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SINGULAR PERTURBATIONS FOR PARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS LEADING TO ULTRAPARABOLIC EQUATIONS

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Abstract. Linear parabolic equations with *coefficients* of the lower-order terms *unbounded*, and with a *small parameter* multiplying some of the second (highest) space derivatives are considered, in the limiting case when such a parameter goes to zero. This yields a degenerate parabolic (*ultraparabolic*) equation with *one* space-like variable, x, and *two* time-like variables, y and t. No boundary-layer is found to be needed in the case of the boundary-value problem on the *x-unbounded* domain $\mathcal{Q}_T = \{(x, y, t) \in \mathbb{R} \times [0, 1] \times [0, T]\}$ with a *periodic* boundary condition in the variable y and initial data at $t = 0$.

Introduction

Mathematical models for a number of natural phenomena can be formulated in terms of partial differential equations of the form

$$\sum_{i=1}^{m} k_i(x,t) v_{t_i} = \sum_{i,j=1}^{n} a_{ij}(x,t) v_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t) v_{x_i} + c(x,t) v + f(x,t), \quad (0.1)$$

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where we refer to $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as to the *space-like* variables, and to $t = (t_1, \dots, t_m) \in \mathbb{R}^m$ as to the *time-like* variables. The right-hand side of equation (0.1) is assumed to be elliptic; that is, $\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \geq a_0 \sum_{i=1}^n \xi_i^2$, where $a_0 > 0$ is a constant, for every $\xi \in \mathbb{R}^n$ and for all values (x,t) in some domain.

As is well-known, when $m = 1$ equation (0.1) is called *parabolic*. When $m \geq 2$, by the general classification of linear second-order partial differential equations, equation (0.1) is called *ultraparabolic*. Thus, in the ultraparabolic case there are several "time-like" variables in the equation. Ultraparabolic equations are encountered for instance within the theory of Brownian motion (diffusion processes with inertia) [13], probability theory (Markov processes) [28], transport theory (Fokker-Planck-type equations) [26], boundary-layer theory [30], atomic physics (processes of dispersion of electrons) [22], biophysics (an integro-differential ultraparabolic equation) [1], chemistry (the Kramers problem) [6], and in certain problems of hydrodynamics [17] and mathematical finance [10]. In particular, the Kolmogorov equation for diffusion with inertia [18], the Fokker-Planck equation [26], the Sonin equation [28], and the linearized Boltzmann equation [22], are of the ultraparabolic type. Also, facing several "times" is not so strange in physical problems, because this may reflect the occurrence of multiscale phenomena.

Special classes of ultraparabolic equations have been investigated by many authors. However, a comprehensive theory for such equations has not been developed to the same extent as that of parabolic equations until now. Here we would like to point out the main difficulties arising in the construction of a general theory for ultraparabolic equations. To be more precise, we consider a special case.

Suppose that $n = 1$, $m = 2$, $k_1(x_1, t_1, t_2) \equiv k(x_1, t_1, t_2)$, and $k_2(x_1, t_1, t_2) \equiv 1$, in equation (0.1); that is, consider the equation

$$v_t + k(x,y,t)v_y = a(x,y,t)v_{xx} + b(x,y,t)v_x + c(x,y,t)v + f(x,y,t) \quad (0.2)$$

with $a(x,y,t) \geq a_0 > 0$, where we set $x_1 = x$, $t_1 = y$, and $t_2 = t$. Here and below, in this section, we assume all smoothness of the coefficients which may be needed, for simplicity. Of course, the quantity $k(x,y,t)v_y$ may also be interpreted, physically, as a (space) transport term. Under this interpretation, the equation can be viewed as a parabolic equation, with space variables x and y, but fully degenerate with respect to y (because diffusion in y is missing). Indeed, this is the case, e.g., of the Fokker-Planck equation considered in transport theory, where the "space variable" y actually represents velocity.

Equation (0.2), with constant coefficients, a, b, and c, as well as with coefficients $a = a(x)$, $b = b(x)$, and $c = c(x)$, and $f(x, y, t) \equiv 0$, has been studied in many papers (see [24, 31], e.g.). Even more often the special cases $k(x, y, t) \equiv \pm x$ have been considered (see [6, 13, 18, 21, 24, 26, 30], e.g.), because the corresponding model equations have a physical interpretation. In [31, 32], the case $k \equiv k(y, t)$ has been considered, while the general case, $k \equiv k(x, y, t)$, has been treated in [14, 15, 16, 23, 25]. For instance, in [25] the equation

$$v_t + k(x, y, t)v_y = \frac{\partial}{\partial x}\left[a(x, y, t)v_x\right] + b(x, y, t)v_x + c(x, y, t)v + f(x, y, t) \quad (0.3)$$

has been studied. As preliminary examples, we now consider three boundary-value problems in the domain $Q_T := \{(x, y, t) \in [0, 1]^2 \times [0, T]\}$, for such an equation. We denote by Γ a certain part of the boundary of Q_T, namely $\Gamma := \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 := \{(x, 0, t) \in Q_T \,:\, k(x, 0, t) > 0\}$ and $\Gamma_2 := \{(x, 1, t) \in Q_T \,:\, k(x, 1, t) < 0\}$. These problems are the following:

(D) The problem with (homogeneous) *Dirichlet* boundary conditions in the variable x,

$$v\,|_{x=0} = 0, \quad v\,|_{x=1} = 0,$$
$$v\,|_{\Gamma} = \psi(x, y, t), \quad v(x, y, 0) = \varphi(x, y).$$

(N) The problem with (homogeneous) *Neumann* boundary conditions in the variable x,

$$v_x\,|_{x=0} = 0, \quad v_x\,|_{x=1} = 0,$$
$$v\,|_{\Gamma} = \psi(x, y, t), \quad v(x, y, 0) = \varphi(x, y).$$

(M) The problem with (homogeneous) *mixed*-type boundary conditions in the variable x,

$$(v_x + \beta(y, t)v)\,|_{x=0} = 0, \quad (v_x + \gamma(y, t)v)\,|_{x=1} = 0,$$
$$v\,|_{\Gamma} = \psi(x, y, t), \quad v(x, y, 0) = \varphi(x, y).$$

Note that the definition of Γ depends *only on the sign* of the coefficient $k(x, y, t)$ on some points of the planes $y = 0$ and $y = 1$. Consider three basic special cases:

- $k(x, y, t) > 0$ in Q_T; then $\Gamma_2 = \emptyset$, $\Gamma \equiv \Gamma_1$, and the boundary condition is
$$v\,|_{y=0} = \psi(x, t);$$
- $k(x, y, t) < 0$ in Q_T; then $\Gamma_1 = \emptyset$, $\Gamma \equiv \Gamma_2$, and the boundary condition is
$$v\,|_{y=1} = \psi(x, t);$$
- $k(x, y, t)\,|_{y=0} \equiv 0$ and $k(x, y, t)\,|_{y=1} \equiv 0$; then $\Gamma = \emptyset$ (so that the boundary condition $v\,|_{\Gamma} = \psi(x, y, t)$ is *ignored*).

Applying the results obtained in [25] to equation (0.3) in the domain Q_T, it follows that (under some smoothness assumptions on the data) there exists a *unique* weak solution to each of the problems **(D)**, **(N)**, and **(M)** above. In other words, the choice of Γ, under which problems **(D)**, **(N)**, and **(M)** are well-posed, *depends only* on the sign of the coefficient $k(x, y, t)$ on some part of the boundary of Q_T; that is, $Q_T \cap \{y = 0\}$ and $Q_T \cap \{y = 1\}$.

Going back to equation (0.1), we stress that similar results have been established for weak, strong, and special solutions to equation (0.1) in general (or in divergence) form. For some results toward a general theory, we refer the reader to [14, 15, 16, 23, 25, 29] and to the recent investigations in [31, 32]. In [15, 25], existence and uniqueness of weak and strong solutions to basic boundary-value problems for equation (0.1) have been proved, similarly to the cases **(D)**, **(N)**, and **(M)**. A stability theory has been given in [31], and a Schauder-type theory for special solutions to (0.1) has been constructed in certain Hölder spaces in [32].

We emphasize below three basic difficulties which affect the theory of ultraparabolic problems (i.e., when $m \geq 2$ in equation (0.1)), in contrast to that of the standard parabolic problems ($m = 1$ and $k_1(x, t) \equiv 1$ in equation (0.1)):

- Depending on some specific properties of the coefficients of equation (0.1) (in particular, on the sign of the coefficient $k(x, y, t)$ in (0.2)), one needs to leave a certain part of the boundary of the domain free from any boundary condition.
- Standard smoothness conditions on data of well-posed parabolic problems do *not* guarantee the same solvability results for well-posed ultraparabolic problems as for the correspondingly similar parabolic ones. Examples of this type, when neither strong nor classical solutions to (0.1) exist, are shown, correspondingly, in [25, 32].
- The applications mentioned above deal with the case that the coefficient $k(x, y, t)$ in (0.2) is *unbounded* as $x \to \pm\infty$ (in fact, $k(x, y, t) = \pm x$ in the model examples).

In this paper, we introduce a new well-posed boundary-value problem for the ultraparabolic equation (0.2), namely,

(P) The problem on the *x-unbounded* domain $\mathcal{Q}_T := \{(x, y, t) \in \mathbb{R} \times [0, 1] \times [0, T]\}$ with a *y-periodic* boundary condition, and initial data (at $t = 0$),

$$v\,|_{y=0} = v\,|_{y=1}, \tag{0.4}$$

$$v(x, y, 0) = \varphi(x, y). \tag{0.5}$$

Such a problem is natural for a number of applications, for instance,

- Considering the time evolution of a distribution function, $\rho(r,\theta,t)$, $t \in [0,T]$, in the plane, in polar coordinates (r,θ), the unbounded radial variable, r, appears in the domain $\{(r,\theta,t) \in [0,\infty) \times [0,2\pi) \times [0,T]\}$, and periodicity in the angle variable, that is, the condition $\rho\,|_{\theta=0} = \rho\,|_{\theta=2\pi}$, is natural. A similar situation is encountered when one considers gas-dynamics problems around the full Earth's globe (approximated by a ball of radius $r_0 > 0$), in spherical coordinates, $(r,\theta,\phi) \in [r_0,\infty) \times [0,2\pi) \times [0,\pi]$.
- An initial–boundary-value problem of such a type (but for a nonlinear integro-differential ultraparabolic equation) has been formulated in [1] and analyzed in [20, 21, 4], to investigate a certain problem arising from biophysics.
- In many problems of hydrodynamics and geophysics, one looks for a wave solution, i.e., a solution periodic in one of the variables.

One of the main results of this paper is to establish that the aforementioned problem **(P)** for the ultraparabolic equation (0.2) is well-posed. It is most important to stress the following:

- The solvability result concerning problem (0.2), (0.4), (0.5) does *not depend on* the sign of the coefficient $k(x,y,t)$ in equation (0.2).
- The coefficients $k(x,y,t)$, $a(x,y,t)$, $b(x,y,t)$, and $c(x,y,t)$ of equation (0.2), as well as its right-hand side, $f(x,y,t)$, are allowed to depend, in general, on *all* variables.
- The coefficients $k(x,y,t)$, $b(x,y,t)$, and $c(x,y,t)$ in equation (0.2) may be *unbounded*.

Such results are obtained on the basis of certain special decay properties of classical solutions to linear parabolic equations with unbounded coefficients. Such decay properties are established in Section 1, and have an independent interest in the qualitative theory of parabolic equations. These results represent some of the high points of the paper (cf. Theorems 1.2 and 1.3). The basic idea is to apply the results of Section 1 to the singularly perturbed *parabolic* problem

$$u_t + k(x,y,t)u_y = \varepsilon u_{yy} + a(x,y,t)u_{xx} + b(x,y,t)u_x + c(x,y,t)u + f(x,y,t), \tag{0.6}$$

$$(u,u_y)\,|_{y=0} = (u,u_y)\,|_{y=1}, \tag{0.7}$$

$$u(x,y,0) = \varphi(x,y), \tag{0.8}$$

in the domain $\mathcal{Q}_T$, whose associated reduced form (obtained formally setting $\varepsilon = 0$ and ignoring the periodicity condition on u_y) is the *ultraparabolic* problem **(P)**, i.e., problem (0.2), (0.4), (0.5). Assumptions on the coefficients will be made later, in Theorem 2.3. In Section 2, we obtain ε-*uniform* estimates for the solutions to the parabolic problem (0.6)–(0.8) (in particular, in the anisotropic Sobolev space $W_2^{3,2,1}(\mathcal{Q}_{\tilde{T}})$), so that such a problem does *not* require any boundary layer (cf. Theorem 2.3). Using this fact, we then establish, in Section 3, existence of strong solutions to the ultraparabolic problem **(P)**.

In [5], we have considered three boundary-value problems for equation (0.6). A method was proposed for the search of suitable "compatibility conditions" on data of a given singular perturbation problem, under which one may expect that *no boundary layer* occurs. The possibility of such an outcome was rather surprising. One of the problems considered in [5] was problem (0.6)–(0.8) in the domain $\mathcal{Q}_T$, with *bounded* coefficients. In case of problem (0.6)–(0.8), such compatibility conditions were the y-periodicity of the data of the problem. In this paper, we are able to establish that, under the compatibility conditions referred to above, again (that is, as in [5]) no boundary layer occurs in the problem (0.6)–(0.8), in the more general case that the coefficients of the lower-order terms are *unbounded* as $x \to \pm\infty$.

1. DECAY PROPERTIES OF SOLUTIONS TO PARABOLIC CAUCHY PROBLEMS

The qualitative theory of partial differential equations of the parabolic type has been well studied for a long time, in the case of bounded coefficients [7, 8, 9, 11, 12, 19]. The case of unbounded coefficients has also been developed to some extent [7, 11], but it is still not fully understood at the present time. In this section, we establish some basic properties of solutions to the Cauchy problem for linear, second-order, parabolic equations of a general form, where the coefficients of lower-order derivatives are unbounded. We consider the case that the right-hand side of the equation and the initial data can be estimated (in absolute value) by a function depending (only) on the norm of a given vector x of some of the space variables, $\xi = (x, y)$. Our only assumption on such a nonnegative estimating function is that it is nonincreasing.

Though this topic is of an independent interest for the qualitative theory of parabolic equations, it has a close connection with the singular perturbation problems under investigation in the present paper, as well as with other applications. The connection with the singularly perturbed problem in (0.6)–(0.8) will be made clear in Section 2. Here we explain only the connection with other applications.

A number of mathematical models have appeared in the recent years [1, 17, 27] describing the time evolution of some distribution function, $u(x, y, t)$, where $x \in \mathbb{R}^n$ is a space-vector variable, while $y \in \mathbb{R}^m$ can be considered as a vector of parameters of the problem. Due to its probabilistic meaning, the sought solution, $u(x, y, t)$, is expected to be nonnegative, to vanish at infinity as $x \to \infty$, and to satisfy the identity

$$\int_{\mathbb{R}^n} u(x, y, t)\, dx = 1$$

for all $y \in \mathbb{R}^m$ and for all $t \geq 0$. This identity represents the normalization of the probability function, $u(x, y, t)$, which should hold for all y and all times. Solvability of such nonlinear (possibly integro-differential) problems can be established through parabolic regularization. For this reason, we investigate properties of solutions to the Cauchy problem for parabolic equations when the right-hand side, $f(x, y, t)$, and the initial data, $\varphi(x, y)$, satisfy the inequalities

$$|f(x, y, t)| \leq G(|x|), \qquad |\varphi(x, y)| \leq G(|x|),$$

where $G(\delta)$ is a nonnegative, nonincreasing function of δ for $\delta \geq 0$. We stress that the case of our main interest is when $\lim_{\delta \to +\infty} G(\delta) = 0$. However, we do *not* impose such an additional condition throughout this section.

As a result of the present investigation, we provide new estimates for the solutions $u(x, y, t)$ to the Cauchy problem with unbounded coefficients. These estimates depend on the function $G(\delta)$ and imply, in particular, that $\lim_{x \to \infty} u(x, y, t) = 0$ in a slab $H_{\tilde{T}} := \{(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^m \times (0, \tilde{T})\}$, with $\tilde{T}$ small enough, whenever $G(\delta) \leq C\mathrm{e}^{-M\delta^2}$, where $C > 0$ and $M > 0$ are arbitrary fixed constants (in general, the constant $\tilde{T}$ depends on M).

Similar (but stronger) results have already been established by the authors for parabolic equations with bounded coefficients; see [20] (in particular, in such a case $\lim_{x \to \infty} u(x, y, t) = 0$ whenever $\lim_{\delta \to +\infty} G(\delta) = 0$). This made it possible to prove existence of *strong* solutions [20, 21], and of *classical* solutions [4], to an initial–boundary-value problem for a certain nonlinear integro-differential Fokker-Planck-type equation.

1.1. **Basic notation.** Let $\alpha \in (0, 1]$ and $T > 0$ be arbitrary, fixed constants. Let $\xi = (\xi_1, \ldots, \xi_{n+m}) \in \mathbb{R}^{n+m}$ be a vector, and $|\xi| := (\xi_1^2 + \cdots + \xi_{n+m}^2)^{1/2}$ its Euclidean norm, $l = (l_1, \ldots, l_{n+m})$ a multi-index, whose modulus is given by $|l| := l_1 + \cdots + l_{n+m}$, $H_T := \{(\xi, t) : \xi \in \mathbb{R}^{n+m},\ t \in (0, T)\}$ be a certain *open* slab in $\mathbb{R}^{n+m+1}$, and $\mathcal{H}_K := \{(\xi, t) : |\xi| \leq K,\ t \in [0, T]\}$ be a *closed* cylinder in $\mathbb{R}^{n+m+1}$. We shall represent a point of $\mathbb{R}^{n+m}$ as $\xi = (x, y)$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$.

We denote by $C^{\alpha,0}_{\xi,t}(\overline{H}_T)$ the set of functions $u(\xi, t)$, continuous in $\overline{H}_T$, equipped with the norm

$$\|u\|_{C^{\alpha,0}_{\xi,t}(\overline{H}_T)} := \sup_{(\xi,t)\in\overline{H}_T} |u(\xi,t)| + \sup_{(\xi,t),(\eta,t)\in\overline{H}_T,\,\xi\neq\eta} \frac{|u(\xi,t)-u(\eta,t)|}{|\xi-\eta|^\alpha}.$$

Denote by $C^{2+\alpha,0}_{\xi,t}(\mathcal{H}_K)$ the set of functions $u(\xi,t)$, continuous in $\mathcal{H}_K$ along with their partial derivatives $D^l_\xi u(\xi,t)$ for $|l|\le 2$, with the norm

$$\|u\|_{C^{2+\alpha,0}_{\xi,t}(\mathcal{H}_K)} := \sum_{|l|\le 2} \sup_{(\xi,t)\in\mathcal{H}_K} \left|D^l_\xi u(\xi,t)\right| + \sum_{|l|=2} \sup_{(\xi,t),(\eta,t)\in\mathcal{H}_K,\,\xi\neq\eta} \frac{\left|D^l_\xi u(\xi,t)-D^l_\xi u(\eta,t)\right|}{|\xi-\eta|^\alpha}.$$

In a similar way, we define the spaces $C^{2+\alpha}(\mathbb{R}^{n+m})$, $C^{2+\alpha,1+\alpha/2}_{\xi,t}(\overline{H}_T)$, and $C^{2+\alpha,1+\alpha/2}_{\xi,t}(\mathcal{H}_K)$ (cf. [19]).

Finally, denote by L the differential operator such that

$$Lu := \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n+m} a_{ij}(\xi,t)\frac{\partial^2 u}{\partial\xi_i\partial\xi_j} - \sum_{i=1}^{n+m} b_i(\xi,t)\frac{\partial u}{\partial\xi_i} - c(\xi,t)u, \tag{1.1}$$

with all coefficients, $a_{ij}(\xi,t)$, $b_i(\xi,t)$, and $c(\xi,t)$, defined in the slab H_T. Throughout the paper we assume that

$$a_0 := \inf_{(\xi,t)\in H_T,\,v\in\mathbb{R}^{n+m}} \frac{\sum_{i,j=1}^{n+m} a_{ij}(\xi,t)v_iv_j}{|v|^2} > 0, \tag{1.2}$$

that is, that the operator L is *uniformly parabolic* in the slab H_T. We shall consider the Cauchy problem for the equation $Lu=f$ in H_T. The assumed smoothness of the coefficients of L and that of the right-hand side, $f(\xi,t)$, will be specified below.

1.2. **Restrictions on the coefficients' growth which guarantee classical solvability of the Cauchy problem.** In this subsection we recall some known results concerning classical solvability of the Cauchy problem for parabolic equations with *unbounded* coefficients. These will be essential below. For convenience, we collect the main restrictions on the coefficients of the uniformly parabolic operator L in the following

Assumption 1.1. All coefficients, $a_{ij}(\xi,t)$, $b_i(\xi,t)$, and $c(\xi,t)$ of the *uniformly parabolic* operator L in (1.1) belong to the spaces $C^{2+\alpha,0}_{\xi,t}(\mathcal{H}_K)$, *for* *every* $K > 0$, *and* are such that

$$\left|D^l_\xi a_{ij}(\xi,t)\right| \le C, \quad \left|D^l_\xi b_i(\xi,t)\right| \le C(1+|\xi|), \quad \left|D^l_\xi c(\xi,t)\right| \le C(1+|\xi|^2),$$

in $\overline{H}_T$, for $i,j = 1,2,\dots,n+m$ and $|l| \le 2$, where $C > 0$ is a constant.

If Assumption 1.1 is satisfied, then, for every given $a > 0$, there exist a constant $A > 0$ and a fundamental solution $Z(\xi,t,\eta,\tau)$ for the equation $Lu = 0$ in the slab H_{T_0}, with $T_0 := \min\{(2aA)^{-1}, T\}$, such that

- the constant T_0 depends on $n+m$ and on the constant C appearing in Assumption 1.1, but it is *independent* of the constant a_0 in (1.2);
- the inequality

$$|D^l_\xi Z(\xi,t,\eta,\tau)| \le \frac{C}{(t-\tau)^{(n+m)/2+|l|/2}} \mathrm{e}^{a|\xi|^2/(1-aAt)-a|\eta|^2/(1-aA\tau)-M|\xi-\eta|^2/(t-\tau)} \tag{1.3}$$

  holds for $\xi,\eta \in \mathbb{R}^{n+m}$, $0 \le \tau < t \le T_0$, and $|l| \le 2$, where $C > 0$ and $M > 0$ are certain constants (cf. [11]).

Moreover, if $\varphi(\xi)$ is a continuous function, bounded in $\mathbb{R}^{n+m}$, and $f(\xi,t)$ is a Hölder-continuous function, bounded in $\overline{H}_{T_0}$, then the function

$$u(\xi,t) = I_1 + I_2 := \int_{\mathbb{R}^{n+m}} Z(\xi,t,\eta,0)\varphi(\eta)d\eta + \int_0^t \int_{\mathbb{R}^{n+m}} Z(\xi,t,\eta,\tau)f(\eta,\tau)d\eta\, d\tau \tag{1.4}$$

is a classical solution to the Cauchy problem

$$Lu = f(\xi,t) \quad \text{in} \quad H_{T_0}, \qquad u(\xi,0) = \varphi(\xi) \quad \text{for} \quad \xi \in \mathbb{R}^{n+m}. \tag{1.5}$$

Note that the function I_1 is a classical solution of (1.5) with $f(\xi,t) \equiv 0$, while I_2 is a classical solution of (1.5) with $\varphi(\xi) \equiv 0$.

In connection with the aforementioned applications, we need to establish some basic properties of the convolutions I_1 and I_2 in (1.4) involving the fundamental solution, $Z(\xi,t,\eta,\tau)$. This amounts to considering the case of a uniformly parabolic operator like L with lower-order terms *unbounded*. At the same time, we are especially interested in the case that the right-hand side of the parabolic equation in (1.5), $f(\xi,t)$, as well as the initial data, $\varphi(\xi,t)$, vanish when some of the space variables $(\xi_1,\dots,\xi_n) = x$ go to infinity.

1.3. **Estimating the function I_1.** For the classical solution I_1 to the homogeneous Cauchy problem with continuous, bounded initial data $\varphi(\xi)$, we establish the following additional properties:

Theorem 1.2. *Suppose that the coefficients of the* uniformly parabolic *operator L satisfy Assumption* 1.1, *and the function $\varphi(\xi)$ is continuous in $\mathbb{R}^{n+m}$ and such that*

$$\sup_{y\in\mathbb{R}^m} |\varphi(x,y)| \le G(|x|), \tag{1.6}$$

where $G(\delta)$ is a nonnegative, monotone, nonincreasing function for $\delta \ge 0$. Then, for the convolution

$$I_1(\xi,t) := \int\limits_{\mathbb{R}^{n+m}} Z(\xi,t,\eta,0)\varphi(\eta)\,d\eta, \tag{1.7}$$

where $Z(\xi,t,\eta,\tau)$ is the fundamental solution above, the estimate

$$|D^l_\xi I_1(\xi,t)| \le \frac{C}{t^{|l|/2}}\Big(\frac{2\pi}{M}\Big)^{(n+m)/2} \mathrm{e}^{2(\frac{2}{M}+A)a^2t|\xi|^2}\big(G(q|x|)+G(0)\,\mathrm{e}^{-M(1-q)^2|x|^2/2t}\big)$$

holds in H_{T_0}, for all $q \in (0,1)$ and $|l| \le 2$, where $C, M, a, A > 0$ are the constants appearing in (1.3)*.*

Proof. The derivatives of the function in (1.7) are given, in H_{T_0}, by

$$D^l_\xi I_1(\xi,t) = \int_{\mathbb{R}^{n+m}} D^l_\xi Z(\xi,t,\eta,0)\varphi(\eta)\,d\eta$$

for $|l| \le 2$. Therefore, we obtain by (1.3)

$$|D^l_\xi I_1(x_0,y_0,t)| \le \frac{C}{t^{(n+m)/2+|l|/2}}\,\mathrm{e}^{a|\xi_0|^2/(1-Aat)} \int_{\mathbb{R}^n} \mathrm{e}^{-a|x|^2-M|x_0-x|^2/t}\,G(|x|)\,dx$$

$$\times \int_{\mathbb{R}^m} \mathrm{e}^{-a|y|^2-M|y_0-y|^2/t}\,dy := \frac{C}{t^{(n+m)/2+|l|/2}}\,\mathrm{e}^{a|\xi_0|^2/(1-Aat)} \cdot I_{11}\cdot I_{12}, \tag{1.8}$$

in H_{T_0}, for $|l| \le 2$. Here we have defined the integrals I_{11} and I_{12} in an obvious way, for convenience. The important auxiliary relation

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-ax^2-M(\omega-x)^2/t}\,dx = \frac{\sqrt{\pi t}}{\sqrt{M+at}}\,\mathrm{e}^{-\frac{a}{1+at/M}\,\omega^2}, \tag{1.9}$$

valid for every $\omega \in \mathbb{R}$ and for all positive constants M, a, and t, can be established by direct integration, using the well-known result $\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,dx = \sqrt{\pi}$. Using formula (1.9),

$$I_{12} = \prod_{i=1}^m \int_{-\infty}^{+\infty} \mathrm{e}^{-ay_i^2-M(y_{0i}-y_i)^2/t}\,dy_i = \Big(\frac{\pi t}{M+at}\Big)^{m/2}\mathrm{e}^{-\frac{a}{1+at/M}\,|y_0|^2}. \tag{1.10}$$

Furthermore, using the monotonicity of $G(\delta)$, we can derive the estimate

$$|I_{11}| \le G(q|x_0|)\int_{|x|\ge q|x_0|} \mathrm{e}^{-a|x|^2-M|x_0-x|^2/t}\,dx + G(0)\int_{|x|\le q|x_0|} \mathrm{e}^{-a|x|^2-M|x_0-x|^2/t}\,dx$$

for every $q \in (0,1)$. If $|x| \le q|x_0|$, then $|x_0 - x| \ge |x_0| - |x| \ge (1-q)|x_0|$, and thus

$$\begin{aligned} |I_{11}| &\le G(q|x_0|) \int_{|x|\ge q|x_0|} \mathrm{e}^{-a|x|^2 - M|x_0-x|^2/t}\,dx \\ &+ G(0)\,\mathrm{e}^{-M(1-q)^2|x_0|^2/2t} \int_{|x|\le q|x_0|} \mathrm{e}^{-a|x|^2 - M|x_0-x|^2/2t}\,dx \\ &\le \Big(G(q|x_0|) + G(0)\,\mathrm{e}^{-M(1-q)^2|x_0|^2/2t} \Big) \int_{\mathbb{R}^n} \mathrm{e}^{-a|x|^2 - M|x_0-x|^2/2t}\,dx. \end{aligned}$$

Using again (1.9), we obtain finally

$$\begin{aligned} |I_{11}| &\le \Big(G(q|x_0|) + G(0)\,\mathrm{e}^{-M(1-q)^2|x_0|^2/2t} \Big) \prod_{i=1}^{n} \int_{-\infty}^{+\infty} \mathrm{e}^{-ax_i^2 - M(x_{0i}-x_i)^2/2t}\,dx_i \\ &= \Big(G(q|x_0|) + G(0)\,\mathrm{e}^{-M(1-q)^2|x_0|^2/2t} \Big) \left(\frac{\pi t}{\frac{M}{2} + at} \right)^{n/2} \mathrm{e}^{-\frac{a}{1+2at/M}|x_0|^2}. \quad (1.11) \end{aligned}$$

Summing up the estimates in (1.8), (1.10), and (1.11), we conclude that Theorem 1.2 is proved. □

1.4. **Estimating the function I_2.** For the integral I_2 in (1.4), which stands (under some assumptions) for a classical solution to the inhomogeneous Cauchy problem with zero initial data, a similar result holds:

Theorem 1.3. *Suppose that the coefficients of the* uniformly parabolic *operator L satisfy Assumption* 1.1, *and the function $f(\xi,t)$ is continuous in H_T and such that*

$$\sup_{y\in\mathbb{R}^m,\, t\in(0,T)} |f(x,y,t)| \le G(|x|),$$

where $G(\delta)$ is a nonnegative, monotone, nonincreasing function for $\delta \ge 0$. Then, for the convolution

$$I_2(\xi,t) := \int_0^t \int_{\mathbb{R}^{n+m}} Z(\xi,t,\eta,\tau) f(\eta,\tau)\,d\eta\,d\tau,$$

where $Z(\xi,t,\eta,\tau)$ is the fundamental solution above, the estimate

$$\begin{aligned} \left| D_\xi^l I_2(\xi,t) \right| &\le 2C \left(\frac{2\pi}{M} \right)^{(n+m)/2} t^{1-|l|/2}\,\mathrm{e}^{2(\frac{2}{M}+A)a^2 t|\xi|^2} \\ &\times \Big(G(q|x|) + G(0)\,\mathrm{e}^{-M(1-q)^2|x|^2/2t} \Big) \end{aligned}$$

holds in H_{T_0}, for every $q \in (0,1)$ and $|l| \le 1$, where $C, M, a, A > 0$ are the constants appearing in (1.3).

Proof. Proceeding as in the proof of Theorem 1.2, we obtain the estimate

$$\left|D_\xi^l I_2(x,y,t)\right| \le \int_0^t \frac{C}{(t-\tau)^{|l|/2}} \left(\frac{2\pi}{M}\right)^{(n+m)/2} \mathrm{e}^{2(\frac{2}{M}+A)a^2(t-\tau)|\xi|^2}$$

$$\times \left(G(q|x|) + G(0)\,\mathrm{e}^{-M(1-q)^2|x|^2/2(t-\tau)}\right) d\tau \le C \left(\frac{2\pi}{M}\right)^{(n+m)/2} \mathrm{e}^{2(\frac{2}{M}+A)a^2 t|\xi|^2}$$

$$\times \left(G(q|x|) + G(0)\,\mathrm{e}^{-M(1-q)^2|x|^2/2t}\right) \int_0^t \frac{d\tau}{(t-\tau)^{|l|/2}}$$

in H_{T_0}, for every $q \in (0,1)$ and $|l| \le 1$. The required inequality then follows, and hence the theorem is proved. □

1.5. **The case of bounded coefficients.** When the coefficients of the operator L in (1.1) are bounded, then stronger results hold. Moreover, in this case it is possible to relax the smoothness assumptions on the coefficients of L, and establish global decay properties of solutions to the Cauchy problem in the original slab, H_T, rather than in H_{T_0} (see [20]). Therefore, in this subsection we replace Assumption 1.1 with the following:

Assumption 1.4. All coefficients, $a_{ij}(\xi,t)$, $b_i(\xi,t)$, and $c(\xi,t)$ of the *uniformly parabolic* operator L in (1.1) belong to the space $C^{\alpha,0}_{\xi,t}(\overline{H}_T)$.

We stress that the decay estimates derived in this subsection are new. If Assumption 1.4 is fulfilled, then there exists a fundamental solution, $\tilde{Z}(\xi,t,\eta,\tau)$, for the equation $Lu = 0$ in the slab H_T, such that

$$\left|D_\xi^l \tilde{Z}(\xi,t,\eta,\tau)\right| \le \frac{C}{(t-\tau)^{(n+m)/2+|l|/2}}\,\mathrm{e}^{-M|\xi-\eta|^2/(t-\tau)} \tag{1.12}$$

for $\xi,\eta \in \mathbb{R}^{n+m}$, $0 \le \tau < t \le T$, and $|l| \le 2$, where $C, M > 0$ are certain constants (cf. [11]). Moreover, if $\varphi(\xi)$ is a continuous function, bounded in $\mathbb{R}^{n+m}$, and $f(\xi,t)$ is a Hölder-continuous function, bounded in $\overline{H}_T$, then the function

$$u(\xi,t) = \tilde{I}_1 + \tilde{I}_2 := \int\limits_{\mathbb{R}^{n+m}} \tilde{Z}(\xi,t,\eta,0)\varphi(\eta)\,d\eta + \int\limits_0^t \int\limits_{\mathbb{R}^{n+m}} \tilde{Z}(\xi,t,\eta,\tau) f(\eta,\tau)\,d\eta\,d\tau$$

is a classical solution to the Cauchy problem

$$Lu = f(\xi,t) \quad \text{in} \quad H_T, \qquad u(\xi,0) = \varphi(\xi) \quad \text{for} \quad \xi \in \mathbb{R}^{n+m}. \tag{1.13}$$

Here the function $\tilde{I}_1$ is a classical solution of (1.13) with $f(\xi,t) \equiv 0$, while $\tilde{I}_2$ is a classical solution of (1.13) with $\varphi(\xi) \equiv 0$. The following two theorems hold:

Theorem 1.5. *Suppose that the coefficients of the* uniformly parabolic *operator L satisfy Assumption* 1.4, *and the function $\varphi(\xi)$ is continuous in $\mathbb{R}^{n+m}$ and such that*

$$\sup_{y\in\mathbb{R}^m} |\varphi(x,y)| \le G(|x|),$$

where $G(\delta)$ is a nonnegative, monotone, nonincreasing function for $\delta \ge 0$. Then, for the convolution

$$\tilde{I}_1(\xi,t) := \int_{\mathbb{R}^{n+m}} \tilde{Z}(\xi,t,\eta,0)\varphi(\eta)\,d\eta,$$

where $\tilde{Z}(\xi,t,\eta,\tau)$ is the fundamental solution above, the estimate

$$\left|D^l_\xi \tilde{I}_1(\xi,t)\right| \le \frac{C}{t^{|l|/2}}\left(\frac{2\pi}{M}\right)^{(n+m)/2}\left(G(q|x|) + G(0)\,\mathrm{e}^{-M(1-q)^2|x|^2/2t}\right)$$

holds in H_T, for all $q \in (0,1)$ and $|l| \le 2$, where $C, M > 0$ are the constants appearing in (1.12).

Theorem 1.6. *Suppose that the coefficients of the* uniformly parabolic *operator L satisfy Assumption* 1.4, *and the function $f(\xi,t)$ is continuous in H_T and such that*

$$\sup_{y\in\mathbb{R}^m,\, t\in(0,T)} |f(x,y,t)| \le G(|x|),$$

where $G(\delta)$ is a nonnegative, monotone, nonincreasing function for $\delta \ge 0$. Then, for the convolution

$$\tilde{I}_2(\xi,t) := \int_0^t \int_{\mathbb{R}^{n+m}} \tilde{Z}(\xi,t,\eta,\tau) f(\eta,\tau)\,d\eta\,d\tau,$$

where $\tilde{Z}(\xi,t,\eta,\tau)$ is the fundamental solution above, the estimate

$$\left|D^l_\xi \tilde{I}_2(\xi,t)\right| \le 2C\left(\frac{2\pi}{M}\right)^{(n+m)/2} t^{1-|l|/2}\left(G(q|x|) + G(0)\,\mathrm{e}^{-M(1-q)^2|x|^2/2t}\right)$$

holds in H_T, for every $q \in (0,1)$ and $|l| \le 1$, where $C, M > 0$ are the constants appearing in (1.12).

The proofs of Theorems 1.5 and 1.6 are similar to those of Theorems 1.2 and 1.3, and therefore are omitted here (cf. [20]).

2. A singularly perturbed problem

Consider now the parabolic problem (0.6)–(0.8) on the unbounded domain $\mathcal{Q}_T = \{(x,y,t) \in \mathbb{R} \times [0,1] \times [0,T]\}$.

2.1. **A preliminary lemma.** We first consider the Cauchy problem (1.5), and establish a special lemma which will be used in the next subsection.

Lemma 2.1. *Suppose that*

(a) *the coefficients of the* uniformly parabolic *operator L satisfy Assumption* 1.1 *and, in addition, $a_{ij}(\xi,t) \in C^{2+\alpha,1+\alpha/2}_{\xi,t}(\overline{H}_T)$, while $b_i(\xi,t)$ and $c(\xi,t)$ belong to the spaces $C^{2+\alpha,1+\alpha/2}_{\xi,t}(\mathcal{H}_K)$, for every $K > 0$;*

(b) *the right-hand side, $f(\xi,t)$, belongs to $C^{2+\alpha,1+\alpha/2}_{\xi,t}(\overline{H}_T)$, and is such that*
$$|D^l_\xi f(\xi,t)| \le C\mathrm{e}^{-M|x|^2}$$
in $\overline{H}_T$ for $|l| \le 2$, where $C, M > 0$ are constants;

(c) *the initial data, $\varphi(\xi)$, belongs to $C^{2+\alpha}(\mathbb{R}^{n+m})$, and is such that*
$$|D^l_\xi \varphi(\xi)| \le C\mathrm{e}^{-M|x|^2}$$
in $\mathbb{R}^{n+m}$ for $|l| \le 2$, where $C, M > 0$ are constants;

(d) *all coefficients of the operator L, as well as the functions $f(\xi,t)$ and $\varphi(\xi)$, are* periodic *in $y \in \mathbb{R}^m$ with a certain given m-dimensional period, y_0.*

Then, there exist some positive constants $\tilde{C}$, $\tilde{M}$, and $\tilde{T}$ (with $\tilde{T} \le T$), and a classical solution, $u(\xi,t)$, to problem (1.5) *in the slab $\overline{H}_{\tilde{T}}$, such that*

(1) *the constant $\tilde{T}$ depends on $n+m$, on the constant C in Assumption* 1.1*, and on other parameters, but is* independent *of the constant a_0 in* (1.2);

(2) *the partial derivatives $D^{k,l}_{t,\xi} u(\xi,t)$ are continuous in $\overline{H}_{\tilde{T}}$ for $2k+|l| \le 2$, and continuous in $H_{\tilde{T}}$ for $2k+|l| \le 4$;*

(3) *the estimates*
$$|D^{k,l}_{t,\xi} u(\xi,t)| \le \tilde{C}\mathrm{e}^{-\tilde{M}|x|^2}, \quad \text{for } 2k+|l| \le 2,$$
$$|D^{k,l}_{t,\xi} u(\xi,t)| \le \frac{\tilde{C}}{\sqrt{t}}\,\mathrm{e}^{-\tilde{M}|x|^2}, \quad \text{for } 2k+|l| = 3,$$
and
$$|D^{k,l}_{t,\xi} u(\xi,t)| \le \frac{\tilde{C}}{t}, \quad \text{for } 2k+|l| = 4,$$
hold in $H_{\tilde{T}}$;

(4) *the solution $u(\xi,t)$ is periodic in y with period y_0.*

The proof of this lemma is based on the general theory of linear parabolic equations [2, 3, 7, 8, 9, 19], and Theorems 1.2 and 1.3. We omit the rather lengthy proof of this lemma because it is quite standard, cf. [7, 19], e.g.

2.2. **Uniform estimates.** From now on, we consider the case $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Therefore, throughout this subsection, $\xi = (x, y) \in \mathbb{R}^2$, $H_T = \{(\xi, t) : \xi \in \mathbb{R}^2,\ t \in (0, T)\}$, and $\mathcal{H}_K = \{(\xi, t) : \sqrt{x^2 + y^2} \le K,\ t \in [0, T]\}$. Concerning equation (0.6), where ε is a positive constant, Assumption 1.1 takes on the following form:

Assumption 2.2. The coefficients $k(\xi, t)$, $a(\xi, t)$, $b(\xi, t)$, and $c(\xi, t)$ of equation (0.6) belong to the spaces $C^{2+\alpha,0}_{\xi,t}(\mathcal{H}_K)$, for every $K > 0$, and are such that

$$\tilde{a}_0 := \inf_{H_T} a(\xi, t) > 0, \quad |D^l_\xi a(\xi, t)| \le C,$$
$$|D^l_\xi k(\xi, t)| + |D^l_\xi b(\xi, t)| \le C(1 + |\xi|), \quad |D^l_\xi c(\xi, t)| \le C(1 + |\xi|^2),$$

in $\overline{H}_T$ for $|l| \le 2$, where $C > 0$ is a constant.

We now establish the following:

Theorem 2.3. *Suppose that*

(a) *the coefficients of equation* (0.6) *satisfy Assumption* 2.2 *and, in addition,* $a(\xi, t) \in C^{2+\alpha,1+\alpha/2}_{\xi,t}(\overline{H}_T)$, *while* $k(\xi, t)$, $b(\xi, t)$, *and* $c(\xi, t)$ *belong to the spaces* $C^{2+\alpha,1+\alpha/2}_{\xi,t}(\mathcal{H}_K)$, *for every* $K > 0$, *and satisfy the inequalities*

$$\sup_{\mathcal{Q}_T} |k_y(x, y, t)| < \infty, \quad \sup_{\mathcal{Q}_T} |b_x(x, y, t)| < \infty, \quad \sup_{\mathcal{Q}_T} c(x, y, t) < \infty;$$

(b) *the right-hand side,* $f(\xi, t)$, *belongs to* $C^{2+\alpha,1+\alpha/2}_{\xi,t}(\overline{H}_T)$, *and is such that*

$$|D^l_\xi f(\xi, t)| \le C\mathrm{e}^{-Mx^2}$$

in $\overline{H}_T$ *for* $|l| \le 2$, *where* $C, M > 0$ *are constants;*

(c) *the initial data,* $\varphi(\xi)$, *belongs to* $C^{2+\alpha}(\mathbb{R}^2)$, *and is such that*

$$|D^l_\xi \varphi(\xi)| \le C\mathrm{e}^{-Mx^2}$$

in $\mathbb{R}^2$ *for* $|l| \le 2$, *where* $C, M > 0$ *are constants;*

(d) *the coefficients* k, a, b, *and* c, *as well as the functions* $f(\xi, t)$ *and* $\varphi(\xi)$, *are* periodic *in* $y \in \mathbb{R}$ *with unit period.*

Then, for every $\varepsilon \in (0, 1)$, *there exist a constant* $\tilde{T} \in (0, T]$, independent of ε, *and a classical solution,* $u(x, y, t) \equiv u^\varepsilon(x, y, t)$, *to problem* (0.6)–(0.8) *on the unbounded domain* $\mathcal{Q}_{\tilde{T}} = \{(x, y, t) \in \mathbb{R} \times [0, 1] \times [0, \tilde{T}]\}$, *such that the estimate*

$$\int_0^1 \int_{-\infty}^{+\infty} x^{2m} \left(u^2 + u_x^2 + u_y^2 + u_{xx}^2 + u_{xy}^2 + u_{yy}^2\right) dx\, dy$$

$$+\int_0^{\tilde{T}}\int_0^1\int_{-\infty}^{+\infty} x^{2m}\left(u_{xxx}^2+u_{yyx}^2+u_t^2\right)dx\,dy\,dt \le C_m \tag{2.1}$$

holds for $t \in [0,\tilde{T}]$ *and for every* $m = 0,1,2,\dots,$ *where the constants* C_m *are* independent of ε.

In the inequality (2.1) we dropped the index ε, for short, as we shall do below. The following two remarks are very important.

Remark 2.4. Under the assumptions of Theorem 2.3, there exists a constant C^* such that the inequalities

$$\begin{aligned}
&\sup|k_y(x,y,t)| + \sup|b_x(x,y,t)| + \sup c(x,y,t) \le C^*,\\
&\sup(|a(x,y,t)| + |a_x(x,y,t)| + |a_y(x,y,t)| + |a_{xx}(x,y,t)| + |a_{xy}(x,y,t)|\\
&\qquad + |a_{yy}(x,y,t)|) \le C^*,\\
&|k(x,y,t)| + |k_x(x,y,t)| + |k_{xx}(x,y,t)| + |b(x,y,t)| + |b_y(x,y,t)|\\
&\qquad + |b_{yy}(x,y,t)| \le C^*(1+|x|),\\
&|c(x,y,t)| + |c_x(x,y,t)| + |c_y(x,y,t)| + |c_{yy}(x,y,t)| \le C^*(1+x^2)
\end{aligned}$$

hold in $\mathcal{Q}_{\tilde{T}}$. Then, the sequence of constants $\{C_m\}_{m=0}^{\infty}$ in (2.1) can be shown to depend *only* on the initial data, φ, the right-hand side, f, the constant $\tilde{T}$, the constant $\tilde{a}_0$ in Assumption 2.2, and the constant C^*.

Remark 2.5. Under the assumptions of Theorem 2.3, there exists a constant C^{**} such that the inequalities

$$\begin{aligned}
&\sup|k_y(x,y,t)| + \sup|b_x(x,y,t)| + \sup c(x,y,t) \le C^{**},\\
&\sup(|a_x(x,y,t)| + |a_y(x,y,t)| + |a_{xx}(x,y,t)| + |a_{xy}(x,y,t)|\\
&\qquad + |a_{yy}(x,y,t)|) \le C^{**},\\
&|k_x(x,y,t)| + |k_{xx}(x,y,t)| + |b_y(x,y,t)| + |b_{yy}(x,y,t)| \le C^{**}(1+|x|),\\
&|c_x(x,y,t)| + |c_y(x,y,t)| + |c_{yy}(x,y,t)| \le C^{**}(1+x^2)
\end{aligned}$$

hold in $\mathcal{Q}_{\tilde{T}}$. Then, the constant C_0 in (2.1) can be shown to depend *only* on the initial data, φ, the right-hand side, f, the constant $\tilde{T}$, the constant $\tilde{a}_0$ in Assumption 2.2, and the constant C^{**}.

Proof of Theorem 2.3. We first consider the Cauchy problem for equation (0.6) in the slab H_T and use Lemma 2.1. Thus, there exists a constant $\tilde{T} \in (0,T]$ such that, for every $\varepsilon \in (0,1)$, there exists a *classical solution*, $u^\varepsilon(x,y,t)$, to problem (0.6)–(0.8) on the unbounded domain $\mathcal{Q}_{\tilde{T}}$. This solution possesses all the properties (2)–(4) listed in Lemma 2.1 (with the period $y_0 = 1$ in (4)). Using these properties, we can prove inequality (2.1). We omit details in the proof of such an inequality, because it is similar to that of estimate (2.14) in [5]. There are however several nontrivial differences in connection with the unboundedness of the coefficients. The inequality (2.1) can be established in three separate steps:

Step 1. One can first prove the estimate

$$\int_0^1 \int_{-\infty}^{+\infty} x^{2m} u^2 \, dx \, dy + \int_0^{\tilde{T}} \int_0^1 \int_{-\infty}^{+\infty} x^{2m} \left(u_x^2 + \varepsilon u_y^2 \right) dx \, dy \, dt \le C_m \quad (2.2)$$

for $t \in [0, \tilde{T}]$ and for all $m = 0, 1, 2, \dots$, where the constants C_m are independent of $\varepsilon \in (0, 1)$.

Step 2. Using the already-established estimate (2.2), one can prove the inequality

$$\int_0^1 \int_{-\infty}^{+\infty} x^{2m} u_y^2 \, dx \, dy + \int_0^{\tilde{T}} \int_0^1 \int_{-\infty}^{+\infty} x^{2m} \left(u_{xy}^2 + \varepsilon u_{yy}^2 \right) dx \, dy \, dt \le C_m \quad (2.3)$$

for $t \in [0, \tilde{T}]$ and for all $m = 0, 1, 2, \dots$, where the constants C_m are independent of $\varepsilon \in (0, 1)$.

Step 3. Finally, using the estimates (2.2) and (2.3) (derived in Steps 1 and 2), one can obtain the relation

$$\begin{aligned} &\int_0^1 \int_{-\infty}^{+\infty} x^{2m} \left(u_x^2 + u_{xx}^2 + u_{xy}^2 + u_{yy}^2 \right) dx \, dy \qquad (2.4) \\ &+ \int_0^{\tilde{T}} \int_0^1 \int_{-\infty}^{+\infty} x^{2m} \left(u_{xxx}^2 + u_{yyx}^2 + u_t^2 \right) dx \, dy \, dt \le C_m \end{aligned}$$

for $t \in [0, \tilde{T}]$ and for all $m = 0, 1, 2, \dots$, where the constants C_m are independent of $\varepsilon \in (0, 1)$.

Each one of the estimates (2.2)–(2.4) can be proved by induction on $m = 0, 1, 2, \dots$ (cf. the proof of estimate (2.14) in [5]). All together, the estimates in (2.2)–(2.4) yield the relation (2.1). This completes the proof. □

3. The ultraparabolic problem (P)

Consider the problem (0.2), (0.4), (0.5) on the unbounded domain $\mathcal{Q}_{\tilde{T}} = \{(x, y, t) \in \mathbb{R} \times [0, 1] \times [0, \tilde{T}]\}$, with the constant $\tilde{T}$ appearing in Theorem 2.3. We are now able to establish one of the main results of the paper. This result verifies Hypothesis 2.1 of [5] concerning the singularly perturbed problem (0.6)–(0.8) with unbounded coefficients and its limiting form (0.2), (0.4), (0.5).

Theorem 3.1. *Suppose that the data of problem* (0.6)–(0.8) *satisfy all assumptions of Theorem* 2.3. *Then there exist a sequence* ε_n*, with* $\lim_{n\to\infty} \varepsilon_n = 0$*, and a* strong solution*,* $v(x,y,t)$*, to problem* (0.2), (0.4), (0.5) *on the unbounded domain* $\mathcal{Q}_{\tilde{T}}$*, such that the functions* $u_n(x,y,t) := u^{\varepsilon_n}(x,y,t)$ *in Theorem* 2.3 *and the function* $v(x,y,t)$ *possess the following properties:*

(1) *For every fixed* $m = 0, 1, 2, \ldots,$ *the sequence* $\{x^m u_n(x,y,t)\}_{n=1}^{\infty}$ *and* $v(x,y,t)$ *belong both to the* anisotropic *Sobolev space* $W_2^{3,2,1}(\mathcal{Q}_{\tilde{T}})$*, as well as to the* anisotropic *Hölder spaces* $C^{\lambda,\lambda,\frac{1}{12}}(\Pi_K)$*, for every* $\lambda \in (0,1)$ *and for every* $K > 0$*, being* $\Pi_K := \mathcal{Q}_{\tilde{T}} \cap \{x \in [-K, K]\}$*. Moreover, for every fixed* $m = 0, 1, 2, \ldots,$ *the sequence* $\{x^m u_n(x,y,t)\}_{n=1}^{\infty}$ *is* uniformly *bounded, as* $n \to \infty$*, in these spaces.*

(2) $v(x,y,t)$ *satisfies equation* (0.2) *almost everywhere in* $\mathcal{Q}_{\tilde{T}}$*, and the boundary data in* (0.4)*, as well as the initial data in* (0.5)*, as a continuous function in* $\mathcal{Q}_{\tilde{T}}$*.*

(3) *For every* $m = 0, 1, 2, \ldots,$ *the function* $v(x,y,t)$ *can be estimated as*

$$|v(x,y,t)| \le \frac{C_m}{1 + |x|^m}$$

in $\mathcal{Q}_{\tilde{T}}$*, for some positive constant* C_m*.*

(4) *The sequence* $\{u_n(x,y,t)\}_{n=1}^{\infty}$ *converges to the function* $v(x,y,t)$ *in the following sense:*

$$\lim_{n\to\infty} \|u_n - v\|_{C(\Pi_K)} = 0, \quad \lim_{n\to\infty} \|u_n - v\|_{W_2^{2,1,0}(\Pi_K)} = 0,$$

$$\frac{\partial u_n}{\partial t} \to \frac{\partial v}{\partial t} \quad \textit{and} \quad \frac{\partial^2 u_n}{\partial y^2} \to \frac{\partial^2 v}{\partial y^2} \quad \textit{weakly in } L^2(\Pi_K), \ \textit{as } n \to \infty,$$

for every $K > 0$*.*

The proof is based on Theorem 2.3 and embedding theorems; however, it is *not* quite standard. We omit details because the existence of strong solutions to problem (0.2), (0.4), (0.5) with *bounded* coefficients has been earlier proved by the authors in [5], following the same lines.

Summary

This paper continues and extends the investigations started in [5], and concerns parabolic equations perturbed by a small parameter and ultraparabolic equations obtained from the previous ones in some formal limits. In contrast with [5], where we dealt with the case of equations with *bounded* coefficients, here we considered the case of *unbounded* coefficients. In spite of this, solvability results *similar* to those obtained in [5] have been obtained.

In addition, new decay properties for solutions to second-order linear parabolic equations with *unbounded* coefficients are established. These properties are of independent interest, from the point of view of the general theory of parabolic partial differential equations.

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