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BLOW-UP FOR SEMILINEAR PARABOLIC PROBLEMS WITH NONCONSTANT COEFFICIENTS

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Abstract. In this paper we study the solutions of some semilinear parabolic problems with nonconstant coefficients. We prove the existence of solutions which blow up at a finite time, and give the behavior near a point of blow-up.

1. Introduction: notation and main results

In this paper we consider the problem

$$\begin{cases} u_t + Lu = \lambda\ a(x)f(u) \quad \text{for} \quad (x,t) \in \mathbb{R}^N \times (0,+\infty) \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}^N \\ u(x,t) \to 0 \quad \text{when} \quad \mid x \mid \to \infty \end{cases} \tag{1.1}$$

with $L = -\Delta + c^2$, $c > 0$, $N > 2$, and $\lambda > 0$. The functions a and u_0 are continuous, bounded, strictly positive, and tend to zero at infinity. The function f is superlinear. We also assume that f is C^2 with nonnegative values, $f'(x) > 0$ for $x > 0$, and $f''(x) > 0$ for $x > 0$.

The operator L appeared earlier in some elliptic problems related with the equation of Klein-Gordon [8], [9]. The first motivation of this work is the study of the relationship with the elliptic problem. The special form of the right-hand side of (1.1) is given for the sake of simplicity. More general forms can be considered.

In this paper (1.1) will be written as

$$\begin{cases} u_t - \Delta u = F(x,u) \quad \text{for} \quad (x,t) \in \mathbb{R}^N \times (0,+\infty) \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}^N \\ u(x,t) \to 0 \quad \text{when} \quad \mid x \mid \to \infty, \end{cases} \tag{1.2}$$

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where $F(x,u) = \lambda\ a(x)f(u) - c^2u$. We consider regular solutions of (1.2) in the sense of Kaplan [5]: let Ω be an open, regular, connected, not necessarily bounded set of $\mathbb{R}^N$ and $Q_T = \Omega \times (0,T]$ for $T > 0$. "The function u is $C^{2,1}(\overline{Q_T})$" means that u, $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$, and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ are defined in Q_T and can be continuously continued up to $\overline{\Omega} \times (0,T]$.

We use extensively the comparison theorem of Kaplan [5], which we denote in the rest of the paper by Kaplan's theorem.

In Section 2, we study the solutions of (1.2) which blow up at a finite time, and get estimates of the time of blow-up. In fact we cannot use the standard methods on (1.2), due to the term "a," which tends to zero at infinity. We work in the ball centered at zero of radius $R > 0$, denoted by B_R, and we consider the problem

$$\begin{cases} u_t - \Delta u = F(x,u) \quad \text{for} \quad (x,t) \in B_R \times (0,+\infty) \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in B_R \\ u(x,t) = 0 \quad \text{when} \quad |\ x\ | = R. \end{cases} \tag{1.3}$$

We prove existence of blow-up for solutions of (1.3), and then we can conclude with Kaplan's theorem. More precisely, define

$$\alpha_R = \min_{x \in B_R} a(x), \tag{1.4}$$

(λ_1, ϕ) depending on R such that

$$\begin{cases} -\Delta\phi = \lambda_1 \phi \quad \text{in} \quad B_R \\ \phi = 0 \quad \text{on} \quad \partial B_R \\ \phi > 0 \quad \text{in} \quad B_R \\ \int_{B_R} \phi\ dx = 1 \end{cases} \tag{1.5}$$

and s_0 the greatest zero of the function g_R defined on $[0,\infty)$ by

$$g_R(s) = \lambda\ \alpha_R f(s) - (\lambda_1 + c^2)s. \tag{1.6}$$

Throughout this paper we denote by T_b the time of blow-up of a function b and by

$$[0, T(b)] \tag{1.7}$$

a closed interval on which b is bounded and regular. The main result of this section is the following:

Theorem 1.1. *Fix $R = R_0 > 0$ such that $\int_A^{+\infty} \frac{ds}{f(s)} < \infty$ for $A > 0$ and $\min\limits_{\overline{B_R}} u_0 > s_0$. Then the solution u of* (1.2) *blows up in a finite time $T_u \neq 0$.*

In Section 3, we study the blow-up rate and prove that the qualitative properties of the solutions of (1.2) near a blow-up point is the same as in the constant coefficient case [2], [3]. More precisely we prove the following theorem:

Theorem 1.2. *Let $p > 1$ and $R_0 > 0$. For $f(t) \geq t^p$ for all $t \geq 0$, we have the estimate*

$$\| u(\cdot, t) \|_\infty \geq [(p-1)\lambda \, \alpha_{R_0}]^{-\frac{1}{p-1}} (T_u - t)^{-\frac{1}{p-1}} \tag{1.8}$$

for $0 < t < T_u$ with

$$\| u(., t) \|_\infty = \sup_{x \in \mathbb{R}^N} u(x, t). \tag{1.9}$$

We also give an upper bound of a solution of (1.2) in a neighborhood of a blow-up point. Let $R_0 > 0$. We introduce the following assumptions:

$$\Delta u_0 - c^2 u_0 + \lambda \, a(x) f(u_0) \geq 0 \quad \text{in} \quad B_{R_0} \tag{1.10}$$

and

$$f'(r) r - f(r) \geq 0 \quad \text{for} \quad r \geq 0. \tag{1.11}$$

Theorem 1.3. *Assume* (1.10) *and* (1.11)*. Let (X, T_u) be a blow-up point for a solution u of* (1.2) *and assume that V is an open neighborhood of X in B_{R_0} such that u is bounded on $\partial V \times [0, T_u)$. Then for every $\eta \in (0, T_u)$, there exists a constant $\delta = \delta(u, \eta)$ such that*

$$u(x, t) \leq \Phi^{-1}[-\delta(T_u - t)] \tag{1.12}$$

for all $x \in V$ and $t \in (\eta, T_u)$, where Φ^{-1} is the inverse function of a primitive Φ of f.

We make precise both of the previous theorems in particular cases:

Theorem 1.4. *Under the assumptions of Theorems* 1.2 *and* 1.3*, for $f(u) = u^p$ or $f(u) = (1 + u)^p$ with $p > 1$, the function w defined by*

$$w(x, t) = (T_u - t)^{\frac{1}{p-1}} u(x, t) \tag{1.13}$$

is bounded on $\mathbb{R}^N \times (0, T_u)$.

In Section 4, we give the asymptotic behavior of the solution u of (1.2) near a blow-up point.

Theorem 1.5. *Assume that $1 < p < \frac{N+2}{N-2}$ and $f(u) = u^p$ or $f(u) = (1+u)^p$ with $p > 1$. Let (X, T_u) be a blow-up point for u satisfying* (1.2)*. Then*

$$\lim_{t \to T_u} (T_u - t)^{\frac{1}{p-1}} u(X + y(T_u - t)^{\frac{1}{2}}, t) = (\lambda \, (p-1) a(X))^{-\frac{1}{p-1}}. \tag{1.14}$$

The limit is independent of $y \in \mathbb{R}^N$ and it is uniform on each compact set $| y | \leq C$.

2. Existence of blow-up for solutions of (1.2)

2.1. **Upper bound for u_R and u.** We assume there exists a regular solution u_R of (1.3). We have

$$a(x) \leq \| a \|_\infty .$$

Consider the differential problem

$$\begin{cases} \dfrac{dz}{dt} = \lambda \ \| a \|_\infty f(z) - c^2 z \\ z(0) \geq \sup\limits_{x \in \mathbb{R}^N} u_0(x), \quad z(t) \geq 0. \end{cases} \tag{2.1}$$

By the change $z(t) = e^{-c^2 t} v(t)$

$$\begin{cases} \dfrac{dv}{dt} = e^{c^2 t} \lambda \ \| a \|_\infty f(e^{-c^2 t} v(t)) \geq 0 \\ v(0) \geq \sup\limits_{x \in \mathbb{R}^N} u_0(x), \end{cases}$$

and then $v(t) \geq 0$ and $z(t) \geq 0$ on their interval of definition. We use the notation introduced in (1.7). Choosing $z(0) \geq \sup_{x \in \mathbb{R}^N} u_0(x)$, we get a solution z of (2.1), regular and bounded on $[0, T(z)]$. By Kaplan's theorem, we obtain

$$u_R(x,t) \leq z(t) \quad \text{for all} \quad x \in \overline{B_R} \quad \text{and} \quad t \in (0, \min(T(u_R), T(z))). \tag{2.2}$$

As z does not depend on R, the inequality (2.2) is true for every R. Now we look at u. Assume there exists a regular solution of (1.2). In B_R, u satisfies

$$\begin{cases} u_t - \Delta u = F(x,u) \\ u(x,0) = u_0(x) \\ u(x,t) \geq 0 \quad \text{for} \quad | x |= R. \end{cases}$$

By Kaplan's theorem in B_R we have

$$u \geq u_R \quad \text{in} \quad B_R \times (0, \min(T(u), T(u_R))). \tag{2.3}$$

As $u_R(x,t) = 0$ outside of B_R, the inequality (2.3) is true in $\mathbb{R}^N$. On the other side, Kaplan's theorem gives

$$u(x,t) \leq z(t) \quad \text{in} \quad \mathbb{R}^N \times [0, \min(T(u), T(z))].$$

Finally, we get

$$\text{For all } R > 0 \quad u_R \leq u \leq z \quad \text{in} \quad \mathbb{R}^N \times (0, \min(T(u), T(u_R), T(z))). \tag{2.4}$$

2.2. **Existence of blowing-up solutions.**

2.2.1. *Blow-up for u_R.* Following the original idea of Kaplan [5], we show that $\sup_{x\in\overline{B_R}} u_R(x,t)$ is bounded from below. This estimate allows us to prove the existence of blow-up for u_R. We assume that u_R exists as a regular solution of (1.3). Define

$$\hat{u}_R(t) = \int_{B_R} u_R(x,t)\phi(x)\ dx, \tag{2.5}$$

where ϕ is defined in (1.5). Then

$$\hat{u}_R(0) \geq \inf_{\overline{B_R}} u_0(x). \tag{2.6}$$

Multiplying the equation (1.3) of u_R by ϕ and integrating over B_R, we find

$$\frac{d\hat{u}_R}{dt} = \lambda \int_{B_R} af(u_R)\phi\ dx - (\lambda_1 + c^2)\hat{u}_R,$$

as

$$\int_{B_R} af(u_R)\phi\ dx \geq \alpha_R \int_{B_R} f(u_R)\phi\ dx \geq \alpha_R f\Big(\int_{B_R} u_R\ \phi\ dx\Big)$$

with α_R defined in (1.4), and by use of Jensen's inequality. Finally we find a differential inequality for $\hat{u}_R$,

$$\frac{d\hat{u}_R}{dt} \geq \lambda\alpha_R f(\hat{u}_R) - (\lambda_1 + c^2)\hat{u}_R \tag{2.7}$$

with (2.6). Consider the differential problem

$$\frac{d\zeta_R}{dt} = \lambda\alpha_R f(\zeta_R) - (\lambda_1 + c^2)\zeta_R, \quad \zeta_R(0) = \inf_{x\in\overline{B_R}} u_0(x). \tag{2.8}$$

Kaplan's theorem gives $\hat{u}_R(t) \geq \zeta_R(t)$ for $t \in [0, \min(T(\hat{u}_R), T(\zeta_R))]$. As $\hat{u}_R(t) \leq \sup_{x\in\overline{B_R}} u_R(x,t)$, we get

$$\sup_{x\in\overline{B_R}} u_R(x,t) \geq \zeta_R(t). \tag{2.9}$$

Inequality (2.9) is true for bounded functions; now if ζ_R blows up for $T_{\zeta_R} > T(\zeta_R)$, by continuity we get blow-up results for u_R. Recall that s_0 and g_R are defined in (1.6). Then

Proposition 2.1. *If $\int_A^{+\infty} \frac{ds}{f(s)} < \infty$ and if $\inf_{\overline{B_R}} u_0 > s_0$, then ζ_R blows up at a finite time*

$$T_{\zeta_R} = \int_{\zeta_R(0)}^{+\infty} \frac{ds}{\lambda\alpha_R f(s) - (\lambda_1 + c^2)s}.$$

Note that the integral is convergent as f is superlinear. The second condition says that u_0 must be "big enough." Now the inequality (2.9) shows that u_R must blow up for a finite time T_{u_R}, if ζ_R does. Using the function z introduced in (2.1), we know that $u_R(x,t) \leq z(t)$ for $t \in (0, \min(T(u_R), T(z)))$. Let $h(s) = \lambda \parallel a \parallel_\infty f(s) - c^2 s$, and denote by s_1 the greatest zero of h. We get

Proposition 2.2. *If $z(0) > s_1$, then z blows up in a finite time:*

$$T_z = \int_{z(0)}^{+\infty} \frac{ds}{\lambda \parallel a \parallel_\infty f(s) - c^2 s}.$$

We can choose $z(0) > s_1$, and then get a function z which blows up at $t = T_z$. Finally we get:

Proposition 2.3. *Under the conditions of Propositions* 2.1 *and* 2.2*, we get blow-up for u_R in a finite time T_{u_R} satisfying*

$$\int_{z(0)}^{+\infty} \frac{ds}{\lambda \parallel a \parallel_\infty f(s) - c^2 s} \leq T_{u_R} \leq \int_{\inf\limits_{\overline{B_R}} u_0}^{+\infty} \frac{ds}{\lambda \alpha_R f(s) - (\lambda_1 + c^2) s}.$$

2.2.2. *Blow-up for u.* We can deduce from the preceding subsection conditions for the explosion of u. We assume that the conditions of proposition 2.3 are satisfied for $R = R_0$. We have

$$u_{R_0}(x,t) \leq u(x,t) \leq z(t) \quad \text{for} \quad (x,t) \in \overline{B_{R_0}} \times (0, \min(T(u_{R_0}), T(u), T(z)))$$

and

$$\sup_{x \in \overline{B_{R_0}}} u(x,t) \geq \sup_{x \in \overline{B_{R_0}}} u_{R_0}(x,t).$$

Then Theorem 1.1 results from the inequality $T_z \leq T_u \leq T_{\zeta_{R_0}}$.

2.2.3. *Special cases.* First we consider $f(u) = u^p$ with $p > 1$. Then (2.8) becomes

$$\frac{d\zeta}{dt} = \lambda \alpha_R \zeta^p - (\lambda_1 + c^2)\zeta, \quad \zeta(0) = \inf_{x \in B_R} u_0(x).$$

By the change $\zeta(t) = e^{-\mu t} g(t)$ with $\mu = \lambda_1 + c^2$, (2.8) becomes

$$\frac{dg}{dt} = \lambda \alpha_R e^{-(p-1)\mu t} g^p, \quad g(0) = \inf_{x \in B_R} u_0(x). \tag{2.10}$$

Integrating the differential equation (2.10) between 0 and t, and taking $t = T_g$, we get

$$1 - e^{-(p-1)\mu T_g} = \frac{\mu}{\lambda \alpha_R g^{p-1}(0)},$$

which gives the condition for existence of blow-up for g,

$$\inf_{x\in B_R} u_0(x) = g(0) > \Big[\frac{\lambda\alpha_R}{\lambda_1 + c^2}\Big]^{\frac{1}{p-1}},$$

and the time of blow-up for g and ζ_R,

$$T_g = T_{\zeta_R} = -\frac{1}{\mu(p-1)} \ln\Big[1 - \frac{\mu}{\lambda\alpha_R g^{p-1}(0)}\Big].$$

We proceed in a similar way for z. In (2.1) we make the change $z(t) = e^{-c^2 t} v(t)$ and get

$$\frac{dv}{dt} = \lambda \parallel a \parallel_\infty e^{-(p-1)c^2 t} v^p, \quad v(0) \geq \sup_{\mathbb{R}^N} u_0.$$

Integrating this differential equation between 0 and t and taking $t = T_v$, we get

$$v(0) > \Big[\frac{\lambda \parallel a \parallel_\infty}{c^2}\Big]^{\frac{1}{p-1}}$$

and

$$T_z = T_v = -\frac{1}{(p-1)c^2} \ln\big(1 - \frac{c^2}{\lambda \parallel a \parallel_\infty v^{p-1}(0)}\big).$$

Then

$$-\frac{1}{(p-1)c^2} \ln\big(1 - \frac{c^2}{\lambda \parallel a \parallel_\infty z^{p-1}(0)}\big) \leq T_{u_R} \leq -\frac{1}{\mu(p-1)} \ln\big(1 - \frac{\mu}{\lambda\alpha_R g^{p-1}(0)}\big).$$

Next, if $f(u) = (1+u)^p$ or $f(u) = 1 + u^p$ with $p > 1$, we have $f(u) > u^p$ for $u > 0$, and so we get the same bound from below of u_R and the same bound from above of T_{u_R}. On the other side, if $f(t) = (1+t)^p$ we can integrate the associated differential equation to get a bound from above of u_R and a bound from below of T_{u_R}. Precisely, we have

$$\frac{dv}{dt} = \lambda \parallel a \parallel_\infty e^{c^2 t}(1 + e^{-c^2 t} v)^p \leq \lambda \parallel a \parallel_\infty e^{c^2 t}(1+v)^p, \quad v(0) \geq \sup_{\mathbb{R}^N} u_0.$$

Considering

$$\frac{d\nu}{dt} = \lambda \parallel a \parallel_\infty e^{c^2 t}(1+\nu)^p, \quad \nu(0) = v(0),$$

we get

$$e^{c^2 T_\nu} = e^{c^2 T_v} = 1 + \frac{c^2}{(p-1)\lambda \parallel a \parallel_\infty} \times \frac{1}{[1+\nu(0)]^{p-1}}.$$

Let us notice that no necessary condition for the blow-up of ν appears in this proof. Finally, we have

$$\frac{1}{c^2} \ln\Big[1 + \frac{c^2}{(p-1)\lambda \parallel a \parallel_\infty} \times \frac{1}{[1+\nu(0)]^{p-1}}\Big]$$

$$\leq T_{u_R} \leq -\frac{1}{(p-1)\mu} \ln\Big[1 - \frac{\mu}{\lambda\alpha_R g^{p-1}(0)}\Big].$$

3. Estimate of blow-up rate

In this section we give an estimate of a solution u of (1.2) with respect to $(T_u - t)$.

3.1. **Lower bound. Proof of Theorem 1.2.** First we give a minoration of u_{R_0}. Recall that $\hat{u}_{R_0}$ defined in (2.5) satisfies (2.6)–(2.7).

First we study the case $f(u) = u^p$. We consider the differential problem (2.8) with the change $\zeta(t) = e^{-\mu t} g(t)$, $\mu = \lambda_1 + c^2$; we get

$$\frac{dg}{dt} = \lambda\alpha_{R_0} e^{-(p-1)\mu t} g^p, \quad g(0) = \inf_{\overline{B_{R_0}}} u_0.$$

By integration,

$$g(t) = \Big(\frac{\mu}{\lambda\alpha_{R_0}}\Big)\Big[e^{-(p-1)\mu t} - e^{-(p-1)\mu T_g}\Big]^{-\frac{1}{p-1}}$$

for $0 < t < T_g$, and by the mean value theorem,

$$g(t) = [(p-1)\lambda\alpha_{R_0}]^{-\frac{1}{p-1}} (T_g - t)^{-\frac{1}{p-1}} e^{\mu\theta} \quad (t < \theta < T_g)$$

for $0 < t < T_g$. Now

$$\hat{u}_{R_0}(t) \geq [(p-1)\lambda\alpha_{R_0}]^{-\frac{1}{p-1}} (T_g - t)^{-\frac{1}{p-1}}.$$

As $T_{\hat{u}_{R_0}} \geq T_g$, we obtain

$$\hat{u}_{R_0}(t) \geq [(p-1)\lambda\alpha_{R_0}]^{-\frac{1}{p-1}} (T_{\hat{u}_{R_0}} - t)^{-\frac{1}{p-1}}$$

for $0 < t < T_{\hat{u}_{R_0}}$. As $\sup_{x \in B_{R_0}} u_{R_0}(x,t) \geq \hat{u}_{R_0}(t)$ with $T_{u_{R_0}} \leq T_{\hat{u}_{R_0}}$, we get

$$\| u_{R_0}(\cdot,t) \|_\infty \geq [(p-1)\lambda\alpha_{R_0}]^{-\frac{1}{p-1}} (T_{u_{R_0}} - t)^{-\frac{1}{p-1}} \tag{3.1}$$

for $0 < t < T_{u_{R_0}}$. As $u(x,t) \geq u_{R_0}(x,t)$ for $x \in \mathbb{R}^N$, we finally obtain

$$\| u(\cdot,t) \|_\infty \geq [(p-1)\lambda\alpha_{R_0}]^{-\frac{1}{p-1}} (T_u - t)^{-\frac{1}{p-1}} \tag{3.2}$$

for $0 < t < T_u$.

Now if $f(u) > u^p$ for $u \geq 0$, we get again

$$\frac{d\hat{u}_{R_0}}{dt} \geq \lambda\alpha_R (\hat{u}_{R_0})^p - (\lambda_1 + c^2)\hat{u}_{R_0}, \quad \hat{u}_{R_0}(0) \geq \inf_{x \in \overline{B_{R_0}}} u_0(x), \tag{3.3}$$

and the proof is still valid.

3.2. **Lower bound of a solution of (1.2) in a neighborhood of a blow-up point.** In this section we prove Theorems 1.3 and 1.4. Let us consider a solution u of (1.2); we restrict ourselves to B_{R_0}, as the blow-up occurs in B_{R_0}. We assume (1.10). For instance if $u_0 = U_0$, where U_0 is a positive constant, and if $f(t) = t^p$, we find that (1.10) is satisfied if

$$U_0 \geq \Big[\frac{c^2}{\lambda \inf_{x\in\overline{B_{R_0}}} a(x)}\Big]^{\frac{1}{p-1}},$$

and furthermore U_0 must satisfy the condition of Proposition 2.1. It is possible to choose U_0 in such a way. In the case $f(t) = (1+t)^p$, it is easy to see that we can choose $u_0 = U_0$ a constant satisfying (1.10), but the value of U_0 is not explicit.

Now we give here a proof widely inspired by Friedmann and Mac Leod [2]. Let δ be a positive real, and consider the function J defined for $(x,t) \in B_{R_0} \times (0, T_u)$ by $J(x,t) = u_t(x,t) - \delta f(u(x,t))$. Because of (1.10), we have $u_t(x,t) > 0$ in $B_{R_0} \times (0, T_u)$ (see [7]).

Lemma 3.1. *Under the condition* (1.11), *J satisfies in $B_{R_0} \times (0, T_u)$ the differential inequality*

$$J_t - \Delta J + c^2 J - \lambda a(x) f'(u) J \geq 0. \tag{3.4}$$

Proof. A direct computation gives

$$J_t - \Delta J + c^2 J - \lambda a(x) f'(u) J = \delta f''(u)|\nabla u|^2 + \delta c^2 [f'(u)u - f(u)],$$

and the result holds.

The condition (1.11) says that f is greater than a linear function. In fact, the solutions of the equation $f'(u)u - f(u) = 0$ are the linear functions (in particular, $f(0) = 0$). We can say also that $xf'(x) - f(x) \geq 0$ is equivalent to the following: the function $x \mapsto \frac{f(x)}{x}$, for $x > 0$, is an increasing function.

We see also that $f(x) = x^p$ satisfies the inequality (1.11) for $x \geq 0$, and that $f(x) = (1+x)^p$ satisfies (1.11) for $x \geq 1$.

Lemma 3.2. *Let (X, T_u) be a blow-up point for u a solution of* (1.2), *and assume that V is an open neighborhood of X in B_{R_0} such that u is bounded on $\partial V \times [0, T_u)$. Then for every $\eta \in (0, T_u)$, there exists $\delta = \delta(\eta, u) > 0$ such that $J \geq 0$ in $V \times (\eta, T_u)$.*

Proof. As $u_t > 0$ in $\overline{V} \times (0, T_u)$, there exists a constant $C_u > 0$ such that $u_t \geq C_u > 0$ in $\overline{V} \times (0, T_u)$. As u is bounded on $\partial V \times [0, T_u)$, so is $f(u)$, and there exists δ_1 such that

$$u_t - \delta_1 f(u) \geq C_u - \delta_1 f(u) \geq 0$$

on $\partial V \times [0, T_u)$. On the other hand, let x be in V and η in $(0, T_u)$. The function $x \mapsto u(x, \eta)$ is bounded on $\overline{V}$ by definition of T_u. Then there exists a constant $\delta_2(u, \eta) > 0$ such that

$$J(x, \eta) = u_t(x, \eta) - \delta_2 f(u(x, \eta)) \geq 0$$

for $x \in \overline{V}$. By Kaplan's theorem, comparing J and 0 on $V \times [\eta, T(u)]$, $T(u) < T_u$, we conclude that $J \geq 0$ in $V \times [\eta, T(u)]$, and the result holds by continuity.

Proof of Theorem 1.3. By Lemma 3.2, we get $u_t \geq \delta f(u)$; assuming $f(u) \neq 0$, we have

$$\frac{u_t}{f(u)} \geq \delta.$$

By integration in the interval $(t, t') \subset (0, T_u)$, and taking $t' = T_u$, we get

$$\Phi(u(t)) \leq -\delta(T_u - t).$$

Φ is a monotone function, which has an inverse function Φ^{-1}, and the result holds.

Proof of Theorem 1.4. If $f(r) = r^p$, then $\Phi(r) = -1/[(p-1)r^{p-1}]$ and $\Phi^{-1}(r) = [-(p-1)r]^{-1/(p-1)}$, $r < 0$.

If $f(r) = (1+r)^p$, then $\Phi(r) = -1/[(p-1)(1+r)^{p-1}]$ and $\Phi^{-1}(r) = -1 + [-(p-1)r]^{-1/(p-1)}$, $r < 0$.

4. ASYMPTOTIC BEHAVIOR

To prove Theorem 1.5 we follow the idea of Giga and Kohn [3]. To study u near a point (X, T_u), we introduce the rescaled function w of Theorem 1.4,

$$w(y, s) = (T_u - t)^{\frac{1}{p-1}} u(x, t), \tag{4.1}$$

with

$$\begin{cases} x - X = (T_u - t)^{\frac{1}{2}} y \\ T_u - t = e^{-s}. \end{cases} \tag{4.2}$$

For $f(u) = u^p$, $p > 1$, the function w solves

$$w_s - \frac{1}{\rho}\nabla \cdot (\rho \nabla w) + \frac{1}{p-1} w + c^2 e^{-s} w = \lambda a(X + e^{-\frac{s}{2}} y) w^p \tag{4.3}$$

in $\mathbb{R}^N \times (\sigma_0, +\infty)$, where $\sigma_0 = -\ln T_u$ and $\rho(y) = \exp(-|y|^2/4)$. And for $f(u) = (1+u)^p$, $p > 1$, equation (4.3) is replaced by

$$w_s - \frac{1}{\rho}\nabla \cdot (\rho \nabla w) + \frac{1}{p-1} w + c^2 e^{-s} w = \lambda a(X + e^{-\frac{s}{2}} y)(e^{-\frac{s}{p-1}} + w)^p. \tag{4.4}$$

Before proving Theorem 1.5, we establish two lemmas which concern L^2 estimates of w_s and ∇w. Note that in Lemma 4.1 the condition $p < (N +$

$2)/(N-2)$ is not needed. We denote by M a bound from above of w, which exists by the preceding theorem.

Lemma 4.1. *There exists a real number $L>0$ which depends only on p, c, λ, a, T_u, M, and $\int_{\mathbb{R}^N} |\nabla w|^2(y,\sigma_0)\rho(y)dy$, such that*

$$\int_{\sigma_0}^{+\infty}\int_{\mathbb{R}^N} w_s^2\rho\ dy\ ds \le L\ . \tag{4.5}$$

Proof. *Case 1:* $f(u)=u^p$, $p>1$. Multiplying equation (4.3) by $w_s\rho$ and integrating on any ball B_R, we obtain for $s>\sigma_0$

$$\begin{aligned}&\int_{B_R} w_s^2\rho\ dy\ +\int_{B_R}\Big[-w_s\nabla\cdot(\rho\nabla w)+\Big(\frac{1}{p-1}+c^2e^{-s}\Big)ww_s\rho\Big]\ dy\\ &=\ \lambda\int_{B_R} a(X+e^{-\frac{s}{2}}y)w^p w_s\rho\ dy\ .\end{aligned} \tag{4.6}$$

Since

$$\int_{B_R}-w_s\nabla\cdot(\rho\nabla w)dy=\int_{B_R}-w\nabla\cdot(\rho\nabla w_s)+\int_{\partial B_R}\nabla w_s\cdot\nu w\rho\, d\sigma-\int_{\partial B_R}\nabla w\cdot\nu w_s\rho\, d\sigma,$$

this implies when R tends to infinity

$$\int_{\mathbb{R}^N}-w_s\nabla\cdot(\rho\nabla w)dy=\int_{\mathbb{R}^N}-w\nabla\cdot(\rho\nabla w_s)dy.$$

Then

$$\frac{1}{2}\frac{d}{ds}\Big(\int_{R^N}|\nabla w|^2\rho\, dy\Big)=-\int_{\mathbb{R}^N} w_s\nabla.(\rho\nabla w)dy.$$

This and relation (4.6) lead us to

$$\begin{aligned}&\int_{\mathbb{R}^N} w_s^2\rho\ dy\ +\frac{1}{2}\frac{d}{ds}\Big(\int_{R^N}|\nabla w|^2\rho\, dy\Big)+\int_{\mathbb{R}^N}\Big(\frac{1}{p-1}+c^2e^{-s}\Big)ww_s\rho\, dy\\ &=\ \lambda\int_{\mathbb{R}^N} a(X+e^{-\frac{s}{2}}y)w^p w_s\rho\, dy.\end{aligned}$$

Now consider $\tau>\sigma_0$ and integrate this relation on $[\sigma_0,\tau]$:

$$\begin{aligned}&\int_{\sigma_0}^{\tau}\int_{\mathbb{R}^N} w_s^2\rho\, dy+\frac{1}{2}\int_{R^N}|\nabla w|^2(y,\tau)\rho\, dy=\frac{1}{2}\int_{R^N}|\nabla w|^2(y,\sigma_0)\rho\, dy\\ &-\int_{\sigma_0}^{\tau}\int_{\mathbb{R}^N}(\frac{1}{p-1}+c^2e^{-s})ww_s\rho\, dy\, ds+\lambda\int_{\sigma_0}^{\tau}\int_{\mathbb{R}^N} a(X+e^{-\frac{s}{2}}y)w^p w_s\rho\, dy\, ds.\end{aligned} \tag{4.7}$$

To obtain (4.5), we have to bound the second and third terms of the right-hand side of (4.7). Using an integration by parts, we have

$$-\int_{\sigma_0}^{\tau}\int_{\mathbb{R}^N}\Big(\frac{1}{p-1}+c^2e^{-s}\Big)ww_s\rho\, dy\, ds$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^N}\Big(\Big[\Big(\frac{1}{p-1}+c^2e^{-s}\Big)\frac{w^2}{2}\Big]_{\sigma_0}^{\tau} - c^2\int_{\sigma_0}^{\tau} e^{-s}\frac{w^2}{2}ds\Big)\rho\,dy \qquad (4.8)\\
&\le \int_{\mathbb{R}^N}\Big(\frac{1}{p-1}+c^2T_u\Big)\frac{w^2}{2}(y,\sigma_0)\rho\,dy + c^2\frac{M^2}{2}T_u\int_{\mathbb{R}^N}\rho\,dy\\
&\le \Big(\frac{1}{2(p-1)}+c^2T_u\Big)M^2\int_{\mathbb{R}^N}\rho\,dy.
\end{aligned}
$$

Finally, we also have

$$
\begin{aligned}
&\lambda\int_{\sigma_0}^{\tau}\int_{\mathbb{R}^N} a(X+e^{-\frac{s}{2}}y)w^p w_s\rho\,dy\,ds = \lambda\int_{\mathbb{R}^N}\Big[a(X+e^{-\frac{s}{2}}y)\frac{w^{p+1}}{p+1}\Big]_{\sigma_0}^{\tau}\rho\,dy\\
&\qquad + \frac{\lambda}{2}\int_{\mathbb{R}^N}\Big(\int_{\sigma_0}^{\tau} e^{-\frac{s}{2}}\nabla a(X+e^{-\frac{s}{2}}y)\cdot y\frac{w^{p+1}}{p+1}ds\Big)\rho\,dy\\
&\le \lambda||a||_\infty\frac{M^{p+1}}{p+1}\int_{\mathbb{R}^N}\rho\,dy + \frac{\lambda}{2}||\nabla a||_\infty\frac{e^{-\frac{\sigma_0}{2}}}{2}\frac{M^{p+1}}{p+1}\int_{\mathbb{R}^N}|y|\rho\,dy. \qquad (4.9)
\end{aligned}
$$

Combining (4.7)–(4.8) and (4.9) we derive (4.5).

Case 2: $f(u) = (1+u)^p$, $p > 1$. The only difference is that (4.9) is replaced by

$$
\begin{aligned}
&\lambda\int_{\sigma_0}^{\tau}\int_{\mathbb{R}^N} a(X+e^{-\frac{s}{2}}y)(e^{-\frac{s}{p-1}}+w)^p w_s\rho\ dy\ ds\\
&= \lambda\int_{\sigma_0}^{\tau}\int_{\mathbb{R}^N} a(X+e^{-\frac{s}{2}}y)(e^{-\frac{s}{p-1}}+w)^p(-\frac{1}{p-1}e^{-\frac{s}{p-1}}+w_s)\rho\ dy\ ds\\
&+\frac{\lambda}{p-1}\int_{\sigma_0}^{\tau}\int_{\mathbb{R}^N} a(X+e^{-\frac{s}{2}}y)(e^{-\frac{s}{p-1}}+w)^p e^{-\frac{s}{p-1}}\rho\ dy\ ds.
\end{aligned}
$$

The first term can be treated integrating by parts as in (4.9) and the second term is bounded.

Now we give an estimate of the gradient of w. We treat only the case $f(u) = u^p$, $p > 1$. The other case can be treated similarly, as in the previous lemma. We introduce the energy function E for w as follows: for $s \ge \sigma_0$,

$$
\begin{aligned}
&E[w](s) =\\
&\int_{\mathbb{R}^N}\Big(\frac{1}{2}|\nabla w|^2 + (\frac{1}{p-1}+c^2e^{-s})\frac{w^2}{2} - \frac{\lambda}{p+1}a(X+e^{-\frac{s}{2}}y)w^{p+1}\Big)\rho|y|^2dy\\
&-\frac{1}{2}\int_{\mathbb{R}^N}(\frac{1}{2}|y|^2-N)w^2\rho\,dy.
\end{aligned}
$$

Lemma 4.2. *Assume that* $1 < p < (N+2)/(N-2)$. *Then there exists a real number* $\tilde{L} > 0$ *which depends only on* p, c, λ, a, T_u, M, *and* $E[w](\sigma_0)$, *such that*

$$\int_{\sigma_0}^{+\infty}\int_{\mathbb{R}^N}|\nabla w|^2(1+|y|^2)\rho\,dy\ ds\le\tilde{L}\ . \tag{4.10}$$

The proof follows the idea of Propositions 4.1, 4.2, and 4.3 of [3], and we give only the derivative of E:

$$\frac{d}{ds}E[w](s)=-\int_{\mathbb{R}^N}w_s^2|y|^2\rho\,dy-\frac{c^2}{2}e^{-s}\int_{\mathbb{R}^N}w^2|y|^2\rho\,dy$$
$$-(p+3)\int_{\mathbb{R}^N}w_s(\nabla w\cdot y)\rho\,dy+\frac{\lambda e^{-\frac{s}{2}}}{2(p+1)}\int_{\mathbb{R}^N}\nabla a(X+e^{-\frac{s}{2}}y)\cdot yw^{p+1}|y|^2\rho\,dy$$
$$-\int_{\mathbb{R}^N}\left[\frac{p-1}{4}|y|^2+\frac{1}{2}(N+2-p(N-2))\right]|\nabla w|^2\rho\,dy$$
$$-\frac{p-1}{2}c^2e^{-s}\int_{\mathbb{R}^N}(\frac{1}{2}|y|^2-N)w^2\rho\,dy-\lambda e^{-\frac{s}{2}}\int_{\mathbb{R}^N}\nabla a(X+e^{-\frac{s}{2}}y)\cdot yw^{p+1}\rho\,dy.$$

Proof of Theorem 1.5. Let (s_j) be any sequence tending to infinity. Consider the function w_j defined on $\mathbb{R}^N\times(\sigma_0-s_j,+\infty)$ by $w_j(y,s)=w(y,s+s_j)$. The function w_j is bounded by M, and it is a respective solution of

$$w_{js}-\frac{1}{\rho}\nabla\cdot(\rho\nabla w_j)+\frac{1}{p-1}w_j+c^2e^{-s-s_j}w_j=\lambda a(X+e^{-\frac{s+s_j}{2}}y)w_j^p$$

for $f(u)=u^p$ and

$$w_{js}-\frac{1}{\rho}\nabla\cdot(\rho\nabla w_j)+\frac{1}{p-1}w_j+c^2e^{-s-s_j}w_j=\lambda a(X+e^{-\frac{s+s_j}{2}}y)\Big(e^{-\frac{s+s_j}{p-1}}+w_j\Big)^p$$

for $f(u)=(1+u)^p$. Using the L^q-regularity theory for parabolic equations (see [6]), we deduce that ∇w_j, D^2w_j, and w_{js} are bounded in $L^q(B_R\times(-R,+\infty))$ for each $q\in(1,+\infty)$ and $R>0$ (when s_j is large enough), the bound being independent of j. By Sobolev's inequality and Schauder's estimates (see [1]) we obtain that (D^2w_j) and (w_{js}) are Hölder-continuous on each $B_R\times(-R,+\infty)$, uniformly with respect to j.

By the Arzela-Ascoli theorem and a diagonal argument, there exists a subsequence, still denoted by w_j, converging uniformly to a function l on each $B_R\times(-R,+\infty)$. This function l is in $C^{2,1}(\mathbb{R}^{N+1})$ and it is a solution of

$$l_s-\frac{1}{\rho}\nabla\cdot(\rho\nabla l)+\frac{1}{p-1}l=\lambda a(X)l^p\ .$$

Because of Lemmas 4.1 and 4.2, we have

$$\int_{-R}^{+\infty}\int_{B_R}|\nabla w_j|^2\rho\,dy\,ds=\int_{-R+s_j}^{+\infty}\int_{B_R}|\nabla w|^2\rho\,dy\,ds\to 0$$

$$\int_{-R}^{+\infty}\int_{B_R} w_{js}^2\rho\,dy\,ds = \int_{-R+s_j}^{+\infty}\int_{B_R} w_{js}^2\rho\,dy\,ds \to 0$$

as $j \to +\infty$, for all $R > 0$. Thus, l is independent of both y and s and satisfies

$$\frac{1}{p-1}l = \lambda a(X)l^p \ .$$

Finally, because of Theorem 2.1 of [4], the limit (1.14) holds.

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