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LOCAL WELL-POSEDNESS FOR DISPERSION-GENERALIZED BENJAMIN-ONO EQUATIONS

J. Colliander [1]
Department of Mathematics, University of Toronto
Toronto, Ontario M5S 3G3, Canada

C. Kenig [2]
Department of Mathematics, University of Chicago
Chicago, Illinois 60637

G. Staffilani [3]
Massachusetts Institute of Technology, Department of Mathematics
Cambridge, MA 02139-4307

(Submitted by: Gustavo Ponce)

Abstract. In this paper we study local well-posedness in the energy space for a family of dispersive equations that can be seen as dispersive "interpolations" between the KdV and the Benjamin-Ono equation.

1. Introduction

We consider the initial-value problem

$$\begin{cases} \partial_t u + \partial_x D_x^{1+a} u + \frac{1}{2}\partial_x(u^2) = 0, \\ u(x,0) = u_0(x) \qquad\qquad x \in \mathbb{R},\, t \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $0 \leq a \leq 1$. Here D_x^{1+a} is the Fourier multiplier operator with symbol $|\xi|^{1+a}$. These equations arise as mathematical models for the weakly nonlinear propagation of long waves in shallow channels. We recall that when $a = 0$ the equation in (1.1) is called the Benjamin-Ono equation, and when $a = 1$ it is called the KdV equation. In the endpoint cases cases ($a = 0$ and $a = 1$), the equations have an infinite number of conserved integrals and are

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integrable by the inverse scattering method [1], [5]. When $0 < a < 1$ there is no integrability, but three integrals are still conserved [25]:

$$I_1 = \int u(x,t)\,dx, \tag{1.2}$$

$$I_2 = \int |u(x,t)|^2\,dx, \tag{1.3}$$

$$I_3 = \frac{1}{6}\int u^3\,dx + \int |D_x^{\frac{1+a}{2}} u|^2\,dx. \tag{1.4}$$

Several papers have been published on the well-posedness for the initial value problem (1.1); below we recall only the most recent results:

$a = 0$: For $a = 0$, (1.1) is the Benjamin-Ono initial-value problem, which is known to have global weak solutions in L^2 ([23], [9], [10], and [26]). Moreover, Ponce [22] proved global well-posedness in $H^{3/2}$ by first proving local well-posedness via the energy method enhanced with dispersive smoothing and globalizing this result with the next conservation law in the hierarchy of conserved quantities for Benjamin-Ono. Koch and Tzvetkov [17] have recently obtained local well-posedness for $s > \frac{5}{4}$.

$0 < a < 1$: Kenig, Ponce, and Vega [11] have shown that (1.1) is locally well-posed for data in H^s provided $s \geq \frac{3}{4}(2-a)$ using the energy method enhanced with the smoothing effect. When $a \geq \frac{4}{5}$ note that $\frac{3}{4}(2-a) \geq \frac{1}{2} + \frac{a}{2}$. Therefore, in the regime $\frac{4}{5} \leq a$, the conservation law (1.4) and the local theory from [11] combine to prove global well-posedness of (1.1).

$a = 1$: For $a = 1$, (1.1) is the KdV initial-value problem, and Bourgain [2] used a fixed-point argument to prove local (and hence global) well-posedness in L^2. Subsequently, Kenig, Ponce, and Vega [16] proved, again using a fixed-point argument, local well-posedness in H^s for $s > -3/4$ and Colliander, Keel, Staffilani, Takaoka, and Tao [8] extended this to a global result. Christ, Colliander, and Tao [4] recently established a local[4] well-posedness result at the $s = -3/4$ endpoint for KdV by conjugating an extension of the $s = \frac{1}{4}$ local theory [13] for the modified KdV equation using the Miura transform (see [18] and [19]).

Recently Molinet, Saut, and Tzvetkov [20] have shown that for $0 \leq a < 1$ an H^s assumption alone on the initial data is insufficient for a proof of local well-posedness via Picard iteration or fixed-point arguments no matter what space of functions of spacetime containing $C([0,T]; H^s)$ is considered for the contraction. Thus, the natural goal of proving local well-posedness of (1.1)

[4]Global well-posedness at $s = -\frac{3}{4}$ remains an open problem.

in the space H^{s_*}, $s_* = \frac{1}{2} + \frac{a}{2}$ appearing in I_3 is not attainable via a fixed-point argument. In particular, the type of approach used by Bourgain [2] and Kenig, Ponce, and Vega ([13], [14], and [16]) for the KdV initial-value problem is not possible for (1.1) in the range $0 \le a < 1$.

Two paths emerge for addressing the well-posedness issues for (1.1). We might first choose to abandon the fixed-point approach to proving local well-posedness and try to further enhance the classical energy method by applying it somehow at lower regularity. Since (1.1) is known ([9], [10]) to have global weak solutions in L^2, the main issue in this approach is uniqueness. The enhancements of the classical energy method using the smoothing effect appearing in [22] and [11] follow this path. As an alternate approach, we might choose to abandon H^s and prove local well-posedness via a fixed-point argument in some other space of initial data. If the norms of the data used in the proof can be shown to be finite for all time we might also prove global well-posedness for such initial data. This paper follows the second path described above by considering initial data in the space F^{s_*} defined below which involves L^2-type integrability with respect to a spatial weight, in addition to H^s regularity.

Remark 1.1. The present paper should be distinguished from [15]. Here, we are considering dispersive generalizations of the Benjamin-Ono and KdV equations all with the same quadratic nonlinearity. In [15], Kenig, Ponce, and Vega considered higher-power generalizations of the nonlinearity within the Benjamin-Ono setting. Note also that the higher-power cases in the dispersion-generalized setting ($a \ge 0$) were already considered in [11].

Before we present our result we observe that from (1.3) and (1.4) it follows that if u is a solution for the IVP (1.1) with $a \ge 0$, then

$$\|u(t)\|_{H^{s_*}} \le C\|u_0\|_{H^{s_*}}, \tag{1.5}$$

where $s_* = 1/2 + a/2$. Indeed, by Sobolev and interpolation, we have the bound

$$\Big|\int u^3 dx\Big| \sim \|u\|^3_{H^{\frac{1}{6}}} \sim \|u\|_{L^2}^{3(1-\frac{1}{3+3a})}\|u\|_{H^{\frac{1}{2}+\frac{a}{2}}}^{3(\frac{1}{3+3a})},$$

which, when combined with (1.4), gives (1.5). If one could prove well-posedness in an interval of time $[0, T]$, where $T = T(\|u_0\|_{H^{s_*}})$, then thanks to (1.5) an iteration in time would give global well-posedness. Unfortunately the standard way of obtaining a local result is by a fixed-point theorem in Sobolev spaces of L^2 type, and the recent result of Molinet, Saut, and Tzvetkov [20] shows that for $0 \le a < 1$ this cannot be done.

Remark 1.2. There is another equation for which Molinet, Saut, and Tzvetkov [21] exhibit a counterexample showing the failure of an iteration method in H^s: the KP-I equation. But, in two recent papers ([6] and [7]), we show that one can still use a fixed-point method to obtain well-posedness results, as long as one considers weighted Sobolev spaces together with the classical H^s space. In this paper, we prove analogous well-posedness results for the IVP (1.1) with $0 < a < 1$ using a fixed-point-theorem method on weighted Sobolev spaces[5].

We shall also see, by example, that our basic estimates, Propositions 3.1 and 3.2, fail when $a = 0$, which explains why our results hold only for the case $0 < a < 1$ (see the Appendix).

Throughout the paper we use the following notation for the Fourier transform:

$$F(f)(\xi) = \hat{f}(\xi) = \int e^{ix\xi} f(x)\, dx,$$

and similarly for the inverse Fourier transform we write

$$F^{-1}(g)(x) = \check{g}(x) = \int e^{-ix\xi} g(\xi)\, d\xi.$$

Definition 1.1. Let

$$s_* = 1/2 + a/2. \tag{1.6}$$

We define the space of functions F^s as the completion of the Schwartz functions through the norm

$$\|f\|_{F^s} = \|f\|_{H^s} + \|xf\|_{H^{s-2s_*}}. \tag{1.7}$$

We forecast that the space F^{s_*} will be the space of the initial data in the IVP (1.1). We also need a space that will contain the evolution under the dynamics dictated by (1.1) of the initial data. To define this space we start by introducing the dispersive function $\omega(\xi) = \xi|\xi|^{1+a}$ associated to the IVP (1.1). We also denote with $\chi_0(z)$ an even, smooth characteristic function of the interval $[-1, 1]$ and with χ an even, smooth characteristic function of the set $\{z : 1/2 < |z| < 2\}$. We use the notation $\chi_j(z) = \chi(2^{-j}z)$ with $j \in \mathbb{N}$, whenever a dyadic decomposition is needed.

Definition 1.2. We define the space X_s^b for $s, b \in \mathbb{R}$, as the closure of the Schwartz functions through the norm

$$\|f\|_{X_s^b} = \sum_{j\geq 0} 2^{jb}\Big(\int_{\mathbb{R}^2} \chi_j(\lambda - \omega(\xi))(1 + |\xi|)^{2s}|\hat{f}|^2(\lambda, \xi)\, d\xi\, d\lambda\Big)^{1/2}, \tag{1.8}$$

[5]The description of these spaces is contained in Definitions 1.1 and 1.2.

and the space $Y^b_{s_0,s_1}$, for $s_0, s_1, b \in \mathbb{R}$ as

$$Y^b_{s_0,s_1} = \{f : xf \in X^b_{s_0} \text{ and } tf \in X^b_{s_1}\}. \tag{1.9}$$

Finally, for $s, b \in \mathbb{R}$, we set

$$Z^b_s = X^b_s \cap Y^b_{s-2s_*,s}. \tag{1.10}$$

If an interval of time $[0,T]$ is fixed, then we say that $f \in Z^b_{s,T}$ if there is an extension $\tilde{f}$ of f on the whole real line such that $\tilde{f} \in Z^b_s$ and $\|f\|_{Z^b_{s,T}} = \inf \|\tilde{f}\|_{Z^b_s}$, where the infimum is taken over all possible extensions $\tilde{f}$. (These norms are adaptations of the spaces introduced in [2] by Bourgain for KdV and NLS to the generalized Benjamin-Ono setting, where, in light of the examples in [20], we are forced into introducing some version of the spatial weight.)

We will prove in Theorem 1 that $Z^{1/2}_{s_*,T}$, where s_* is as in (1.6), is a space of evolution of the initial data in F^{s_*} through the IVP (1.1).

We are now ready to state the main result of this paper.

Theorem 1. *Assume that $a \in (0,1)$. Then for any $u_0 \in F^s$ with $s \geq s_* = 1/2 + a/2$, there exist $T = T(\|u_0\|_{F^{s_*}})$ and a unique solution u for the IVP* (1.1), *such that $u \in Z^{1/2}_{s,T} \cap C([0,T], F^s)$. Moreover, given any $T' \in (0,T)$, there exists a neighborhood U_{u_0} of $u_0 \in F^s$ such that the map from U_{u_0} to $Z^{1/2}_{s,T'}$, that associates to the initial data the unique solution of the IVP* (1.1), *is continuous, and in fact, real analytic.*

Remark 1.3. If $u(x,t)$ solves (1.1), then $u_\sigma(x,t) = \sigma^{1+a} u(\sigma x, \sigma^{2+a} t)$ is also a solution, at least formally. This dilation invariance distinguishes the scaling-invariant Sobolev index for (1.1) to be $s = \frac{1}{2} - a$. We observe therefore that Theorem 1 is close to the scaling-invariant result near $a = 0$ and assumes much more regularity on the data than scaling suggests is required for a near 1. Note also that the results obtained here in the $a \geq \frac{4}{5}$ are inferior to those obtained by Kenig, Ponce, and Vega in [11] in the sense that we require more spatial regularity of the data as well as integrability against a weight. Further remarks on the near optimality of our results near $a = 0$ are made at the end of Section 3.

Remark 1.4. Theorem 1 could be iterated to obtain a global well-posedness result for (1.1) for initial data in the space F^{s_*} if one could prove that $\|u(t)\|_{F^{s_*}}$ is finite for all time. Recalling (1.7), (1.4), and (1.5), all that remains to be shown is the finiteness for all time of $\|xu(t)\|_{H^{-s_*}}$. It may be possible that a Gronwall-type argument like that presented in Theorem

3.3 of [24] for the KP equation will prove this. Note that the results in [11] already provide global well-posedness for (1.1) in the energy space identified using (1.4) in the regime $a \geq \frac{4}{5}$.

The paper is organized as follows: in Section 2 we present some estimates regarding the solution of the linear homogeneous and inhomogeneous IVP associated with (1.1). In Section 3 we present some bilinear estimates needed in Section 4, where we prove Theorem 1 using a contraction method. The paper ends with Section 5, an appendix in which we prove some estimates involving the spaces X_s^b and Y_{s_0,s_1}^b and cut-off functions in time.

2. The linear estimates

We denote with $W(t)u_0$ the solution of the linear IVP

$$\partial_t u + \partial_x D_x^{1+a} u = 0, \qquad u(x,0) = u_0(x) \qquad\qquad x \in \mathbb{R},\, t \in \mathbb{R}. \tag{2.1}$$

The first lemma we present contains a priori estimates for the solution $W(t)u_0$.

Lemma 2.1. *For any* $(\theta, \beta) \in [0,1] \times [0, a/2]$, *we have*

$$\Big(\int_{\mathbb{R}} \|D_x^{\theta\beta/2} W(t) u_0\|_p^q \, dt \Big)^{1/q} \quad \leq \quad C \|u_0\|_{L^2} \tag{2.2}$$

$$\Big(\int_{\mathbb{R}} \|D_x^{\theta\beta} W(t-t') f(\cdot, t') \, dt'\|_{L_x^p}^q \, dt \Big)^{1/q} \quad \leq \quad C \|f\|_{L_t^{q'} L_x^{p'}}, \tag{2.3}$$

where $(q,p) = (2(2+a)/(\theta(\beta+1)), 2/(1-\theta))$ *and* $1/p + 1/p' = 1/q + 1/q' = 1$.

The proof of this lemma is due to Kenig, Ponce, and Vega [12].

A consequence of this lemma is the following Strichartz inequality:

Corollary 1. *Let*[6] $b_0 = (3+a)/(4(2+a))$; *then for any* $f \in X_0^{b_0}$

$$\|f\|_{L^4} \leq C \|f\|_{X_0^{b_0}}. \tag{2.4}$$

Proof. We first show that

$$\|f\|_{L^2} \leq \|f\|_{X_0^0}. \tag{2.5}$$

In fact, for a sequence of numbers c_j, $\|c_j\|_{l^2} \leq \|c_j\|_{l^1}$, and it's enough to write

$$\|f\|_{L^2} = \Big(\sum_{j \geq 0} \int \chi_j(\lambda - \omega(\xi)) |\hat{f}|^2(\lambda, \xi) \, d\xi \, d\lambda \Big)^{1/2}$$

[6]Notice that for $a \in [0,1]$ we have $b_0 \in (0, 1/2)$.

and set $c_j = \big(\int \chi_j(\lambda-\omega(\xi))|\hat f|^2(\lambda,\xi)\,d\xi\,d\lambda\big)^{1/2}$. Next we take $\beta=0$ and $p=q$ in (2.2); hence, $\theta=(2+a)/(3+a)$ and $p=q=2(3+a)$. A standard argument based on a layer decomposition along the translates of the surface $S=\{(\xi,\omega(\xi))\}$ (see for example [7]) gives

$$\|f\|_{L^{2(3+a)}} \le C\|f\|_{X_0^{1/2}}. \tag{2.6}$$

We now interpolate (2.5) and (2.6). We write $1/4=\theta/2+(1-\theta)/2(3+a)$, $b_0=(1-\theta)/2$, and after a simple calculation it follows that $\theta=(1+a)/(2(2+a))$ and $b_0=(3+a)/(4(2+a))$. □

We proceed now to the estimate of the group $W(t)$ in the spaces X_s^b and $Y^b_{s_0,s_1}$. In the following we will always assume that $\psi(t)$ is a smooth cutoff function supported in the interval $[-1,1]$.

Lemma 2.2. *For any $\delta\in(0,1)$ and $u_0\in H^s$, $s\in\mathbb{R}$, we have*

$$\|\psi(t/\delta)W(t)u_0\|_{X_s^{1/2}} \le C\|u_0\|_{H^s},$$

where $C>0$ is independent of δ.

Proof. Observe that

$$F(\psi(t/\delta)W(t)u_0)(\xi,\lambda)=\delta\hat\psi(\delta(\lambda-\omega(\xi)))\widehat{u_0}(\xi). \tag{2.7}$$

We thus need to estimate

$$\sum_{j\ge0}2^{j/2}\Big(\int_{\mathbb{R}^2}(1+|\xi|)^{2s}|\widehat{u_0}(\xi)|^2\delta^2|\hat\psi(\delta(\lambda-\omega(\xi)))|^2\chi_j(\lambda-\omega(\xi))\,d\xi\,d\lambda\Big)^{1/2}.$$

We first integrate the λ variable, so we need to evaluate

$$I_j=\delta\Big(\int_{\mathbb{R}}|\hat\psi(\delta\lambda)|^2\chi_j(\lambda)\,d\lambda\Big)^{1/2}.$$

For $j=0$

$$I_0=\delta\Big(\int_{|\lambda|\le1}|\hat\psi(\delta\lambda)|^2\,d\lambda\Big)^{1/2}=\delta\Big(\int_{|r|\le\delta}|\hat\psi(r)|^2\frac{dr}{\delta}\Big)^{1/2}=C\delta.$$

For $j>0$

$$\begin{aligned}
I_j &= \delta\Big(\int_{2^{j-1}\le|\lambda|\le2^j}|\hat\psi(\delta\lambda)|^2\,d\lambda\Big)^{1/2}=\delta^{1/2}\Big(\int_{\delta2^{j-1}\le|r|\le\delta2^j}|\hat\psi(r)|^2\,dr\Big)^{1/2}\\
&\le \delta^{1/2}\Big(\int_{\delta2^{j-1}\le|r|\le\delta2^j}|\hat\psi(r)|^2(1+|r|)^{2N}\frac{dr}{(1+|r|)^{2N}}\Big)^{1/2}\\
&= C\|\hat\psi(r)(1+|r|)^N\|_{L^\infty}\delta^{1/2}\frac{(\delta2^j)^{1/2}}{(1+\delta2^j)^N},
\end{aligned}$$

for any $N \in \mathbb{N}$, $N > 1$. We then obtain

$$\begin{aligned}\|\psi(t/\delta)W(t)u_0\|_{X_s^{1/2}} &\leq \delta\|u_0\|_{H^s}\\ &+ \Big(\sum_{j\geq 1}\frac{\delta 2^j}{(1+\delta 2^j)^N}\Big)\|\hat{\psi}(r)(1+|r|)^N\|_{L^\infty}\|u_0\|_{H^s}.\end{aligned}$$

But it may be shown that $\sum_{j\geq 1}\frac{\delta 2^j}{(1+\delta 2^j)^N} \leq C$, uniformly for $N > 1$. Hence the lemma follows. □

The companion of the estimate we just proved is in the following lemma.

Lemma 2.3. *For any $\delta \in (0,1)$ and $u_0 \in H^s$, $s \in \mathbb{R}$, we have*

$$\|\psi(t/\delta)W(t)u_0\|_{Y^{1/2}_{s-2s_*,s}} \leq C\|u_0\|_{F^s},$$

where $C > 0$ is independent of δ.

Proof. Using again (2.7) we write

$$\frac{\partial}{\partial\lambda}F(\psi(t/\delta)W(t)u_0)(\xi,\lambda) = \delta^2\hat{\psi}'(\delta(\lambda-\omega(\xi))\widehat{u_0}(\xi),$$

so that by Lemma 2.2

$$\|t\psi(t/\delta)W(t)u_0\|_{X_s^{1/2}} \leq C\delta\|u_0\|_{H^s}.$$

On the other hand,

$$\begin{aligned}&\frac{\partial}{\partial\xi}F(\psi(t/\delta)W(t)u_0)(\xi,\lambda)\\ &= \delta(\hat{\psi}(\delta(\lambda-\omega(\xi))))\frac{\partial}{\partial\xi}\widehat{u_0}(\xi) - \delta^2\hat{\psi}'(\delta(\lambda-\omega(\xi)))\omega'(\xi)\widehat{u_0}(\xi).\end{aligned}$$

We recall that $|\omega'(\xi)| \sim |\xi|^{1+a}$; hence, by Lemma 2.2,

$$\|x\psi(t/\delta)W(t)u_0\|_{X^{1/2}_{s-2s_*}} \leq C\|xu_0\|_{H^{s-2s_*}} + C\delta\|u_0\|_{H^s},$$

as desired. □

We now need a version of Lemmas 2.2 and 2.3 for the solution of the inhomogeneous linear problem.

Lemma 2.4. *For any $\epsilon \in (0,1)$ and for any $\delta \in (0,1)$,*

$$\Big\|\psi(t/\delta)\int_0^t W(t-t')h(t')\,dt'\Big\|_{X_s^{1/2}} \leq C_\epsilon\delta^{-\epsilon}\|h\|_{X_s^{-1/2}}.$$

Proof. We follow the arguments of Kenig, Ponce, and Vega in [14]. We write

$$\begin{aligned} & \psi(t/\delta)\int_0^t W(t-t')h(t')\,dt' \\ = \; & \psi(t/\delta)\int_{\mathbb{R}^2} e^{ix\xi}\hat{h}(\xi,\lambda)\psi(\lambda-\omega(\xi))\frac{e^{i\lambda t}-e^{i\omega(\xi)t}}{\lambda-\omega(\xi)}\,d\lambda\,d\xi \\ + \; & \psi(t/\delta)\int_{\mathbb{R}^2} e^{ix\xi}\hat{h}(\xi,\lambda)[1-\psi(\lambda-\omega(\xi))]\frac{e^{i\lambda t}-e^{i\omega(\xi)t}}{\lambda-\omega(\xi)}\,d\lambda\,d\xi = I + II. \end{aligned}$$

To estimate I we first perform a Taylor expansion:

$$I = \sum_{k=1}^{\infty}\frac{i^k}{k!}t^k\psi(t/\delta)\int_{\mathbb{R}} e^{ix\xi+t\omega(\xi)}\Big(\int_{\mathbb{R}}\hat{h}(\xi,\lambda)\psi(\lambda-\omega(\xi))(\lambda-\omega(\xi))^{k-1}\,d\lambda\Big)d\xi.$$

Now let $t^k\psi(t/\delta) = \delta^k(t/\delta)^k\psi(t/\delta) = \delta^k\psi_k(t/\delta)$, for $k \in \mathbb{N}$. Then

$$I = \sum_{k=1}^{\infty}\frac{i^k}{k!}\delta^k\psi_k(t/\delta)W(t)G(x),$$

where

$$\hat{G}(\xi) = \int_{\mathbb{R}}\hat{h}(\xi,\lambda)\psi(\lambda-\omega(\xi))(\lambda-\omega(\xi))^{k-1}\,d\lambda. \tag{2.8}$$

We want to use Lemma 2.2 together with its proof. To do so we first need to estimate $|\widehat{\psi_k}(s)|$ and $|\widehat{\psi_k}(s)|(1+|s|)^N$ for $N=2$, uniformly with respect to $k \in \mathbb{N}$. Note that

$$|\widehat{\psi_k}(s)| \le \int_{|t|\le 1}|t|^k|\psi(t)|\,dt \le C,$$

uniformly in k. On the other hand,

$$\begin{aligned} \widehat{\psi_k}(s) &= \int_{|t|\le 1} e^{ist}t^k\psi(t)\,dt \\ &= (is)^{-2}\int_{|t|\le 1}\partial_t^2(e^{ist})t^k\psi(t)\,dt = (is)^{-2}\int_{|t|\le 1}(e^{ist})\partial_t^2(t^k\psi(t))\,dt, \end{aligned}$$

and for $|s| \ge 1$ it follows that

$$|\widehat{\psi_k}(s)| \le C\frac{(1+k)^2}{(1+|s|)^2},$$

uniformly with respect to k. Then by Lemma 2.2 $\|I\|_{X_s^{1/2}} \le C\|G\|_{H^s}$, and

$$\|G\|_{H^s} = \Big(\int_{\mathbb{R}}(1+|\xi|)^{2s}\Big|\int_{\mathbb{R}}\hat{h}(\xi,\lambda)\psi(\lambda-\omega(\xi))(\lambda-\omega(\xi))^{k-1}\,d\lambda\Big|^2 d\xi\Big)^{1/2}$$

$$\leq \quad \Big(\int_{\mathbb{R}} (1+|\xi|)^{2s} \int_{|\lambda-\omega(\xi)|\leq 1} |\hat{h}(\xi,\lambda)|^2 \, d\lambda \, d\xi \Big)^{1/2} \leq \|h\|_{X_s^{-1/2}},$$

as desired. This takes care of I. To estimate II we write $II = II_1 + II_2$, where

$$\begin{aligned} II_1 &= -\psi(t/\delta) \int_{\mathbb{R}^2} e^{ix\xi} e^{i\omega(\xi)t} [1-\psi(\lambda-\omega(\xi))] \frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)} \, d\lambda \, d\xi, \\ II_2 &= \psi(t/\delta) \int_{\mathbb{R}^2} e^{ix\xi} e^{i\lambda t} [1-\psi(\lambda-\omega(\xi))] \frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)} \, d\lambda \, d\xi. \end{aligned}$$

In view of Lemma 2.2, to estimate II_1 we need only to show that

$$\Big\| F^{-1}\Big(\int_{\mathbb{R}} [1-\psi(\lambda-\omega(\xi))] \frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)} \, d\lambda \Big) \Big\|_{H^s} \leq C\|h\|_{X_s^{-1/2}}. \tag{2.9}$$

In fact,

$$\begin{aligned} &\Big(\int_{\mathbb{R}} (1+|\xi|)^{2s} \Big(\int_{\mathbb{R}} [1-\psi(\lambda-\omega(\xi))] \frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)} \, d\lambda \Big)^2 d\xi \Big)^{1/2} \\ \leq \; &\Big(\int_{\mathbb{R}} (1+|\xi|)^{2s} \Big(\sum_{j\geq 1} \int_{\mathbb{R}} |\hat{h}(\xi,\lambda)| \frac{\chi_j(\lambda-\omega(\xi))}{2^j} \, d\lambda \Big)^2 d\xi \Big)^{1/2} \\ \leq \; &\sum_{j\geq 1} \Big(\int_{\mathbb{R}^2} (1+|\xi|)^{2s} 2^{-j} |\hat{h}(\xi,\lambda)|^2 \chi_j(\lambda-\omega(\xi)) \, d\lambda \, d\xi \Big)^{1/2}, \end{aligned}$$

and (2.9) follows. Finally, for II_2 we use Lemma 5.1, and we reduce the matter to estimating

$$\begin{aligned} &\Big\| \int_{\mathbb{R}^2} e^{ix\xi+i\lambda t} [1-\psi(\lambda-\omega(\xi))] \frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)} \, d\lambda \, d\xi \Big\|_{X_s^{1/2}} \\ &\leq \sum_{j\geq 0} 2^{j/2} \Big(\int_{\mathbb{R}^2} \frac{|\hat{h}(\xi,\lambda)|^2}{(1+|\lambda-\omega(\xi)|)^2} (1+|\xi|)^{2s} \chi_j(\lambda-\omega(\xi)) \, d\lambda \, d\xi \Big)^{\frac{1}{2}} \leq C\|h\|_{X_s^{-\frac{1}{2}}}, \end{aligned}$$

and this conclues the proof of the lemma. □

We now prove a corresponding lemma for the space $Y^{1/2}_{s-2s_*,s}$.

Lemma 2.5. *For any $\epsilon \in (0,1)$ and for any $\delta \in (0,1)$,*

$$\Big\| \psi(t/\delta) \int_0^t W(t-t') h(t') \, dt' \Big\|_{Y^{1/2}_{s-2s_*,s}} \leq C_\epsilon \delta^{-\epsilon} (\|h\|_{Y^{-1/2}_{s-2s_*,s}} + \|h\|_{X_s^{-1/2}}).$$

Proof. We perform a decomposition into I, II_1, and II_2 as in Lemma 2.4. For I we use Lemma 2.3, and we need only to estimate

$$\|G\|_{F^s} = \|G\|_{H^s} + \|xG\|_{H^{s-2s_*}},$$

where G is defined in (2.8). In Lemma 2.4 we proved that $\|G\|_{H^s} \leq C\|h\|_{X_s^{-\frac{1}{2}}}$. To estimate $\|xG\|_{H^{s-2s_*}}$, we first observe that

$$\begin{aligned}\frac{\partial}{\partial\xi}\hat{G}(\xi) &= \int_{\mathbb{R}} \frac{\partial}{\partial\xi}\hat{h}(\xi,\lambda)\psi(\lambda-\omega(\xi))(\lambda-\omega(\xi))^{k-1}\,d\lambda \\ &- (k-1)\omega'(\xi)\int_{\mathbb{R}} \hat{h}(\xi,\lambda)\psi(\lambda-\omega(\xi))(\lambda-\omega(\xi))^{k-2}\,d\lambda \\ &- \omega'(\xi)\int_{\mathbb{R}} \hat{h}(\xi,\lambda)\psi'(\lambda-\omega(\xi))(\lambda-\omega(\xi))^{k-1}\,d\lambda = \widehat{J}_1+\widehat{J}_2+\widehat{J}_3.\end{aligned}$$

We can now write

$$\|xG\|_{H^{s-2s_*}} \leq \sum_{i=1}^{3} \|J_i\|_{H^{s-2s_*}}.$$

Following the argument in the proof of Lemma 2.4, it is easy to see that

$$\|J_1\|_{H^{s-2s_*}} \leq C\|xh\|_{X^{-1/2}_{s-2s_*}} \leq C\|h\|_{Y^{-1/2}_{s-2s_*,s}}.$$

On the other hand, because $|\omega'(\xi)| \sim |\xi|^{1+a}$, it follows that $(1+|\xi|)^{s-2s_*}|\omega'(\xi)| \lesssim (1+|\xi|)^s$; hence,

$$\sum_{i=2,3} \|J_i\|_{H^{s-2s_*}} \leq Ck\|h\|_{X_s^{-1/2}},$$

which suffices.

For II_1, we use again Lemma 2.3, and we have to show that

$$\left\|F^{-1}\Big(\int_{\mathbb{R}} [1-\psi(\lambda-\omega(\xi))]\frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)}\,d\lambda\Big)\right\|_{F^s} \leq C(\|h\|_{Y^{-1/2}_{s-2s_*,s}} + \|h\|_{X_s^{-1/2}}). \tag{2.10}$$

We observe that

$$\begin{aligned}&\frac{\partial}{\partial\xi}\Big(\int_{\mathbb{R}} [1-\psi(\lambda-\omega(\xi))]\frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)}\,d\lambda\Big) = \int_{\mathbb{R}} \frac{\partial}{\partial\xi}\hat{h}(\xi,\lambda)\frac{[1-\psi(\lambda-\omega(\xi))]}{\lambda-\omega(\xi)}\,d\lambda \\ &+ \omega'(\xi)\int_{\mathbb{R}} \hat{h}(\xi,\lambda)\frac{\psi'(\lambda-\omega(\xi))}{\lambda-\omega(\xi)}\,d\lambda + \omega'(\xi)\int_{\mathbb{R}} \hat{h}(\xi,\lambda)\frac{[1-\psi(\lambda-\omega(\xi))]}{(\lambda-\omega(\xi))^2}\,d\lambda \\ &= \widehat{M_1}+\widehat{M_2}+\widehat{M_3},\end{aligned}$$

and using both the argument in the proof of Lemma 2.4 and the observations above we have

$$\begin{aligned} \|M_1\|_{H^s} &\le C\|xh\|_{X^{-1/2}_{s-2s_*}} \le C\|h\|_{Y^{-1/2}_{s-2s_*,s}}, \\ \sum_{i=2,3} \|M_i\|_{H^{s-2s_*}} &\le C\|h\|_{X^{-1/2}_s}, \end{aligned}$$

which combined with (2.9) gives (2.10). Finally we turn to II_2. We use Lemma 5.4, and we obtain

$$\begin{aligned} &\left\|\psi(t/\delta)\int_{\mathbb{R}^2} e^{ix\xi+i\lambda t}[1-\psi(\lambda-\omega(\xi))]\frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)}\,d\lambda\,d\xi\right\|_{Y^{1/2}_{s-2s_*,s}} \\ &\le C_\epsilon\delta^{-\epsilon}\left\|\int_{\mathbb{R}^2} e^{ix\xi+i\lambda t}[1-\psi(\lambda-\omega(\xi))]\frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)}\,d\lambda\,d\xi\right\|_{Y^{1/2}_{s-2s_*,s}}. \end{aligned}$$

We have that

$$\begin{aligned} &\frac{\partial}{\partial\xi}\Big(\hat{h}(\xi,\lambda)\frac{[1-\psi(\lambda-\omega(\xi))]}{\lambda-\omega(\xi)}\Big) = \frac{\partial}{\partial\xi}\hat{h}(\xi,\lambda)\frac{[1-\psi(\lambda-\omega(\xi))]}{\lambda-\omega(\xi)} \\ &+\omega'(\xi)\hat{h}(\xi,\lambda)\frac{\psi'(\lambda-\omega(\xi))}{\lambda-\omega(\xi)} + \omega'(\xi)\hat{h}(\xi,\lambda)\frac{[1-\psi(\lambda-\omega(\xi))]}{(\lambda-\omega(\xi))^2}, \end{aligned}$$

so we obtain

$$\begin{aligned} &\left\|x\int_{\mathbb{R}^2} e^{ix\xi+i\lambda t}[1-\psi(\lambda-\omega(\xi))]\frac{\hat{h}(\xi,\lambda)}{\lambda-\omega(\xi)}\,d\lambda\,d\xi\right\|_{X^{1/2}_{s-2s_*}} \\ &\le C(\|h\|_{Y^{-1/2}_{s-2s_*,s}} + \|h\|_{X^{-1/2}_s}). \end{aligned}$$

Next we turn to ∂_λ:

$$\begin{aligned} &\frac{\partial}{\partial\lambda}\Big(\hat{h}(\xi,\lambda)\frac{[1-\psi(\lambda-\omega(\xi))]}{\lambda-\omega(\xi)}\Big) = \frac{\partial}{\partial\lambda}\hat{h}(\xi,\lambda)\frac{[1-\psi(\lambda-\omega(\xi))]}{\lambda-\omega(\xi)} \\ &-\hat{h}(\xi,\lambda)\frac{\psi'(\lambda-\omega(\xi))}{\lambda-\omega(\xi)} + \hat{h}(\xi,\lambda)\frac{[1-\psi(\lambda-\omega(\xi))]}{(\lambda-\omega(\xi))^2} = \sum_{i=1,2,3}\widehat{W}_i, \end{aligned}$$

and following the proof in Lemma 2.4 it is easy to see that

$$\|W_1\|_{X^{1/2}_s} \le C\|th\|_{X^{-1/2}_s}, \qquad \sum_{i=2,3}\|(W_i)\|_{X^{1/2}_s} \le C\|h\|_{X^{-1/2}_s}.$$

The proof of the lemma is now complete. □

Remark 2.1. The results in the Appendix, Lemma 2.4, and Lemma 2.5 are probably also true with $\epsilon = 0$.

3. The bilinear estimates

This section represents the heart of the matter of this paper. We start with two propositions in which we prove bilinear estimates in the spaces $X^b_{s_*}$ and $Y^{-b}_{-s_*,s_*}$. After the proofs of these estimates, we analyze in detail certain low-/high-frequency interactions which reveal the near optimality of our bilinear estimates.

Proposition 3.1. *If $a \in (0,1)$ and $s_* = 1/2 + a/2$, there exists $0 < b < 1/2$ such that*

$$\|\partial_x(uv)\|_{X^{-b}_{s_*}} \leq C\|u\|_{X^{1/2}_{s_*}}\big(\|v\|_{X^{1/2}_{s_*}} + \|v\|^{1/2}_{X^{1/2}_{s_*}}\|v\|^{1/2}_{Y^{1/2}_{-s_*,s_*}}\big) \quad (3.1)$$
$$+ C\|v\|_{X^{1/2}_{s_*}}\big(\|u\|_{X^{1/2}_{s_*}} + \|u\|^{1/2}_{X^{1/2}_{s_*}}\|u\|^{1/2}_{Y^{1/2}_{-s_*,s_*}}\big).$$

The companion estimate in the space $Y^{-b}_{-s_*,s_*}$ takes the following form:

Proposition 3.2. *If $a \in (0,1)$ and $s_* = 1/2 + a/2$, there exists $0 < b < 1/2$ such that*

$$\|\partial_x(uv)\|_{Y^{-b}_{-s_*,s_*}} \leq C\|u\|_{Y^{1/2}_{-s_*,s_*}}\big(\|v\|_{X^{1/2}_{s_*}} + \|v\|^{1/2}_{X^{1/2}_{s_*}}\|v\|^{1/2}_{Y^{1/2}_{-s_*,s_*}}\big) \quad (3.2)$$
$$+ C\|v\|_{Y^{1/2}_{-s_*,s_*}}\big(\|u\|_{X^{1/2}_{s_*}} + \|u\|^{1/2}_{X^{1/2}_{s_*}}\|u\|^{1/2}_{Y^{1/2}_{-s_*,s_*}}\big) + C\|u\|_{X^{1/2}_{s_*}}\|v\|_{X^{1/2}_{s_*}}.$$

Proof. [Proof of Proposition 3.1] We write

$$|F(\partial_x(uv))(\xi,\lambda)| = |\xi|\Big|\int_{\mathbb{R}^2}\hat{u}(\xi_1,\lambda_1)\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1\Big|,$$

and we split the set of integration into the two regions where $|\xi-\xi_1| \leq |\xi_1|$ and $|\xi-\xi_1| > |\xi_1|$. When we integrate in the first region we will obtain the first half of the right-hand side of (3.1); when we integrate on the second region we will obtain the second half. So without loss of generality we can assume that $|\xi-\xi_1| \leq |\xi_1|$. Using duality and setting $\xi_2 = \xi - \xi_1$ allows us to rewrite the left-hand side of (3.1) as

$$\sup_{\|g_j\|_{L^2}\leq 1}\sum_{j\geq 0}\sum_{j_1,j_2\geq 0}2^{-bj}\int_{|\xi_2|\leq|\xi_1|} g_j(\xi_1+\xi_2,\lambda_1+\lambda_2)\chi_j(\lambda_1+\lambda_2-\omega(\xi_1+\xi_2)) \quad (3.3)$$
$$\times\frac{|\xi_1+\xi_2|\max(1,|\xi_1+\xi_2|)^{s_*}}{\max(1,|\xi_1|)^{s_*}\max(1,|\xi_2|)^{s_*}}\phi_{1,j_1}(\xi_1,\lambda_1)\phi_{2,j_2}(\xi_2,\lambda_2)\,d\xi_1\,d\lambda_1\,d\xi_2\,d\lambda_2,$$

where, if we set $u = u_1$ and $v = u_2$, then

$$\phi_{i,j_i}(\xi_i,\lambda_i) = |\hat{u}_i(\xi_i,\lambda_i)|\chi_{j_i}(\lambda_i-\omega(\xi_i))\max(1,|\xi_i|)^{s_*}$$

and

$$\|u_i\|_{X^{1/2}_{s_*}} = \sum_{j_i \geq 0} 2^{j_1/2} \|\phi_{i,j_i}\|_{L^2}.$$

The analysis of (3.3) is obtained by considering different cases.

Case 1: $|\xi_1| \leq 1$. Then $|\xi_2| \leq 1$ and $|\xi_2 + \xi_1| \leq 2$. We observe that if $G_j = F^{-1}(g_j(\xi,\lambda)\chi_j(\lambda - \omega(\xi)))$, then $\|G_j\|_{L^2} \leq 1$. We use the Strichartz inequality (2.4) to obtain the bound

$$\begin{aligned}(3.3) \quad &\leq \sum_{j\geq 0} \sum_{j_i, j_2 \geq 0} 2^{-jb} \|G_j\|_{L^2} \|F^{-1}(\phi_{1,j_1})\|_{L^4} \|F^{-1}(\phi_{2,j_2})\|_{L^4} \\ &\leq C \sum_{j\geq 0} \sum_{j_i, j_2 \geq 0} 2^{-jb} 2^{j_1 b_0} \|\phi_{1,j_1}\|_{L^2} 2^{j_2 b_0} \|\phi_{2,j_2}\|_{L^2},\end{aligned}$$

and this gives the desired estimate as long as $b > 0$.

Case 2: $|\xi_1 + \xi_2| \leq 1$ and $|\xi_1| \geq 1$. This case is treated like Case 1.

Case 3: $|\xi_1| \geq 1$, $|\xi_1 + \xi_2| \geq 1$, and $1/4|\xi_1| \leq |\xi_2| \leq |\xi_1|$. Then by simple arguments

$$(3.3) \leq \sum_{j\geq 0} \sum_{j_1, j_2 \geq 0} 2^{-jb} \int_{\mathbb{R}^2} g_j \chi_j |\xi_1 + \xi_2|^{1-s_*} \phi_{1,j_1} \phi_{2,j_2} \, d\xi_1 \, d\lambda_1 \, d\xi_2 \, d\lambda_2. \quad (3.4)$$

We next perform a dyadic decomposition of ξ_1, hence ξ_2, by setting $\xi_i \sim 2^{m_1}$, $m_1 \sim m_2$. We then continue with

$$(3.4) \leq \sum_{m_1 \geq 0} \sum_{j\geq 0} \sum_{j_1, j_2 \geq 0} 2^{-jb} \int_{\mathbb{R}^2} g_j \chi_j |\xi_1 + \xi_2|^{1-s_*} \phi_{1,m_1,j_1} \phi_{2,m_1,j_2} \, d\xi_1 \, d\lambda_1 \, d\xi_2 \, d\lambda_2, \quad (3.5)$$

where we used the notation $\phi_{i,m_i,j_i} = \phi_{i,j_i} \chi_{\{|\xi_i| \sim 2^{m_i}\}}$. We have to consider two subcases

Case 3a: $j \geq (1 - s_*)m_1/b$. We use again the Strichartz inequality (2.4) to obtain the following bound:

$$(3.5) \leq \sum_{m_1 \geq 0} \sum_{j \geq (1-s_*)m_1/b} \sum_{j_1, j_2 \geq 0} 2^{-jb} 2^{(1-s_*)m_1} 2^{j_1 b_0} 2^{j_2 b_0} \|\phi_{1,m_1,j_1}\|_{L^2} \|\phi_{2,m_1,j_2}\|_{L^2}. \quad (3.6)$$

We then sum in j to get

$$(3.6) \leq \sum_{m_1 \geq 0} \sum_{j_1, j_2 \geq 0} 2^{j_1 b_0} 2^{j_2 b_0} \|\phi_{1,m_1,j_1}\|_{L^2} \|\phi_{2,m_1,j_2}\|_{L^2},$$

and Cauchy-Schwarz in m_1 concludes the argument also in this case.

Case 3b: $j \leq (1 - s_*)m_1/b$. We change variables by setting

$$\lambda_i = \theta_i + \omega(\xi_i). \quad (3.7)$$

Then we can continue the inequality in (3.4) with

$$\begin{aligned}&\leq \sum_{m_1\geq 0}\sum_{0\leq j\leq (1-s_*)m_1/b}\sum_{j_1,j_2\geq 0} 2^{-jb}\int_{\mathbb{R}^2}|\xi_1+\xi_2|^{1-s_*} \\ &\times g_j(\xi_1+\xi_2,\theta_1+\omega(\xi_1)+\theta_2+\omega(\xi_2)) \\ &\times \chi_j(\theta_1+\omega(\xi_1)+\theta_2+\omega(\xi_2)-\omega(\xi_1+\xi_2))\phi_{1,m_1,j_1}\phi_{2,m_1,j_2}\,d\xi_1\,d\xi_2\,d\theta_1\,d\theta_2.\end{aligned} \tag{3.8}$$

Note now that

$$|\omega(\xi_1)+\omega(\xi_2)-\omega(\xi_1+\xi_2)|\leq C2^{\max(j_1,j_2,j)}. \tag{3.9}$$

At this point it is important to understand this expression in order to obtain extra constraints that will allow a summation in the expression given by (3.8).

We recall that we are analyzing the region where $1/4|\xi_1|\leq|\xi_2|\leq|\xi_1|$. Assume in addition that $\xi_1\geq 0$ and $\xi_2\geq 0$. Then we can write $\xi_2=\beta\xi_1$, where $1/4\leq\beta\leq 1$. Then

$$\begin{aligned}\omega(\xi_1+\xi_2)-\omega(\xi_1)-\omega(\xi_2)&=(\xi_1+\xi_2)^{2+a}-\xi_1^{2+a}-\xi_2^{2+a}\\ &=((1+\beta)^{2+a}-1-\beta^{2+a})\xi_1^{2+a}.\end{aligned}$$

We now consider the function $f(\beta)=(1+\beta)^{2+a}-1-\beta^{2+a}$, and we observe that $f'(\beta)=(2+a)((1+\beta)^{1+a}-\beta^{1+a})\geq 0$; because $f(1/4)=c>0$ it follows that

$$|\omega(\xi_1)+\omega(\xi_2)-\omega(\xi_1+\xi_2)|\geq c|\xi_1|^{2+a}.$$

If $\xi_1\leq 0$ and $\xi_2\leq 0$ the argument can be reapplied thanks to the presence of the absolute value.

If $\xi_1\geq 0$ and $\xi_2\leq 0$, then $\xi_1+\xi_2\geq 0$. If we set $\xi_2=-\beta\xi_1$, then again $1/4\leq\beta\leq 1$ and

$$\begin{aligned}\omega(\xi_1+\xi_2)-\omega(\xi_1)-\omega(\xi_2)&=(\xi_1+\xi_2)^{2+a}-\xi_1^{2+a}+|\xi_2|^{2+a}\\ &=((1-\beta)^{2+a}-1+\beta^{2+a})\xi_1^{2+a}.\end{aligned}$$

In this case we define the function $f(\beta)=1-(1-\beta)^{2+a}-\beta^{2+a}$, and $f'(\beta)=(2+a)((1-\beta)^{1+a}-\beta^{1+a})$. We first analyze f in the range $1/2\leq\beta\leq 1$. Here $f'(\beta)\leq 0$ and $f(1)=0$, so that $f(\beta)\geq 0$ as $\beta\to 1$, and by Taylor expansion $f(\beta)=-(2+a)(\beta-1)+O((\beta-1)^2)$ and $f(\beta)\geq c(1-\beta)$ for $1/2\leq\beta\leq 1$. Next, for $1/4\leq\beta\leq 1/2$, $f'(\beta)\geq 0$, $f(1/4)>0$; hence, also in this case $f(\beta)\geq c(1-\beta)$, and so

$$|\omega(\xi_1)+\omega(\xi_2)-\omega(\xi_1+\xi_2)|\geq c|\xi_1|^{2+a}(1-\beta)=c|\xi_1|^{1+a}|\xi_1+\xi_2|.$$

Finally, we consider the case $\xi_1 < 0$ and $\xi_2 \geq 0$. Then $\xi_1 + \xi_2 \leq 0$, and we have

$$\omega(\xi_1+\xi_2) - \omega(\xi_1) - \omega(\xi_2) = -|\xi_1+\xi_2|^{2+a} + |\xi_1|^{2+a} - |\xi_2|^{2+a},$$

which behaves as in the previous case. We summarize our findings as follows: if $1/4|\xi_1| \leq |\xi_2| \leq |\xi_1|$, then

$$|\omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1+\xi_2)| \geq \begin{cases} c|\xi_1|^{2+a} & \text{if } \ \xi_1\xi_2 \geq 0 \\ c|\xi_1|^{1+a}|\xi_1+\xi_2| & \text{if } \ \xi_1\xi_2 \leq 0. \end{cases} \tag{3.10}$$

Assume now that $j = \max(j, j_1, j_2)$. Then, using (3.9) and (3.10), we have that

$$2^{(1+a)m_1} \leq C2^j \leq C2^{(1-s_*)m_1/b},$$

which is not possible since $(1+a) > (1-s_*)/b$ if we take b such that $\frac{1-a}{2(1+a)} < b < 1/2$. Thus, $j < \max(j_1, j_2)$. Assume $j_1 \geq j_2$; the other case is identical. We then use our lower bound in (3.10) to conclude that

$$|\xi_1+\xi_2|^{1-s_*} \leq C2^{j_1(1-s_*)}2^{-m_1(1+a)(1-s_*)}.$$

We insert this estimate in (3.8), and we use the Strichartz estimate (2.4) to obtain

$$\begin{aligned}(3.8) \leq \sum_{m_1\geq 0}\sum_{j_1,j_2\geq 0}\sum_{0\leq j\leq (1-s_*)m_1/b} & 2^{-jb}2^{jb_0}2^{j_1(1-s_*)}2^{-m_1(1+a)(1-s_*)}2^{j_2 b_0} \\ & \times \|\phi_{1,m_1,j_1}\|_{L^2}\|\phi_{2,m_1,j_2}\|_{L^2}.\end{aligned}$$

We now take $b > b_0, b < 1/2$ and do the sum in j. Because $(1-s_*) = 1/2 - a/2 < 1/2$, we sum in m_1, and we obtain the desired result.

Case 4: $|\xi_1| \geq 1$, $|\xi_1+\xi_2| \geq 1$, and $|\xi_2| \leq 1/4|\xi_1|$. In this case $|\xi_1+\xi_2| \sim |\xi_1|$ and

$$(3.3) \leq \sum_{j\geq 0}\sum_{j_1,j_2\geq 0} 2^{-jb}\int_{\mathbb{R}^2} g_j\chi_j \frac{|\xi_1|}{\max(1,|\xi_2|)^{s_*}}\phi_{1,j_1}\phi_{2,j_2}\, d\xi_1\, d\lambda_1\, d\xi_2\, d\lambda_2. \tag{3.11}$$

We first use the change of variables (3.7), then we perform a dyadic decomposition so that $|\xi_1| \sim 2^{m_1}$ (and hence $|\xi_1+\xi_2| \sim 2^{m_1}$), and we can continue with

$$(3.11) \leq \sum_{m_1\geq 1}\sum_{j\geq 0}\sum_{j_1,j_2\geq 0} 2^{-jb}\int_{\mathbb{R}^2} g_{j,m_1}\chi_j 2^{m_1}\phi_{1,m_1,j_1}\phi_{2,j_2}\, d\xi_1\, d\xi_2\, d\theta_1\, d\theta_2. \tag{3.12}$$

We analyze two subcases.

Case 4a: $m_1 \leq (2b/(1-a) - \epsilon)j$, for some $\epsilon > 0$ to be chosen later.

Using an argument similar to the one used in [7], we make the following change of variables:

$$(u, w) = T(\xi_1, \xi_2), \tag{3.13}$$

where $u = T_1(\xi_1, \xi_2) = \xi_1 + \xi_2$, $w = T_2(\xi_1, \xi_2) = \theta_1 + \theta_2 + \omega(\xi_1) + \omega(\xi_2)$. It is easy to see that if J is the Jacobian of this change of variables, then

$$|J| = |\omega'(\xi_1) - \omega'(\xi_2)| \sim |\xi_1|^{1+a}. \tag{3.14}$$

Now we define $H(\theta_1, \theta_2, u, w) = \chi_j \phi_{1,m_1,j_1} \phi_{2,j_2} \circ T^{-1}(\theta_1, \theta_2, u, w)$, and we write

$$\begin{aligned}(3.12) \leq &\sum_{j\geq 0} \sum_{0\leq m_1 \leq j(2b/(1-a)-\epsilon)} \sum_{j_1,j_2\geq 0} 2^{-jb} 2^{m_1} \\ &\times \int_{\mathbb{R}^2} g_{j,m_1}(u,w) \frac{H(\theta_1,\theta_2,u,w)}{|J|}\, du\, dw\, d\theta_1\, d\theta_2 \\ \leq &\sum_{j\geq 0} \sum_{0\leq m_1 \leq j(2b/(1-a)-\epsilon)} \sum_{j_1,j_2\geq 0} 2^{-jb} 2^{m_1} 2^{-m_1(1+a)/2} \|g_{j,m_1}\|_{L^2} \\ &\times \int_{|\theta_i|\sim 2^{j_i}} \Big(\int\int \frac{H(\theta_1,\theta_2,u,w)^2}{|J|}\, du\, dw \Big)^{1/2} d\theta_1\, d\theta_2.\end{aligned}$$

Now we invert the change of variable in the last integration in u and w to continue with

$$\begin{aligned}\leq &\sum_{j\geq 0} \sum_{0\leq m_1 \leq j(2b/(1-a)-\epsilon)} \sum_{j_1,j_2\geq 0} 2^{-jb} 2^{m_1(1/2-a/2)} 2^{j_1/2} 2^{j_2/2} \\ &\times \|g_{j,m_1}\|_{L^2} \|\phi_{1,m_1,j_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2},\end{aligned}$$

and if we sum with respect to m_1 we obtain

$$\leq \sum_{j\geq 0} \sum_{j_1,j_2\geq 0} 2^{-jb} 2^{(2b/(1-a)-\epsilon)j(1/2-a/2)} 2^{j_1/2} 2^{j_2/2} \|\phi_{1,,j_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2},$$

and we finish by summing with respect to j, since

$$-b + (2b/(1-a) - \epsilon)(1/2 - a/2) < 0.$$

Case 4b: $m_1 > j(2b/(1-a) - \epsilon)$, for some $\epsilon > 0$ to be chosen later. This is the most delicate part of the proof. We return to (3.3), and we keep the function v in it. Then we use the change of variables (3.7), and we write

$$\begin{aligned}(3.3) \leq &\sum_{j\geq 0} \sum_{m_1 \geq j(2b/(1-a)-\epsilon)} \sum_{j_1,j_2\geq 0} 2^{-jb} 2^{m_1} \\ &\times \int g_{j,m_1} \chi_j \phi_{1,m_1,j_1} |\hat{v}| \chi_{j_2}\, d\xi_1\, d\xi_2\, d\theta_1\, d\theta_2.\end{aligned} \tag{3.15}$$

Next, for fixed ξ_1, θ_1, and θ_2, we estimate the measure of the set Δ_{ξ_2} such that $|\theta_1+\theta_2+\omega(\xi_1)+\omega(\xi_2)-\omega(\xi_1+\xi_2))| \sim 2^j$. Thus, let

$$f(\xi_2)=\theta_1+\theta_2+\omega(\xi_1)+\omega(\xi_2)-\omega(\xi_1+\xi_2);$$

then

$$f'(\xi_2)=\omega'(\xi_2)-\omega'(\xi_1+\xi_2)=(a+2)(|\xi_2|^{1+a}-|\xi_1+\xi_2|^{1+a}).$$

But in the region that we are analyzing $|\xi_2|\le 1/4|\xi_1|$ and $|\xi_1+\xi_2|\ge 3/4|\xi_1|$; hence, $|f'(\xi_2)|\ge C|\xi_1|^{1+a}$, and by the mean-value theorem the measure of Δ_{ξ_2} can be estimated by

$$|\Delta_{\xi_2}|\le C2^j2^{-m_1(1+a)}.$$

We then use the change of variables (3.13) and we obtain

$$\begin{aligned}(3.15) \le \sum_{j\ge 0}\sum_{m_1\ge j(2b/(1-a)-\epsilon)}\sum_{j_1,j_2\ge 0} & 2^{-jb}2^{m_1(1-a)/2}2^{j_1/2}2^{j_2/2}2^{(j-m_1(1+a))/2} \\ & \times \|g_{j,m_1}\|_{L^2}\|\phi_{1,m_1,j_1}\|_{L^2}\||\hat v|\chi_{j_2}\|_{L^2_{\theta_2},L^\infty_{\xi_2}} \\ \le \sum_{j\ge 0}\sum_{m_1\ge j(2b/(1-a)-\epsilon)}\sum_{j_1,j_2\ge 0} & 2^{j(1/2-b)}2^{-m_1a}2^{j_1/2}2^{j_2/2} \\ & \times \|g_{j,m_1}\|_{L^2}\|\phi_{1,m_1,j_1}\|_{L^2}\||\hat v|(\xi_2,\theta_2+\omega(\xi_2))\chi_{j_2}(\theta_2)\|_{L^2_{\theta_2},L^\infty_{\xi_2}}.\end{aligned} \tag{3.16}$$

We now sum in m_1 to obtain

$$\begin{aligned}(3.16) \le & \sum_{j\ge 0}\sum_{j_1,j_2\ge 0} 2^{j(1/2-b)}2^{-aj(2b/(1-a)-\epsilon)}2^{j_1/2}2^{j_2/2} \\ \times & \|g_j\|_{L^2}\|\phi_{1,j_1}\|_{L^2}\||\hat v|(\xi_2,\theta_2+\omega(\xi_2))\chi_{j_2}(\theta_2)\|_{L^2_{\theta_2},L^\infty_{\xi_2}}.\end{aligned} \tag{3.17}$$

Now observe that there exists $\epsilon>0$ such that

$$(1/2-b)-a(2b/(1-a)-\epsilon)<0 \tag{3.18}$$

if and only if $(1/2-b)-a2b/(1-a)<0$ if and only if $b>\frac{1-a}{2(1+a)}$, and because for $0<a<1$ it follows that $\frac{1-a}{2(1+a)}<1/2$, there exists b such that $\frac{1-a}{2(1+a)}<b<1/2$ and ϵ such that (3.18) is satisfied. We go back to (3.17) to sum in j and we obtain

$$(3.17)\le \sum_{j_1,j_2\ge 0}2^{j_1/2}2^{j_2/2}\|\phi_{1,j_1}\|_{L^2}\||\hat v|(\xi_2,\theta_2+\omega(\xi_2))\chi_{j_2}(\theta_2)\|_{L^2_{\theta_2},L^\infty_{\xi_2}}. \tag{3.19}$$

Next we note that if we set

$$|\hat v|(\xi_2,\theta_2+\omega(\xi_2))\chi_{j_2}(\theta_2)=|\hat v_{j_2}|(\xi_2,\theta_2+\omega(\xi_2)),$$

then

$$|\hat{v}_{j_2}(\xi_2, \theta_2 + \omega(\xi_2))|^2 \leq \int_{-\infty}^{\xi_2} |\hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2))| \Big| \frac{\partial}{\partial \nu_2} \hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2)) \Big| \, d\nu_2$$

$$\leq \int_{-\infty}^{\xi_2} |\hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2))| (1 + |\nu_2|)^{s_*} \frac{\Big| \frac{\partial}{\partial \nu_2} \hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2)) \Big|}{(1 + |\nu_2|)^{s_*}} \, d\nu_2$$

$$\leq \Big(\int_{-\infty}^{\infty} |\hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2))|^2 (1 + |\nu_2|)^{2s_*} \, d\nu_2 \Big)^{1/2}$$

$$\times \Big(\int_{-\infty}^{\infty} \Big| \frac{\partial}{\partial \nu_2} \hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2)) \Big|^2 \frac{d\nu_2}{(1 + |\nu_2|)^{2s_*}} \Big)^{1/2}.$$

Remark 3.1. We pause to make a technical remark regarding the use of the spatial weight in the present analysis. Observe that the $L^\infty_{\xi_2}$ norm first appeared in (3.16) and was just estimated using a Sobolev-type inequality. The appearance of the differentiation operator ∂_{ν_2} with respect to the Fourier variable corresponds with the weight x appearing in the definition (1.7) of the space F^s and the associated spacetime space Y defined in (1.9).

Because this estimate is independent of ξ_2 we can write

$$\begin{aligned} &\|\hat{v}_{j_2}(\xi_2, \theta_2 + \omega(\xi_2))\|_{L^2_{\theta_2} L^\infty_{\xi_2}} \\ &\leq \Big(\int_{\mathbb{R}} \Big(\int_{-\infty}^{\infty} |\hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2))|^2 (1 + |\nu_2|)^{2s_*} \, d\nu_2 \Big)^{1/2} \\ &\times \Big(\int_{-\infty}^{\infty} \Big| \frac{\partial}{\partial \nu_2} \hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2)) \Big|^2 \frac{d\nu_2}{(1 + |\nu_2|)^{2s_*}} \Big)^{1/2} d\theta_2 \Big)^{1/2}, \end{aligned}$$

and after Cauchy-Schwarz in θ_2 we can continue with

$$\begin{aligned} &\leq \Big(\int_{\mathbb{R}^2} |\hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2))|^2 (1 + |\nu_2|)^{2s_*} \, d\nu_2 \, d\theta_2 \Big)^{1/4} \\ &\times \Big(\int_{\mathbb{R}^2} \Big| \frac{\partial}{\partial \nu_2} \hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2)) \Big|^2 \frac{1}{(1 + |\nu_2|)^{2s_*}} \, d\nu_2 \, d\theta_2 \Big)^{1/4}. \end{aligned}$$

If we substitute this last estimate in (3.19), we sum in j_1, and we use Cauchy-Schwarz in j_2, we obtain

$$\begin{aligned} (3.19) \leq \|u\|_{X^{1/2}_{s_*}} &\Big(\sum_{j_2 \geq 0} \|\hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2))(1 + |\nu_2|)^{s_*}\|_{L^2} 2^{j_2/2} \Big)^{1/2} \\ &\times \Big(\sum_{j_2 \geq 0} \Big\| \frac{\partial}{\partial \nu_2} \hat{v}_{j_2}(\nu_2, \theta_2 + \omega(\nu_2))(1 + |\nu_2|)^{-s_*} \Big\|_{L^2} 2^{j_2/2} \Big)^{1/2}. \end{aligned} \tag{3.20}$$

Now observe that

$$\begin{aligned}&\Big|\frac{\partial}{\partial \nu_2}\hat{v}_{j_2}(\nu_2,\theta_2+\omega(\nu_2))\Big|\\&=\Big|\frac{\partial \hat{v}_{j_2}}{\partial \nu_2}(\nu_2,\theta_2+\omega(\nu_2))+\omega'(\nu_2)\frac{\partial \hat{v}_{j_2}}{\partial \lambda_2}(\nu_2,\theta_2+\omega(\nu_2))\Big|\\&\leq \Big|\frac{\partial \hat{v}_{j_2}}{\partial \nu_2}(\nu_2,\theta_2+\omega(\nu_2))\Big|+C|\nu_2|^{1+a}\Big|\frac{\partial \hat{v}_{j_2}}{\partial \lambda_2}(\nu_2,\theta_2+\omega(\nu_2))\Big|.\end{aligned}$$

Because $s_*=1/2+a/2$, it follows that

$$\frac{|\nu_2|^{1+a}}{(1+|\nu_2|)^{s_*}}\leq |\nu_2|^{1/2+a/2}\leq (1+|\nu_2|)^{1/2+a/2},$$

and our desired inequality holds. □

We next turn to the proof of the corresponding fact in the Y space.

Proof. [Proof of Proposition 3.2] We first estimate

$$\frac{\partial}{\partial \lambda}F(\partial_x(uv))=\frac{\partial}{\partial \lambda}\Big(\xi\int_{\mathbb{R}}\hat{u}(\xi_1,\lambda_1)\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1\Big). \tag{3.21}$$

We again split the integration into the region $|\xi-\xi_1|\leq|\xi_1|$ and $|\xi-\xi_1|\geq|\xi_1|$. For symmetry reasons explained at the beginning of the proof of Proposition 3.1, it is enough to estimate the integral on the first region. In this case we replace (3.21) with

$$\frac{\partial}{\partial \lambda}F(\partial_x(uv))=\xi\int_{\mathbb{R}}\frac{\partial \hat{u}}{\partial \lambda_1}(\xi_1,\lambda_1)\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1.$$

Since $F^{-1}(\frac{\partial \hat{u}}{\partial \lambda})\in X^{1/2}_{s_*}$, we obtain the desired estimate using the proof of Proposition 3.1.

Next we turn to

$$\frac{\partial}{\partial \xi}F(\partial_x(uv))=\frac{\partial}{\partial \xi}(\xi(\hat{u}*\hat{v})(\xi,\lambda))=\hat{u}*\hat{v}(\xi,\lambda)+\xi\frac{\partial}{\partial \xi}(\hat{u}*\hat{v})(\xi,\lambda), \tag{3.22}$$

and we recall that we have to estimate it in $X^{-b}_{-s_*}$. We start with the term $\hat{u}*\hat{v}(\xi,\lambda)$. Our usual duality argument leads us to rewriting the left-hand side of (3.2) as

$$\begin{aligned}&\sup_{\|g_j\|_{L^2}\leq 1}\sum_{j\geq 0}2^{-jb}\int_{\mathbb{R}^4}\frac{g_j(\xi_1+\xi_2,\lambda_1+\lambda_2)\chi_j(\lambda_1+\lambda_2-\omega(\xi_1+\xi_2))}{\max(1,|\xi_1+\xi_2|)^{s_*}}\\&\quad\times \hat{u}(\xi_1,\lambda_1)\hat{v}(\xi_2,\lambda_2)\,d\xi_1\,d\xi_2\,d\lambda_1\,d\lambda_2,\end{aligned}$$

and one can simply use the Strichartz inequality (2.4), just as we did in Case 1 of Proposition 3.1. This leads to an estimate by $C\|u\|^{1/2}_{X^{1/2}_{s_*}}\|v\|^{1/2}_{X^{1/2}_{s_*}}$.

We now turn to the term $\xi\frac{\partial}{\partial\xi}(\hat{u}*\hat{v})(\xi,\lambda)$. In order to treat this term we write $1=\theta_1(\rho)+\theta_2(\rho)+\theta_3(\rho)$, where $supp\,\theta_1\subset\{|\rho|<1\}$, $\theta_1=1$ on $\{|\rho|<1/2\}$, $supp\,\theta_2\subset\{1/2<|\rho|<2\}$, and $supp\,\theta_3\subset\{|\rho|>2\}$, $\theta_3=1$ on $\{|\rho|>4\}$. Then

$$\begin{aligned}\xi\frac{\partial}{\partial\xi}(\hat{u}*\hat{v})(\xi,\lambda)&=\sum_{i=1}^{3}\xi\frac{\partial}{\partial\xi}\int\hat{u}(\xi_1,\lambda_1)\theta_i\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1\\&=I_1+I_2+I_3.\end{aligned}$$

We start with I_1 by writing

$$\begin{aligned}I_1&=\xi\int\hat{u}(\xi_1,\lambda_1)\frac{\partial}{\partial\xi}\Big(\theta_1\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\Big)\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1\\&\quad+\xi\int\hat{u}(\xi_1,\lambda_1)\theta_1\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\frac{\partial}{\partial\xi}\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1=I_{11}+I_{12}.\end{aligned}\tag{3.23}$$

Clearly,

$$I_{11}=\xi\int\hat{u}(\xi_1,\lambda_1)\frac{sign\,(\xi-\xi_1)}{|\xi_1|}\theta_1'\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1.$$

The contribution of I_{11} leads us to estimating

$$\begin{aligned}&\sum_{j\geq0}2^{-jb}\int_{\mathbb{R}^4}|g_j(\xi_1+\xi_2,\lambda_1+\lambda_2)\chi_j(\lambda_1+\lambda_2-\omega(\xi_1+\xi_2))|\\&\times\quad\frac{|\xi_1+\xi_2|}{\max(1,|\xi_1+\xi_2|)^{s_*}|\xi_1|}\Big|\theta_1'\Big(\frac{|\xi_2|}{|\xi_1|}\Big)\Big||\hat{u}(\xi_1,\lambda_1)||\hat{v}(\xi_2,\lambda_2)|\,d\xi_1\,d\xi_2\,d\lambda_1\,d\lambda_2,\end{aligned}\tag{3.24}$$

and because in the support of θ_1' we have that $|\xi_2|\sim|\xi_1|$, it follows that $|\xi_1+\xi_2|\leq|\xi_1|$, and hence (3.24) can be handled just by using the Strichartz inequality (2.4).

In I_{12} we can write

$$\frac{\partial}{\partial\xi}\hat{v}(\xi-\xi_1,\lambda-\lambda_1)=-\frac{\partial}{\partial\xi_1}\hat{v}(\xi-\xi_1,\lambda-\lambda_1),$$

and we integrate by parts in ξ_1 to get

$$\begin{aligned}I_{12}&=\xi\int\frac{\partial}{\partial\xi_1}\hat{u}(\xi_1,\lambda_1)\theta_1\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1\\&\quad+\xi\int\hat{u}(\xi_1,\lambda_1)\frac{\partial}{\partial\xi_1}\theta_1\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1=I_{121}+I_{122}.\end{aligned}\tag{3.25}$$

Now

$$\frac{\partial}{\partial\xi_1}\theta_1\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)=-\frac{sign\,(\xi-\xi_1)}{|\xi_1|}\theta_1'\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)+\frac{sign\,\xi_1|\xi-\xi_1|}{|\xi_1|^2}\theta_1'\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big);$$

hence,

$$\Big|\frac{\partial}{\partial\xi_1}\theta_1\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\Big| \le \frac{C}{|\xi_1|}\Big|\theta_1'\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\Big|,$$

and we go back to (3.24). We turn our attention to I_{121} and notice that $|\xi-\xi_1| \le |\xi_1|$ in the region of integration. We thus are led to estimate

$$\begin{aligned}&\sum_{j\ge 0} 2^{-jb}\int_{|\xi_2|\le|\xi_1|} g_j(\xi_1+\xi_2,\lambda_1+\lambda_2)\chi_j(\lambda_1+\lambda_2-\omega(\xi_1+\xi_2)) \\ &\times \frac{|\xi_1+\xi_2|}{\max(1,|\xi_1+\xi_2|)^{s_*}}\frac{\max(1,|\xi_1|)^{s_*}}{\max(1,|\xi_2|)^{s_*}}\frac{|\partial_{\xi_1}\hat{u}(\xi_1,\lambda_1)|}{\max(1,|\xi_1|)^{s_*}} \\ &\times \max(1,|\xi_2|)^{s_*}|\hat{v}(\xi_2,\lambda_2)|\,d\xi_1\,d\xi_2\,d\lambda_1\,d\lambda_2.\end{aligned} \tag{3.26}$$

We now consider several cases:

Case 1: $|\xi_1|\le 1$. Then $|\xi_2|\le 1$ and $|\xi_1+\xi_2|\le 2$, and we proceed just as in Case 1 in the proof of Proposition 3.1.

Case 2: $|\xi_1|\ge 1$, $|\xi_1+\xi_2|\le 1/2$. Then $|\xi_2|\sim|\xi_1|$, and the multiplier in (3.26) can be bounded by

$$\frac{|\xi_1+\xi_2|}{\max(1,|\xi_1+\xi_2|)^{s_*}}\frac{\max(1,|\xi_1|)^{s_*}}{\max(1,|\xi_2|)^{s_*}} \le C,$$

and we are back in Case 1.

Case 3: $|\xi_1|\ge 1$, $|\xi_1+\xi_2|\ge 1/2$ and $1/4|\xi_1|\le|\xi_2|\le|\xi_1|$. Then

$$\frac{|\xi_1+\xi_2|}{\max(1,|\xi_1+\xi_2|)^{s_*}}\frac{\max(1,|\xi_1|)^{s_*}}{\max(1,|\xi_2|)^{s_*}} \le C|\xi_1+\xi_2|^{1-s_*},$$

and the estimate follows as in Case 3 of Proposition 3.1.

Case 4: $|\xi_1|\ge 1$, $|\xi_1+\xi_2|\ge 1/2$, and $1/4|\xi_1|\ge|\xi_2|$. Then $|\xi_1|\sim|\xi_1+\xi_2|$ and

$$\frac{|\xi_1+\xi_2|}{\max(1,|\xi_1+\xi_2|)^{s_*}}\frac{\max(1,|\xi_1|)^{s_*}}{\max(1,|\xi_2|)^{s_*}} \le C\frac{|\xi_1|}{\max(1,|\xi_2|)^{s_*}},$$

and the result follows as in Case 4 of Proposition 3.1. This concludes the estimate involving I_1.

We next turn to I_2. Just as in (3.23), we can write $I_2 = I_{21}+I_{22}$, where now both integrals involve θ_2. Because $supp\,\theta_2'\subset\{1/2<\rho<2\}$, for I_{21} we can repeat the argument presented for I_{11}. Now using the same splitting described in (3.25), we can write $I_{22}=I_{221}+I_{222}$, where again θ_1 is replaced by θ_2. It is easy to see that I_{222} can be treated just like I_{11}. We are left with I_{221}, which leads to

$$\sum_{j\ge 0} 2^{-jb}\int_{|\xi_2|\le|\xi_1|} g_j(\xi_1+\xi_2,\lambda_1+\lambda_2)\chi_j(\lambda_1+\lambda_2-\omega(\xi_1+\xi_2))\theta_2\Big(\frac{|\xi_2|}{|\xi_1|}\Big)$$

$$\times \frac{|\xi_1+\xi_2|}{\max(1,|\xi_1+\xi_2|)^{s_*}} \frac{\max(1,|\xi_1|)^{s_*}}{\max(1,|\xi_2|)^{s_*}} \frac{|\partial_{\xi_1}\hat{u}(\xi_1,\lambda_1)|}{\max(1,|\xi_1|)^{s_*}} \tag{3.27}$$
$$\times \max(1,|\xi_2|)^{s_*}|\hat{v}(\xi_2,\lambda_2)|\,d\xi_1\,d\xi_2\,d\lambda_1\,d\lambda_2.$$

We note that $|\xi_2|\sim|\xi_1|$ on $supp\,\theta_2$. So if $|\xi_1|\le 1$, then $|\xi_2|,|\xi_2+\xi_1|\le C$, and the bound follows as in Case 1 of Proposition 3.1. If $|\xi_1|\ge 1$ and $|\xi_2+\xi_1|\le 1$, then again the bound follows because

$$\theta_2\Big(\frac{|\xi_2|}{|\xi_1|}\Big)\frac{|\xi_1+\xi_2|}{\max(1,|\xi_1+\xi_2|)^{s_*}}\frac{\max(1,|\xi_1|)^{s_*}}{\max(1,|\xi_2|)^{s_*}}\le C,$$

and we go back to the previous case. Finally, if $|\xi_1|\sim|\xi_2|\ge 1$ and $|\xi_2+\xi_1|\ge 1$ we have

$$\theta_2\Big(\frac{|\xi_2|}{|\xi_1|}\Big)\frac{|\xi_1+\xi_2|}{\max(1,|\xi_1+\xi_2|)^{s_*}}\frac{\max(1,|\xi_1|)^{s_*}}{\max(1,|\xi_2|)^{s_*}}\le C|\xi_2+\xi_1|^{1-s_*},$$

and we proceed as in Case 3 of Proposition 3.1. Thus also for I_2 we obtain the desired bound.

We finally turn to I_3, which we split in the usual way into $I_3=I_{31}+I_{33}$. Note that $supp\,\theta_3'\subset\{2<\rho<4\}$, and so I_{31} is handled like I_{11}. Now observe that

$$I_{32}=\xi\int\hat{u}(\xi_1,\lambda_1)\theta_3\Big(\frac{|\xi-\xi_1|}{|\xi_1|}\Big)\frac{\partial}{\partial_\xi}\hat{v}(\xi-\xi_1,\lambda-\lambda_1)\,d\xi_1\,d\lambda_1,$$

and if we make the change of variables $\nu_1=\xi-\xi_1$, $\tau_1=\lambda-\lambda_1$, we obtain

$$I_{32}=\xi\int\hat{u}(\xi-\nu_1,\lambda-\tau_1)\theta_3\Big(\frac{|\nu_1|}{|\xi-\nu_1|}\Big)\frac{\partial}{\partial_{\nu_1}}\hat{v}(\nu_1,\tau_1)\,d\nu_1\,d\tau_1,$$

and on $supp\,\theta_3$ we have $|\nu_1|\ge 2|\xi-\nu_1|>|\xi-\nu_1|$. We then treat this term as I_{121}, with the roles of v and u interchanged.

Analysis of certain low/high frequency interactions. We first show that if the Y-norms in (3.1) are ignored, the resulting estimate fails. This analysis shows that a contraction-mapping argument in the space X^b_s alone will fail, which is a special case of the negative result in [20]. Next, we show, by analyzing a more refined low/high frequency interaction, that the natural extension of (3.1) when $a=0$ fails to hold. This demonstrates the near optimality of (3.1) near $a=0$.

The following proposition is contained in [20].

Proposition 3.3. *Consider the space X^b_s adapted to the dispersive function $\omega_a(\xi)=|\xi|^{1+a}\xi$ defined in* (1.8) *for $a\in[0,1)$. The bilinear estimate*

$$\|\partial_x(u_1u_2)\|_{X^{-b}_{s_*}}\le C\|u_i\|_{X^b_{s_*}}\|u_j\|_{X^b_{s_*}} \tag{3.28}$$

where $(i,j)=(1,2)$ FAILS to hold for any $b\in\mathbb{R}$.

Proof. As in [20], consider

$$\widehat{u_1}(\xi,\lambda) = \chi_{[\alpha/2,\alpha]}(\xi)\chi_{\{|\lambda-\omega_a(\xi)|\le 1\}}(\lambda),$$

$$\widehat{u_2}(\xi,\lambda) = \chi_{[N,N+\alpha]}(\xi)\chi_{\{|\lambda-\omega_a(\xi)|\le 1\}}(\lambda),$$

where χ_S denotes a smoothed version of the characteristic function of the set S and $\alpha \sim N^{-1-a}$.

Remark 3.2. The choice $\alpha \sim N^{-1-a}$ may be explained geometrically. Consider the Taylor expansion of the dispersive function $\omega_a(\xi)$ at $\xi = N$ for some $N \gg 1$:

$$\omega_a(N+x) = N^{2+a} + (2+a)N^{1+a}x + (2+a)(1+a)N^a x^2 + \cdots .$$

Note that the horizontal line $\lambda = N^{2+a}$ is the zeroth order approximation to the curve $\lambda = \omega_a(\xi)$ at the point (N, N^{2+a}). The curve $\lambda = \omega(N+x)$ separates a distance ~ 1 from the horizontal line $\lambda = N^{2+a}$ when $x \sim N^{-1-a}$. A similar geometric observation regarding the tangent line to the curve at (N, N^{2+a}) explains the choice $\beta \sim N^{-a/2}$ below.

By analyzing a convolution, it may be shown that

$$[\partial_x \widehat{(u_1 u_2)}](\xi,\lambda) \sim N\alpha\chi_{\{|\lambda-\omega_a(\xi)|\lesssim 1\}}(\lambda)\chi_{[N,N+\alpha]}(\xi). \tag{3.29}$$

Thus, the left side of (3.28) is of size $N^{s_*}N\alpha\alpha^{\frac{1}{2}}$. The right side of (3.28) is of the size $N^{s_*}\alpha$. For (3.28) to hold, we must have $N^{s_*}\alpha\alpha^{\frac{1}{2}} \lesssim N^{s_*}\alpha$ for all $N \gg 1$. But $\alpha \sim N^{-1-a}$, so this requires $\frac{1}{2} - \frac{a}{2} \le 0$, forcing $a \ge 1$. Therefore (3.28) fails. □

Next, we consider a more refined low-/high-frequency interaction between

$$\widehat{u_1}(\xi,\lambda) = \chi_{[-\beta,\beta]}(\xi)\chi_{\{|\lambda-\omega_a(\xi)|\le 1\}}(\lambda),$$

$$\widehat{u_2}(\xi,\lambda) = \chi_{[N,N+\beta]}(\xi)\chi_{\{|\lambda-\omega_a(\xi)|\le 1\}}(\lambda),$$

where $\beta = N^{-a/2}$. Recall that this choice of β may be explained geometrically by noting that the tangent to the graph of the dispersive function at (N, N^{2+a}) separates ~ 1 from the graph. Note that the support of $\widehat{u_1}$ intersects lines $\lambda = constant$ in intervals of length at most $\beta = N^{-\alpha/2}$. Note also that the support of $\widehat{u_2}$ intersects lines $\lambda = constant$ in intervals of size at most N^{-1-a} and that the projection of the support of $\widehat{u_2}$ along the λ-axis is an interval of length $N^{1+a/2}$. A convolution analysis shows that

$$[\partial_x \widehat{(u_1 u_2)}](\xi,\lambda) \sim NN^{-1-a}\chi_{\{|\lambda-N^{2+a}|\lesssim N^{1+a/2}\}}(\lambda)\chi_{\{|\xi-N|\lesssim\beta\}}(\xi). \tag{3.30}$$

We may now calculate the left and right sides of (3.1) and find

$$\text{left side of (3.1)} \gtrsim N^{s_*} N N^{-1-a} \beta^{\frac{1}{2}} \sum_{0 \le j \le 10^{-10} \log N} 2^{j(-b)} 2^{j/2}. \tag{3.31}$$

The right side of (3.1) involves various terms of three sizes:

$$N^{s_*}\beta,\ \beta N^{\frac{1+a}{2}},\ \beta N^{s_*} N^{\frac{a}{4}}.$$

The last of these is the largest since $s_* = \frac{1}{2} + \frac{a}{2}$ and $a \in [0, 1)$. For (3.1) to hold, we must have

$$N^{s_*} N N^{-1-a} \beta^{\frac{1}{2}} \sum_{0 \le j \le 10^{-10} \log N} 2^{j(-b)} 2^{j/2} \lesssim \beta N^{s_*} N^{\frac{a}{4}}, \tag{3.32}$$

for all $N \gg 1$. The sum contributes $\sim N^{\epsilon}$ if $b = \frac{1}{2} - \epsilon$ and contributes $\sim \log N$ if $b = \frac{1}{2}$. Recall that $b \in (\frac{1-a}{2(1+a)}, \frac{1}{2})$ was used in the proof of Proposition 3.1. Since $\beta \sim N^{-a/2}$, (3.32) requires $N^{-a} \log N \lesssim 1$, which fails to hold if $a = 0$. Note also that for $a > 0$ there is a bit of slack allowing us to take $b < \frac{1}{2}$ for this interaction, which is consistent with Proposition 3.1.

We summarize the negative result just obtained with the following statement.

Proposition 3.4. *Consider the space X_s^b adapted to the dispersive function $\omega_a(\xi) = |\xi|^{1+a}\xi$ defined in (1.8) for $a = 0$. The bilinear estimate*

$$\|\partial_x(u_1 u_2)\|_{X_{\frac{1}{2}}^{-\frac{1}{2}}} \le C \|u_i\|_{X_{\frac{1}{2}}^{\frac{1}{2}}} \left(\|u_j\|_{X_{\frac{1}{2}}^{\frac{1}{2}}} + \|u_j\|_{X_{\frac{1}{2}}^{\frac{1}{2}}}^{\frac{1}{2}} \|u_j\|_{Y_{-\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}}}^{\frac{1}{2}} \right) + (i \text{ switched with } j) \tag{3.33}$$

where $(i, j) = (1, 2)$ FAILS to hold.

4. Proof of Theorem 1

We now turn to the local well-posedness result for the IVP (1.1) stated in Theorem 1. With the results proved in Section 2, Section 3, and Section 5, the proof becomes standard, similar to the one presented in [14]. For completeness we present it in detail.

We first assume that $s = s_*$ and that $u_0 \in F^{s_*}$, and we choose $\psi \in C_0^\infty(\mathbb{R})$, a cutoff function for the interval $[-1, 1]$. For δ small, to be chosen, we consider the nonlinear mapping

$$\Phi_{u_0}(v) = \psi(t/\delta) W(t) u_0 + \psi(t/\delta) \int_0^t W(t - t') \partial_x(v^2) dt'.$$

We then consider the ball $B_a = \{v \in Z^{1/2}_{s_*} : \|v\|_{Z^{1/2}_{s_*}} \leq a\}$. Theorem 1 follows from proving that there exist $a > 0$ and $\delta = \delta(\|u_0\|_{F^{s_*}})$ such that

$$\Phi_{u_0} : B_a \longrightarrow B_a \tag{4.1}$$

and

$$\|\Phi_{u_0}(v) - \Phi_{u_0}(w)\|_{Z^{1/2}_{s_*}} \leq 1/2\|v - w\|_{Z^{1/2}_{s_*}}. \tag{4.2}$$

By the linear estimates in Lemma 2.2 and Lemma 2.3 we have that

$$\|\psi(t/\delta)W(t)u_0\|_{Z^{1/2}_{s_*}} \leq C\|u_0\|_{F^{s_*}}.$$

Choose $a = 2C\|u_0\|_{F^{s_*}}$. We need to estimate the nonlinear part of the mapping Φ_{u_0}. We choose a new cutoff function in time $\tilde{\psi}$ such that $\tilde{\psi} = 1$ on the support of ψ. Then

$$NL(t,x) = \psi(t/\delta)\int_0^t W(t-t')\partial_x(v^2)dt' = \psi(t/\delta)\int_0^t W(t-t')\partial_x(\tilde{v}^2)dt',$$

where $\tilde{v} = \tilde{\psi}(t/\delta)v$. Thus, by Lemma 2.4 and Lemma 2.5,

$$\|NL\|_{Z^{1/2}_{s_*}} \leq C_\epsilon \delta^{-\epsilon}\|\partial_x(\tilde{v}^2)\|_{Z^{-1/2}_{s_*}},$$

and by Lemma 5.6 we can continue with

$$\leq C_\epsilon \delta^{-\epsilon+\theta}\|\partial_x(v^2)\|_{Z^{-b}_{s_*}},$$

and by the bilinear estimates (3.1) and (3.2),

$$\leq C_\epsilon \delta^{-\epsilon+\theta}\|v\|^2_{Z^{1/2}_{s_*}} \leq C_\epsilon \delta^{-\epsilon+\theta}a^2.$$

We fix now b as in the bilinear estimates (3.1) and (3.2), θ corresponding to b, and we choose $\epsilon = \theta/2$. Then we choose δ small enough such that $C_\theta \delta^{\theta/2}a < 1/2$. We then obtain (4.1). To prove (4.2) we argue similarly, using $\partial_x(v^2) - \partial_x(w^2) = \partial_x((v+w)(v-w))$ and the bilinear estimates to obtain

$$\|\Phi_{u_0}(v) - \Phi_{u_0}(w)\|_{Z^{1/2}_{s_*}} \leq 2aC_\theta\delta^\theta\|u - w\|_{Z^{1/2}_{s_*}},$$

and we now choose δ so small that $2C_\theta\delta^\theta a < 1/2$. This finishes the proof of the theorem when $s = s_*$. If $s > s_*$, then we set $\tilde{s} = s - s_*$, and we use the following higher-order bilinear estimate:

$$\|\partial_x v^2\|_{X^{-b}_s} \quad \leq \quad \|\partial_x(vD^{\tilde{s}}v)\|_{X^{-b}_{s_*}} \leq C\|v\|_{Z^{1/2}_{s_*}}\|v\|_{Z^{1/2}_s}$$

and similarly for the space $Y^{-b}_{s-s_*,s}$. Now it is easy to see that by repeating the chain of inequalities for $s > s_*$, one can prove that also in this case the time interval δ depends only on $\|u_0\|_{F^{s_*}}$.

5. Appendix: Time cutoffs and the X and Y norms

In this section we first present a few lemmas that quantify how the norm of a function f changes in the spaces $X_s^{1/2}$ and $Y_{s-2s_*,s}^{1/2}$ when multiplied by a time cutoff function relative to an interval of size δ.

A good estimate would be

$$\|\psi(t/\delta)f\|_{X_s^{1/2}} \leq C\|f\|_{X_s^{1/2}}, \tag{5.1}$$

where C is independent of δ and s. Such an estimate is a bit cumbersome to prove. So, we replace (5.1) with a weaker estimate that in any case will suffice for what we need.

Lemma 5.1. *Given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\|\psi(t/\delta)f\|_{X_s^{1/2}} \leq C_\epsilon \delta^{-\epsilon}\|f\|_{X_s^{1/2}},$$

for any $s \in \mathbb{R}$.

If we are willing to relax a bit the regularity of the left-hand side in (5.1), then we have

Lemma 5.2. *Given $b \in (0, 1/2)$, there exists $\theta = \theta(b) > 0$ such that*

$$\|\psi(t/\delta)f\|_{X_s^b} \leq C\delta^{\theta}\|f\|_{X_s^{1/2}},$$

for any $s \in \mathbb{R}$.

To prove Lemma 5.1 we need to introduce the auxiliary space $\tilde{X}_s^b$ defined as the closure of the Schwartz functions with respect to the norm[7]

$$\|f\|_{\tilde{X}_s^b} = \Big(\int_{\mathbb{R}^2}(1+|\lambda-\omega(\xi)|)^{2b}(1+|\xi|)^{2s}|\hat{f}(\xi,\lambda)|^2 d\xi\, d\lambda\Big)^{1/2}.$$

Lemma 5.1 will be proved as an interpolation between two estimates in the space $\tilde{X}_s^b$, one with $b < 1/2$ and one with $b > 1/2$. More precisely, we use the following results:

Lemma 5.3. *If $b \in (0, 1/2)$, then*

$$\|\psi(t/\delta)f\|_{\tilde{X}_s^b} \leq C\|f\|_{\tilde{X}_s^b}, \tag{5.2}$$

for any $s \in \mathbb{R}$.

[7] One can also write

$$\|f\|_{\tilde{X}_s^b} = \Big(\sum_{j\geq 0} 2^{2jb}\int_{\mathbb{R}^2}\chi_j(\lambda-\omega(\xi))(1+|\xi|)^{2s}|\hat{f}(\xi,\lambda)|^2 d\xi\, d\lambda\Big)^{1/2}.$$

If $b \in (1/2, 1)$, *then*

$$\|\psi(t/\delta)f\|_{\tilde{X}_s^b} \leq C\delta^{1/2-b}\|f\|_{\tilde{X}_s^b}, \tag{5.3}$$

for any $s \in \mathbb{R}$.

In both cases C *is independent of* δ.

Proof. The proof of this lemma follows the proof of Lemma 3.1 in [14]. We first observe that using the notation

$$\widehat{J^s h}(\xi) = (1+|\xi|)^s \hat{h}(\xi) \quad \text{and} \quad \widehat{\Lambda^b m}(\lambda) = (1+|\lambda|)^b \hat{m}(\lambda)$$

we have

$$\|f\|_{\tilde{X}_s^b} \sim \|\Lambda^b W(t) J^s f\|_{L^2}.$$

We also note that for any s and any $b \geq 0$

$$\Big(\int_{\mathbb{R}^2} (1+|\xi|)^{2s} |\widehat{\psi_\delta f}|^2 \, d\xi \, d\lambda\Big)^{1/2} = \|\psi(\delta^{-1}\cdot) J^s f\|_{L^2} \leq C\|J^s f\|_{L^2} \leq \|f\|_{\tilde{X}_s^b},$$

where $\psi_\delta(t) = \psi(\delta^{-1}t)$. If we also use the notation $\widehat{D^b m}(\lambda) = |\lambda|^b \hat{m}(\lambda)$, then

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\lambda - \omega(\xi)|^{2b} (1+|\xi|)^{2s} |\widehat{\psi_\delta f}|^2 \, d\xi \, d\lambda \\
&= \int_{\mathbb{R}} (1+|\xi|)^{2s} \int_{\mathbb{R}} |\lambda - \omega(\xi)|^{2b} |\widehat{\psi_\delta f}|^2 \, d\lambda \, d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|)^{2s} \int_{\mathbb{R}} |\lambda|^{2b} |\hat{f} * (\delta\hat{\psi}(\delta\cdot))(\lambda - \omega(\xi))|^2 \, d\lambda \, d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|)^{2s} \Big(\int_{-\infty}^{\infty} |D^b(e^{i\omega(\xi)t}\tilde{f}(t)\psi(t/\delta))|^2 \, dt\Big) \, d\xi,
\end{aligned}$$

where in the last step we denoted with $\tilde{f}$ the partial Fourier transform with respect to ξ, and we used the identity

$$\|D^b(e^{iat}f(t))\|_{L^2}^2 = \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 |\lambda - a|^{2b} d\lambda. \tag{5.4}$$

We thus need to prove that for any $a \in \mathbb{R}$

$$\int_{-\infty}^{\infty} |D^b(e^{iat}f(t)\psi(t/\delta))|^2 \, dt \leq C \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 |\lambda - a|^{2b} d\lambda. \tag{5.5}$$

We recall the Leibniz rule for fractional derivative [13]: for any $\alpha \in (0,1)$

$$\|D_t^\alpha(fg) - gD_t^\alpha(f) - fD_t^\alpha(g)\|_{L^2} \leq C\|D_t^{\alpha_1}(f)\|_{L^{p_1}}\|D_t^{\alpha_2}(g)\|_{L^{p_2}}, \tag{5.6}$$

where $\alpha = \alpha_1 + \alpha_2$, $\alpha_i \geq 0$, and $1/2 = 1/p_1 + 1/p_2$. In our case we use (5.6) with $\alpha_1 = b$ and $\alpha_2 = 0$. Then

$$\|D^b(e^{iat}f(t)\psi(t/\delta))\|_{L^2} \leq C\|D^b(e^{iat}f(t))\psi(t/\delta)\|_{L^2} + \|e^{iat}f(t)D^b\psi(t/\delta)\|_{L^2}$$

$$= I_1 + I_2.$$

To estimate I_1 we use the fact that $\|\psi\|_{L^\infty} \leq C$ and the identity (5.4). We are then left with the term I_2. By the Sobolev embedding theorem

$$\|e^{iat} f\|_{L^q} \leq C\|D^b(e^{iat} f)\|_{L^2},$$

provided $1/q = 1/2 - b$. We now set $2r = q$; it follows that $1/r = 1 - 2b > 0$, and we can apply Hölder's inequality to obtain

$$I_2 = \|e^{iat} f(t) D^b\psi(t/\delta)\|_{L^2} \leq \|e^{iat} f\|_{L^{2r}} \|D^b\psi(t/\delta)\|_{L^{2r'}}. \tag{5.7}$$

Now observe that $1/r' = 1 - 1/r = 2b$; hence,

$$\|D^b\psi(t/\delta)\|_{L^{2r'}} = \delta^{1/2r'-b}\|D^b\psi(t)\|_{L^{2r'}} = C.$$

It follows that

$$I_2 \leq C\|D^b(e^{iat} f)\|_{L^2},$$

and (5.5) follows thanks to (5.4).

The proof of (5.3) is exactly the same as the one given to prove Lemma 3.1 in [14]; hence, we decided to omit it. □

Proof. [Proof of Lemma 5.1] We first observe the following facts about real interpolation. Let A be a Banach space, and

$$l_s^q(A) = \Big\{(f_j) : f_j \in A, \Big(\sum_{j\geq 0} (2^{js}\|f_j\|_A)^q\Big)^{1/q} < \infty\Big\}.$$

Then by Theorem 5.6.1 in [3]

$$l_{s_\theta}^q(A) = (l_{s_0}^{q_0}(A), l_{s_1}^{q_1}(A))_{\theta,q} \ \ (s_0 \neq s_1), \tag{5.8}$$

where $s_\theta = \theta s_0 + (1-\theta)s_1$, $1 \leq q \leq \infty$, and $1 \leq q_i \leq \infty$, $i = 1, 2$. Using (5.8) we see that for any $s \in \mathbb{R}$,

$$(\tilde{X}_s^{b_1}, \tilde{X}_s^{b_2},)_{\theta,1} = X_s^b,$$

where $b = \theta b_1 + (1-\theta)b_2$, and $b_1 \neq b_2$. Now Lemma 5.1 follows from this observation, interpolation, and Lemma 5.3. □

Proof. [Proof of Lemma 5.2] Let us fix $b < b_1 < b_2 < 1/2$. Assume one can prove

$$\|\psi(t/\delta) f\|_{\tilde{X}_s^{b_1}} \leq C\delta^\theta \|f\|_{\tilde{X}_s^{b_2}}. \tag{5.9}$$

Then, since $\|\psi(t/\delta) f\|_{X_s^b} \leq \|\psi(t/\delta) f\|_{\tilde{X}_s^{b_1}}$ and $\|f\|_{\tilde{X}_s^{b_2}} \leq \|f\|_{X_s^{1/2}}$, we will be done. By complex interpolation, (5.9) will follow from the two estimates below:

$$\|\psi(t/\delta) f\|_{\tilde{X}_s^{b_2}} \leq C\|f\|_{\tilde{X}_s^{b_2}}, \tag{5.10}$$

and

$$\|\psi(t/\delta)f\|_{\tilde{X}_s^0} \leq C\delta^{b_2}\|f\|_{\tilde{X}_s^{b_2}}. \tag{5.11}$$

Clearly (5.10) is true by (5.2) of Lemma 5.3. So we only have to show (5.11). By repeating some of the arguments in the proof of Lemma 5.3, one can prove that

$$\|\psi(t/\delta)f\|^2_{\tilde{X}_s^0} \sim \int_{\mathbb{R}} \Big(\int_{-\infty}^{\infty} |(e^{i\omega(\xi)t}J^s f(t)\psi(t/\delta))|^2\, dt\Big)\, d\xi. \tag{5.12}$$

Now we observe that

$$\|e^{iat}J^s f(t)\psi(t/\delta)\|_{L^2} \leq \|e^{iat}J^s f\|_{L^{2r}}\|\psi(t/\delta)\|_{L^{2r'}},$$

where $1/q = 1/2r = 1/2 - b_2$. Then we can continue with

$$\leq C\|e^{iat}J^s f\|_{L^{2r}}\delta^{1/2r'} \leq C\|D^{b_2}(e^{iat}J^s f)\|_{L^2}\delta^{b_2}.$$

If we insert this in (5.12) and use the identity (5.4), we then obtain (5.11). □

Lemmas 5.1 and 5.2 can also be proved if one replaces the spaces X_s^b with the spaces $Y^b_{s-2s_*,s}$.

Lemma 5.4. *Given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\|\psi(t/\delta)f\|_{Y^{1/2}_{s-s_*,s}} \leq C_\epsilon\delta^{-\epsilon}\|f\|_{Y^{1/2}_{s-s_*,s}},$$

for any $s \in \mathbb{R}$.

Proof. Let $\psi_\delta(t) = \psi(t/\delta)$, so that $F(\psi(t/\delta)f)(\xi,\lambda) = \widehat{\psi_\delta} *_\lambda \hat{f}(\xi,\lambda)$; hence,

$$\begin{aligned}\frac{\partial}{\partial\xi}F(\psi(t/\delta)f)(\xi,\lambda) &= \widehat{\psi_\delta} *_\lambda \frac{\partial}{\partial\xi}\hat{f}(\xi,\lambda),\\ \frac{\partial}{\partial\lambda}F(\psi(t/\delta)f)(\xi,\lambda) &= \widehat{\psi_\delta} *_\lambda \frac{\partial}{\partial\lambda}\hat{f}(\xi,\lambda).\end{aligned}$$

Then Lemma 5.1 gives

$$\begin{aligned}\|x\psi(t/\delta)f\|_{X^{1/2}_{s-2s_*}} &= C_\epsilon\delta^{-\epsilon}\|xf\|_{X^{1/2}_{s-2s_*}}\\ \|t\psi(t/\delta)f\|_{X^{1/2}_s} &= C_\epsilon\delta^{-\epsilon}\|tf\|_{X^{1/2}_s},\end{aligned}$$

and this concludes the proof. □

With a similar argument we can also prove, from Lemma 5.2,

Lemma 5.5. *Given $b \in (0,1/2)$, there exists $\theta = \theta(b) > 0$ such that*

$$\|\psi(t/\delta)f\|_{Y^b_{s-s_*,s}} \leq C\delta^\theta\|f\|_{Y^b_{s-s_*,s}},$$

for any $s \in \mathbb{R}$.

We conclude this section with the following lemma.

Lemma 5.6. *For any $b \in (0, 1/2)$, there exists $\theta = \theta(b) > 0$ such that* [8]

$$\|\psi(t/\delta)v\|_{Z_s^{-1/2}} \leq \delta^\theta \|v\|_{Z_s^{-b}},$$

for any $s \in \mathbb{R}$.

Proof. We estimate only $\|\psi(t/\delta)v\|_{X_s^{-1/2}}$ because the estimate for $\|\psi(t/\delta)v\|_{Y^{-1/2}_{s-s_*,s}}$ will then follow as Lemma 5.4 followed from Lemma 5.1. It is easy to see that if $C^{1/2}_{-s}$ is the space defined through the norm

$$\|g\|_{C^{1/2}_{-s}} = \sup_j 2^{j/2} \Big(\int_{\mathbb{R}^2} (1+|\xi|)^{2s} \|\hat{g}(\xi,\lambda)\|^2 \chi_j(\lambda - \omega(\xi))\, d\lambda\, d\xi \Big)^{1/2},$$

then $C^{1/2}_{-s} = (X_s^{-1/2})^*$, with duality pairing $\int_{\mathbb{R}^2} v\bar{g}\, d\lambda\, d\xi$. Thus,

$$\begin{aligned} \|\psi(t/\delta)v\|_{X_s^{-1/2}} &= \sup_{\|g\|_{C^{1/2}_{-s}} \leq 1} \Big| \int_{\mathbb{R}^2} \psi(t/\delta) v \bar{g}\, d\lambda\, d\xi \Big| \\ &= \sup_{\|g\|_{C^{1/2}_{-s}} \leq 1} \Big| \int_{\mathbb{R}^2} v \overline{\psi(t/\delta) g}\, d\lambda\, d\xi \Big|. \end{aligned}$$

We thus need to prove that

$$\|\psi(t/\delta)g\|_{C^b_{-s}} \leq C\delta^\theta \|g\|_{C^{1/2}_{-s}}.$$

We go back to the proof of Lemma 5.2, where we showed that for $0 < b_1 < b_2 < 1/2$, we have $\|\psi(t/\delta)g\|_{\tilde{X}^{b_1}_{-s}} \leq C\delta^\theta \|g\|_{\tilde{X}^{b_2}_{-s}}$. Then

$$\|\psi(t/\delta)g\|_{C^b_{-s}} \leq \|\psi(t/\delta)g\|_{\tilde{X}^{b_1}_{-s}} \leq C\delta^{b_2} \|g\|_{\tilde{X}^{b_2}_{-s}} \leq C\delta^{b_2} \|g\|_{C^{1/2}_{-s}},$$

as desired. □

References

[1] A.S. Fokas and M. Ablowitz, *The inverse scattering transform for the Benjamin-Ono equation—a pivot to multidimensional problems* Stud. Appl. Math., 68 (1984), 1–10.

[2] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations II. The KdV equation*, GAFA, 3 (1993), 209–262.

[3] J. Bergh and J. Löfström, "Interpolation Spaces," Springer-Verlag, 1976.

[4] M. Christ, J. Colliander, and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, to appear in American J. Math., 2003.

[8] The space Z_s^b is defined in (1.10).

[5] R. Coifman and V. Wickerhauser, *The scattering transform for the Benjamin-Ono equation*, Inverse Problems, 6 (1990), 825–861.

[6] J. Colliander, C. Kenig, and G. Staffilani, *Small solutions for the Kadomtsev-Petviashvili I equation*, Mosc. Math. J., 1 (2001), 491–520.

[7] J. Colliander, C. Kenig, and G. Staffilani, *Low regularity solutions for the Kadomtsev-Petviashvili I equation*, to appear, Geom. Funct. Anal., 2003.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc., 16 (2003), 705–749.

[9] J. Ginibre and G. Velo, *Propriétés de lissage et existence de solutions pour léquation de Benjamin-Ono généralisée*, C.R. Acad. Sci. Paris, 308 (1989), 309–314.

[10] J. Ginibre and G. Velo, *Smoothing properties and existence of solutions for the generalized Benjamin-Ono equation*, J. Diff. Eq., 93 (1991), 150–232.

[11] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc., 4 (1991), 323–347.

[12] C.E. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J., 40 (1991), 33–69.

[13] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math., 46 (1993), 527–620.

[14] C.E. Kenig, G. Ponce, and L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J., 71 (1993), 1–21.

[15] C.E. Kenig, G. Ponce, and L. Vega, *On the generalized Benjamin-Ono equation,* Trans. Amer. Math. Soc., 342 (1994), 155–172.

[16] C.E. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc., 9 (1996), 573–603.

[17] H. Koch and N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equation in* $H^s(\mathbb{R})$, preprint, to appear in Int. Math. Res. Notices, 2003.

[18] R.M. Miura, *Korteweg-de Vries equation and generalizations I. A remarkable explicit nonlinear transformation,* J. Math. Phys., 9 (1968), 1202–1204.

[19] R.M. Miura, *The Korteweg-de Vries equation, a survey of results,* SIAM Review, 18 (1976), 412–459.

[20] L. Molinet, J.-C. Saut, and N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, SIAM J. Math. Anal., 33 (2001), 982–988.

[21] L. Molinet, J.-C. Saut, and N. Tzvetkov, *Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation*, Duke Math. J., 115 (2002), 353–384.

[22] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Diff. Integral Eq., 4 (1991), 527–542.

[23] J.-C. Saut, *Sur quelques généralizations de l'équation de Korteweg-de Vries*, J. Math. Pures Appl., 58 (1979), 21–61.

[24] J.-C. Saut, *Remarks on the generalized Kadomtsev-Petviashvili equations*, Indiana U. Math. J., 42 (1993), 1011–1026.

[25] A. Sidi, C. Sulem, and P.-L. Sulem, *On the long time behavior of a generalized KdV equation*, Acta Applicandae Math. J., 7 (1986), 35–47.

[26] M. Tom, *Smoothing properties of some weak solutions of the Benjamin-Ono equation*, Diff. Integral Eq., 3 (1990), 683–694.