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THE SUB-SUPERSOLUTION METHOD AND EXTREMAL SOLUTIONS FOR QUASILINEAR HEMIVARIATIONAL INEQUALITIES

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Abstract. We generalize the sub-supersolution method known for weak solutions of single and multivalued equations to quasilinear elliptic hemivariational inequalities. To this end we first introduce our basic notion of sub- and supersolutions on the basis of which we then prove existence, comparison, compactness, and extremality results for the hemivariational inequalities under considerations.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. In this paper we deal with the following quasilinear hemivariational inequality:

$$u \in V_0: \quad \langle Au - f, v - u\rangle + \int_\Omega j^o(u; v-u)\, dx \geq 0, \quad \forall\, v \in V_0, \tag{1.1}$$

where $j^o(s;r)$ denotes the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ at s in the direction r given by

$$j^o(s;r) = \limsup_{y \to s,\ t \downarrow 0} \frac{j(y + t\, r) - j(y)}{t}; \tag{1.2}$$

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cf., e.g., [5, Chapter 2]. The operator $A : V \to V_0^*$ is assumed to be a second-order quasilinear differential operator in divergence form:

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)), \quad \text{with} \;\; \nabla u = \Big(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\Big). \tag{1.3}$$

Let $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ denote Clarke's generalized gradient of j defined by

$$\partial j(s) := \{\zeta \in \mathbb{R} : j^o(s; r) \geq \zeta\, r, \quad \forall r \in \mathbb{R}\}. \tag{1.4}$$

A method of super-subsolutions has been established recently in [3, 4] for quasilinear elliptic and parabolic differential inclusion problems in the form

$$Au + \partial j(u) \ni f, \;\; \text{in} \;\; \Omega, \quad u = 0, \;\; \text{on} \;\; \partial\Omega. \tag{1.5}$$

One can show that any solution of (1.5) is a solution of the hemivariational inequality (1.1). The reverse is true only if the function j is regular in the sense of Clarke, which means that the one-sided directional derivative and the generalized directional derivative coincide; cf. [5, Chapter 2.3].

The main goal of this paper is to generalize the super-subsolution method to the general case of hemivariational inequalities (1.1). This extension is by no means a straightforward generalization of the theory developed for the multivalued problems (1.5) because of the intrinsic asymmetry of hemivariational inequalities compared with the symmetric structure of the multivalued equation (1.5). In this paper we introduce an appropriate notion of sub- and supersolutions for inequalities in the form (1.1) in a unified and coherent way which is inspired by recent papers on the sub-supersolution method for variational inequalities; see [7, 8].

The plan of the paper is as follows: In Section 2 we introduce our basic notion of sub-supersolution, and in Section 3 we prove an existence and comparison result in terms of sub- and supersolutions. In Section 4 the solution set within the interval formed by sub- and supersolutions is characterized as a compact set which possesses extremal elements. Finally, in Section 5 we deal with hemivariational inequalities when the operator A is perturbed by a general noncoercive lower-order term.

The theory developed in this paper can be extended to hemivariational inequalities involving even more general quasilinear elliptic operators of Leray-Lions type and functions $j : \Omega \times \mathbb{R} \to \mathbb{R}$ depending, in addition, on the space variable x, which, however, have been omitted in order to avoid too many technicalities and in order to emphasize the main ideas.

2. Notation and hypotheses

We assume $f \in V_0^*$ and impose the following hypotheses of Leray-Lions type on the coefficient functions a_i, $i = 1, \dots, N$, of the operator A:

(A1) Each $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $a_i(x, \xi)$ is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in ξ for almost all $x \in \Omega$. There exist a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega), 1/p + 1/q = 1$, such that

$$|a_i(x, \xi)| \le k_0(x) + c_0 \, |\xi|^{p-1} \,,$$

for almost every $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

(A2) $\sum_{i=1}^N (a_i(x, \xi) - a_i(x, \xi'))(\xi_i - \xi_i') > 0$ for almost every $x \in \Omega$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) $\sum_{i=1}^N a_i(x, \xi)\xi_i \ge \nu |\xi|^p - k_1(x)$ for almost every $x \in \Omega$, and for all $\xi \in \mathbb{R}^N$ with some constant $\nu > 0$ and some function $k_1 \in L^1(\Omega)$.

As a consequence of (A1) and (A2) the semilinear form a associated with the operator A by

$$\langle Au, \varphi \rangle := a(u, \varphi) = \int_\Omega \sum_{i=1}^N a_i(x, \nabla u) \frac{\partial \varphi}{\partial x_i} \, dx, \qquad \forall \; \varphi \in V_0$$

is well-defined for any $u \in V$, and the operator $A : V_0 \to V_0^*$ is continuous, bounded, and monotone. For functions $w, z : \Omega \to \mathbb{R}$ and sets W and Z of functions defined on Ω we use the notation $w \wedge z = \min\{w, z\}$, $w \vee z = \max\{w, z\}$, $W \wedge Z = \{w \wedge z : w \in W, \; z \in Z\}$, $W \vee Z = \{w \vee z : w \in W, \; z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$.

Definition 2.1. A function $\underline{u} \in V$ is called a *subsolution* of (1.1) if the following holds:

(i) $\underline{u} \le 0$ on $\partial\Omega$,
(ii) $\langle A\underline{u} - f, v - \underline{u} \rangle + \int_\Omega j^o(\underline{u}; v - \underline{u}) \, dx \ge 0, \quad \forall \; v \in \underline{u} \wedge V_0$.

Definition 2.2. $\bar{u} \in V$ is a *supersolution* of (1.1) if the following holds:

(i) $\bar{u} \ge 0$ on $\partial\Omega$,
(ii) $\langle A\bar{u} - f, v - \bar{u} \rangle + \int_\Omega j^o(\bar{u}; v - \bar{u}) \, dx \ge 0, \quad \forall \; v \in \bar{u} \vee V_0$.

We assume the following hypothesis for j:

(H) The function $j : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and its Clarke's generalized gradient ∂j satisfies the following growth conditions:
  (i) there exists a constant $c_1 \ge 0$ such that

$$\xi_1 \le \xi_2 + c_1(s_2 - s_1)^{p-1}$$

for all $\xi_i \in \partial j(s_i),\ i = 1, 2$, and for all s_1 and s_2 with $s_1 < s_2$.

(ii) there is a constant $c_2 \geq 0$ such that

$$\xi \in \partial j(s): \quad |\xi| \leq c_2\,(1 + |s|^{p-1}), \quad \forall\, s \in \mathbb{R}.$$

Let $L^p(\Omega)$ be equipped with the natural partial ordering of functions defined by $u \leq w$ if and only if $w - u$ belongs to the positive cone $L^p_+(\Omega)$ of all non-negative elements of $L^p(\Omega)$. This induces a corresponding partial ordering also in the subspace V of $L^p(\Omega)$, and if $u, w \in V$ with $u \leq w$, then

$$[u, w] = \{z \in V : u \leq z \leq w\}$$

denotes the order interval formed by u and w.

In the proofs of our main results we make use of the cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ related to an ordered pair of functions $\underline{u}$ and $\bar{u}$, and given by

$$b(x, s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if} \quad s > \bar{u}(x), \\ 0 & \text{if} \quad \underline{u}(x) \leq s \leq \bar{u}(x), \\ -(\underline{u}(x) - s)^{p-1} & \text{if} \quad s < \underline{u}(x). \end{cases} \tag{2.1}$$

One readily verifies that b is a Carathéodory function satisfying the growth condition

$$|b(x, s)| \leq k_2(x) + c_3\,|s|^{p-1} \tag{2.2}$$

for almost every $x \in \Omega$, for all $s \in \mathbb{R}$, with some function $k_2 \in L^q_+(\Omega)$. Moreover, one has the following estimate:

$$\int_\Omega b(x, u(x))\,u(x)\,dx \geq c_4\,\|u\|^p_{L^p(\Omega)} - c_5, \quad \forall u \in L^p(\Omega), \tag{2.3}$$

where c_4 and c_5 are some positive constants. In view of (2.2) the Nemytskij operator $B : L^p(\Omega) \to L^q(\Omega)$ defined by

$$Bu(x) = b(x, u(x))$$

is continuous and bounded, and thus due to the compact embedding $V \subset L^p(\Omega)$ it follows that $B : V_0 \to V_0^*$ is compact.

Remarks 2.1. The notion of sub-supersolution introduced here extends that for inclusions of hemivariational type introduced in [3, 4]. To see this let $\bar{u}$ be a supersolution of the inclusion (1.5); i.e., $\bar{u} \in V$ and there is a function $\eta \in L^q(\Omega)$ such that $\bar{u} \geq 0$ on $\partial\Omega$, $\eta(x) \in \partial j(\bar{u}(x))$, and the following inequality holds:

$$\langle A\bar{u} - f, \varphi\rangle + \int_\Omega \eta(x)\varphi(x)\,dx \geq 0, \quad \forall\, \varphi \in V_0 \cap L^p_+(\Omega). \tag{2.4}$$

Thus (2.4), in particular, holds for φ in the form $\varphi = (w - \bar{u})^+$, for any $w \in V_0$, which yields, by applying the definition of Clarke's generalized gradient, the following inequality:

$$\langle A\bar{u} - f, (w - \bar{u})^+ \rangle + \int_\Omega j^o(\bar{u}(x); (w - \bar{u})^+(x))\, dx \geq 0, \quad \forall\, w \in V_0, \qquad (2.5)$$

which is equivalent to Definition 2.2. In the case that j is regular in the sense of Clarke (see [5, Chapter 2.3]) one can prove that the reverse is true; i.e., in this case any supersolution of (1.1) according to Definition 2.2 is also a supersolution of the associated inclusion (1.5). Analogous results hold for subsolutions.

3. Existence and comparison result

One of the main tools we are going to use in later proofs is the following surjectivity result for multivalued pseudomonotone and coercive mappings; cf., e.g., [9, Theorem 2.6], [6, Theorem 1.3.70, p. 62], or [10, Chapter 32].

Proposition 3.1. *Let X be a real, reflexive Banach space with dual space X^*, and let the multivalued operator $\mathcal{A} : X \to 2^{X^*}$ be pseudomonotone and coercive. Then $\mathcal{A}$ is surjective; i.e., range $\mathcal{A} = X^*$.*

For convenience let us recall the notions of multivalued pseudomonotone and generalized pseudomonotone operators and their relation to each other; cf., e.g., [9, Chapter 2]

Definition 3.1. Let X be a real, reflexive Banach space. The operator $\mathcal{A} : X \to 2^{X^*}$ is called *pseudomonotone* if the following conditions hold:

(i) The set $\mathcal{A}(u)$ is nonempty, bounded, closed, and convex for all $u \in X$.

(ii) $\mathcal{A}$ is upper semicontinuous from each finite-dimensional subspace of X to the weak topology on X^*.

(iii) If $(u_n) \subset X$ with $u_n \rightharpoonup u$, and if $u_n^* \in \mathcal{A}(u_n)$ is such that

$$\limsup \langle u_n^*, u_n - u \rangle \leq 0,$$

then for each element $v \in X$ there exists $u^*(v) \in \mathcal{A}(u)$ with

$$\liminf \langle u_n^*, u_n - v \rangle \geq \langle u^*(v), u - v \rangle.$$

Definition 3.2. Let X be a real, reflexive Banach space. The operator $\mathcal{A} : X \to 2^{X^*}$ is called *generalized pseudomonotone* if the following holds:

Let $(u_n) \subset X$ and $(u_n^*) \subset X^*$ with $u_n^* \in \mathcal{A}(u_n)$. If $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* and if $\limsup \langle u_n^*, u_n - u \rangle \leq 0$, then the element u^* lies in $\mathcal{A}(u)$ and

$$\langle u_n^*, u_n \rangle \to \langle u^*, u \rangle.$$

Proposition 3.2. *Let X be a real, reflexive Banach space. If the operator $\mathcal{A}: X \to 2^{X^*}$ is pseudomonotone then $\mathcal{A}$ is generalized pseudomonotone.*

Proposition 3.3. *Let X be a real, reflexive Banach space, and let $\mathcal{A}: X \to 2^{X^*}$ be a bounded, generalized pseudomonotone operator. If for each $u \in X$ we have that $\mathcal{A}(u)$ is a nonempty, closed, and convex subset of X^*, then $\mathcal{A}$ is pseudomonotone.*

The main result of this section is the following existence and comparison theorem.

Theorem 3.1. *Assume hypotheses* (A1)–(A3) *and* (H), *and let $\bar{u}$ and $\underline{u}$ be super- and subsolutions of* (1.1), *respectively, satisfying $\underline{u} \le \bar{u}$. Then there exist solutions of* (1.1) *within the order interval $[\underline{u}, \bar{u}]$.*

Proof. Let us consider the auxiliary hemivariational inequality

$$u \in V_0: \ \langle Au - f + \lambda B(u), v - u\rangle + \int_\Omega j^o(u; v-u)\, dx \ge 0, \ \forall\, v \in V_0, \quad (3.1)$$

where $\lambda \ge 0$ is some constant which is at our disposal and which will be specified later. We note that any solution $u \in [\underline{u}, \bar{u}]$ of (1.1) is a solution of (3.1), and vice versa.

Let us introduce the function $J: L^p(\Omega) \to \mathbb{R}$ by

$$J(v) = \int_\Omega j(v(x))\, dx, \quad \forall\, v \in L^p(\Omega).$$

Using the growth condition (H) (ii) and Lebourg's mean-value theorem, we note that the function J is well-defined and Lipschitz continuous on bounded sets in $L^p(\Omega)$, thus locally Lipschitz. Moreover, the Aubin-Clarke theorem (see [5, p. 83]) ensures that, for each $u \in L^p(\Omega)$, we have

$$\xi \in \partial J(u) \Longrightarrow \xi \in L^q(\Omega) \text{ with } \xi(x) \in \partial j(u(x)) \text{ for a.e. } x \in \Omega.$$

Consider now the multivalued operator $F: V_0 \to 2^{V_0^*}$ defined by

$$F(v) = Av + \lambda B(v) + \partial(J|_{V_0})(v), \quad \forall\, v \in V_0,$$

where $J|_{V_0}$ denotes the restriction of J to V_0. One readily verifies that the operator $A + \lambda B: V_0 \to V_0^*$ is continuous, bounded, strictly monotone, and thus, in particular, pseudomonotone. Let us check that the multivalued operator $\partial(J|_{V_0})$ is generalized pseudomonotone in the sense of Definition 3.2; see also [6, Definition 1.3.63 (b), p. 58]. To this end, let $v_n \rightharpoonup v$ in V_0 and $w_n \rightharpoonup w$ in V_0^* with $w_n \in \partial(J|_{V_0})(v_n)$. The compactness of the embedding $W_0^{1,p}(\Omega) \subset L^p(\Omega)$ implies that we have $v_n \to v$ in $L^p(\Omega)$. Using the density of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$ we know that $w_n \in \partial J(v_n)$ (see [5, p. 47]), so (w_n) is bounded in $L^q(\Omega)$. Thus we obtain that $w_n \rightharpoonup w$ in $L^q(\Omega)$. This yields $w \in \partial J(v)$ and

$$\langle w_n, v_n\rangle_{V_0^*,V_0} = \langle w_n, v_n\rangle_{L^q(\Omega),L^p(\Omega)} \to \langle w, v\rangle_{L^q(\Omega),L^p(\Omega)} = \langle w, v\rangle_{V_0^*,V_0},$$

which proves the generalized pseudomonotonicity of the operator $\partial(J|_{V_0})$. The growth condition (H) (ii) ensures that the multivalued operator $\partial(J|_{V_0})$ is bounded. Then we get by Proposition 3.3 that $\partial(J|_{V_0})$ is pseudomonotone (see also [6, Proposition 1.3.66, p. 59]). Since $A + \lambda B$ and $\partial(J|_{V_0})$ are pseudomonotone, it follows that the multivalued operator F is pseudomonotone. Thus in view of Proposition 3.1 the operator F is surjective provided F is coercive, which can readily be seen as follows: For any $v \in V_0$ and any $w \in \partial(J|_{V_0})(v)$ we obtain by applying (A3), (H) (ii), and (2.3) the estimate

$$\begin{aligned}
&\frac{1}{\|v\|_{V_0}}\langle Av + \lambda B(v) + w, v\rangle_{V_0^*,V_0}\\
&= \frac{1}{\|v\|_{V_0}}\Big[\int_\Omega \sum_{i=1}^N a_i(x,\nabla v)\frac{\partial v}{\partial x_i}\,dx + \lambda\langle B(v), v\rangle_{V_0^*,V_0} + \int_\Omega wv\,dx\Big]\\
&\geq \frac{1}{\|v\|_{V_0}}\Big[\nu\int_\Omega |\nabla v|^p\,dx - \int_\Omega k_1\,dx + c_4\lambda\|v\|^p_{L^p(\Omega)}\\
&- c_5\lambda - c_2\int_\Omega (1+|v|^{p-1})|v|\,dx\Big] \geq \frac{1}{\|v\|_{V_0}}\Big[\nu\|v\|^p_{V_0} - C_0\Big],
\end{aligned}$$

for some constant $C_0 > 0$, by choosing the constant λ to satisfy $c_4\lambda > c_2$. Since $p > 1$, it follows that $F : V_0 \to 2^{V_0^*}$ is coercive. Applying Proposition 3.1 (see also, e.g., [6, Theorem 1.3.70, p. 62]), we obtain that there exists $u \in V_0$ such that $f \in F(u)$; i.e., there is a $\xi \in \partial J(u)$ such that $\xi \in L^q(\Omega)$ with $\xi(x) \in \partial j(u(x))$ for almost every $x \in \Omega$ and

$$Au - f + \lambda B(u) + \xi = 0 \quad \text{in} \quad V_0^*, \tag{3.2}$$

where

$$\langle \xi, \varphi\rangle = \int_\Omega \xi(x)\,\varphi(x)\,dx \quad \forall \ \ \varphi \in V_0, \tag{3.3}$$

and thus by definition of Clarke's generalized gradient ∂j from (3.3) we get

$$\langle \xi, \varphi\rangle = \int_\Omega \xi(x)\,\varphi(x)\,dx \leq \int_\Omega j^o(u(x);\varphi(x))\,dx \quad \forall \ \ \varphi \in V_0. \tag{3.4}$$

Due to (3.2) and (3.4) we conclude that $u \in V_0$ is a solution of the auxiliary hemivariational inequality (3.1). To complete the proof we need only to show that there are solutions of (3.1) which belong to the interval $[\underline{u}, \bar{u}]$. In fact we are going to prove that any solution u of (3.1) belongs to this interval.

Let us show that $u \le \bar{u}$. By Definition 2.2 we have

$$\langle A\bar{u} - f, v - \bar{u}\rangle + \int_\Omega j^o(\bar{u}; v - \bar{u})\, dx \ge 0, \quad \forall\, v \in \bar{u} \vee V_0,$$

which implies, due to the fact that $v = \bar{u} \vee \varphi = \bar{u} + (\varphi - \bar{u})^+$ with $\varphi \in V_0$ and $w^+ = w \vee 0$, the following inequality:

$$\langle A\bar{u} - f, (\varphi - \bar{u})^+\rangle + \int_\Omega j^o(\bar{u}; (\varphi - \bar{u})^+)\, dx \ge 0, \quad \forall\, \varphi \in V_0. \tag{3.5}$$

Taking the special functions $v = u - (u - \bar{u})^+$ in (3.1) and $\varphi = u$ in (3.5) and adding the resulting inequalities we obtain

$$\begin{aligned} &\langle Au - A\bar{u}, (u - \bar{u})^+\rangle + \lambda \langle B(u), (u - \bar{u})^+\rangle \\ &\le \int_\Omega \Big(j^o(\bar{u}; (u - \bar{u})^+) + j^o(u; -(u - \bar{u})^+) \Big)\, dx. \end{aligned} \tag{3.6}$$

Next we estimate the right-hand side of (3.6) by using the following facts from nonsmooth analysis; cf. [5, Chapter 2]:

The function $r \mapsto j^o(s; r)$ is finite and positively homogeneous, $\partial j(s)$ is a nonempty, convex, and compact subset of $\mathbb{R}$, and one has

$$j^o(s; r) = \max\{\xi\, r : \xi \in \partial j(s)\}. \tag{3.7}$$

For functions w and v we denote $\{w > v\} = \{x \in \Omega : w(x) > v(x)\}$. By using (H) and the properties on j^o and ∂j we get, for certain $\bar{\xi}(x) \in \partial j(\bar{u}(x))$ and $\xi(x) \in \partial j(u(x))$, the following estimate:

$$\begin{aligned} &\int_\Omega \Big(j^o(\bar{u}; (u - \bar{u})^+) + j^o(u; -(u - \bar{u})^+) \Big)\, dx \\ &= \int_{\{u > \bar{u}\}} \Big(j^o(\bar{u}; u - \bar{u}) + j^o(u; -(u - \bar{u})) \Big)\, dx \\ &= \int_{\{u > \bar{u}\}} \Big(\bar{\xi}(x)(u(x) - \bar{u}(x)) + \xi(x)(-(u(x) - \bar{u}(x))) \Big)\, dx \\ &= \int_{\{u > \bar{u}\}} (\bar{\xi}(x) - \xi(x))(u(x) - \bar{u}(x))\, dx \le \int_{\{u > \bar{u}\}} c_1\, (u(x) - \bar{u}(x))^p\, dx. \end{aligned} \tag{3.8}$$

Since $\langle Au - A\bar{u}, (u - \bar{u})^+\rangle \ge 0$ and

$$\langle B(u), (u - \bar{u})^+\rangle = \int_{\{u > \bar{u}\}} (u - \bar{u})^p\, dx,$$

we get from (3.6) and (3.8) the estimate

$$(\lambda - c_1) \int_{\{u > \bar{u}\}} (u - \bar{u})^p\, dx \le 0. \tag{3.9}$$

Selecting the free parameter $\lambda \geq 0$ in such a way that $\lambda - c_1 > 0$, (3.9) yields

$$\int_\Omega \big((u-\bar{u})^+\big)^p\,dx \leq 0,$$

which implies $(u-\bar{u})^+ = 0$ and thus $u \leq \bar{u}$. The proof for the inequality $\underline{u} \leq u$ can be carried out in a similar way, which completes the proof of the theorem. □

4. Extremal solutions

Let $\mathcal{S}$ denote the set of all solutions of (1.1) within the interval $[\underline{u}, \bar{u}]$ of an ordered pair of sub- and supersolutions. In this section we are going to show that $\mathcal{S}$ possesses the smallest and greatest elements with respect to the given partial ordering. The smallest and greatest elements of $\mathcal{S}$ are called the *extremal solutions* of (1.1) within $[\underline{u}, \bar{u}]$. First, we introduce the following notion.

Definition 4.1. Let $(\mathcal{P}, \leq)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be *upward directed* if for each pair $x, y \in \mathcal{C}$ there is a $z \in \mathcal{C}$ such that $x \leq z$ and $y \leq z$, and $\mathcal{C}$ is *downward directed* if for each pair $x, y \in \mathcal{C}$ there is a $w \in \mathcal{C}$ such that $w \leq x$ and $w \leq y$. If $\mathcal{C}$ is both upward and downward directed it is called *directed*.

Lemma 4.1. *The solution set $\mathcal{S}$ is a directed set.*

Proof. By Theorem 3.1 we have $\mathcal{S} \neq \emptyset$. Given $u_1, u_2 \in \mathcal{S}$ we shall show that there is a $u \in \mathcal{S}$ such that $u_k \leq u$, $k = 1, 2$, which means $\mathcal{S}$ is upward directed. To this end we consider the following auxiliary hemivariational inequality:

$$u \in V_0 : \ \langle Au - f + \lambda B(u), v - u\rangle + \int_\Omega j^o(u; v-u)\,dx \geq 0, \ \forall\, v \in V_0, \tag{4.1}$$

where $\lambda \geq 0$ is a free parameter to be chosen later, but unlike the proof of Theorem 3.1 the operator B is now given by the following cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$:

$$b(x,s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if} \quad s > \bar{u}(x), \\ 0 & \text{if} \quad u_0(x) \leq s \leq \bar{u}(x), \\ -(u_0(x) - s)^{p-1} & \text{if} \quad s < u_0(x), \end{cases} \tag{4.2}$$

where $u_0 = \max(u_1, u_2)$. By arguments similar to those in the proof of Theorem 3.1 we get the existence of solutions of (4.1). The set $\mathcal{S}$ is shown to be upward directed provided that any solution u of (4.1) satisfies $u_k \leq u \leq \bar{u}$, $k = 1, 2$, because then $Bu = 0$ and thus $u \in \mathcal{S}$ exceeding u_k.

By assumption $u_k \in \mathcal{S}$, which means $u_k \in [\underline{u}, \bar{u}]$ satisfies

$$u_k \in V_0: \ \langle Au_k - f, v - u_k\rangle + \int_\Omega j^o(u_k; v - u_k)\, dx \geq 0, \ \forall\, v \in V_0. \tag{4.3}$$

Taking the special functions $v = u + (u_k - u)^+$ in (4.1) and $v = u_k - (u_k - u)^+$ in (4.3) and adding the resulting inequalities we obtain

$$\begin{aligned} &\langle Au_k - Au, (u_k - u)^+\rangle - \lambda\langle B(u), (u_k - u)^+\rangle \\ &\leq \int_\Omega \Big(j^o(u; (u_k - u)^+) + j^o(u_k; -(u_k - u)^+)\Big)\, dx. \end{aligned} \tag{4.4}$$

Similarly to (3.8) we get for the right-hand side of (4.4) the estimate

$$\int_\Omega \Big(j^o(u; (u_k - u)^+) + j^o(u_k; -(u_k - u)^+)\Big)\, dx \leq \int_{\{u_k > u\}} c_1\, (u_k(x) - u(x))^p\, dx. \tag{4.5}$$

For the terms on the left-hand side we have

$$\langle Au_k - Au, (u_k - u)^+\rangle \geq 0, \tag{4.6}$$

and (4.2) yields

$$\begin{aligned} \langle B(u), (u_k - u)^+\rangle &= -\int_{\{u_k > u\}} (u_0(x) - u(x))^{p-1}(u_k(x) - u(x))\, dx \\ &\leq -\int_{\{u_k > u\}} (u_k(x) - u(x))^p\, dx. \end{aligned} \tag{4.7}$$

By means of (4.5), (4.6), and (4.7) we get from (4.4) the inequality

$$(\lambda - c_1) \int_{\{u_k > u\}} (u_k(x) - u(x))^p\, dx \leq 0. \tag{4.8}$$

Selecting λ such that $\lambda > c_1$ from (4.8) it follows $u_k \leq u$. The proof for $u \leq \bar{u}$ follows similar arguments, and thus $\mathcal{S}$ is upward directed. By obvious modifications of the auxiliary problem one can show analogously that $\mathcal{S}$ is also downward directed. □

Lemma 4.2. *The solution set $\mathcal{S}$ is compact in V_0.*

Proof. First we prove that $\mathcal{S}$ is bounded in V_0. Since any $u \in \mathcal{S}$ belongs to the interval $[\underline{u}, \bar{u}]$ it follows that $\mathcal{S}$ is bounded in $L^p(\Omega)$. Moreover, any $u \in \mathcal{S}$ solves (1.1); i.e., we have

$$u \in V_0: \quad \langle Au - f, v - u\rangle + \int_\Omega j^o(u; v - u)\, dx \geq 0, \quad \forall\, v \in V_0,$$

and thus by taking $v = 0$ we get

$$\langle Au, u\rangle \le \langle f, u\rangle + \int_\Omega j^o(u; -u)\, dx, \tag{4.9}$$

which yields by applying (A3), (H) (ii), and Young's inequality

$$\nu \|\nabla u\|^p_{L^p(\Omega)} \le \|k_1\|_{L^1(\Omega)} + c(\varepsilon)\, \|f\|^q_{V_0^*} + \varepsilon\, \|u\|^p_{V_0} + \tilde{\alpha}\, (\|u\|_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}), \tag{4.10}$$

for any $\varepsilon > 0$, and hence the boundedness of $\mathcal{S}$ in V_0 follows by choosing ε sufficiently small and by taking into account that $\mathcal{S}$ is bounded in $L^p(\Omega)$.

Let $(u_n) \subset \mathcal{S}$. Then there is a subsequence (u_k) of (u_n) with

$$u_k \rightharpoonup u \text{ in } V_0, \quad u_k \to u \text{ in } \quad L^p(\Omega), \quad \text{and} \quad u_k(x) \to u(x) \;\text{ a.e. in }\; \Omega. \tag{4.11}$$

Obviously $u \in [\underline{u}, \bar{u}]$. Since u_k solves (1.1) we get with $v = u$ in (1.1)

$$\langle Au_k - f, u - u_k\rangle + \int_\Omega j^o(u_k; u - u_k)\, dx \ge 0,$$

and thus

$$\langle Au_k, u_k - u\rangle \le \langle f, u_k - u\rangle + \int_\Omega j^o(u_k; u - u_k)\, dx. \tag{4.12}$$

Due to (4.11) and due to the fact that $(s, r) \mapsto j^o(s; r)$ is upper semicontinuous we get by applying Fatou's lemma

$$\limsup_k \int_\Omega j^o(u_k; u - u_k)\, dx \le \int_\Omega \limsup_k j^o(u_k; u - u_k)\, dx = 0. \tag{4.13}$$

In view of (4.13) we thus obtain from (4.11) and (4.12)

$$\limsup_k \langle Au_k, u_k - u\rangle \le 0. \tag{4.14}$$

Since the operator A enjoys the (S_+)-property, the weak convergence of (u_k) in V_0 along with (4.14) imply the strong convergence $u_k \to u$ in V_0; see, e.g., [2, Theorem D.2.1]. Moreover, the limit u belongs to $\mathcal{S}$ as can be seen by passing to the lim sup on the left-hand side of the following inequality,

$$\langle Au_k - f, v - u_k\rangle + \int_\Omega j^o(u_k; v - u_k)\, dx \ge 0, \tag{4.15}$$

where we have used Fatou's lemma and the strong convergence of (u_k) in V_0. This completes the proof of the lemma. □

By means of Lemma 4.1 and Lemma 4.2 we are able to prove the following extremality result.

Theorem 4.1. *The solution set $\mathcal{S}$ possesses extremal elements.*

Proof. We show the existence of the greatest element of $\mathcal{S}$. Since V_0 is separable we have that $\mathcal{S} \subset V_0$ is separable too, so there exists a countable, dense subset $Z = \{z_n : n \in \mathbb{N}\}$ of $\mathcal{S}$. By Lemma 4.1, $\mathcal{S}$ is upward directed, so we can construct an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$. Select $u_{n+1} \in \mathcal{S}$ such that $\max\{z_n, u_n\} \le u_{n+1} \le \overline{u}$. The existence of u_{n+1} is due to Lemma 4.1. By Lemma 4.2 we find a subsequence of (u_n), denoted again (u_n), and an element $u \in \mathcal{S}$ such that $u_n \to u$ in V_0, and $u_n(x) \to u(x)$ almost everywhere in Ω. This last property of (u_n) combined with its increasing monotonicity implies that the entire sequence is convergent in V_0 and, moreover, $u = \sup_n u_n$. By construction, we see that $\max\{z_1, z_2, \ldots, z_n\} \le u_{n+1} \le u$, $\forall n$; thus, $Z \subset [\underline{u}, u]$. Since the interval $[\underline{u}, u]$ is closed in V_0, we infer $\mathcal{S} \subset \overline{Z} \subset \overline{[\underline{u}, u]} = [\underline{u}, u]$, which in conjunction with $u \in \mathcal{S}$ ensures that u is the greatest solution of (1.1). The existence of the least solution of (1.1) can be proved in a similar way. □

5. Hemivariational inequalities with lower-order terms

As noted previously, the discussion above can be extended to hemivariational inequalities with general quasilinear elliptic operator of Leray-Lions type and the functional j depending on the space variable x. In this section, we outline its extension to the case where the principal operator Au is perturbed by a lower-order term Gu. The inequality (1.1) is extended to

$$u \in V_0 : \quad \langle Au + Gu - f, v - u\rangle + \int_\Omega j^o(u; v-u)\,dx \ge 0, \quad \forall\, v \in V_0, \tag{5.1}$$

where G is the Nemytskij operator associated with a Carathéodory function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$:

$$\langle Gu, v\rangle = \int_\Omega g(\cdot, u, \nabla u)v\,dx, \ \forall u, v \in V. \tag{5.2}$$

For the integral in (5.2) to be defined, we need some growth condition on g, which will be specified later. Note that the operator $A + G$ is not coercive in general. The definition of supersolutions of (1.1) now becomes

Definition 5.1. A function $\bar{u} \in V$ is called a supersolution of (5.1) if the following holds:

(i) $\bar{u} \ge 0$ on $\partial\Omega$,
(ii) $G\bar{u} \in L^q(\Omega)$,
(iii) $\langle A\bar{u} + G\bar{u} - f, v - \bar{u}\rangle + \int_\Omega j^o(\bar{u}; v - \bar{u})\,dx \ge 0, \quad \forall\, v \in \bar{u} \vee V_0$.

We have a similar definition for subsolutions related to Definition 2.1. That the solution set of (5.1) is a nonempty directed set follows from the following result, which is an analog of Theorem 3.1 and Lemma 4.1:

Theorem 5.1. *Assume the hypotheses* (A1)–(A3), (H), *and that* (5.1) *has subsolutions* $\underline{u}_1, \dots, \underline{u}_k$ *and supersolutions* $\bar{u}_1, \dots, \bar{u}_m$ *such that*

$$\underline{u} := \max\{\underline{u}_1, \dots, \underline{u}_k\} \le \bar{u} := \min\{\bar{u}_1, \dots, \bar{u}_m\}. \tag{5.3}$$

Suppose furthermore g *has the growth condition*

$$|g(x, u, \xi)| \le c_7(x) + c_8|\xi|^{p-1}$$

for almost every $x \in \Omega$, *all* $\xi \in \mathbb{R}^N$, *and all* u *in the interval*

$$[\min\{\underline{u}_1(x), \dots, \underline{u}_k(x)\}, \max\{\bar{u}_1(x), \dots, \bar{u}_m(x)\}],$$

where $c_7 \in L^q(\Omega)$ *and* $c_8 > 0$. *Then there exists a solution* u *of* (5.1) *such that*

$$\underline{u} \le u \le \bar{u}.$$

Proof. The proof follows the same lines as those of Theorem 3.1 and Lemma 4.1 with the following modifications for an appropriate truncated auxiliary hemivariational inequality whose solvability yields the assertion of the theorem by comparison. To this end we consider the following auxiliary hemivariational inequality:

$$u \in V_0 : \ \langle Au - f + \gamma B(u) + C(u), v - u\rangle + \int_\Omega j^o(u; v - u)\, dx \ge 0, \ \forall\, v \in V_0, \tag{5.4}$$

where b and B are defined as in the proof of Theorem 3.1 (with $\underline{u}$ and $\bar{u}$ given in (5.3)), and $C : V_0 \to V_0^*$ is defined by

$$C(u) = G(Tu) + \sum_{i=1}^{k}\sum_{j=1}^{m} |G(T_{ij}u) - G(Tu)| \ (u \in V_0),$$

where T and T_{ij} are truncation operators given by

$$Tu(x) = \begin{cases} \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) \le u(x) \le \bar{u}(x) \\ \bar{u}(x) & \text{if} \quad u(x) > \bar{u}(x), \end{cases}$$

$$T_{ij}u(x) = \begin{cases} \underline{u}_i(x) & \text{if} \quad u(x) < \underline{u}_i(x) \\ u(x) & \text{if} \quad \underline{u}_i(x) \le u(x) \le \bar{u}_j(x) \\ \bar{u}_j(x) & \text{if} \quad u(x) > \bar{u}_j(x), \end{cases}$$

for $1 \le i \le k$, $1 \le j \le m$, $x \in \Omega$, and

$$\langle |G(T_{ij}u) - G(Tu)|, v\rangle = \int_\Omega |g(\cdot, T_{ij}u, \nabla T_{ij}u) - g(\cdot, Tu, \nabla Tu)|\, v\, dx,$$

for all $u, v \in V_0$. One can easily check that $A + \gamma B + C : V_0 \to V_0^*$ is continuous, bounded, and pseudomonotone; cf., e.g., [2, Theorem D.2.1].

Introducing the same function J as in the proof of Theorem 3.1 we can show that the multivalued operator $A + \gamma B + C + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is pseudomonotone and coercive with γ chosen sufficiently large. Hence, by arguments similar to those in the proof of Theorem 3.1 we infer that (5.4) has a solution u. To prove that u satisfies $u \le \bar{u}$, we just check that $u \le \bar{u}_j$ for every $j \in \{1, \dots, m\}$. This is a direct consequence of (5.4) with $v = u - (u - \bar{u}_j)^+$, inequality (iii) in Definition 5.1 with $\bar{u}_j$ and $v = \bar{u}_j \vee u$, and the fact that

$$\Big\langle G(Tu) - G\bar{u}_j + \sum_{i=1}^{k} \sum_{l=1}^{m} |G(T_{il}u) - G(Tu)|, (u - \bar{u}_j)^+ \Big\rangle \ge 0.$$

Similarly, one can prove that $u \ge \underline{u}$. As a consequence, we have $Bu = 0$ and $Tu = T_{ij}u = u$. Thus, u is a solution of (5.1). □

Remarks 5.1. Using arguments similar to those in Lemma 4.2 and Theorem 4.1 we can also show that the set $\mathcal{S}$ of solutions of (5.1) within the interval $[\underline{u}, \bar{u}]$ is a compact set of V_0 and has smallest and greatest elements, which are the extremal solutions of (5.1) in that interval.

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