \frac{1}{p} b(u_n + w_n) \\
&= J(u_n) + J(w_n) = 2\alpha(\Omega) + o(1).
\end{aligned}
$$

Moreover, for $\varphi \in C_c^\infty(\Omega)$ with y symmetry, we have

$$
\begin{aligned}
&\left|\left\langle J'(u_n + w_n), \varphi \right\rangle\right| = \Big| \int_\Omega \nabla(u_n + w_n)\nabla\varphi + (u_n + w_n)\varphi \\
&- \int_\Omega |u_n + w_n|^{p-2}(u_n + w_n)\varphi \Big| \\
&= \Big| \int_\Omega \nabla u_n \nabla\varphi + u_n\varphi + \int_\Omega \nabla w_n \nabla\varphi + w_n\varphi - \int_\Omega |u_n|^{p-2} u_n\varphi - \int_\Omega |w_n|^{p-2} w_n\varphi \Big| \\
&= \left|\left\langle J'(u_n), \varphi\right\rangle\right| + \left|\left\langle J'(w_n), \varphi\right\rangle\right| \le \left\|J'(u_n)\right\|_{H^{-1}} + \left\|J'(w_n)\right\|_{H^{-1}}.
\end{aligned}
$$

Therefore, $\|J'(u_n + w_n)\|_{H_s^{-1}} = o(1)$. We conclude that $\{u_n + w_n\}$ is a $(\mathrm{PS})_{2\alpha(\Omega)}$ sequence in $H_s(\Omega)$ for J. By Theorem 7 and Theorem 8, we have that $\alpha_s(\Omega) \le 2\alpha(\Omega)$. □

Then we have the following symmetric Palais-Smale condition.

Theorem 20. *Suppose that Ω is a y-symmetric, large domain in $\mathbb{R}^N$ separated by a y-symmetric, bounded domain Q. Then $\alpha_s(\Omega) < 2\alpha(\Omega)$ if and only if J satisfies the $(PS)_{\alpha_s(\Omega)}$ condition in $H_s(\Omega)$.*

Proof. Let $\alpha_s(\Omega) < 2\alpha(\Omega)$. Suppose J does not satisfy the $(\mathrm{PS})_{\alpha_s(\Omega)}$ condition. By Theorem 13, there exists a $(\mathrm{PS})_{\alpha_s(\Omega)}$ sequence $\{u_n\}$ in $H_s(\Omega)$ for J such that $\{\xi_n u_n\}$ is also a $(\mathrm{PS})_{\alpha_s(\Omega)}$ sequence in $H_s(\Omega)$ for J, where ξ_n is as in (2.1). Let $w_n = \xi_n u_n$; then by Lemma 2, we obtain

$$
J(w_n) = \alpha_s(\Omega) + \mathrm{o}(1), \quad J'(w_n) = \mathrm{o}(1) \text{ in } H^{-1}(\Omega). \tag{3.1}
$$

Since Ω is a y-symmetric domain in $\mathbb{R}^N$ separated by a bounded domain Q, there is $n_0 > 0$, such that $w_n = 0$ in $\overline{Q}$ for $n \ge n_0$ and there exist two disjoint subdomains Ω_1 and Ω_2 of Ω such that

$$
(x, y) \in \Omega_2 \text{ if and only if } (x, -y) \in \Omega_1, \qquad \Omega \setminus \overline{Q} = \Omega_1 \cup \Omega_2.
$$

Note that, for $n \geq n_0$, $w_n = w_n^1 + w_n^2$, and $w_n^1(x,y) = w_n^2(x,-y)$, where for $i = 1, 2$,

$$w_n^i(x) = \begin{cases} w_n(x) & \text{for } x \in \Omega_i, \\ 0 & \text{for } x \notin \Omega_i. \end{cases}$$

Then $w_n^i \in H_0^1(\Omega_i)$. We obtain $J\left(w_n^1\right) = J\left(w_n^2\right)$ and

$$\alpha_s(\Omega) + o(1) = J(w_n) = J\left(w_n^1\right) + J\left(w_n^2\right) = 2J\left(w_n^i\right) \text{ for } i = 1, 2,$$

or

$$J\left(w_n^i\right) = \frac{1}{2}\alpha_s(\Omega) + o(1) \text{ for } i = 1, 2.$$

By (3.1), we have

$$J'(w_n^i) = \mathrm{o}(1) \text{ in } H_0^1(\Omega_i) \text{ for } i = 1, 2.$$

Therefore $\frac{1}{2}\alpha_s(\Omega)$ is a (PS) value in $H_0^1(\Omega)$ for J. By Theorem 7 and Theorem 8,

$$\frac{1}{2}\alpha_s(\Omega) \geq \alpha(\Omega_i).$$

Since Ω and Ω_i are large domains of $\mathbb{R}^N$, by Theorem 17, we have

$$\alpha(\Omega_i) = \alpha\left(\mathbb{R}^N\right) = \alpha(\Omega).$$

Thus $\alpha_s(\Omega) \geq 2\alpha(\Omega)$, a contradiction.

Conversely, suppose that J satisfies the $(\mathrm{PS})_{\alpha_s(\Omega)}$ condition in $H_s(\Omega)$. By Theorem 19, we have $\alpha_s(\Omega) \leq 2\alpha(\Omega)$. Suppose that $\alpha_s(\Omega) = 2\alpha(\Omega)$. By the definition of large domain in $\mathbb{R}^N$, we may take a domain $\tilde{\Omega} = \Omega \setminus \overline{B^N(0;\tilde{r})}$ for some $\tilde{r} > 0$ such that $\tilde{\Omega} \subsetneqq \Omega$ and $\tilde{\Omega}$ is a proper, y-symmetric, large domain in $\mathbb{R}^N$ separated by a y-symmetric, bounded domain. By Theorem 11, we have $2\alpha\left(\mathbb{R}^N\right) = 2\alpha(\Omega) = \alpha_s(\Omega) < \alpha_s(\tilde{\Omega})$. By Theorem 19, $\alpha_s(\tilde{\Omega}) \leq 2\alpha(\tilde{\Omega}) = 2\alpha(\mathbb{R}^N)$. Hence we have $2\alpha(\mathbb{R}^N) < 2\alpha(\mathbb{R}^N)$, a contradiction. □

As a consequence of Theorem 20, we have the following result.

Theorem 21. *If Ω is a y-symmetric, large domain in $\mathbb{R}^N$ separated by a y-symmetric, bounded domain, then $\alpha(\Omega) < \alpha_s(\Omega)$.*

Proof. Since $\Omega \subsetneqq \mathbb{R}^N$, we have $\alpha_s\left(\mathbb{R}^N\right) \leq \alpha_s(\Omega)$. Assume that $\alpha_s\left(\mathbb{R}^N\right) = \alpha_s(\Omega)$; then by Theorem 11, J does not satisfy the $(\mathrm{PS})_{\alpha_s(\Omega)}$ condition in $H_s(\Omega)$ for J. By Theorem 17, $\alpha\left(\mathbb{R}^N\right) = \alpha(\Omega)$; by Theorem 18, $\alpha_s\left(\mathbb{R}^N\right) = \alpha\left(\mathbb{R}^N\right)$; and by Theorem 20, $2\alpha(\Omega) \leq \alpha_s(\Omega)$. We conclude that

$$2\alpha\left(\mathbb{R}^N\right) = 2\alpha(\Omega) \leq \alpha_s(\Omega) = \alpha_s\left(\mathbb{R}^N\right) = \alpha\left(\mathbb{R}^N\right),$$

a contradiction. □

Consider the y-symmetric, large domain Ω_R in $\mathbb{R}^N$ separated by a y-symmetric, bounded domain, where

$$\Omega_R = [\mathbf{P}^+ + (0, \frac{R}{2})] \cup B^N(0; R) \cup [\mathbf{P}^- - (0, \frac{R}{2})].$$

Then we have the following existence result.

Theorem 22. *There exists an $R_0 > 0$, such that for $R \geq R_0$, there is a positive-y-symmetric, positive solution of equation* (1.1) *in Ω_R.*

Proof. By Lien-Tzeng-Wang [17], we have that $\alpha\left(B^N(0,R)\right)$ is strictly decreasing as R is strictly increasing and $\alpha\left(B^N(0,R)\right) \searrow \alpha\left(\mathbb{R}^N\right)$ as $R \to \infty$. By Theorem 18, $\alpha\left(B^N(0,R)\right) = \alpha_s\left(B^N(0,R)\right)$ for each R. Thus, there is an $R_0 > 0$ such that $\alpha_s(\Omega_R) \leq \alpha_s\left(B^N(0,R)\right) < 2\alpha\left(\mathbb{R}^N\right) = 2\alpha(\Omega_R)$ for each $R \geq R_0$. By Theorem 10 and Theorem 20, we have that there is a y-symmetric, positive solution of the equation (1.1) in Ω_R for each $R \geq R_0$. □

4. Two-bump domains

We need the following definitions.

Definition 23. A domain in $\mathbb{R}^N$ is periodic if there exist a partition $\{P_n\}$ of Ω and points $\{y_n\}$ in $\mathbb{R}^N$ satisfying the following conditions:
(i) $\{y_n\}$ forms a subgroup of $\mathbb{R}^N$;
(ii) P_0 is bounded;
(iii) $P_n = y_n + P_0$.

Typical examples of periodic domains are the infinite strip $\mathbf{A}^r$ and the whole space $\mathbb{R}^N$.

Definition 24. We say that Ω is an achieved domain in $\mathbb{R}^N$, if there is a weak solution u of equation (1.1) in Ω such that $J(u) = \alpha(\Omega)$.

Let the interior flask domain $\mathbf{F}_s^r$ be as follows:

$$B^N(z_0; s) = \{z \in \mathbb{R}^N : |z - z_0| < s\};$$
$$\mathbf{A}^r = \{(x, y) \in \mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R} : |x| < r\};$$
$$\mathbf{A}_s^r = \{(x, y) \in \mathbf{A}^r : s < y\}; \quad \mathbf{F}_s^r = \mathbf{A}_0^r \cup B^N(0; s).$$

Example 25. (i) The bounded domains in $\mathbb{R}^N$ are the achieved domains;
(ii) $\mathbb{R}^N$ is an achieved domain;
(iii) The periodic domains in $\mathbb{R}^N$ are the achieved domains. In particular, the infinite strip $\mathbf{A}^r$ is an achieved domain in $\mathbb{R}^N$;
(iv) There exists an $s_0 > 0$ such that $\mathbf{F}_s^r$ is an achieved domain in $\mathbb{R}^N$ if $s > s_0$.

Proof. (i) By Theorem 12. (ii) and (iii) follow from Lien-Tzeng-Wang [17]. (iv) follows from Chen-Wang [7]. □

Throughout this section, let Θ be a proper, achieved, y-symmetric domain in $\mathbb{R}^N$ bounded in the x direction, such as bounded domains and the infinite strip $\mathbf{A}^r$. For $R>0$, let Ω_R^1 and Ω_R^2 be two disjoint, bounded domains in $\mathbb{R}^N$ such that Ω_R^1 contains a ball of radius R and $\Omega_R^2 = \left\{(x,y) : (x,-y) \in \Omega_R^1\right\}$. Let Θ be a proper, achieved, y-symmetric domain in $\mathbb{R}^N$ bounded in the x direction such that $\Theta \cap \Omega_R^1 \neq \emptyset$. Let $D_R = \Omega_R^1 \cup \Theta \cup \Omega_R^2$. Then D_R is called a two-bump domain.

Here are some examples of two-bump domains.

Example 26. (i) Let $t>R>r>0$. The bounded dumbbell domain D_R^1 is a two-bump domain, where $D_R^1 = B^N\left((0,-t),R\right) \cup \mathbf{A}^r_{-t,t} \cup B^N\left((0,t),R\right).$
(ii) Let $t>R>r>0$. The unbounded dumbbell domain D_R^2 is a two-bump domain, where $D_R^2 = B^N\left((0,-t),R\right) \cup \mathbf{A}^r \cup B^N\left((0,t),R\right).$

Then we have

Theorem 27. *For all* $R>0$, *we have*
(i) $\alpha(\Theta) \geq \alpha(D_R) > \alpha\left(\mathbb{R}^N\right);$
(ii) *J satisfies the* $(PS)_{\alpha_X(D_R)}$ *condition in* $X(D_R)$.

Proof. By Theorem 11 and Theorem 12, it suffices to assume that Θ is unbounded.

(i) Since $\Theta \subset D_R \subsetneqq \mathbb{R}^N$, we have $\alpha(\Theta) \geq \alpha(D_R) \geq \alpha(\mathbb{R}^N)$. Suppose that $\alpha(D_R) = \alpha(\mathbb{R}^N)$; by Theorem 11, J does not satisfy the $(\mathrm{PS})_{\alpha(D_R)}$ condition. By Theorem 13, there exists a sequence $\{u_n\}$ in $H_0^1(D_R)$ such that $\{u_n\}$ and $\{\xi_n u_n\}$ are the $(\mathrm{PS})_{\alpha(D_R)}$ sequence $\{u_n\}$ for J, where ξ_n is as in (2.1). Let $w_n = \xi_n u_n$; then

$$J(w_n) = \alpha(D_R) + \mathrm{o}(1), \quad J'(w_n) = \mathrm{o}(1) \text{ in } H^{-1}(D_R).$$

Since $D_R = \Omega_R^1 \cup \Theta \cup \Omega_R^2$ is a y-symmetric domain in $\mathbb{R}^N$ separated by a bounded domain, there exists $n_0 > 0$ such that for $n \geq n_0$, $w_n \in H_0^1(\Theta)$, $J(w_n) = \alpha(D_R) + \mathrm{o}(1)$, and $a(w_n) = b(w_n) + o(1)$. By Lemma 4, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n w_n\}$ is in $\mathbf{M}(\Theta)$ and $\{s_n w_n\}$ is a $(\mathrm{PS})_{\alpha(D_R)}$ sequence in $X(\Theta)$ for J. Thus $\alpha(\Theta) \leq \alpha(D_R)$. We then conclude that $\alpha(\Theta) = \alpha(D_R) = \alpha\left(\mathbb{R}^N\right)$. However, since Θ is a proper, achieved, y-symmetric domain in $\mathbb{R}^N$, by Theorem 11, $\alpha(\Theta) > \alpha\left(\mathbb{R}^N\right)$. This is a contradiction. Thus $\alpha(\Theta) \geq \alpha(D_R) > \alpha\left(\mathbb{R}^N\right)$ for all $R>0$.

(*ii*) It suffices to prove the case $X(D_R) = H_0^1(D_R)$. Since Ω_R^1, Θ, and Ω_R^2 are achieved, by Theorem 11,

$$\alpha(D_R) < \min\left\{\alpha(\Omega_R^1), \alpha(\Theta), \alpha(\Omega_R^2)\right\}.$$

By Theorem 14, J satisfies the $(\mathrm{PS})_{\alpha(D_R)}$ condition in $H_0^1(D_R)$. □

We apply Theorem 21 and Theorem 27 to prove the following result.

Theorem 28. *There is an* $R_0 > 0$ *such that for* $R > R_0$ *the equation* (1.1) *on* D_R *has three positive solutions in which one is* y*-symmetric and other two are nonaxially symmetric.*

Proof. Take $\rho > 0$ such that $\Omega = (\mathbb{R}^N \setminus \overline{\mathbb{R}^N_{-\rho,\rho}}) \cup \Theta$ is connected. Then Ω is a y-symmetric, large domain in $\mathbb{R}^N$ separated by a bounded domain. By Theorem 21, we have $\alpha\left(\mathbb{R}^N\right) = \alpha(\Omega) < \alpha_s(\Omega)$. By Lien-Tzeng-Wang [17], we have that $\alpha\left(B^N(0,R)\right)$ is strictly decreasing as R is strictly increasing and $\alpha\left(B^N(0,R)\right) \searrow \alpha\left(\mathbb{R}^N\right)$ as $R \to \infty$. Take $R_0 > 0$, such that for $R > R_0$

$$\alpha\left(\mathbb{R}^N\right) < \alpha\left(B^N(0,R)\right) < \alpha_s(\Omega). \tag{4.1}$$

Take a ball $B^N((x_R, y_R), R)$ such that $B^N((x_R, y_R), R) \subsetneq D_R$; by Theorem 11 and Theorem 27, we conclude that

$$\alpha\left(\mathbb{R}^N\right) < \alpha(D_R) < \alpha(B^N((x_R, y_R), R)) = \alpha\left(B^N(0,R)\right). \tag{4.2}$$

Therefore, by (4.1)–(4.2) and $D_R \subset \Omega$, we have

$$\alpha(D_R) < \alpha\left(B^N(0,R)\right) < \alpha_s(\Omega) \le \alpha_s(D_R). \tag{4.3}$$

Thus,

$$\alpha(D_R) < \alpha_s(D_R) \tag{4.4}$$

By Theorem 27 (*ii*), there are a y-symmetric, positive solution u_1 and a positive solution u_2 of equation (1.1) in the domain D_R for $R > R_0$ such that $J(u_1) = \alpha_s(D_R)$ and $J(u_2) = \alpha(D_R)$. Let $u_3(x,y) = u_2(x,-y)$; then u_3 is the third positive solution. By (4.4), u_1, u_2, and u_3 are different. Moreover, u_1 is a y-symmetric, positive solution while both u_2, and u_3 are nonaxially symmetric, positive solutions of equation (1.1) in the domain D_R. □

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