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GENERATION AND METASTABILITY OF PATTERNS FOR A CLASS OF LOCAL AND NONLOCAL EVOLUTION EQUATIONS

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Abstract. In this paper we use the comparison method to study the generation and metastability of patterns for a class of local and nonlocal nonlinear evolution equations. We show that for a typical initial datum, the pattern generated can last for a very long time, but is eventually destroyed.

1. Introduction

In this paper, we consider the one-space-dimensional evolution equation

$$u_t(x,t) = \mathcal{A}[u(\cdot,t)](x)\ , \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

where $\mathcal{A}$ is a nonlinear operator which is independent of the time t, maps functions of a space variable $\cdot$ to functions of x.

Assume that $\mathcal{A}$ is translation invariant; namely, for any $h \in \mathbb{R}$ and any function $u(x)$

$$\mathcal{A}[u(\cdot + h)](x) = \mathcal{A}[u(\cdot)](h + x), \quad x \in \mathbb{R}.$$

With this translation invariance, $\mathcal{A}$ maps constant functions to constant functions, so that, denoting by $\mathbf{1}$ the function identically equal to 1, there is a function $F(\cdot)$ such that

$$\mathcal{A}[c\mathbf{1}] = F(c)\mathbf{1}\ , \quad c \in \mathbb{R}. \tag{1.2}$$

We assume that F has the following properties:

$$F \in C^1(\mathbb{R}), \quad F(0) = F(1) = 0, \quad F'(0) < 0, \quad F'(1) < 0. \tag{1.3}$$

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Model (1.1) was first introduced by Chen in [6]. As pointed out by him, this model includes many important models such as the Allen-Cahn equation (e.g., see [5], [8], [9], [10], [11], and [12])

$$u_t = \triangle u - f(u), \tag{1.4}$$

the neural network model (see [7])

$$u_t = -u + J * S(u),$$

the Ising model (see [14], [15], [16], and [17])

$$u_t = -u + \tanh(\beta(J * u)),$$

and the model proposed in [1]

$$u_t = J * u - u - g(u). \tag{1.5}$$

What is common to these models is the "bistable" character; i.e., $u = \mathbf{0}$ and $\mathbf{1}$ are stable equilibrium solutions. Indeed, (1.3) implies that if we consider only x-independent solutions of (1.1), these equilibrium solutions are local attractors. Bearing this in mind, one can easily imagine the following scenario: for an initial function $\phi(x)$, which takes values between 0 and 1, the solution $u(x,t)$ of (1.1) is pulled towards 0 and 1, respectively, as much as allowed by the "smoothing effect" of the operator $\mathcal{A}$ (such as the familiar Laplace $\triangle$ in (1.4)), generating or exaggerating a spatial structure which we call a **pattern**. The purpose of this paper is to prove this generation and the evolution of patterns of (1.1). To detect patterns more easily and to study their evolution more conveniently, we rescale space via $y = \epsilon x$, $\epsilon > 0$ small, and consider $v(y,t) = u(x,t)$. Then if u satisfies (1.1), v satisfies

$$v_t(y,t) = \mathcal{A}[v(\epsilon\cdot,t)](\frac{y}{\epsilon}). \tag{1.6}$$

For this reason, we shall focus on the solution $u_\epsilon(x,t)$ of the following initial value problem:

$$\begin{cases} u_t(x,t) &= \mathcal{A}[u(\epsilon\cdot,t)](\frac{x}{\epsilon}), \\ u(x,0) &= \phi(x), \quad 0 \le \phi(x) \le 1, \end{cases} \tag{1.7}$$

where $\epsilon > 0$ is small and $\phi(x)$ does not depend on ϵ and is reasonably smooth.

Under certain conditions on the operator $\mathcal{A}$, we shall prove

- **Generation of patterns.** For generic initial value ϕ, at time $t = O(|\ln\epsilon|)$, the shape of u_ϵ is close to a step function: $u_\epsilon \approx 0$ or 1 except in the "interfacial regions" whose thickness is of order $O(\sqrt{\epsilon}|\ln\epsilon|)$ (see Theorem 2.1 and Remark 2.3).

At this moment, we reset $t = 0$ and say that the (new) initial datum has a "transition layer structure."

Now a natural question arises:

(Q) Assuming that the initial datum has a "transition layer structure" as defined above, what will happen to the solution $u_\epsilon(x,t)$ of (1.7) for ϵ small enough?

It turns out that the answer depends crucially on the traveling-wave solution of (1.1). A traveling-wave solution is one of the form $U(x - ct)$, c being a constant, that satisfies

$$\lim_{\xi\to-\infty} U(\xi) = 0, \lim_{\xi\to\infty} U(\xi) = 1. \tag{1.8}$$

Under the standing assumptions (A1–A4) (see below), Chen [6] proved the existence of traveling-wave solutions which also satisfy

$$U \in C^1(\mathbb{R}), \quad U'(\xi) > 0, \quad \lim_{|\xi|\to\infty} U'(\xi) = 0. \tag{1.9}$$

We assume $c = 0$ ($c \neq 0$ is presumably the easy case, and we shall not deal with this case here due to the limitation of space).

To answer (Q), we prove

- **Persistence of patterns.** If the stationary traveling-wave solution U of (1.1) has an algebraic convergence rate at $x = \pm\infty$, then the pattern generated can change only slightly in the time scale $O(\epsilon^{-\delta})$ for some $\delta > 0$. If the stationary traveling-wave solution has an exponential convergence rate at $x = \pm\infty$, then the pattern generated can change only slightly in the time scale $O(e^{\frac{c}{\epsilon}})$ for some $c > 0$ (see Theorems 3.1 and 3.3).
- **Annihilation of patterns.** The life span of patterns is at most of order $O(e^{\frac{c}{\epsilon}})$ for some $c > 0$, and in some cases (e.g., the "pure nonlocal" case), it is of the algebraic order $O(\epsilon^{-\sigma})$ for some $\sigma > 0$ (see Theorems 4.1 and 4.3, Remarks 4.2 and 4.5).

Thus we know that the pattern has only *metastability*.

To study the metastability of patterns, two methods have been widely used, i.e., the energy method (e.g., see [3]) and the geometric method (e.g., see [4]). In this paper, we use another method—the comparison method, which is developed by X. Wang in [18] to study the metastability of the convolution model (1.5). For detailed discussion of this model, see [1], [13], and [18]. The mechanism of this method is more elementary compared with the previous two methods we just mentioned. Roughly speaking, if for an equation the comparison principle holds, then this method will apply. But the construction of appropriate sub- and supersolutions is not an easy task.

Moreover, different sub- and supersolutions will lead to different conclusions even for the same equation. Good construction of sub- and supersolutions can reveal some new phenomena of the model. Our method is to construct sub- and supersolutions by using the stationary traveling-wave solutions of (1.1). So the existence and properties of the traveling-wave solutions of (1.1) play an important role in our method. Such properties will appear as assumptions for (1.1). But for some special models such as the Allen-Cahn equation and the models we will discuss in Section 5, these properties are true.

Following [6], we make the following assumptions:

(A1) $\mathcal{A}$ is translation invariant, and the function F in (1.2) satisfies, for some $\alpha \in (0,1)$,

$$F > 0 \text{ in } (-1,0)\cup(\alpha,1), \quad F < 0 \text{ in } (0,\alpha)\cup(1,2),$$
$$F'(0) < 0, \quad F'(1) < 0, \quad F'(\alpha) > 0.$$

(A2) There exists a positive, continuous function $\varkappa(x,t)$ defined on $[0,\infty)\times(0,\infty)$ such that if $u(x,t)$ and $v(x,t)$ satisfy $-1 \le u, v \le 2$, $u_t \ge \mathcal{A}[u]$, $v_t \le \mathcal{A}[v]$, and $u(\cdot,0) \ge v(\cdot,0)$, then

$$u(x,t) - v(x,t) \ge \varkappa(|x|,t)\int_0^1 [u(y,0)-v(y,0)]\,dy\ , \quad x\in\mathbb{R},\ t>0.$$

(A3) There exist positive constants K_1, K_2, and K_3, and a probability measure ν, such that for any $u, v \in L^\infty(\mathbb{R})$ with $-1 \le u, v \le 2$

$$|\mathcal{A}[u+v](x) - \mathcal{A}[u](x)| \le K_1 \int_{\mathbb{R}} |v(x-y)|\nu(dy)$$
$$+ K_2 \min\{\|v_{xx}\|_{C^0([x-1,x+1])}, \|v(x+\cdot)\|_{C^0([-1,1])}\}, \quad x\in\mathbb{R};$$
$$|\mathcal{A}[u+v] - \mathcal{A}[u] - \mathcal{A}'[u](v)| \le K_3\|v\|^2_{C^0(\mathbb{R})};$$
$$|\mathcal{A}'[u+v](\mathbf{1})(x) - \mathcal{A}'[u](\mathbf{1})(x)| \le K_1 \int_{\mathbb{R}} |v(x-y)|\nu(dy)$$
$$+ K_2 \min\{\|v_{xx}\|_{C^0([x-1,x+1])}, \|v(x+\cdot)\|_{C^0([-1,1])}\}, \quad x\in\mathbb{R},$$

where $\mathcal{A}'[\cdot](\cdot)$ is the Fréchet derivative of $\mathcal{A}$ defined by

$$\mathcal{A}'[u](v) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\{\mathcal{A}[u+\epsilon v] - \mathcal{A}[u]\}.$$

(A4) For any function $\phi(\cdot)$ satisfying $0 \le \phi \le 1$ and $\|\phi\|_{C^2(\mathbb{R})} < \infty$, (1.1) with initial condition $u(\cdot,0) = \phi(\cdot)$ has a unique solution satisfying

$$\sup_{t\in[0,\infty)} \|u(\cdot,t)\|_{C^2(\mathbb{R})} < \infty.$$

This paper is organized as follows. In Section 2, we prove the generation of patterns. In Section 3, we prove that the pattern generated will last for a very long time. In Section 4, we prove that the pattern will disappear eventually. In Section 5, we study a nonlocal model in detail.

2. Generation of patterns

We will show in this section that, at time $t = O(|\ln \epsilon|)$, the solution $u_\epsilon(x,t)$ of (1.7) will display a "transition layer structure" for $\epsilon > 0$ small enough. To do this, we make the following assumption about the operator $\mathcal{A}$:

(A5) There exist constants $D_1 \geq 0$ and $m_1 \geq 0$, and smooth functions g and ρ, with $g'(0) > m_1$, $g'(1) > m_1$, $\rho \geq 0$, $\rho(x) = \rho(-x)$, $\int_{\mathbb{R}} \rho \, dx = 1$, and $\int_{-\infty}^{\infty} x^2 \rho(x) \, dx < \infty$, such that

$$\mathcal{A}[u(\cdot)+v(\cdot)](x) - \mathcal{A}[u(\cdot)](x) \geq D_1 v_{xx}(x) - [g(u(x)+v(x)) - g(u(x))] + m_1 \rho * v, \tag{2.1}$$

where $\rho * v = \int_{\mathbb{R}} \rho(x-y) v(y) \, dy$. Then we have

Theorem 2.1. *Suppose that $\|\phi\|_{C^2}$ is finite. Let $u_\epsilon(x,t)$ be the unique solution of* (1.7). *Then there exist constants $\tau_0 > 0$ (depending only on $\mathcal{A}$) and $M_0 > 0$ and $M_1 > 0$, both depending on $\mathcal{A}$ and $\|\phi\|_{C^2}$, such that for small $\epsilon > 0$, if x is such that $\phi(x) \geq \alpha + M_0 \sqrt{\epsilon} |\ln \epsilon|$, then*

$$u_\epsilon(x, \tau_0 |\ln \epsilon|) \geq 1 - M_1 \epsilon. \tag{2.2}$$

To prove Theorem 2.1, we use the method in [5] and [13] with modifications. Define $F_\epsilon(u)$ by

$$F_\epsilon(u) = (1 - \kappa_\epsilon(u)) F(u) - \kappa_\epsilon(u) \frac{\alpha + \epsilon^{1/2} |\ln \epsilon| - u}{|\ln \epsilon|}, \tag{2.3}$$

where κ_ϵ is a nonnegative C_0^∞ cutoff function satisfying the following:

$\kappa_\epsilon(u) \equiv 0$, for $u \leq \alpha - \frac{\sqrt{\epsilon}}{c_1}$ and $u \geq \alpha + 3\sqrt{\epsilon}|\ln \epsilon|$,
$c_1 = \|F'\|_{L^\infty([0,1])}$, F is given by (1.2),
$\kappa_\epsilon(u) \equiv 1$, for $\alpha \leq u \leq \alpha + 2\sqrt{\epsilon}|\ln \epsilon|$,
$0 \leq \kappa_\epsilon'(u) \leq \frac{2c_1}{\sqrt{\epsilon}}$, for $u < \alpha$,
$0 \geq \kappa_\epsilon'(u) \geq -\frac{2}{\sqrt{\epsilon}|\ln \epsilon|}$, for $u > \alpha$.

Then $F_\epsilon(u)$ possesses the following properties:

(i) F_ϵ has exactly three zeroes: 0, $\alpha + \sqrt{\epsilon}|\ln \epsilon|$, and 1,
(ii) $F_\epsilon(u) \leq F(u)$, for $u \in [-1, 2]$,
(iii) $|F_\epsilon'(u)| \leq c_2$ for $u \in [-1, 2]$ and all small $\epsilon > 0$ ($c_2 > 0$ is a constant),
(iv) There exists a positive constant c_3 independent of ϵ such that
$-\, F_\epsilon(u) \leq -c_3 \sqrt{\epsilon}$, for $u \in [\alpha - \sqrt{\epsilon}/c_1, \alpha]$,

$$-F_\epsilon(u) \geq c_3\sqrt{\epsilon},\ \text{for } u \in [\alpha + 2\sqrt{\epsilon}|\ln\epsilon|, \alpha + 3\sqrt{\epsilon}|\ln\epsilon|].$$

Now, let $V(\eta, t)$ be the solution of the ordinary differential equation

$$V_t(\eta, t) = F_\epsilon(V(\eta, t)), \quad V(\eta, 0) = \eta \in \mathbb{R}^1. \tag{2.4}$$

Then we have

Lemma 2.2. (*i*) *There exists a positive constant* τ_0 *such that for all small* $\epsilon > 0$

$$V(\eta, t) \geq 1 - \epsilon^2,\ \textit{for } t \geq \tau_0|\ln\epsilon|,\ \eta \geq \alpha + 3\sqrt{\epsilon}|\ln\epsilon|. \tag{2.5}$$

(*ii*) $V(\eta, t)$ *is* C^2 *smooth in both* η *and* t, *and*

$$V_\eta > 0,\ \textit{for all } \eta \in [-1, 2],\ t \geq 0. \tag{2.6}$$

(*iii*) *There exists a positive constant* c_4 *such that for* $0 \leq t \leq \tau_0|\ln\epsilon|$, $\eta \in [-1, 2]$,

$$|V_{\eta\eta}(\eta, t)| \leq c_4/\epsilon. \tag{2.7}$$

(*iv*) *If for some* $\delta \in (0, 1)$, $\delta \leq V(\eta, t) \leq 1 - \delta$, $t \geq 0$, *then there exists a positive constant* c_5 *such that*

$$V_\eta(\eta, t) \geq c_5 > 0. \tag{2.8}$$

This lemma follows directly from Lemma 2 and Lemma 3 in [13].

Proof of Theorem 2.1. We consider a subsolution of (1.7) of the form $\underline{u}(x, t) = V(\phi(x) - \epsilon Mt, t) - h(t)$, where $M > 0$ is large and is to be determined and $h(t) > 0$ is small with $h(0) = 0$.

First we see that $\underline{u}(x, 0) = V(\phi(x), 0) = \phi(x)$. Now we want to prove that for $0 \leq t \leq \tau_0|\ln\epsilon|$, $\mathcal{L}\underline{u} \leq 0$, where τ_0 is given in Lemma 2.2 and

$$\begin{aligned}
\mathcal{L}\underline{u} =& \underline{u}_t - \mathcal{A}[\underline{u}(\epsilon\cdot, t)](\frac{x}{\epsilon}) = V_t(\phi(x) - \epsilon Mt, t) - \epsilon MV_\eta(\phi(x) - \epsilon Mt, t) - h'(t) \\
& - \mathcal{A}[V(\phi(\epsilon\cdot) - \epsilon Mt, t) - h(t)](\frac{x}{\epsilon}) \\
& = F_\epsilon(V(\phi(x) - \epsilon Mt, t)) - \epsilon MV_\eta(\phi(x) - \epsilon Mt, t) - h'(t) \\
& - \mathcal{A}[V(\phi(\epsilon\cdot) - \epsilon Mt, t) - h(t)](\frac{x}{\epsilon}) \\
& \leq F(V(\phi(x) - \epsilon Mt, t)) - \epsilon MV_\eta(\phi(x) - \epsilon Mt, t) - h'(t) \\
& - \mathcal{A}[V(\phi(\epsilon\cdot) - \epsilon Mt, t) - h(t)](\frac{x}{\epsilon}) \\
& = \mathcal{A}[V(\phi(x) - \epsilon Mt, t)\mathbf{1}](x) - \epsilon MV_\eta(\phi(x) - \epsilon Mt, t) - h'(t) \\
& - \mathcal{A}[V(\phi(\epsilon\cdot) - \epsilon Mt, t) - h(t)](\frac{x}{\epsilon}) \\
& = \mathcal{A}[V(\phi(x) - \epsilon Mt, t)\mathbf{1}](\frac{x}{\epsilon}) - \epsilon MV_\eta(\phi(x) - \epsilon Mt, t) - h'(t)
\end{aligned}$$

$$- \mathcal{A}[V(\phi(\epsilon\cdot) - \epsilon Mt, t) - h(t)](\frac{x}{\epsilon}).$$

From (A5), we have

$$\begin{aligned} &\mathcal{A}[u(\epsilon\cdot)](\frac{x}{\epsilon}) - \mathcal{A}[u(\epsilon\cdot) + v(\epsilon\cdot)](\frac{x}{\epsilon}) \\ \le & - D_1\epsilon^2 v_{xx}(x) + g(u(x) + v(x)) - g(u(x)) - m_1\rho_\epsilon * v(x), \end{aligned}$$

where $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\frac{x}{\epsilon})$. So

$$\begin{aligned} &\mathcal{A}[V(\phi(x) - \epsilon Mt, t)\mathbf{1}](\frac{x}{\epsilon}) - \mathcal{A}[V(\phi(\epsilon\cdot) - \epsilon Mt, t) - h(t)](\frac{x}{\epsilon}) \\ \le & - D_1\epsilon^2 V_{\eta\eta}(\phi(x) - \epsilon Mt, t)\phi'^2(x) - \epsilon^2 D_1 V_\eta(\phi(x) - \epsilon Mt, t)\phi''(x) \\ & + g(V(\phi(x) - \epsilon Mt, t) - h(t)) - g(V(\phi(x) - \epsilon Mt, t)) \\ & - m_1\rho_\epsilon * V(\phi(x) - \epsilon Mt, t) + m_1 h(t) + m_1 V(\phi(x) - \epsilon Mt, t). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}\underline{u} \le & - D_1\epsilon^2 V_{\eta\eta}(\phi(x) - \epsilon Mt, t)\phi'^2(x) - \epsilon^2 D_1 V_\eta(\phi(x) - \epsilon Mt, t)\phi''(x) \\ & + g(V(\phi(x) - \epsilon Mt, t) - h(t)) - g(V(\phi(x) - \epsilon Mt, t)) \\ & + m_1(V(\phi(x) - \epsilon Mt, t) - \rho_\epsilon * V(\phi(x) - \epsilon Mt, t)) \\ & + m_1 h(t) - h'(t) - \epsilon M V_\eta(\phi(x) - \epsilon Mt, t) \\ \le & D_1 c_4 \epsilon \phi'^2(x) + \epsilon(\epsilon D_1|\phi''(x)| - M)V_\eta(\phi(x) - \epsilon Mt, t) \\ & + g(V(\phi(x) - \epsilon Mt, t) - h(t)) - g(V(\phi(x) - \epsilon Mt, t)) \\ & + m_1(V(\phi(x) - \epsilon Mt, t) - \rho_\epsilon * V(\phi(x) - \epsilon Mt, t)) + m_1 h(t) - h'(t). \end{aligned} \tag{2.9}$$

But

$$\begin{aligned} & V(\phi(x) - \epsilon Mt, t) - \rho_\epsilon * V(\phi(x) - \epsilon Mt, t) \\ = & V(\phi(x) - \epsilon Mt, t) - \int_{-\infty}^{\infty} \rho_\epsilon(x - y)V(\phi(y) - \epsilon Mt, t)dy \\ = & V(\phi(x) - \epsilon Mt, t) - \int_{-\infty}^{\infty} \frac{1}{\epsilon}\rho(\frac{x - y}{\epsilon})V(\phi(y) - \epsilon Mt, t)dy \\ = & V(\phi(x) - \epsilon Mt, t) - \int_{-\infty}^{\infty} \rho(y)V(\phi(x - \epsilon y) - \epsilon Mt, t)dy \\ = & \int_{-\infty}^{\infty} \rho(y)[V(\phi(x) - \epsilon Mt, t) - V(\phi(x - \epsilon y) - \epsilon Mt, t)]dy. \end{aligned} \tag{2.10}$$

By Taylor's theorem,

$$V(\phi(x) - \epsilon Mt, t) - V(\phi(x - \epsilon y) - \epsilon Mt, t)$$

$$= V_\eta(\phi(x) - \epsilon Mt, t)(\phi(x) - \phi(x - \epsilon y)) + \frac{1}{2}V_{\eta\eta}(\theta, t)(\phi(x) - \phi(x - \epsilon y))^2$$
$$= V_\eta(\phi(x) - \epsilon Mt, t)(\epsilon\phi'(x)y + O(\epsilon^2 y^2)) + V_{\eta\eta}(\theta, t)O(\epsilon^2 y^2). \tag{2.11}$$

So, from (2.10), (2.11), Lemma 2.2, and the fact that ρ is even, we have

$$\begin{aligned} &V(\phi(x) - \epsilon Mt, t) - \rho_\epsilon * V(\phi(x) - \epsilon Mt, t) \\ =&V_\eta(\phi(x) - \epsilon Mt, t)\int_{-\infty}^{\infty} \rho(y)O(\epsilon^2 y^2)dy + O(\epsilon)\int_{-\infty}^{\infty} \rho(y)y^2\, dy \\ =&V_\eta(\phi(x) - \epsilon Mt, t)O(\epsilon^2) + O(\epsilon). \end{aligned} \tag{2.12}$$

Hence, from (2.9), we have

$$\begin{aligned} \mathcal{L}\underline{u} \leq& \epsilon[D_1 c_4 \phi'^2(x) + (\epsilon D_1|\phi''(x)| + O(\epsilon) - M)V_\eta(\phi(x) - \epsilon Mt, t)] \\ &+ g(V(\phi(x) - \epsilon Mt, t) - h(t)) - g(V(\phi(x) - \epsilon Mt, t)) \\ &+ O(\epsilon) + m_1 h(t) - h'(t) \\ \leq& -\frac{M\epsilon}{2}V_\eta(\phi(x) - \epsilon Mt, t) + g(V(\phi(x) - \epsilon Mt, t) - h(t)) \\ &- g(V(\phi(x) - \epsilon Mt, t)) + O(\epsilon) + m_1 h(t) - h'(t) \end{aligned}$$

(for M large enough, depending on ρ and ϕ). So, in order to have $\mathcal{L}\underline{u} \leq 0$, we need only

$$\begin{aligned} &-\frac{M\epsilon}{2}V_\eta(\phi(x) - \epsilon Mt, t) + g(V(\phi(x) - \epsilon Mt, t) - h(t)) \\ &- g(V(\phi(x) - \epsilon Mt, t)) + O(\epsilon) + m_1 h(t) - h'(t) \leq 0. \end{aligned} \tag{2.13}$$

We assert that h may be chosen so that it satisfies, for small $\tau > 0$,

$$h(0) = 0,\ 0 \leq h(t) \leq \tau, \quad \text{for } 0 \leq t \leq \tau_0|\ln\epsilon|. \tag{2.14}$$

Since $g'(0) > m_1$ and $g'(1) > m_1$, there exist a small constant $\delta > 0$ and $\nu > m_1$, such that if $0 \leq h \leq \tau$ and $|V| \leq \delta$ or $|V - 1| \leq \delta$, then

$$g(V - h) - g(V) \leq -\nu h. \tag{2.15}$$

So if $0 \leq V(\phi(x) - \epsilon Mt, t) \leq \delta$ or $1 - \delta \leq V(\phi(x) - \epsilon Mt, t) \leq 1$, from (2.13), in order to have $\mathcal{L}\underline{u} \leq 0$, it suffices to have

$$-\frac{M\epsilon}{2}V_\eta(\phi(x) - \epsilon Mt, t) - (\nu - m_1)h(t) - h'(t) + O(\epsilon) \leq 0. \tag{2.16}$$

So we take $h(t)$ such that

$$h'(t) + (\nu - m_1)h(t) = O(\epsilon), \quad h(0) = 0; \tag{2.17}$$

i.e.,

$$h(t) = O(\epsilon)(1 - e^{-(\nu - m_1)t}) \leq O(\epsilon).$$

With this $h(t)$, (2.14) is satisfied.
If $\delta \leq V(\phi(x) - \epsilon Mt, t) \leq 1 - \delta$, from Lemma 2.2, we have

$$V_\eta(\phi(x) - \epsilon Mt, t) \geq c_5$$

and

$$g(V(\phi(x) - \epsilon Mt, t) - h(t)) - g(V(\phi(x) - \epsilon Mt, t)) \leq \|g'\|_{L^\infty} h.$$

So, from (2.13), we need

$$-\frac{M\epsilon c_5}{2} + (\|g'\|_{L^\infty} + m_1)h - h'(t) + O(\epsilon) \leq 0.$$

In view of (2.17), we need only

$$-\frac{M\epsilon c_5}{2} + (\|g'\|_{L^\infty} + \nu)h \leq 0;$$

i.e.,

$$\frac{M\epsilon c_5}{2} \geq (\|g'\|_{L^\infty} + \nu)h = (\|g'\|_{L^\infty} + \nu)O(\epsilon).$$

This is true if M is large, independent of ϵ. So we have thus proved that $\underline{u}(x,t) = V(\phi(x) - Mt, t) - h(t)$ is a subsolution of (1.7) for $0 \leq t \leq \tau_0|\ln\epsilon|$. Hence, by the comparison principle, we have

$$u_\epsilon(x,t) \geq V(\phi(x) - \epsilon Mt, t) - h(t), \text{ for } 0 \leq t \leq \tau_0|\ln\epsilon|.$$

Let $M_0 = 3 + M\tau_0$; then if x is such that

$$\phi(x) \geq \alpha + M_0\sqrt{\epsilon}|\ln\epsilon| = \alpha + (3 + M\tau_0)\sqrt{\epsilon}|\ln\epsilon|,$$

for $0 \leq t \leq \tau_0|\ln\epsilon|$,

$$\phi(x) - \epsilon Mt \geq \alpha + 3\sqrt{\epsilon}|\ln\epsilon|.$$

So by Lemma 2.2, we have

$$u(x, \tau_0|\ln\epsilon|) \geq 1 - M_1\epsilon,$$

for some constant $M_1 > 0$. This completes the proof of (2.2). □

Remark 2.3. If there exist constants $D_2 \geq 0$ and $m_2 \geq 0$, and a smooth function g with $g'(0) > m_2$, $g'(1) > m_2$, and $\rho \in C^2(\mathbb{R})$ as in Theorem 2.1 such that

$$\mathcal{A}[u(\cdot)+v(\cdot)](x) - \mathcal{A}[u(\cdot)](x) \leq D_2 v_{xx}(x) - [g(u(x)+v(x)) - g(u(x))] + m_2\rho * v, \tag{2.18}$$

then there exist constants $\tau_0 > 0$ (depending only on $\mathcal{A}$), and $M_0 > 0$ and $M_1 > 0$, both depending on $\mathcal{A}$ and $\|\phi\|_{C^2}$, such that for small $\epsilon > 0$, if x is such that $\phi(x) \leq \alpha - M_0\sqrt{\epsilon}|\ln\epsilon|$, then

$$u_\epsilon(x, \tau_0|\ln\epsilon|) \leq M_1\epsilon. \tag{2.19}$$

If both (2.1) and (2.18) hold, then we know that after some time, $u_\epsilon(x,t)$ will display a "transition layer structure" for ϵ small enough.

3. Persistence of patterns

From Theorem 2.1 and Remark 2.3, we know that if $u_\epsilon(x,t)$ is the solution of (1.7), then interfaces of thickness at the order of $O(\sqrt{\epsilon}|\ln\epsilon|)$ are generated in a time scale $O(|\ln\epsilon|)$. Now we are interested in how long the pattern generated can last. To do this, we reset time so that the new time $t=0$ corresponds to the old time $t=\tau_0|\ln\epsilon|$. Thus the new initial datum has a "transition layer structure." We prove that for such initial data , the pattern can change only slightly in a very long time. The life span of the patterns depends on the probability measure ν in assumption (A3), as well as the convergence rate of $U(x)$ at $x=\pm\infty$.

Theorem 3.1. *Suppose that there exist positive constants $c>0$ and $d>1$ such that*

$$0 \le 1 - U(x) \le cx^{1-d}, \quad \text{for } x \gg 1, \tag{3.1}$$

$$0 \le U(x) \le c|x|^{1-d}, \quad \text{for } x \ll -1, \tag{3.2}$$

and

$$\int_{|y|\ge x} \nu(dy) \le cx^{1-d}, \quad \text{for } x \gg 1. \tag{3.3}$$

Let $u_\epsilon(x,t)$ be the unique solution of (1.7).

(i) *Suppose there exists a constant $\mu \in (0, d-1)$ such that $\phi(x) \ge 1-\epsilon^\mu$ on some interval (l,∞); then for any $\gamma \in (0,1)$ satisfying $(1-\gamma)(d-1) > \mu$, there exists a constant $\epsilon_0 > 0$ depending only on $\mathcal{A}$, μ, and γ such that for $0<\epsilon\le\epsilon_0$,*

$$u_\epsilon(x,t) \ge 1 - 2\epsilon^\mu, \ \text{for } x \ge l + 2\epsilon^\gamma, \ t \ge 0. \tag{3.4}$$

A similar conclusion holds if $\phi(x) \ge 1-\epsilon^\mu$ on $(-\infty, l)$ or if $\phi(x) \le \epsilon^\mu$ on (l,∞) or $(-\infty,l)$.

(ii) *Suppose there exist positive constants $\delta>0$ and $\mu \in (0, d-1)$ such that $\phi(x) \ge 1-\epsilon^\mu$ on some interval (a,b) with width $b-a \ge 2\delta$; then for any $\gamma\in(0,1)$ satisfying $(1-\gamma)(d-1)>\mu$, there exist a constant $\epsilon_0>0$ depending only on $\mathcal{A}$, μ, γ, and δ, and a constant T depending only on $\mathcal{A}$, such that for $0<\epsilon\le\epsilon_0$*

$$\begin{aligned} u_\epsilon(x,t) \ge 1 - 3\epsilon^\mu, \ \text{for } x \in (a+3\epsilon^\gamma, b-3\epsilon^\gamma), \\ 0 \le t \le T\epsilon^{\gamma-d}(b-a)^{d-1}. \end{aligned} \tag{3.5}$$

A similar conclusion holds if $\phi(x) \le \epsilon^\mu$ on (a,b).

Remark 3.2. From (ii), we know that if $\phi(x)$ has a "transition layer structure," then $u_\epsilon(x,t)$ will maintain this structure at least $O(\epsilon^{\gamma-d})$ long.

Proof. (i). Consider a subsolution of (1.7) of the form

$$\underline{u}(x,t) = U(\frac{x-\zeta(t)}{\epsilon}) - \omega(t). \tag{3.6}$$

We want to choose $\zeta(t)$ and $\omega(t)$ such that

$$\underline{u}(x,0) \le \phi(x),\ x \in (-\infty,\infty), \tag{3.7}$$

and

$$\underline{u}_t(x,t) \le \mathcal{A}[\underline{u}(\epsilon\cdot,t)](\frac{x}{\epsilon}); \tag{3.8}$$

i.e.,

$$\begin{aligned} -\omega'(t) - \frac{\zeta'(t)}{\epsilon}U'(\frac{x-\zeta(t)}{\epsilon}) &\le \mathcal{A}[U(\cdot - \frac{\zeta(t)}{\epsilon}) - \omega(t)\mathbf{1}](\frac{x}{\epsilon}) \\ &= \mathcal{A}[U(\cdot) - \omega(t)\mathbf{1}](\frac{x-\zeta(t)}{\epsilon}). \end{aligned} \tag{3.9}$$

We first consider (3.7). We take $\zeta(t)$ and $\omega(t)$ such that they at least satisfy

$$\begin{cases} \omega(0) = \epsilon^\mu,\ \omega'(t) \le 0,\ \omega(t) \ge 0, \\ \zeta(0) = l + \epsilon^\gamma, \quad \zeta'(t) > 0. \end{cases} \tag{3.10}$$

Then, for $\epsilon > 0$ small enough, (3.7) is true.

Now, we show that, by choosing $\zeta(t)$ and $\omega(t)$, (3.8) is true. First, in view of (3.10), we take $\omega(t) = \epsilon^\mu e^{-\beta t}$, where $\beta = \frac{1}{2}\min\{-F'(0), -F'(1)\} > 0$, and F is defined in (1.2). We define a small constant δ_0 and a large positive constant $M_0 \ge 1$ by

$$\delta_0 = \min\{\frac{1}{3}, \frac{\beta}{8(K_1+K_2)}\},$$

$$\nu(\{|y| \ge M_0\}) := \int_{|y|\ge M_0} \nu(dy) \le \frac{\beta}{8K_1},$$

where K_1 and K_2 are defined in assumption (A3). Let $M_1 = M_1(U) > 0$ be a constant such that

$$U(\eta) > 1 - \delta_0,\ \text{for all } \eta \ge M_1, \quad U(\eta) < \delta_0,\ \text{for all } \eta \le -M_1. \tag{3.11}$$

We define σ_1 by

$$\sigma_1 = (\|F'\|_{c^0([-1,2])} + \beta + K_1 + K_2)(\min_{\eta\in[-M_1-M_0, M_1+M_0]} \beta U'(\eta))^{-1}. \tag{3.12}$$

Denoting $\frac{x-\zeta(t)}{\epsilon}$ by η, we can write (3.9) as

$$-\omega'(t) - \frac{\zeta'(t)}{\epsilon} U'(\eta) \le \mathcal{A}[U(\cdot) - \omega(t)\mathbf{1}](\eta). \tag{3.13}$$

With $\omega(t) = \epsilon^\mu e^{-\beta t}$, (3.13) becomes

$$\epsilon^\mu \beta e^{-\beta t} - \frac{\zeta'(t)}{\epsilon} U'(\eta) - \mathcal{A}[U(\cdot) - \epsilon^\mu e^{-\beta t}\mathbf{1}](\eta) \le 0.$$

Observe that $\mathcal{A}[U(\cdot)](\eta) = 0$; we can write this as

$$\epsilon^\mu \beta e^{-\beta t} - \frac{\zeta'(t)}{\epsilon} U'(\eta) + \epsilon^\mu e^{-\beta t} \int_0^1 \mathcal{A}'[U(\cdot) - \theta \epsilon^\mu e^{-\beta t}\mathbf{1}](\mathbf{1})(\eta)\, d\theta \le 0. \tag{3.14}$$

Now, we consider three separate cases:

(i) $|\eta| \le M_1 + M_0$.

(ii) $\eta > M_1 + M_0$.

(iii) $\eta < -M_1 - M_0$.

Case (i) By (1.2), for any constant $\alpha \in \mathbb{R}$, we have $\mathcal{A}'[\alpha\mathbf{1}](\mathbf{1}) = F'(\alpha)\mathbf{1}$. Taking $\alpha = U(\eta) - \theta\epsilon^\mu e^{-\beta t}$, we obtain, from assumption (A3),

$$\begin{aligned} &|\mathcal{A}'[U(\cdot) - \theta\epsilon^\mu e^{-\beta t}\mathbf{1}](\mathbf{1})(\eta) - F'(U(\eta) - \theta\epsilon^\mu e^{-\beta t})| \\ &\le K_1 \int_{\mathbb{R}} |U(\eta - y) - U(\eta)|\, \nu(dy) + K_2 \|U(\cdot + \eta) - U(\eta)\|_{C^0([-1,1])} \\ &\le K_1 + K_2. \end{aligned}$$

It then follows that

$$|\int_0^1 \mathcal{A}'[U(\cdot) - \theta\epsilon^\mu e^{-\beta t}\mathbf{1}](\mathbf{1})(\eta)\, d\theta| \le K_1 + K_2 + \|F'\|_{c^0([-1,2])}.$$

So, in order to make (3.14) true, we need $\zeta(t)$ to satisfy

$$\frac{\zeta'(t)}{\epsilon} U'(\eta) \ge \epsilon^\mu e^{-\beta t}[\beta + K_1 + K_2 + \|F'\|_{c^0([-1,2])}],$$

but for $|\eta| \le M_1 + M_0$, we have, from (3.12),

$$U'(\eta) \ge \frac{\beta + K_1 + K_2 + \|F'\|_{c^0([-1,2])}}{\beta\sigma_1}.$$

Hence, we need only that $\zeta(t)$ satisfies

$$\zeta'(t) \ge \beta\sigma_1 \epsilon^{1+\mu} e^{-\beta t}. \tag{3.15}$$

Combining (3.10) and (3.15), we choose $\zeta(t) = l + \epsilon^\gamma + \sigma_1\epsilon^{1+\mu}(1 - e^{-\beta t})$; then (3.14) holds.

Case (ii) Note that we always have $\zeta'(t) > 0$ and $U' > 0$, so in order to have (3.14), we need only have

$$\int_0^1 \mathcal{A}'[U(\cdot) - \theta\epsilon^k e^{-\mu t}\mathbf{1}](\mathbf{1})(\eta)\, d\theta \le -\beta. \tag{3.16}$$

Observe that $\mathcal{A}'[\mathbf{1}](\mathbf{1}) = F'(1)\mathbf{1}$, so from (A3), we have

$$|\mathcal{A}'[U(\cdot) - \theta\epsilon^\mu e^{-\beta t}\mathbf{1}](\mathbf{1})(\eta) - F'(1)| \le \beta.$$

It then follows from the assumption $F'(1) \le -2\beta$ that

$$-\int_0^1 \mathcal{A}'[U(\cdot) - \theta\epsilon^\mu e^{-\beta t}\mathbf{1}](\mathbf{1})(\eta) \ge \beta;$$

i.e., (3.16) holds. Hence, (3.14) is true.

Case (iii) Just as in Case (ii), we can show that (3.14) holds. From the above analysis, we know that

$$\underline{u}(x,t) = U(\frac{x - l - \epsilon^\gamma - \sigma_1\epsilon^{1+\mu}(1 - e^{-\beta t})}{\epsilon}) - \epsilon^\mu e^{-\beta t}$$

is a subsolution of (1.7). So by the comparison principle, we have $u_\epsilon(x,t) \ge \underline{u}(x,t)$ for $x \in \mathbb{R}$, $t \ge 0$, and all small $\epsilon > 0$. Hence, for $x \ge l + 2\epsilon^\gamma$, $t \ge 0$, we have

$$\begin{aligned} u_\epsilon(x,t) &\ge U(\frac{x - l - \epsilon^\gamma - \sigma_1\epsilon^{1+\mu}(1 - e^{-\beta t})}{\epsilon}) - \epsilon^\mu e^{-\beta t} \\ &\ge U(\frac{1}{2\epsilon^{1-\gamma}}) - \epsilon^\mu \ge 1 - 2\epsilon^\mu. \end{aligned}$$

This completes the proof of (3.4).

(ii) Without loss of generality, assume that $[a,b] = [-\delta,\delta]$. Consider a subsolution of (1.7) of the form

$$\underline{u}(x,t) = U(\frac{x - \zeta(t)}{\epsilon}) + U(\frac{-x - \zeta(t)}{\epsilon}) - 1 - \omega(t), \tag{3.17}$$

where $\zeta(t)$ and $\omega(t)$ satisfy at least the following:

$$\begin{cases} \omega(0) = \epsilon^\mu, \quad 0 \le \omega(t) \le 2\epsilon^\mu, \\ -\delta + \epsilon^\gamma = \zeta(0) \le \zeta(t) \le -\delta + 2\epsilon^\gamma, \quad \zeta'(t) > 0. \end{cases} \tag{3.18}$$

We want to choose $\zeta(t)$ and $\omega(t)$ such that $\underline{u}_t \le \mathcal{A}[\underline{u}(\epsilon\cdot,t)](\frac{x}{\epsilon})$; i.e.,

$$\begin{aligned} &-\omega'(t) - \frac{\zeta'(t)}{\epsilon}\left[U'(\eta) + U'(\frac{-x-\zeta(t)}{\epsilon})\right] \\ \le&\mathcal{A}\left[U(\cdot - \frac{\zeta(t)}{\epsilon}) + U(-\cdot - \frac{\zeta(t)}{\epsilon}) - \mathbf{1} - \omega(t)\mathbf{1}\right](\frac{x}{\epsilon}) \\ =&\mathcal{A}\left[U(\cdot - \frac{\zeta(t)}{\epsilon}) + U(-(\cdot - \frac{\zeta(t)}{\epsilon}) - \frac{2\zeta(t)}{\epsilon}) - \mathbf{1} - \omega(t)\mathbf{1}\right](\frac{x}{\epsilon}) \\ =&\mathcal{A}\left[U(\cdot) + U(-\frac{2\zeta(t)}{\epsilon} - \cdot) - \mathbf{1} - \omega(t)\mathbf{1}\right](\frac{x-\zeta(t)}{\epsilon}), \end{aligned} \tag{3.19}$$

and

$$\underline{u}(x,0) \le \phi(x),\ x \in \mathbb{R} \tag{3.20}$$

(where $\eta = \frac{x-\zeta(t)}{\epsilon}$). Now we prove (3.19). We first consider $x \le 0$. Note that for $\eta \in \mathbb{R}$, we have $\mathcal{A}[U(\cdot)](\eta) = 0$. So (3.19) can be written as

$$\begin{aligned} &-\omega'(t) - \frac{\zeta'(t)}{\epsilon}\left[U'(\eta) + U'(\frac{-x-\zeta(t)}{\epsilon})\right] \\ &+\omega(t)\int_0^1 \mathcal{A}'\left[U(\cdot) - \theta\omega(t)\mathbf{1}\right](\mathbf{1})(\eta)\,d\theta \\ \le&\mathcal{A}\left[U(\cdot) + U(\frac{-2\zeta(t)}{\epsilon} - \cdot) - \mathbf{1} - \omega(t)\mathbf{1}\right](\eta) - \mathcal{A}\left[U(\cdot) - \omega(t)\mathbf{1}\right](\eta). \end{aligned} \tag{3.21}$$

From assumption (A3), we have

$$\begin{aligned} &\mathcal{A}\left[U(\cdot) + U(\frac{-2\zeta(t)}{\epsilon} - \cdot) - \mathbf{1} - \omega(t)\mathbf{1}\right](\eta) - \mathcal{A}\left[U(\cdot) - \omega(t)\mathbf{1}\right](\eta) \\ &\ge -K_1\int_{\mathbb{R}}(1 - U(\frac{-x-\zeta(t)}{\epsilon} + y))\,\nu(dy) - K_2(1 - U(\frac{-x-\zeta(t)}{\epsilon} - 1)). \end{aligned}$$

Observe that $-\zeta(t) \ge \delta - 2\epsilon^\gamma$, so for $x \le 0$, we have

$$U(\frac{-x-\zeta(t)}{\epsilon} + y) \ge U(\frac{\delta - 2\epsilon^\gamma}{\epsilon} + y).$$

Hence,

$$\begin{aligned} &\mathcal{A}\left[U(\cdot) + U(\frac{-2\zeta(t)}{\epsilon} - \cdot) - \mathbf{1} - \omega(t)\mathbf{1}\right](\eta) - \mathcal{A}\left[U(\cdot) - \omega(t)\mathbf{1}\right](\eta) \\ &\ge -K_1\int_{\mathbb{R}}(1 - U(\frac{\delta - 2\epsilon^\gamma}{\epsilon} + y))\,\nu(dy) - K_2(1 - U(\frac{\delta - 2\epsilon^\gamma}{\epsilon} - 1)). \end{aligned} \tag{3.22}$$

But (3.1) implies that, for $\epsilon > 0$ small enough,

$$1 - U(\frac{\delta - 2\epsilon^\gamma}{\epsilon} - 1) \leq c(\frac{\delta}{2})^{1-d}\epsilon^{d-1}, \tag{3.23}$$

and

$$\int_{\mathbb{R}} (1 - U(\frac{\delta - 2\epsilon^\gamma}{\epsilon} + y))\,\nu(dy) \leq 2c(\frac{\delta}{3})^{1-d}\epsilon^{d-1}. \tag{3.24}$$

Combining (3.22), (3.23), and (3.24), we have

$$\mathcal{A}\Big[U(\cdot) + U(\frac{-2\zeta(t)}{\epsilon} - \cdot) - \mathbf{1} - \omega(t)\mathbf{1}\Big](\eta) - \mathcal{A}\left[U(\cdot) - \omega(t)\mathbf{1}\right](\eta) \geq -c_\delta\epsilon^{d-1},$$

where $c_\delta = c[2K_1(\frac{\delta}{3})^{1-d} + K_2(\frac{\delta}{2})^{1-d}]$.

Thus, in order to have (3.21), we need only to have

$$\begin{aligned} &-\omega'(t) - \frac{\zeta'(t)}{\epsilon}\Big[U'(\frac{x-\zeta(t)}{\epsilon}) + U'(\frac{-x-\zeta(t)}{\epsilon})\Big] \\ &+ \omega(t)\int_0^1 \mathcal{A}'\left[U(\cdot) - \theta\omega(t)\mathbf{1}\right](\mathbf{1})(\eta)\,d\theta + c_\delta\epsilon^{d-1} \leq 0. \end{aligned} \tag{3.25}$$

To prove (3.25), we consider three separate cases as before:

(i) $\eta > M_0 + M_1$.
(ii) $\eta < -M_0 - M_1$.
(iii) $|\eta| \leq M_0 + M_1$.

Case (i). Observe that we always have $\zeta' > 0$ and $U' > 0$, so in order to make (3.25) come true, we need only to show that

$$-\omega'(t) + \omega(t)\int_0^1 \mathcal{A}'[U(\cdot) - \theta\omega(t)\mathbf{1}](\mathbf{1})(\eta)\,d\theta + c_\delta\epsilon^{d-1} \leq 0. \tag{3.26}$$

Since $\mathcal{A}'[\mathbf{1}](\mathbf{1}) = F'(1)\mathbf{1}$, we have, for $\epsilon > 0$ small enough,

$$\begin{aligned} &|\mathcal{A}'[U(\cdot) - \theta\omega(t)\mathbf{1}](\mathbf{1})(\eta) - F'(1)| \\ \leq &K_1\Big(\int_{|y|\geq M_0} + \int_{|y|<M_0}\Big)|U(\eta - y) - \theta\omega(t) - 1|\,\nu(dy) \\ &+ K_2\|U(\eta + \cdot) - \theta\omega(t) - 1\|_{C^0([-1,1])}. \end{aligned}$$

Since $0 < \omega(t) \leq 2\epsilon^\mu$, we have $|U(\eta - y) - \theta\omega(t) - 1| \leq 2$. Therefore,

$$\int_{|y|\geq M_0} |U(\eta - y) - \theta\omega(t) - 1|\,\nu(dy) \leq 2\int_{|y|\geq M_0} \nu(dy) \leq \frac{\beta}{4K_1}.$$

For $|y| < M_0$ and $\eta > M_1 + M_0$, we have $\eta - y > M_1$. Hence $U(\eta - y) > 1 - \delta_0$. So

$$|U(\eta - y) - \theta\omega(t) - 1| \leq \delta_0 + \theta\omega(t) \leq 2\delta_0 \leq \frac{\beta}{4(K_1 + K_2)}.$$

It follows that

$$\int_{|y|<M_0} |U(\eta - y) - \theta\omega(t) - 1|\, \nu(dy) \le \frac{\beta}{4(K_1 + K_2)}.$$

Thus, we have $|\mathcal{A}'[U(\cdot) - \theta\omega(t)\mathbf{1}](\mathbf{1})(\eta) - F'(1)| \le \beta$. Hence,

$$\mathcal{A}'[U(\cdot) - \theta\omega(t)\mathbf{1}](\mathbf{1})(\eta) \le F'(1) + \beta.$$

By the definition of β, we have $F'(1) \le -2\beta$. Therefore

$$\int_0^1 \mathcal{A}'[U(\cdot) - \theta\omega(t)\mathbf{1}](\mathbf{1})(\eta)\, d\theta \le -\beta.$$

Hence, in order to realize (3.26), we need

$$-\omega'(t) - \beta\omega(t) + c_\delta \epsilon^{d-1} \le 0.$$

So we take $\omega(t)$ such that

$$\omega'(t) + \beta\omega(t) = c_\delta \epsilon^{d-1}, \quad \omega(0) = \epsilon^\mu.$$

Solving it gives

$$\omega(t) = \epsilon^\mu e^{-\beta t} + \frac{c_\delta \epsilon^{d-1}}{\beta}(1 - e^{-\beta t}). \tag{3.27}$$

It is easy to verify that $\omega(t) \le 2\epsilon^\mu$.

With $\omega(t)$ given by (3.27), we know that (3.25) is true for $\eta > M_0 + M_1$.

Case (ii). $\eta < -M_0 - M_1$. In this case, the proof is similar to that in Case (i).

Case(iii). $|\eta| \le M_0 + M_1$. In this case, observe that $\zeta' > 0$ and $U'(\frac{-x-\zeta(t)}{\epsilon}) > 0$, so we need only to show that by choosing $\zeta(t)$, we have

$$-\omega'(t) - \frac{\zeta'(t)}{\epsilon} U'(\eta) + \omega(t) \int_0^1 \mathcal{A}'[U(\cdot) - \theta\omega(t)\mathbf{1}](\mathbf{1})(\eta)\, d\theta + c_\delta \epsilon^{d-1} \le 0. \tag{3.28}$$

Taking $\alpha = U(\eta) - \theta\omega(t)$ in $\mathcal{A}'[\alpha\mathbf{1}](\mathbf{1}) = F'(\alpha)\mathbf{1}$, we obtain, by assumption (A3),

$$\begin{aligned}
&|\mathcal{A}'[U(\cdot) - \theta\omega(t)\mathbf{1}](\mathbf{1})(\eta) - F'(U(\eta) - \theta\omega(t))| \\
&\le K_1 \int_{\mathbb{R}} |U(\eta - y) - U(\eta)|\, \nu(dy) + K_2 \|U(\cdot + \eta) - U(\eta)\|_{C^0([-1,1])} \\
&\le K_1 + K_2.
\end{aligned}$$

So, we have

$$\int_0^1 \mathcal{A}'[U(\cdot) - \theta\omega(t)\mathbf{1}](\mathbf{1})(\eta)\, d\theta \le K_1 + K_2 + \|F'\|_{c^0([-1,2])}.$$

So, in order to have (3.28), we need only to have

$$\zeta'(t)U'(\eta) \geq \epsilon\{\omega(t)(K_1 + K_2 + \|F'\|_{c^0([-1,2])}) + c_\delta \epsilon^{d-1} - \omega'(t)\}.$$

But $\omega'(t) = -\beta\omega(t) + c_\delta \epsilon^{d-1}$, i.e., $c_\delta \epsilon^{d-1} - \omega'(t) = \beta\omega(t)$, so we need

$$\zeta'(t)U'(\eta) \geq \epsilon\{\omega(t)(K_1 + K_2 + \|F'\|_{c^0([-1,2])}) + \beta\omega(t)\}.$$

From the definition of σ_1, we need only to have $\zeta'(t) \geq \sigma_1 \beta\epsilon\omega(t)$. Therefore, we take $\zeta(t)$ such that

$$\zeta'(t) = \beta\epsilon\sigma_1\omega(t), \quad \zeta(0) = -\delta + \epsilon^\gamma.$$

Solving it gives $\zeta(t) = -\delta + \epsilon^\gamma + (\epsilon^{1+\mu}\sigma_1 - \frac{\sigma_1 c_\delta \epsilon^d}{\beta})(1 - e^{-\beta t}) + c_\delta \sigma_1 \epsilon^d t$. So we have $\zeta(t) \leq -\delta + \frac{3}{2}\epsilon^\gamma + c_\delta\sigma_1\epsilon^d t$. Thus, to have $\zeta(t) \leq -\delta + 2\epsilon^\gamma$, we need

$$-\delta + \frac{3}{2}\epsilon^\gamma + c_\delta\sigma_1\epsilon^d t \leq -\delta + 2\epsilon^\gamma;$$

i.e.,

$$t \leq \frac{1}{2\sigma_1 c_\delta}\epsilon^{\gamma-d} = M\epsilon^{\gamma-d},$$

where $M = T(b-a)^{d-1}$ with $T = \frac{1}{2^d\sigma_1 c[2K_1(\frac{1}{3})^{1-d} + K_2(\frac{1}{2})^{1-d}]} > 0$.

So far, we have proved that for our choices of $\zeta(t)$ and $\omega(t)$, for $x \leq 0$ and $0 \leq t \leq T(b-a)^{d-1}\epsilon^{\gamma-d}$, (3.19) is true. By taking $\eta = \frac{-x-\zeta(t)}{\epsilon}$, by symmetry, we can show that (3.19) is also true for $x \geq 0$. So we know that (3.19) holds for $x \in \mathbb{R}$ and $0 \leq t \leq T(b-a)^{d-1}\epsilon^{\gamma-d}$.

It is easy to verify (3.20). So, we know that $\underline{u}(x,t)$ is a subsolution of (1.7). Then, by the comparison principle, we have

$$\underline{u}(x,t) \leq u_\epsilon(x,t) \text{ for } x \in \mathbb{R},\ 0 \leq t \leq T(b-a)^{d-1}\epsilon^{\gamma-d}.$$

In particular, for $x \in [-\delta + 3\epsilon^\gamma, \delta - 3\epsilon^\gamma]$, $0 \leq t \leq T(b-a)^{d-1}\epsilon^{\gamma-d}$,

$$\begin{aligned} u_\epsilon(x,t) &\geq U\Big(\frac{-\delta + 3\epsilon^\gamma - (-\delta + 2\epsilon^\gamma)}{\epsilon}\Big) + U\Big(\frac{-\delta + 3\epsilon^\gamma - (-\delta + 2\epsilon^\gamma)}{\epsilon}\Big) - 1 - 2\epsilon^\mu \\ &\geq 2(1 - O(\epsilon^{(1-\gamma)(d-1)})) - 1 - 2\epsilon^\mu \geq 1 - 3\epsilon^\mu. \end{aligned}$$

This completes the proof of (ii). □

Similarly, we can prove the following theorem.

Theorem 3.3. *Suppose there exist positive constants c and σ such that*

$$0 \leq 1 - U(x) \leq ce^{-\sigma x}, \quad \textit{for } x \gg 1, \tag{3.29}$$

$$0 \leq U(x) \leq ce^{\sigma x}, \quad \textit{for } x \ll -1, \tag{3.30}$$

and

$$\int_{|y|\geq x} \nu(dy) \leq ce^{-\sigma x}, \quad \textit{for } x \gg 1. \tag{3.31}$$

Let $u_\epsilon(x,t)$ be the unique solution of (1.7).

(i) *Suppose there is a constant $\mu > 0$ such that $\phi(x) \geq 1 - \epsilon^\mu$ on some interval (l, ∞); then for any $\gamma \in (0,1)$, there exists a constant $\epsilon_0 > 0$ depending only on $\mathcal{A}$, μ, and γ such that for $0 < \epsilon \leq \epsilon_0$,*

$$u_\epsilon(x,t) \geq 1 - 2\epsilon^\mu, \ \textit{for } x \geq l + 2\epsilon^\gamma, \ t \geq 0. \tag{3.32}$$

A similar conclusion holds if $\phi(x) \geq 1-\epsilon^\mu$ on $(-\infty, l)$ or if $\phi(x) \leq \epsilon^\mu$ on (l,∞) or $(-\infty, l)$.

(ii) *Suppose that there exist positive constants $\delta > 0$ and $\mu > 0$ such that $\phi(x) \geq 1 - \epsilon^\mu$ on some interval (a,b) with width $b - a \geq 2\delta$; then for any $\gamma \in (0,1)$, there exists a constant $\epsilon_0 > 0$ depending only on $\mathcal{A}$, μ, γ, and δ, and constants $T > 0$ and $\overline{\sigma} > 0$ depending only on $\mathcal{A}$, such that for $0 < \epsilon \leq \epsilon_0$*

$$u_\epsilon(x,t) \geq 1 - 3\epsilon^\mu$$
$$\textit{for } x \in (a + 3\epsilon^\gamma, b - 3\epsilon^\gamma), \ 0 \leq t \leq T\epsilon^{\gamma-1} e^{\frac{\overline{\sigma}(b-a)}{2\epsilon}}. \tag{3.33}$$

A similar conclusion holds if $\phi(x) \leq \epsilon^\mu$ on (a,b).

Remark 3.4. From (ii), we know that if $\phi(x)$ has a "transition layer structure," then $u_\epsilon(x,t)$ will maintain this structure at least exponentially long.

4. Annihilation of patterns

In this section, we show that if $\limsup_{|x|\to\infty} \phi(x) < \alpha$ or $\liminf_{|x|\to\infty} \phi(x) > \alpha$, then the pattern will be annihilated after a long time.

Now, we consider two different cases:

(1) $D_1 > 0$ (*the "local-nonlocal" case*),
(2) $D_1 = 0$ (*the "pure nonlocal" case*),

where D_1 is the constant given in assumption (A5).

From Section 3, we know that if (3.1) and (3.2) hold, then the pattern can last at least algebraically long; if (3.29) and (3.30) hold, then the pattern can last at least exponentially long. The following theorem shows that if $D_1 > 0$ in (A5), then the pattern can last only exponentially long. So we only have metastability.

Theorem 4.1. *Suppose that $D_1 > 0$ in assumption (A5) and there exist positive constants $\alpha_0 < \alpha$ and $M > 0$ such that $\phi(x) \leq \alpha_0$ for $|x| \geq M$; then* *there exist constants $\epsilon_0 > 0$ depending only on $\mathcal{A}$, and α_0, $c > 0$ depending only on $\mathcal{A}$ such that*

$$u_\epsilon(x, e^{\frac{c(M+2)}{\epsilon}}) \le \alpha_1 < \alpha, \ \textit{for } x \in (-\infty, \infty), \ 0 < \epsilon \le \epsilon_0. \tag{4.1}$$

Furthermore, after time $t = e^{\frac{c(M+2)}{\epsilon}}$, $u_\epsilon(\cdot, t) \to 0$ in L^∞ norm exponentially fast as $t \to \infty$.

Proof. Step 1. It is easy to show that there exists a constant $\epsilon_0 > 0$ such that for $0 < \epsilon \le \epsilon_0$

$$\phi(x) \le U(\frac{\pm x + M + 1}{\epsilon}) + \alpha_0, \ x \in \mathbb{R}. \tag{4.2}$$

Step 2. We show that

$$u_\epsilon(x,t) \le \begin{cases} \overline{u}_1 & \equiv & U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1 - e^{-\beta t})) + \alpha_0 e^{-\beta t}, \\ \overline{u}_2 & \equiv & U(\frac{-x+M+1}{\epsilon} + \sigma_1\alpha_0(1 - e^{-\beta t})) + \alpha_0 e^{-\beta t}, \end{cases} \tag{4.3}$$

where σ_1 and β are defined in the proof of Theorem 3.1. We prove only the first inequality of (4.3). The second one can be proved in a similar way.

From Step 1, we know that (4.3) holds at $t = 0$. So, by the comparison principle, in order to prove that it is true for $t > 0$ we need to show that

$$\overline{u}_{1t} \ge \mathcal{A}[\overline{u}_1(\epsilon\cdot, t)](\frac{x}{\epsilon});$$

i.e.,

$$\begin{aligned} &\sigma_1\alpha_0\beta e^{-\beta t}U'(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1 - e^{-\beta t})) - \alpha_0\beta e^{-\beta t} \\ \ge \ \ &\mathcal{A}[U(\cdot) + \alpha_0 e^{-\beta t}\mathbf{1}](\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1 - e^{-\beta t})). \end{aligned} \tag{4.4}$$

Let $\eta = \frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1 - e^{\beta t})$; then, in view of $\mathcal{A}[U(\cdot)](\eta) = 0$, (4.4) can be written as

$$\alpha_0 e^{-\beta t}\{\beta\sigma_1 U'(\eta) - \beta - \int_0^1 \mathcal{A}'[U(\cdot) + \theta\alpha_0 e^{-\beta t}\mathbf{1}](\mathbf{1})(\eta)\, d\theta\} \ge 0.$$

So we need only to show that

$$\beta\sigma_1 U'(\eta) - \beta - \int_0^1 \mathcal{A}'[U(\cdot) + \theta\alpha_0 e^{-\beta t}\mathbf{1}](\mathbf{1})(\eta)\, d\theta \ge 0.$$

This can be done in the same way as the proof of (3.25).

Step 3. Let $w_1(x,t) = \overline{u}_1(x,t) - u_\epsilon(x,t) \ge 0$. Then, from (A5), we have

$$w_{1t} \ge D_1\epsilon^2 w_{1xx} - B_1 w_1,$$

where $B_1 = \|g'\|_{L^\infty}$. Let $z_1(x,t) = e^{B_1 t}w_1(x,t)$; then we have $z_{1t} \ge D_1\epsilon^2 z_{1xx}$. So $z_1(x,t)$ satisfies $z_{1t} - D_1\epsilon^2 z_{1xx} \ge 0$ and $z_1(x,0) \ge 0$. Therefore, by the explicit expressions of the solutions of the initial-value problem for $u_t = D_1\epsilon^2 u_{xx}$, we have

$$z_1(x,t) \geq \frac{1}{2\epsilon\sqrt{D_1\pi t}} \int_{2M+4}^{\infty} \Big(U(\frac{\xi+M+1}{\epsilon}) - U(\frac{-\xi+M+1}{\epsilon})\Big) e^{-\frac{(x-\xi)^2}{4D_1\epsilon^2 t}}\, d\xi.$$

Note that for $\epsilon > 0$ small enough and $\xi \geq 2M+4$, we have

$$U(\frac{\xi+M+1}{\epsilon}) - U(\frac{-\xi+M+1}{\epsilon}) > \frac{1}{2}.$$

Hence,

$$z_1(x,t) \geq \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\frac{x-(2M+4)}{2\epsilon\sqrt{D_1 t}}} e^{-\eta^2}\, d\eta.$$

So, for $x \geq -3(M+2)$ and $t > 0$, we have

$$z_1(x,t) \geq \frac{1}{2\sqrt{\pi}} \int_{\frac{5(M+2)}{2\epsilon\sqrt{D_1 t}}}^{\infty} e^{-\eta^2}\, d\eta.$$

Thus, for any positive integer k, we have

$$z_1(x,k) \geq \frac{1}{2\sqrt{\pi}} \frac{\epsilon\sqrt{D_1 k}}{10(M+2)} e^{\frac{-25(M+2)^2}{4\epsilon^2 D_1 k}}.$$

Let k be the smallest integer greater than $\frac{5(M+2)}{\epsilon}$. Then

$$z_1(x,k) \geq \frac{c\sqrt{\epsilon}}{\sqrt{M+2}} e^{-\frac{3(M+2)}{2D_1\epsilon}},$$

and so

$$e^{B_1 k}(\overline{u}_1(x,k) - u_\epsilon(x,k)) \geq \frac{c\sqrt{\epsilon}}{\sqrt{M+2}} e^{-\frac{3(M+2)}{2D_1\epsilon}}.$$

Hence, for $x \geq -3(M+2)$, we have

$$\begin{aligned} u_\epsilon(x,k) &\leq \overline{u}_1(x,k) - \frac{c\sqrt{\epsilon}}{\sqrt{M+2}} e^{-B_1 k} e^{-\frac{3(M+2)}{2D_1\epsilon}} \\ &\leq U\Big(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta k}) - \lambda e^{-\frac{c_1(M+2)}{\epsilon}}\Big) + \alpha_0 e^{-\beta k}, \end{aligned} \tag{4.5}$$

where $\lambda = \frac{1}{\max_{x\in\mathbb{R}} U'(x)} > 0$.

On the other hand, for $x \leq -3(M+2)$, by using the mean-value theorem, we have

$$\begin{aligned} u_\epsilon(x,k) \leq\, & U\Big(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta k}) - \lambda e^{-\frac{c_1(M+2)}{\epsilon}}\Big) + \alpha_0 e^{-\beta k} \\ & + U'(A_1)\lambda e^{-\frac{c_1(M+2)}{\epsilon}}, \end{aligned}$$

where $A_1 \leq -1/\epsilon$. So by (1.9), we have $U'(A_1) = o(1)$ as $\epsilon \to 0$. Therefore, for $x \leq -3M - 6$,

$$\begin{aligned} u_\epsilon(x,k) \leq U\Big(&\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta k}) - \lambda e^{-\frac{c_1(M+2)}{\epsilon}}\Big) \\ &+ \alpha_0 e^{-\beta k} + o(e^{-\frac{c_1(M+2)}{\epsilon}}). \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we know that (4.6) holds for any $x \in \mathbb{R}$. Similarly, we can show that

$$\begin{aligned} u_\epsilon(x,k) \leq U\Big(&\frac{-x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta k}) - \lambda e^{-\frac{c_1(M+2)}{\epsilon}}\Big) \\ &+ \alpha_0 e^{-\beta k} + o(e^{-\frac{c_1(M+2)}{\epsilon}}). \end{aligned}$$

Let $\alpha_1 = \alpha_0 e^{-\beta k} + o(e^{-\frac{c_1(M+2)}{\epsilon}})$ and $\lambda_1 = \sigma_1\alpha_0(1-e^{-\beta k}) - \lambda e^{-\frac{c_1(M+2)}{\epsilon}}$. Thus, we have proved that

$$u_\epsilon(x,k) \leq U\Big(\frac{\pm x+M+1}{\epsilon} + \lambda_1\Big) + \alpha_1.$$

Now, for $n = 2, 3, \dots,$ define

$$\begin{aligned} \alpha_n &= \alpha_{n-1}e^{-\beta k} + o(\exp[-\frac{c_1(M+2)}{\epsilon}]), \\ \lambda_n &= \lambda_{n-1} + \sigma_1\alpha_{n-1}(1-e^{-\beta k}) - \lambda \exp\Big[-\frac{c_1(M+2)}{\epsilon}\Big]. \end{aligned}$$

Then, by mathematical induction, we can show that

$$u_\epsilon(x,nk) \leq U\Big(\frac{\pm x+M+1}{\epsilon} + \lambda_n\Big) + \alpha_n \quad \text{for } x \in \mathbb{R}, \tag{4.7}$$

provided $\alpha_i \leq \alpha_0$ and $|\lambda_i| \leq \frac{2(M+2)}{\epsilon}$, for all $1 \leq i \leq n$.

Now, by the definition of α_n and λ_n, we have for ϵ small enough, $\alpha_n \leq \alpha_0$ and

$$\lambda_n \leq -\frac{n\lambda}{1.2}\exp\Big[-\frac{c_1(M+2)}{\epsilon}\Big] + \sigma_1\alpha_0.$$

Let N_ϵ be the largest integer that is less than or equal to $\frac{1.5(M+2)e^{\frac{c_1(M+2)}{\epsilon}}}{\lambda\epsilon}$; then

$$\frac{1.5(M+2)e^{\frac{c_1(M+2)}{\epsilon}}}{\lambda\epsilon} - 1 \leq N_\epsilon \leq \frac{1.5(M+2)e^{\frac{c_1(M+2)}{\epsilon}}}{\lambda\epsilon}.$$

So, for $1 \leq n \leq N_\epsilon$, $\epsilon > 0$ small enough, we have $|\lambda_n| \leq \frac{2(M+2)}{\epsilon}$; i.e., the restrictions for (4.7) hold. In particular, for $n = N_\epsilon$ it is true. Hence, we have

$$u_\epsilon(x,N_\epsilon k) \leq U\Big(\frac{\pm x+M+1}{\epsilon} + \lambda_{N_\epsilon}\Big) + \alpha_{N_\epsilon} \quad \text{for } x \in \mathbb{R}.$$

Now, by easy computations, we have $\alpha_{N_\epsilon} = o(\exp[-\frac{c_1(M+2)}{\epsilon}])$ and $\lambda_{N_\epsilon} \leq -\frac{M+2}{\epsilon}$. So we have

$$u_\epsilon(x, N_\epsilon k) \leq U\Big(\frac{\pm x + M + 1}{\epsilon} - \frac{M+2}{\epsilon}\Big) + o(\exp\Big[-\frac{c_1(M+2)}{\epsilon}\Big]).$$

It follows that, for $\epsilon > 0$ small enough,

$$u_\epsilon(x, N_\epsilon k) \leq U(\frac{-1}{\epsilon}) + o(\exp\Big[-\frac{c_1(M+2)}{\epsilon}\Big]) \leq \alpha_1 < \alpha.$$

Observe that

$$N_\epsilon k \leq \frac{1.5(M+2)e^{\frac{c_1(M+2)}{\epsilon}}}{\lambda\epsilon}\Big(\frac{5(M+2)}{\epsilon} + 1\Big) \leq e^{\frac{c_2(M+2)}{\epsilon}} \quad (\text{for } 0 < \epsilon \leq \epsilon_0),$$

where ϵ_0 depends on F and α_0. Hence $N_\epsilon k = \exp[\frac{c(M+2)}{\epsilon}]$ for some constant $c > 0$. Therefore (4.1) is true.

Now, let $v(x,t) = u_\epsilon(x, N_\epsilon k + t)$ and compare it with the solution of the ordinary differential equation

$$u_t = F(u), \quad u(0) = \alpha_1 \geq v(x, 0).$$

We know that $u_\epsilon(x, N_\epsilon k + t) \to 0$ exponentially fast as $t \to \infty$.

This completes the proof of Theorem 4.1. □

Remark 4.2. If (2.18) holds with $D_2 > 0$, and there exist positive constants $\alpha_0 > \alpha$ and $M > 0$ such that $\phi(x) \geq \alpha_0$ for $|x| \geq M$, then there exist constants $\epsilon_0 > 0$ depending only on $\mathcal{A}$ and α_0 and $c > 0$ depending only on $\mathcal{A}$ such that

$$u_\epsilon(x, e^{\frac{c(M+2)}{\epsilon}}) \geq \alpha_1 > \alpha, \quad \text{for} \quad x \in (-\infty, \infty), \ 0 < \epsilon \leq \epsilon_0. \tag{4.8}$$

Furthermore, after time $t = e^{\frac{c(M+2)}{\epsilon}}$, $u_\epsilon(\cdot, t) \to 1$ in the L^∞ norm exponentially fast as $t \to \infty$.

Now we consider the case $D_1 = 0$. In this case, the following theorem shows that for general ρ as in (A5), the pattern will last only exponentially long. If the decay rate of ρ is slow such that

$$\rho(x) \geq c_3|x|^{-d} \quad \text{for } |x| \gg 1 \text{ and } \rho(\pm\infty) = 0, \tag{4.9}$$

then the pattern can last only $O(\epsilon^{-d})$ long. These can be compared with Theorems 3.1 and 3.3

Theorem 4.3. *Assume that $D_1 = 0$ in assumption (A5).*

(i) *Assume that there exist positive constants* $\alpha_0 < \alpha$ *and* $M > 0$ *such that* $\phi(x) \leq \alpha_0$ *for* $|x| \geq M$*; then*

$$u_\epsilon(x, e^{\frac{T(M+2)}{\epsilon}}) \leq \alpha_1 < \alpha, \ \text{for } x \in (-\infty, \infty), \ 0 < \epsilon \leq \epsilon_0, \tag{4.10}$$

where $T > 0$ *is a constant depending only on* $\mathcal{A}$ *and* ϵ_0 *depends on* $\mathcal{A}$ *and* α_0*. Furthermore, after time* $t = e^{\frac{T(M+2)}{\epsilon}}$*,* $u_\epsilon(\cdot, t) \to 0$ *in the* L^∞ *norm exponentially fast as* $t \to \infty$*.*

(ii) *If, in addition to the conditions stated in (i),* (4.9) *holds, then*

$$u_\epsilon(x, T(M+2)^d \epsilon^{-d}) \leq \alpha_1 < \alpha, \ \text{for } x \in (-\infty, \infty), \ 0 < \epsilon \leq \epsilon_0, \tag{4.11}$$

where $T > 0$ *is a constant depending only on* $\mathcal{A}$*, and* ϵ_0 *depends on* $\mathcal{A}$ *and* α_0*. Furthermore, after time* $t = T(M+2)^d \epsilon^{-d}$*,* $u_\epsilon(\cdot, t) \to 0$ *in the* L^∞ *norm exponentially fast as* $t \to \infty$*.*

To prove this theorem, we first cite the following lemma from [18].

Lemma 4.4. *Let* $\rho \in \mathfrak{L}^1(\mathbb{R})$ *be nonnegative on* $\mathbb{R}$ *with* $\int_{-\infty}^0 \rho(x)\,dx = \frac{1}{2}$*. Then there exists a positive constant* θ *such that*

$$\int_{-\infty}^{-\theta k} \Big(\overbrace{\rho * \rho * \cdots * \rho}^{k}\Big)(x)\,dx \geq \frac{1}{3^k}, \quad k = 1, 2, 3, \ldots.$$

Proof of Theorem 4.3. (i) Just as in the proof of Theorem 4.1, we still have

$$\phi(x) \leq U(\frac{\pm x + M + 1}{\epsilon}) + \alpha_0, \ x \in \mathbb{R}, \tag{4.12}$$

and

$$u_\epsilon(x,t) \leq \begin{cases} \overline{u}_1 & \equiv & U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1 - e^{-\beta t})) + \alpha_0 e^{-\beta t}, \\ \overline{u}_2 & \equiv & U(\frac{-x+M+1}{\epsilon} + \sigma_1\alpha_0(1 - e^{-\beta t})) + \alpha_0 e^{-\beta t}. \end{cases} \tag{4.13}$$

Now, we define w_1 and z_1 as before. Then, from (A5), we have $w_{1t} \geq m_1 \rho_\epsilon * w_1 - B_1 w_1$, and $z_{1t} \geq 0$. Without loss of generality, we may assume that $m_1 < 1$. Then for any positive integer k, we have

$$z_1(x,k) - z_1(x,k-1) = z_{1t}(x,\xi) \quad \text{with } \xi \in (k-1,k).$$

So $z_1(x,k) \geq m_1^k H_\epsilon * z_1(x,0)$, where $H_\epsilon = \frac{1}{\epsilon} H(\frac{x}{\epsilon})$ and $H = \overbrace{\rho * \rho * \cdots * \rho}^{k}$. So, for $x \geq -3(M+2)$, we have

$$z_1(x,k) \geq \frac{m_1^k}{2} \int_{-\infty}^{-\frac{5(M+2)}{\epsilon}} H(\eta)\,d\eta.$$

Let k be the smallest integer greater than $\frac{5(M+2)}{\epsilon\theta}$ (θ is the constant in Lemma 4.4). Then for $x \geq -3(M+2)$, we have $z_1(x,k) \geq e^{-\frac{c(M+2)}{\epsilon\theta}}$, for some positive constant c independent of ϵ. Thus, for $x \geq -3(M+2)$, we have

$$\overline{u}_1(x,k) - u_\epsilon(x,k) \geq e^{-\frac{c(M+2)}{\epsilon\theta}} e^{-B_1 k}.$$

Therefore, for ϵ small enough, we have

$$u_\epsilon(x,k) \leq U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta k}) - \lambda e^{-\frac{c_1(M+2)}{\epsilon}}) + \alpha_0 e^{-\beta k},$$

where $c_1 > 0$ is a constant independent of ϵ and λ is the same as before.

The rest of the proof is similar to that of Theorem 4.1.

(ii). Just as in the proof of Theorem 4.1, (4.12) and (4.13) are true. Define w_1 and z_1 as before; we have $z_{1t} \geq m_1 \rho_\epsilon * z_1(x,0)$. Thus, we have

$$z_1(x,1) \geq \frac{m_1}{2}\int_{-\infty}^{\frac{x-(2M+4)}{\epsilon}} \rho(\eta)\,d\eta.$$

So, by the condition $\rho(y) \geq c_3|y|^{-d}$, we have, for $x \geq -(3M+6)$,

$$z_1(x,1) \geq \frac{m_1}{2}\int_{-\infty}^{\frac{-5(M+2)}{\epsilon}} \rho(\eta)\,d\eta \geq c\epsilon^{d-1}(M+2)^{1-d}.$$

Therefore, for $x \geq -(3M+6)$,

$$u_\epsilon(x,1) \leq U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta}) - \lambda\epsilon^{d-1}) + \alpha_0 e^{-\beta} \tag{4.14}$$

(where $\lambda = \frac{c(M+2)^{1-d}}{\max_{x\in\mathbb{R}} U'(x)} > 0$).

On the other hand, for $x \leq -(3M+6)$, we have

$$U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta})) - U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta}) - \lambda\epsilon^{d-1}) = U'(A_1)\lambda\epsilon^{d-1},$$

where $A_1 \leq -\frac{1}{\epsilon}$. So, from (1.9), we have $U'(A_1) = o(1)$ as $\epsilon \to 0$. Therefore, for $x \leq -3M-6$,

$$\begin{aligned} &U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta})) \\ &\quad \leq U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta}) - \lambda\epsilon^{d-1}) + o(\epsilon^{d-1}). \end{aligned}$$

Hence, for $x \leq -3M-6$,

$$u_\epsilon(x,1) \leq U(\frac{x+M+1}{\epsilon} + \sigma_1\alpha_0(1-e^{-\beta}) - \lambda\epsilon^{d-1}) + \alpha_0 e^{-\mu} + o(\epsilon^{d-1}). \tag{4.15}$$

This and (4.14) imply that this is true for all $x \in \mathbb{R}$. Similarly, we can show that, for $x \in \mathbb{R}$,

$$u_\epsilon(x,1) \le U(\frac{-x+M+1}{\epsilon}+\sigma_1\alpha_0(1-e^{-\beta})-\lambda\epsilon^{d-1})+\alpha_0 e^{-\beta}+o(\epsilon^{d-1}). \quad (4.16)$$

Now, let $\lambda_1 = \sigma_1\alpha_0(1-e^{-\beta})-\lambda\epsilon^{d-1}$ and $\alpha_1 = \alpha_0 e^{-\beta}+o(\epsilon^{d-1})$. Then (4.15) and (4.16) can be written as

$$u_\epsilon(x,1) \le U(\frac{\pm x+M+1}{\epsilon}+\lambda_1)+\alpha_1.$$

Define $\lambda_k = \lambda_{k-1}+\sigma_1\alpha_{k-1}(1-e^{-\beta})-\lambda\epsilon^{d-1}$ and $\alpha_k = \alpha_{k-1}e^{-\beta}+o(\epsilon^{d-1})$. We can show that, for $k = 1, 2, 3, \ldots$,

$$u_\epsilon(x,k) \le U(\frac{\pm x+M+1}{\epsilon}+\lambda_k)+\alpha_k, \quad (4.17)$$

provided $0 < \alpha_i \le \alpha_0$ and $|\lambda_i| \le \frac{2(M+2)}{\epsilon}$ for all $1 \le i \le k$.

By the definition of α_k and λ_k, we have for ϵ small enough $\alpha_k \le \alpha_0$, and

$$-k\lambda\epsilon^{d-1} \le \lambda_k \le \sigma_1(\alpha_0+(k-1)o(\epsilon^{d-1}))-k\lambda\epsilon^{d-1}.$$

Now, let K_ϵ be the largest integer less than or equal to $\frac{1.5(M+2)}{\lambda\epsilon^d}$; then we have $|\lambda_k| \le \frac{2(M+2)}{\epsilon}$. Therefore, the condition for λ_i and α_i in (4.17) for $1 \le i \le K_\epsilon$ is satisfied. In particular, it is true for $k = K_\epsilon$. Thus, we have

$$u_\epsilon(x,K_\epsilon) \le U(\frac{\pm x+M+1}{\epsilon}+\lambda_{K_\epsilon})+\alpha_{K_\epsilon}.$$

It can be shown that $\alpha_{K_\epsilon} = o(\epsilon^{d-1})$, and $\lambda_{K_\epsilon} \le -\frac{M+2}{\epsilon}$ for $\epsilon > 0$ small. So we have $u_\epsilon(x,K_\epsilon) \le \alpha_1 < \alpha$ (for $\epsilon > 0$ small enough), which proves (4.11).

Then just as before we know that $u_\epsilon(x,K_\epsilon+t) \to 0$ exponentially fast as $t \to \infty$. Thus, we have completed the proof of (ii) of Theorem 4.3. □

Remark 4.5. Suppose (2.18) holds with $D_2 = 0$.

(i) Assume that $\rho(\pm\infty) = 0$, and suppose that there exist positive constants $\alpha_0 > \alpha$ and $M > 0$ such that $\phi(x) \ge \alpha_0$ for $|x| \ge M$; then

$$u_\epsilon(x, e^{\frac{T(M+2)}{\epsilon}}) \ge \alpha_1 > \alpha, \text{ for } x \in (-\infty,\infty),\ 0 < \epsilon \le \epsilon_0, \quad (4.18)$$

where $T > 0$ is a constant depending only on $\mathcal{A}$ and ϵ_0 depends on $\mathcal{A}$ and α_0. Furthermore, after time $t = e^{\frac{T(M+2)}{\epsilon}}$, $u_\epsilon(\cdot,t) \to 1$ in the L^∞ norm exponentially fast as $t \to \infty$.

(ii) If, in addition the conditions in (i), (4.9) holds, then

$$u_\epsilon(x, T(M+2)^d\epsilon^{-d}) \ge \alpha_1 > \alpha, \text{ for } x \in (-\infty,\infty),\ 0 < \epsilon \le \epsilon_0, \quad (4.19)$$

where $T > 0$ is a constant depending only on $\mathcal{A}$ and ϵ_0 depends on $\mathcal{A}$ and α_0. Furthermore, after time $t = T(M+2)^d\epsilon^{-d}$, $u_\epsilon(\cdot,t) \to 1$ in the L^∞ norm exponentially fast as $t \to \infty$.

5. A nonlocal differential equation

In this section, we apply the general theory developed in previous sections to the nonlocal differential equation,

$$u_t = -u + G(\rho * S(u)), \tag{5.1}$$

where

$$\rho * S(u) = \int_{-\infty}^{\infty} \rho(x-y) S(u(y,t))\, dy,$$

with $\rho \in C^2((-\infty,\infty))$, $\rho(-x) = \rho(x)$, $\rho(x) \geq 0$, $\int_{\mathbb{R}} \rho(x)\, dx = 1$, and S and G smooth such that

- (B1) $F(u) = G(S(u)) - u$ has precisely three zeroes: $u = 0$, $u = \alpha$, and $u = 1$, $\alpha \in (0,1)$.
- (B2) $S' > 0$, $S(0) = 0$, $S(\alpha) = \alpha$, $S(1) = 1$, $S'(0) < 1$, $S'(1) < 1$, and $S'(\alpha) > 1$.
- (B3) $G' > 0$, $G'' \geq 0$ on $(0,\alpha)$ and $G'' \leq 0$ on $(\alpha,1)$, $G'(0) < 1$, $G'(1) < 1$, and $G'(\alpha) > 1$.

Model (5.1) includes the model $u_t = -u + \rho * S(u)$, which is studied in [7] and the model $u_t = -u + \tanh(\beta(\rho * u))$, which is studied in [14], [15], [16], and [17].

For (5.1), the stationary traveling-wave solution, denoted by U, satisfies

$$U(x) = G(\rho * S(U(x))), \text{ for } x \in (-\infty,\infty). \tag{5.2}$$

Now we study the decay rate of $U(x)$.

Lemma 5.1. *Let $U(x)$ be a stationary traveling-wave solution of* (5.1); *then*

$$\lim_{x\to-\infty} \rho * S(U(x)) = 0, \tag{5.3}$$

$$\lim_{x\to\infty} \rho * S(U(x)) = 1. \tag{5.4}$$

This can be proved by the assumptions about ρ and S. We omit the details here.

Proposition 5.2. *If there exist constants $d > 1$ and $c > 0$ such that for $|x| \gg 1$, $\rho(x) \leq c|x|^{-d}$, then there exist constants $c_i > 0$, $i = 1,2,3$, such that*

$$0 \leq 1 - U(x) \leq c_1 x^{1-d}, \text{ for } x \gg 1; \tag{5.5}$$

$$0 \leq U(x) \leq c_2 |x|^{1-d}, \text{ for } x \ll -1; \tag{5.6}$$

$$0 < U'(x) \le c_3|x|^{1-d}, \ \textit{for } |x| \gg 1. \tag{5.7}$$

Remark 5.3. From this proposition, we know that (3.1) and (3.2) in Theorem 3.1 hold.

Proof. From (5.2), we have

$$\begin{aligned}
0 < 1 - U(x) &= 1 - G(\rho * S(U(x))) = G(1) - G\Big(\int_{-\infty}^{\infty} \rho(x-y) S(U(y))\, dy\Big) \\
&= G'(\theta)\Big\{\int_{bx}^{2x} \rho(x-y)(1 - S(U(y)))\, dy \\
&+ \quad \Big(\int_{-\infty}^{bx} + \int_{2x}^{\infty}\Big) \rho(x-y)(1 - S(U(y)))\, dy\Big\} \\
&\le G'(\theta) \int_{bx}^{2x} \rho(x-y)(1 - S(U(y)))\, dy + O\Big(\int_{-\infty}^{bx} + \int_{2x}^{\infty}\Big)\rho(x-y)\, dy \\
&\le G'(\theta) \int_{bx}^{2x} \rho(x-y)(1 - S(U(y)))\, dy + cx^{1-d} \text{ (for } x \gg 1 \text{ and some } c > 0) \\
&= G'(\theta) \int_{bx}^{2x} \rho(x-y) S'(\nu)(1 - U(y))\, dy + cx^{1-d},
\end{aligned} \tag{5.8}$$

where $0 < b < 1$ is to be determined, $\int_{-\infty}^{\infty} \rho(x-y) S(U(y))\, dy < \theta < 1$, and $U(y) < \nu < 1$. From Lemma 5.1, we know that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} \rho(x-y) S(U(y))\, dy = 1.$$

So, by the condition $G'(1) < 1$ and

$$\int_{-\infty}^{\infty} \rho(x-y) S(U(y))\, dy < \theta < 1,$$

we know that for $x \gg 1$, $G'(\theta) < 1$. Also, for $y > bx$, $U(bx) < U(y) < \nu < 1$, and observe that $\lim_{x\to\infty} U(bx) = 1$ and $S'(1) < 1$; we know that for $x \gg 1$, $S'(\nu) < 1$. So from (5.8), we know that there exist constants $\delta < 1$ and $M > 0$ large enough such that for $x \ge M$, we have $0 < 1 - U(x) \le \delta(1 - U(bx)) + cx^{1-d}$. So, we have $1 - U(bx) \ge \eta(1 - U(x)) - cx^{1-d}$, where $\eta = \frac{1}{\delta} > 1$. Then, similar to the proof of Lemma 1.1 in [18], we can prove the lemma. □

Proposition 5.4. *If there exist positive constants c and σ such that for $x \ge 0$, $\rho(x) \le ce^{-\sigma x}$, then there exist positive constants $\overline{c}$ and $\overline{\sigma}$ such that*

$$0 < 1 - U(x) \le \overline{c} e^{-\overline{\sigma} x}, \ \textit{for } x \gg 1; \tag{5.9}$$

$$0 < U(x) \le \overline{c} e^{\overline{\sigma} x}, \ \textit{for } x \ll -1; \tag{5.10}$$

$$0 < U'(x) \le \bar{c}e^{-\bar{\sigma}|x|}, \ \textit{for}\ |x| \gg 1. \tag{5.11}$$

Remark 5.5. This proposition implies that (3.29) and (3.30) in Theorem 3.3 hold.

Proof. Just as in the proof of Proposition 5.2, we have, for $x \gg 1$,

$$\begin{aligned}
0 < 1 - U(x) &= G(1) - G\Big(\int_{-\infty}^{\infty} \rho(x-y) S(U(y))\,dy\Big) \\
&= G'(\theta)\Big(\int_{-\infty}^{M} + \int_{M}^{\infty}\Big)\rho(y)(1 - S(U(x-y)))\,dy \\
&< \Big(\int_{-\infty}^{M} + \int_{M}^{\infty}\Big)\rho(y)(1 - S(U(x-y)))\,dy \\
&= \int_{-\infty}^{M} \rho(y)(1 - S(U(x-y)))\,dy + \int_{M}^{\infty} \rho(y)(1 - S(U(x-y)))\,dy \\
&\le 1 - S(U(x-M)) + \int_{M}^{\infty} \rho(y)(1 - S(U(x-y)))\,dy,
\end{aligned}$$

where M is large and to be determined.

Now, for $k = 2, 3, \dots$, taking $x = kM$, we have

$$\begin{aligned}
0 &< 1 - U(kM) \\
&\le 1 - S(U((k-1)M)) + \int_{M}^{\infty} \rho(y)(1 - S(U(kM-y)))\,dy \\
&= 1 - S(U((k-1)M)) + \int_{M}^{2M} \rho(y)(1 - S(U(kM-y)))\,dy \\
&\quad + \cdots + \int_{(k-1)M}^{\infty} \rho(y)(1 - S(U(kM-y)))\,dy \\
&\le 1 - S(U((k-1)M)) + (1 - S(U((k-2)M))) \int_{M}^{2M} \rho(y)\,dy \\
&\quad + (1 - S(U((k-3)M))) \int_{2M}^{3M} \rho(y)\,dy + \cdots \\
&\quad + (1 - S(U(M))) \int_{(k-2)M}^{(k-1)M} \rho(y)\,dy + \int_{(k-1)M}^{\infty} \rho(y)\,dy.
\end{aligned}$$

Since $S'(1) < 1$, there exists an $M > 0$ large enough such that for $x \ge M$

$$\frac{S(U(x)) - 1}{U(x) - 1} \le \delta < 1.$$

So $1 - S(U(x)) \le \delta(1 - U(x))$. Hence, we have

$$\begin{aligned} 1 - U(kM) &\le \delta(1 - U((k-1)M)) + \delta(1 - U((k-2)M)) \int_M^{2M} ce^{-\sigma y}\,dy \\ &+ \delta(1 - U((k-3)M)) \int_{2M}^{3M} ce^{-\sigma y}\,dy + \cdots + \\ &+ \delta(1 - U(M)) \int_{(k-2)M}^{(k-1)M} ce^{-\sigma y}\,dy + \int_{(k-1)M}^{\infty} ce^{-\sigma y}\,dy \\ &\le \delta(1 - U((k-1)M)) + \frac{c\delta}{\sigma} e^{-\sigma M}(1 - U((k-2)M)) \\ &+ \frac{c\delta}{\sigma} e^{-2\sigma M}(1 - U((k-3)M)) + \cdots \\ &+ \frac{c\delta}{\sigma} e^{(k-2)\sigma M}(1 - U(M)) + \frac{c\delta}{\sigma} e^{-(k-1)\sigma M}. \end{aligned}$$

Then, the proof of the lemma is similar to that of Lemma 1.2 in [18]. □

Now we verify that (5.1) satisfies (A1)–(A5). For (5.1), $\mathcal{A}$ is given by

$$\mathcal{A}[u(\cdot, t)](x) = -u(x, t) + G(\rho * S(u)),$$

which maps constants to constants, and the function in (1.2) is given by $F(c) = G(S(c)) - c$. So (B1), (B2), and (B3) imply (A1). Now we verify (A2) as follows:

Assume that $u(x, t)$ and $v(x, t)$ satisfy $-1 \le u, v \le 2$, $u_t \ge -u + G(\rho * S(u))$, $v_t \le -v + G(\rho * S(v))$, and $u(x, 0) \ge v(x, 0)$. Let $w = u - v$; then we have

$$w_t \ge -w + G'(\theta) \int_{-\infty}^{\infty} \rho(x - y)[\int_0^1 S'(\theta u + (1 - \theta)v) d\theta] w\,dy,$$

and $w_0(x) = w(x, 0) \ge 0$. Then it first can be shown that $w \ge 0$ (e.g., see [6] and [18]). Let $\overline{w} = we^t$; then we have

$$\overline{w}_t \ge G'(\theta) \int_{-\infty}^{\infty} \rho(x - y)[\int_0^1 S'(\theta u + (1 - \theta)v) d\theta] \overline{w}\,dy \ge \delta\rho * \overline{w} \ge 0,$$

where $\delta = \min_{0 \le x \le 1} G'(x) \min_{0 \le y \le 1} S'(y) > 0$, so that $\overline{w} \ge w_0(x)$ on $\mathbb{R} \times [0, \infty)$. It then follows that $\overline{w}_t \ge \delta\rho * w_0$, which implies that, for any $t_1 > 0$, $\overline{w}(\cdot, t) \ge \delta t\rho * w_0$. Now repeating the same process on $[t_1, 2t_1], \ldots, [(N-1)t_1, Nt_1]$, we have $\overline{w}(\cdot, Nt_1) \ge (\delta t_1)^N \rho * \rho * \cdots * \rho * w_0$ for all $N = 1, 2, \ldots$.

Let $T > 0$ and $M > 0$ be any fixed numbers. Since $\rho \ge 0$ and $\int_{-\infty}^{\infty} \rho(x) dx = 1$, there exists a positive integer $N = N(M)$ such that

$$c_1(M) = \min_{x \in [-M-1, M+1]} (\rho * \cdots * \rho)(x) > 0.$$

Therefore, taking $t_1 = \frac{T}{N}$, we have

$$\overline{w}(x,T) \geq (\delta T/N)^N \rho * \rho * \cdots * \rho * w_0 \geq (\delta T/N)^N c_1(M) \int_0^1 w_0(y)\, dy,$$

for all $x \in [-M, M]$. That is, (A2) holds. By the smoothness of G and S, we know that (A3) holds. Also it can be shown that (A4) holds (e.g., see [6]). By direct computations, we have

$$\mathcal{A}[u(\cdot) + v(\cdot)](x) - \mathcal{A}[u(\cdot)](x) = -v + \sigma\rho * v,$$

for some constant σ; i.e., (A5) holds with $D_1 = 0$ and $g(u) = u$. Now let $u_\epsilon(x,t)$ be the unique solution of the problem

$$u_t(x,t) = -u(x,t) + G(\rho_\epsilon * S(u)), \quad u(x,0) = \phi(x). \tag{5.12}$$

Then from our previous results, we get the following theorem.

Theorem 5.6. *Let $u_\epsilon(x,t)$ be the unique solution of the problem* (5.12).

(I) ***Generation of patterns.*** *Suppose that $\|\phi\|_{C^2}$ is finite; then there exist constants $\tau_0 > 0$ depending on G, S, and ρ, and $M_0 > 0$ and $M_1 > 0$, both depending on G, S, ρ, and $\|\phi\|_{C^2}$, such that for all small $\epsilon > 0$,*

- *If x is such that $\phi(x) \geq \alpha + M_0\sqrt{\epsilon}|\ln\epsilon|$, then*

$$u_\epsilon(x, \tau_0|\ln\epsilon|) \geq 1 - M_1\epsilon.$$

- *If x is such that $\phi(x) \geq \alpha - M_0\sqrt{\epsilon}|\ln\epsilon|$, then*

$$u_\epsilon(x, \tau_0|\ln\epsilon|) \leq M_1\epsilon.$$

(II) ***Persistence of patterns.***

(A) *Suppose that $\rho(x) \leq cx^{-d}$ for $x \gg 1$ and some constants $c > 0$, $d > 1$.*

* *Suppose there exists a constant $\mu \in (0, d-1)$ such that $\phi(x) \geq 1 - \epsilon^\mu$ on some interval (l, ∞); then for any $\gamma \in (0,1)$ satisfying $(1-\gamma)(d-1) > \mu$, there exists a constant $\epsilon_0 > 0$ depending on G, S, ρ, μ, and γ such that for $0 < \epsilon \leq \epsilon_0$,*

$$u_\epsilon(x,t) \geq 1 - 2\epsilon^\mu, \text{ for } x \geq l + 2\epsilon^\gamma,\ t \geq 0.$$

A similar conclusion holds if $\phi(x) \geq 1 - \epsilon^\mu$ on $(-\infty, l)$ or if $\phi(x) \leq \epsilon^\mu$ on (l, ∞) or $(-\infty, l)$.

* *Suppose there exist positive constants $\delta > 0$ and $\mu \in (0, d-1)$ such that $\phi(x) \geq 1 - \epsilon^\mu$ on some interval (a,b) with width $b - a \geq 2\delta$; then for any $\gamma \in (0,1)$ satisfying $(1-\gamma)(d-1) > \mu$, there exist a constant $\epsilon_0 > 0$ depending only*

on G, S, ρ, μ, γ, and δ, and a constant $T > 0$ depending only on G, S, and ρ, such that for $0 < \epsilon \le \epsilon_0$

$$u_\epsilon(x,t) \ge 1 - 3\epsilon^\mu, \text{ for } x \in (a + 3\epsilon^\gamma, b - 3\epsilon^\gamma),$$
$$0 \le t \le T\epsilon^{\gamma - d}(b-a)^{d-1}.$$

A similar conclusion holds if $\phi(x) \le \epsilon^\mu$ on (a,b).

(B) *Suppose that $\rho(x) \le ce^{-\sigma x}$ for $x \gg 1$ and some positive constants c and σ.*

- *Suppose there is a constant $\mu > 0$ such that $\phi(x) \ge 1 - \epsilon^\mu$ on some interval (l, ∞); then for any $\gamma \in (0,1)$, there exists a constant $\epsilon_0 > 0$ depending only on G, S, ρ, μ, and γ such that for $0 < \epsilon \le \epsilon_0$,*

$$u_\epsilon(x,t) \ge 1 - 2\epsilon^\mu, \text{ for } x \ge l + 2\epsilon^\gamma, \ t \ge 0.$$

A similar conclusion holds if $\phi(x) \ge 1 - \epsilon^\mu$ on $(-\infty, l)$ or if $\phi(x) \le \epsilon^\mu$ on (l, ∞) or $(-\infty, l)$.

- *Suppose that there exist constants $\delta > 0$ and $\mu > 0$ such that $\phi(x) \ge 1 - \epsilon^\mu$ on some interval (a,b) with width $b - a \ge 2\delta$; then for any $\gamma \in (0,1)$ there exists a constant $\epsilon_0 > 0$ depending only on G, S, ρ, μ, γ, and δ and constants $T > 0$ and $\overline{\sigma} > 0$ depending only on G, S, and ρ such that for $0 < \epsilon \le \epsilon_0$*

$$u_\epsilon(x,t) \ge 1 - 3\epsilon^\mu, \text{ for } x \in (a + 3\epsilon^\gamma, b - 3\epsilon^\gamma),$$
$$0 \le t \le T\epsilon^{\gamma - 1} e^{\frac{\overline{\sigma}(b-a)}{2\epsilon}}.$$

A similar conclusion holds if $\phi(x) \le \epsilon^\mu$ on (a,b).

(III) ***Annihilation of patterns.***

(A) *Assume that there exist positive constants $\alpha_0 < \alpha$ and $M > 0$ such that $\phi(x) \le \alpha_0$ for $|x| \ge M$; then*

$$u_\epsilon(x, e^{\frac{T(M+2)}{\epsilon}}) \le \alpha_1 < \alpha, \text{ for } x \in (-\infty, \infty), \ 0 < \epsilon \le \epsilon_0,$$

where $T > 0$ is a constant depending only on G, S, and ρ and ϵ_0 depends on G, S, ρ, and α_0. Furthermore, after time $t = e^{\frac{T(M+2)}{\epsilon}}$, $u_\epsilon(\cdot, t) \to 0$ in the L^∞ norm exponentially fast as $t \to \infty$.

(B) *If, in addition to the conditions given in (A), (4.9) holds, then*

$$u_\epsilon(x, T(M+2)^d \epsilon^{-d}) \le \alpha_1 < \alpha, \text{ for } x \in (-\infty, \infty), \ 0 < \epsilon \le \epsilon_0,$$

where $T > 0$ is a constant depending only on G, S, and ρ and ϵ_0 depends on G, S, ρ, and α_0. Furthermore, after time $t =$

$T(M+2)^d\epsilon^{-d}$, $u_\epsilon(\cdot,t)\to 0$ in the L^∞ norm exponentially fast as $t\to\infty$.

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