Differential and Integral Equations Volume 17, Numbers 1–2, January/February 2004, Pages 1–16

EXACTLY TWO ENTIRE POSITIVE SOLUTIONS FOR A CLASS OF NONHOMOGENEOUS ELLIPTIC EQUATIONS

Kuan-Ju Chen
Department of Applied Science, Chinese Naval Academy
No.669, Jyunsiao Rd., Zuoying District, Kaohsiung City, 813, Taiwan, R.O.C.

(Submitted by: Reza Aftabizadeh)

Abstract. In this paper, we study the nonhomogeneous elliptic equations $-\Delta u(x)+u(x)=\lambda(f(x,u)+h(x))$ in $\mathbb{R}^N$, where $f(x,u)$ and $h(x)$ satisfy some assumptions. We use variational methods to obtain the existence results and give a clever argument to establish the exact number of solutions. In addition to a lack of compactness the main difficulty to overcome is the degenerated structure of the set of possible critical points.

1. Introduction

We consider the existence and the multiplicity of positive solutions of the equation

$$\begin{cases} -\Delta u(x)+u(x)=\lambda(f(x,u)+h(x)) \text{ in } \mathbb{R}^N, \\ u\in H^1(\mathbb{R}^N),\ u>0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\lambda>0$ and $N\geq 3$. We seek solutions of equation (1.1) as critical points of the functional I associated with equation (1.1) and given by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)dx-\lambda\int_{\mathbb{R}^N}F(x,u^+)dx-\lambda\int_{\mathbb{R}^N}h(x)u\,dx,$$

where $F(x,t)=\int_0^t f(x,s)ds$.

It is assumed that $h(x)\in L^2(\mathbb{R}^N)\cap L^{(N+\beta)/2}(\mathbb{R}^N)$ ($\beta>0$ if $N\geq 4$ and $\beta=0$ if $N=3$), $h(x)\geq 0$, $h(x)\not\equiv 0$, and the basic assumptions for the function $f(x,t)$ are

(f1) $f(x,\cdot)\in C^1[0,\infty)$, $f(x,t)$ is measurable in $x\in\mathbb{R}^N$, $f(x,t)\equiv 0$ if $t\leq 0$, and $\lim_{t\to 0}\frac{f(x,t)}{t}=0$ uniformly in $x\in\mathbb{R}^N$;

(f2) there exists a positive constant C, such that for all $x\in\mathbb{R}^N$ and $t\in\mathbb{R}$, $0<\frac{\partial}{\partial t}f(x,t)\leq C(1+|t|^{p-1})$, where $1<p<\frac{N+2}{N-2}$ if $N\geq 3$, $1<p<\infty$ if $N=2$;

Accepted for publication: September 2002.
AMS Subject Classifications: 35A15, 35J20, 35J25, 35J65.

(f3) there exists $\theta \in (0, \frac{1}{2})$ such that $\theta t f(x,t) \geq F(x,t) > 0$, for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}\backslash\{0\}$;
(f4) there exists $\bar{f} : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|x|\to\infty} f(x,t) = \bar{f}(t)$, $f(x,t) \geq \bar{f}(t)$, for all $x \in \mathbb{R}^N$ and $t \geq 0$, and $\lim_{t\to\infty} \frac{f(x,t)}{t} = \infty$ uniformly in $x \in \mathbb{R}^N$;
(f5) $f(x,\cdot) \in C^2(0,\infty)$ and $\frac{\partial^2}{\partial t^2} f(x,t) \geq 0$ for $x \in \mathbb{R}^N$ and $t \geq 0$.

Given $\varepsilon > 0$, by (f1) and (f2), there exists a $C_\varepsilon > 0$ such that

$$0 \leq f(x,u) \leq \varepsilon u + C_\varepsilon |u|^p \tag{1.2}$$

and

$$0 \leq F(x,u) \leq \varepsilon u^2 + C_\varepsilon |u|^{p+1}. \tag{1.3}$$

Let $H^1(\mathbb{R}^N)$ be the Sobolev space of the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $||\cdot||$, where

$$||u|| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2))^{\frac{1}{2}}.$$

Throughout this paper, we denote the universal positive constant by C unless some special statement is given, and denote $\frac{\partial}{\partial t} f(x,t)$ and $\frac{\partial^2}{\partial t^2} f(x,t)$ by $f'(x,t)$ and $f''(x,t)$, respectively, in what follows.

To the author's knowledge, there seems to have been very little progress on existence theory for an equation without trivial solutions. Recently, Zhu-Zhou [17] and Zhu [15] established the existence of multiple positive solutions of equations similar to equation (1.1). In the present paper, the two main difficulties are, of course, first to deal with the entire solutions where we use the appropriate Palais-Smale conditions, and second to show the exact number of solutions; that is to say, we prove that there exists a $\lambda^* > 0$ such that equation (1.1) has exactly two positive solutions if $\lambda \in (0, \lambda^*)$ and equation (1.1) has no positive solution when $\lambda \in (\lambda^*, \infty)$.

2. PRELIMINARIES

In this section, several technical results will be established. Let us recall that a sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ is called a $(\mathrm{PS})_c$-sequence if $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. If any $(\mathrm{PS})_c$-sequence possesses a convergent subsequence, we say the $(\mathrm{PS})_c$-condition is satisfied.

Lemma 2.1. *If $u \in H^1(\mathbb{R}^N)$ is a critical point of I, then u is a nonnegative solution of equation* (1.1). *Moreover, if $u \not\equiv 0$ or $h \not\equiv 0$, then u is a positive solution of equation* (1.1).

Proof. Suppose that $I'(u) = 0$; then for all $\psi \in H^1(\mathbb{R}^N)$, $(I'(u), \psi) = 0$. Thus u is a weak solution of

$$-\Delta u + u = \lambda(f(x, u^+) + h(x)) \text{ in } \mathbb{R}^N. \tag{2.1}$$

By (f3) and the fact that $h(x) \geq 0$, the right-hand side of equation (2.1) is nonnegative, and then by the maximal principle we have that $u(x)$ is nonnegative. If $u \not\equiv 0$ or $h \not\equiv 0$, we can see that the right-hand side of equation (2.1) is nonnegative and not equivalently equal to 0. Thus $u(x)$ is a positive solution of equation (1.1). □

Next we prove the boundedness of Palais-Smale sequences.

Lemma 2.2. *If $\{u_n\}$ is a $(PS)_c$-sequence for I, then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a Palais-Smale sequence for I at level $c \in \mathbb{R}$. By (f3) and if n is large, then

$$\begin{aligned} & c + o(1)(1 + ||u_n||) = I(u_n) - \theta(I'(u_n), u_n) \\ & = (\frac{1}{2} - \theta)||u_n||^2 - \lambda \int_{\mathbb{R}^N} [F(x, u_n^+) - \theta f(x, u_n^+)u_n^+] - \lambda \int_{\mathbb{R}^N} (1 - \theta) h u_n \\ & \geq (\frac{1}{2} - \theta)||u_n||^2 - \lambda(1 - \theta)||h||_{L^2}||u_n||. \end{aligned}$$

Thus, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

Let us now introduce the equation at infinity associated with equation (1.1):

$$\begin{cases} -\Delta u + u = \lambda \bar{f}(u) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ u > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{2.2}$$

P.L. Lions has studied the following minimization problem closely related to equation (2.2):

$$S^\infty = \inf\{I^\infty(u) : u \in H^1(\mathbb{R}^N), u \not\equiv 0, I^{\infty\prime}(u) = 0\} > 0,$$

where $I^\infty(u) = \frac{1}{2}||u||^2 - \lambda \int_{\mathbb{R}^N} \bar{F}(u)$ and $\bar{F}(t) = \int_0^t \bar{f}(s)ds$. For future reference note also that a minimum exists and is realized by a ground-state solution $w > 0$ in $\mathbb{R}^N$ such that $S^\infty = I^\infty(w) = \sup_{s \geq 0} I^\infty(sw)$.

Next we study the breakdown of the Palais-Smale condition for I. The ground-state solution w of equation (2.2) plays an important role in describing the asymptotic behavior of the Palais-Smale sequence for I.

Proposition 2.3. *Assume (f1)–(f4) hold. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a (PS)-sequence for I. Then there exists a subsequence (still denoted $\{u_n\}$) for which the following holds: there exist an integer $m \geq 0$, a sequence $\{x_n^i\} \subset \mathbb{R}^N$ for*

$1 \le i \le m$, a solution u_0 of equation (1.1), *and solutions u^i, for $1 \le i \le m$, of equation* (2.2) *such that as $n \to \infty$,*

$$\begin{aligned}
&u_n \rightharpoonup u_0 \text{ weakly in } H^1(\mathbb{R}^N),\\
&u_n - u_0 - \textstyle\sum_{i=1}^m u^i(x - x_n^i) \to 0 \text{ strongly in } H^1(\mathbb{R}^N),\\
&I(u_n) \to I(u_0) + \textstyle\sum_{i=1}^m I^\infty(u^i),\\
&|x_n^i| \to \infty,\ |x_n^i - x_n^j| \to \infty,\ \text{for } 1 \le i \ne j \le m,
\end{aligned}$$

where we agree that in the case $m = 0$ the above holds without u^i and x_n^i.

This is a standard result that we give here without proof (see [4, 5, 12, and 16] for analogous statements).

3. Local minimal solution

We now prove the existence of positive solutions of equation (1.1).

Lemma 3.1. *Assume (f1) and (f2) hold; then for any given $\rho > 0$, there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, we have $I(u) > 0$ for all $u \in S_\rho = \{u \in H^1(\mathbb{R}^N) : ||u|| = \rho\}$. Moreover, for any $\varepsilon \ge 0$, there exists $\delta > 0$ ($\delta \le \rho$) such that $I(u) \ge -\varepsilon$ for all $u \in \{u \in H^1(\mathbb{R}^N) : \rho - \delta \le ||u|| \le \rho\}$.*

Proof. By (1.3), again using Sobolev embeddings and Hölder's inequality, we see that, for all $u \in S_\rho$,

$$\begin{aligned}
I(u) &= \frac{1}{2}||u||^2 - \lambda\int_{\mathbb{R}^N} F(x, u^+)dx - \lambda\int_{\mathbb{R}^N} hu\, dx\\
&\ge \frac{1}{2}||u||^2 - \lambda\int_{\mathbb{R}^N} (\varepsilon|u|^2 + C_\varepsilon|u|^{p+1}) - \lambda||h||_{L^2}||u||_{L^2}\\
&\ge \frac{1}{2}||u||^2 - \lambda C(||u||^2 + ||u||^{p+1}) - \lambda||h||_{L^2}||u||\\
&\ge \rho(\frac{1}{2}\rho - \lambda C(\rho + \rho^p) - \lambda||h||_{L^2}),
\end{aligned} \tag{3.1}$$

where $C > 0$ is a constant which is independent of λ and ρ. Hence, by (3.1), there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, we have $I(u) > 0$ for all $u \in S_\rho$.

Moreover, we can choose $\lambda_0 > 0$ small enough so that

$$\frac{\partial}{\partial\rho}(\frac{1}{2}\rho - \lambda C(\rho + \rho^p)) = \frac{1}{2} - \lambda C(1 + p\rho^{p-1}) > 0 \text{ for } \lambda \in (0, \lambda_0);$$

then for any $\varepsilon \ge 0$, there exists $\delta > 0$ ($\delta \le \rho$) such that $I(u) \ge -\varepsilon$ for all $u \in \{u \in H^1(\mathbb{R}^N) : \rho - \delta \le ||u|| \le \rho\}$. □

Lemma 3.2. *Assume (f1)–f(4) hold. Let λ_0 be chosen as in Lemma* 3.1 *and $\lambda \in (0, \lambda_0)$; then there exists $u_0 \in B_\rho = \{u \in H^1(\mathbb{R}^N) : ||u|| < \rho\}$, where ρ is* *the number given in Lemma* 3.1*, such that* $I(u_0) = \inf\{I(u) : u \in \bar{B}_\rho\} < 0$*, and* u_0 *is a positive solution of equation* (1.1)*.*

Proof. Since $h(x) \not\equiv 0$ and $h \geq 0$, we can choose a function $\psi \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} h\psi > 0$. For $t \in (0, \infty)$, by (1.3),

$$\begin{aligned} I(t\psi) &= \frac{t^2}{2}||\psi||^2 - \lambda \int_{\mathbb{R}^N} F(x, t\psi^+)dx - \lambda t \int_{\mathbb{R}^N} h\psi\, dx \\ &\leq \frac{t^2}{2}||\psi||^2 + \lambda C t^2 \int_{\mathbb{R}^N} (|\psi|^2 + t^{p-1}|\psi|^{p+1})dx - \lambda t \int_{\mathbb{R}^N} h\psi\, dx. \end{aligned}$$

Then for t small enough, $I(t\psi) < 0$. So $\inf\{I(u) : u \in \bar{B}_\rho\} < 0$, clearly $\inf\{I(u) : u \in \bar{B}_\rho\} > -\infty$. By Lemma 3.1, there exists $\rho' < \rho$ such that $\inf\{I(u) : u \in \bar{B}_\rho\} = \inf\{I(u) : u \in \bar{B}_{\rho'}\}$. Thus the general perturbation principle due to Ekeland guarantees the existence of a (PS)-sequence $\{u_n\} \subset \bar{B}_{\rho'}$ for I at level $\inf\{I(u) : u \in \bar{B}_{\rho'}\}$. By Proposition 2.3, there exist a subsequence (still denote by $\{u_n\}$), an integer $m \geq 0$, a sequence $\{x_n^i\}$, $1 \leq i \leq m$, a solution u_0 of equation (1.1), and solutions u^i $(1 \leq i \leq m)$ of equation (2.2) such that as $n \to \infty$,

$$u_n \rightharpoonup u_0 \text{ weakly in } H^1(\mathbb{R}^N), \tag{3.2}$$

$$\inf\{I(u) : u \in \bar{B}_\rho\} = I(u_0) + \textstyle\sum_{i=1}^m I^\infty(u^i). \tag{3.3}$$

By (3.2), $u_0 \in \bar{B}_\rho$ and $I(u_0) \geq \inf\{I(u) : u \in \bar{B}_\rho\}$; then by (3.3) and the fact that $I^\infty(u^i) \geq S^\infty > 0$ for $1 \leq i \leq m$, we conclude that $m = 0$, $I(u_0) = \inf\{I(u) : u \in \bar{B}_\rho\} < 0$, and $I'(u_0) = 0$. □

By a supersolution of equation (1.1), we mean a function $v \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \nabla v \cdot \nabla \psi + \int_{\mathbb{R}^N} v\psi \geq \lambda(\int_{\mathbb{R}^N} f(x, v)\psi + \int_{\mathbb{R}^N} h(x)\psi)$$

for all $\psi \in H^1(\mathbb{R}^N)$, $\psi \geq 0$ in $\mathbb{R}^N$.

By the standard barrier method, we prove the following theorem:

Theorem 3.3. *If (f1)–(f4) hold, then there exists* $\lambda^* > 0$ *such that*

(i) *for any* $\lambda \in (0, \lambda^*)$*, equation* (1.1) *has a minimal positive solution* u_λ *(i.e., for any positive solution* u *of equation* (1.1)*,* $u \geq u_\lambda$*);*

(ii) *if* $\lambda > \lambda^*$*, equation* (1.1) *has no positive solution.*

Proof. Set $Q_\lambda = \{0 < \lambda < \infty : \text{equation (1.1) is solvable}\}$; by Lemma 3.2, Q_λ is nonempty. Denoting $\lambda^* = \sup Q_\lambda > 0$, we claim that equation (1.1) is solvable for all $\lambda \in (0, \lambda^*)$. Indeed, for any $\lambda \in (0, \lambda^*)$, by the definition of

λ^*, we know that there exists $\lambda' > 0$ and $0 < \lambda < \lambda' < \lambda^*$ such that equation (1.1) with λ' has a solution $u_{\lambda'} > 0$; i.e.,

$$-\Delta u_{\lambda'} + u_{\lambda'} = \lambda'(f(x, u_{\lambda'}) + h(x)) > \lambda(f(x, u_{\lambda'}) + h(x)).$$

Thus $u_{\lambda'}$ is a supersolution of equation (1.1). Since $h(x) \geq 0$ and $h(x) \not\equiv 0$, 0 is a subsolution of equation (1.1). By the standard barrier method, there exists a solution u_λ of equation (1.1) such that $0 \leq u_\lambda \leq u_{\lambda'}$. From the fact that $h(x) \not\equiv 0$, we know that 0 is not a solution of equation (1.1). Then by the fact that $\lambda' > \lambda$ and the strong maximum principle we infer that $0 < u_\lambda < u_{\lambda'}$. Again using a result of Amann (see [2, Theorem 9,4]), we can assume u_λ is a minimal positive solution of equation (1.1) in the interval $[0, u_{\lambda'}]$. □

Lemma 3.4. *Let Ω be a domain in $\mathbb{R}^N$ $(N \geq 1)$, f satisfy (f1) and (f2), $h \in L^2(\Omega) \cap L^{\frac{N}{2}}(\Omega)$, and $u \in H_0^1(\Omega)$ be a weak solution of equation* (1.1). *Then $u \in L^q(\Omega)$ for $q \in [2, \infty)$.*

Proof. The proof follows by the classical regularity theory based on a result of Brezis-Kato [7]. We will write it in detail for the reader's convenience.

For $s \geq 0$ and $l \geq 1$, let $\varphi = \varphi_{s,l} = u \min\{|u|^{2s},\ l^2\} \in H_0^1(\Omega)$. Then by (1.2) we have

$$\begin{aligned} &\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, l^2\} + 2s \int_{\{x\in\Omega | |u|^s \leq l\}} |\nabla u|^2 |u|^{2s} \\ \leq \quad &\int_\Omega |u|^{2+2s} + C\int_\Omega |u|^{2+2s} + C\int_\Omega |u|^{2s+p+1} + C\int_\Omega |h||u| \min\{|u|^{2s}, l^2\}. \end{aligned}$$

Suppose $u \in L^{2s+p+1}(\Omega)$.

$$\begin{aligned} &\int_\Omega |\nabla(u \min\{|u|^s, l\})|^2 \leq \int_\Omega |\nabla u|^2 \min\{|u|^{2s}, l^2\} + (s^2+2s)\int_{\{|u|^s\leq l\}} |\nabla u|^2 |u|^{2s} \\ &\leq C + C\int_\Omega |h||u| \min\{|u|^{2s}, l^2\} \\ &\leq C + C\int_\Omega |h||u| + C\int_\Omega |h||u|^2 \min\{|u|^{2s}, l^2\} \\ &\leq C + CK\int_\Omega |u|^{2s+2} + C\int_{\{|h|\geq K\}} |h||u \min\{|u|^s, l\}|^2 \\ &\leq C(1+K) + C(\int_{\{|h|\geq K\}} |h|^{\frac{N}{2}})^{\frac{2}{N}} (\int_\Omega |u \min\{|u|^s, l\}|^{\frac{2N}{N-2}})^{\frac{N-2}{N}} \\ &\leq C(1+K) + C\varepsilon(K)\int_\Omega |\nabla(u \min\{|u|^s, l\})|^2, \end{aligned}$$

where

$$\varepsilon(K) = (\int_{\{|h|\geq K\}} |h|^{\frac{N}{2}})^{\frac{2}{N}} \to 0 \text{ as } K \to \infty.$$

Fix K such that $\varepsilon(K) = 1/2C$ and observe that for this choice of K (and s as above) we now may conclude that

$$\int_{\{|u|^s \leq l\}} |\nabla(|u|^{s+1})|^2 \leq C \int_\Omega |\nabla(u \min\{|u|^s, l\})|^2 \leq C(1+K)$$

for any $l \geq 1$. Hence we may let $l \to \infty$ to derive that

$$|u|^{s+1} \in H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega).$$

It is easy to see that $u \in L^{\frac{(s+1)2N}{N-2}}(\Omega)$.

Now iterate, letting $s_0 = 0$ and $2s_i + p + 1 = \frac{(s+1)2N}{N-2}$ for $i \geq 1$. It is easy to see $s_i \to \infty$ as $i \to \infty$. Therefore, $u \in L^q(\Omega)$ for $2 \leq q < \infty$. □

Lemma 3.5. *If u is a positive solution of equation* (1.1), *then*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Proof. Since $h(x) \in L^2(\mathbb{R}^N) \cap L^{(N+\beta)/2}(\mathbb{R}^N)$ ($\beta > 0$ if $N \geq 4$ and $\beta = 0$ if $N = 3$), this implies $h(x) \in L^2(\mathbb{R}^N) \cap L^{N/2}(\mathbb{R}^N)$ for $N \geq 3$. By Lemma 3.4, we have $u \in L^q(\Omega)$ for $2 \leq q < \infty$; then by (1.2), $\lambda(f(x,u) + h(x)) \in L^2(\mathbb{R}^N) \cap L^{(N+\beta)/2}(\mathbb{R}^N)$. Hence by the Calderon-Zygmund inequality and [8, Chapter II, Section 8, Proposition 27] (or [14, Proposition 4.3]), we have

$$u \in W^{2,2}(\mathbb{R}^N) \cap W^{2,(N+\beta)/2}(\mathbb{R}^N) \quad (\beta > 0 \text{ if } N \geq 4 \text{ and } \beta = 0 \text{ if } N = 3).$$

By the Sobolev embedding theorem, since $N < 2 \cdot \frac{N+\beta}{2}$ if $N \geq 4$ and $2 > \frac{N}{2}$ if $N = 3$, we obtain that $u \in C(\mathbb{R}^N)$. It is well-known that the Sobolev embedding constants are independent of domains ([1, Lemma 5.15]). Thus there exists a constant C such that for $R > 0$,

$$||u||_{L^\infty(\mathbb{R}^N \setminus B_R)} \leq C||u||_{W^{2,(N+\beta)/2}(\mathbb{R}^N \setminus B_R)}, \text{ if } N \geq 4,$$

$$||u||_{L^\infty(\mathbb{R}^N \setminus B_R)} \leq C||u||_{W^{2,2}(\mathbb{R}^N \setminus B_R)}, \text{ if } N = 3.$$

From this we get $\lim_{|x|\to\infty} u(x) = 0$. □

Let u_λ be the minimal positive solution of equation (1.1) for $\lambda \in (0, \lambda^*)$. We study the following eigenvalue problem:

$$\begin{cases} -\Delta v + v = \mu f'(x, u_\lambda) v \text{ in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N),\ v > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{3.4}$$

Lemma 3.6. *If (f1)–(f5) hold, then the minimization problem*

$$\mu = \inf \left\{ \int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) : v \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} f'(x, u_\lambda) v^2 = 1 \right\}$$

can be achieved by some $v_0 > 0$. Furthermore, $\mu > \lambda$.

Proof. It is easy to see that $\mu < \infty$. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence of μ, i.e.,

$$\int_{\mathbb{R}^N} f'(x, u_\lambda) v_n^2 = 1,\ \int_{\mathbb{R}^N} (|\nabla v_n|^2 + v_n^2) \to \mu, \text{ as } n \to \infty;$$

$\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Without loss of generality (at least by choosing a subsequence) we can assume, for some $v_0 \in H^1(\mathbb{R}^N)$, that

$$\begin{aligned} v_n &\rightharpoonup v_0 \text{ weakly in } H^1(\mathbb{R}^N), \\ v_n &\to v_0 \text{ almost everywhere in } \mathbb{R}^N, \\ v_n &\to v_0 \text{ strongly in } L^s_{loc}(\mathbb{R}^N) \text{ for } 2 \le s < \frac{2N}{N-2}. \end{aligned}$$

Hence,

$$\int_{\mathbb{R}^N} (|\nabla v_0|^2 + v_0^2) \le \varliminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla v_n|^2 + v_n^2) = \mu.$$

By Lemma 3.5 and (f1), we have $f'(x, u_\lambda) \to 0$ as $|x| \to \infty$. Consequently, there exists a constant $C > 0$ such that

$$|f'(x, u_\lambda)| \le C \text{ for } x \in \mathbb{R}^N.$$

Furthermore, for any $\varepsilon > 0$, there exists $R > 0$ such that for $x \in \mathbb{R}^N$, $|x| \ge R$, $|f'(x, u_\lambda)| < \varepsilon$. Then

$$\begin{aligned} &\int_{\mathbb{R}^N} f'(x, u_\lambda)|v_n - v_0|^2 = \int_{B_R} f'(x, u_\lambda)|v_n - v_0|^2 + \int_{\mathbb{R}^N \setminus B_R} f'(x, u_\lambda)|v_n - v_0|^2 \\ &\le C \int_{B_R} |v_n - v_0|^2 + \varepsilon \int_{\mathbb{R}^N \setminus B_R} |v_n - v_0|^2. \end{aligned}$$

Since $v_n \to v_0$ strongly in $L^s_{loc}(\mathbb{R}^N)$ for $2 \le s < 2N/N-2$, $\{v_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$; taking $n \to \infty$, then $\varepsilon \to 0$, we obtain $\int_{\mathbb{R}^N} f'(x, u_\lambda) v_0^2 = 1$. Therefore v_0 achieves μ. Clearly $|v_0|$ also achieves μ. Hence, we may assume $v_0 \ge 0$ in $\mathbb{R}^N$, and v_0 satisfies equation (3.4). Once again, by the maximum principle for weak solutions we deduce that $v_0 > 0$ in $\mathbb{R}^N$.

We will now prove that $\mu > \lambda$. Setting $0 < \lambda < \lambda'$ and $\lambda' \in (0, \lambda^*)$, by Theorem 3.3, equation (1.1) with λ' has a positive solution $u_{\lambda'}$. Since u_λ is the minimum positive solution of equation (1.1), $u_{\lambda'} > u_\lambda$ as $\lambda' > \lambda$. Noting that $\lambda' > \lambda$, $h(x) \geq 0$ and $f(x, u_\lambda) > 0$, $f''(x,t) \geq 0$ for all $t > 0$, we get

$$\begin{aligned} &- \Delta(u_{\lambda'} - u_\lambda) + (u_{\lambda'} - u_\lambda) \\ &= \lambda' f(x, u_{\lambda'}) - \lambda f(x, u_\lambda) + (\lambda' - \lambda)h(x) \\ &= (\lambda' - \lambda) f(x, u_\lambda) + \lambda'(f(x, u_{\lambda'}) - f(x, u_\lambda)) + (\lambda' - \lambda)h(x) \\ &> \lambda f'(x, u_\lambda)(u_{\lambda'} - u_\lambda). \end{aligned} \tag{3.5}$$

Multiplying (3.5) by v_0 (the characteristic function of equation (3.4)) and integrating it over $\mathbb{R}^N$, we get

$$\mu \int_{\mathbb{R}^N} f'(x, u_\lambda)(u_{\lambda'} - u_\lambda)v_0 > \lambda \int_{\mathbb{R}^N} f'(x, u_\lambda)(u_{\lambda'} - u_\lambda)v_0,$$

which implies that $\mu > \lambda$. □

4. Existence of second solution

When $\lambda \in (0, \lambda^*)$, we have shown that equation (1.1) has a minimal positive solution u_λ by Theorem 3.3; then we need only to prove that equation (1.1) has another positive solution in the form of $U_\lambda = u_\lambda + \bar{v}$, where $\bar{v}$ is a positive solution of the following auxiliary equation:

$$\begin{cases} -\Delta v + v = \lambda(f(x, u_\lambda + v) - f(x, u_\lambda)) \text{ in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N),\ v > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{4.1}$$

For equation (4.1), we define the energy functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ as follows:

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) - \lambda \int_{\mathbb{R}^N} (F(x, u_\lambda + v^+) - F(x, u_\lambda) - f(x, u_\lambda)v^+).$$

Lemma 4.1. *If $v \in H^1(\mathbb{R}^N)$ is a critical point of J, then v is a positive solution of equation* (4.1).

Proof. Suppose that $J'(v) = 0$; then for all $\psi \in H^1(\mathbb{R}^N)$, $(J'(v), \psi) = 0$. Thus v is a weak solution of

$$-\Delta v + v = \lambda(f(x, u_\lambda + v^+) - f(x, u_\lambda)) \text{ in } \mathbb{R}^N.$$

Using the monotonicity of f and the maximum principle, we know that the nontrivial critical points of energy functional J are the positive solutions of equation (4.1). □

Lemma 4.2. *If $\{v_n\}$ is a (PS)$_c$-sequence for J, then $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a Palais-Smale sequence for J at level $c \in \mathbb{R}$. By (f3) and if n is large, then letting $\tilde{\theta}$ satisfy $0 < \theta < \tilde{\theta} < 1/2$, we find that

$$\begin{aligned}
& c + o(1)(1 + ||v_n||) = J(v_n) - \tilde{\theta}(J'(v_n), v_n) \\
& \geq \frac{1}{2}||v_n||^2 - \lambda\theta \int_{\mathbb{R}^N} f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+) \\
& - \tilde{\theta}||v_n||^2 + \lambda\tilde{\theta} \int_{\mathbb{R}^N} (f(x, u_\lambda + v_n^+) - f(x, u_\lambda))v_n^+ \\
& \geq (\frac{1}{2} - \tilde{\theta})||v_n||^2 + \lambda(\tilde{\theta} - \theta) \int_{\mathbb{R}^N} f(x, u_\lambda + v_n^+)(v_n^+ - \tau u_\lambda) \\
& - \lambda\tilde{\theta} \int_{\mathbb{R}^N} f(x, u_\lambda)v_n^+ \\
& \geq (\frac{1}{2} - \tilde{\theta})||v_n||^2 + \lambda(\tilde{\theta} - \theta) \int_{\{v_n^+ \leq \tau u_\lambda\}} (v_n^+ - \tau u_\lambda) f(x, u_\lambda + \tau u_\lambda) \\
& - \lambda\tilde{\theta} \int_{\mathbb{R}^N} f(x, u_\lambda)v_n^+ \\
& \geq (\frac{1}{2} - \tilde{\theta})||v_n||^2 - \lambda\theta \int_{\mathbb{R}^N} f(x, u_\lambda + \tau u_\lambda)u_\lambda - \lambda\tilde{\theta} \int_{\mathbb{R}^N} f(x, u_\lambda)v_n^+ \\
& \geq (\frac{1}{2} - \tilde{\theta})||v_n||^2 - \lambda\theta C - \lambda\tilde{\theta}C||v_n^+||,
\end{aligned}$$

where $\tau = \theta/(\tilde{\theta} - \theta)$. Therefore, we deduce that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

Lemma 4.3. *If (f1), (f2), and (f5) hold, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$f(x, u_\lambda + t) - f(x, u_\lambda) - f'(x, u_\lambda)t \leq \varepsilon t + C_\varepsilon t^p,\ t \geq 0,$$

where $1 < p < (N - 2)/(N + 2)$.

Proof. By the mean-value theorem, there exists $\kappa \in [0, 1]$ such that

$$f(x, u_\lambda + t) - f(x, u_\lambda) - f'(x, u_\lambda)t = (f'(x, u_\lambda + \kappa t) - f'(x, u_\lambda))t.$$

Since $f'(x, \cdot) \in C[0, \infty)$ and setting $M = \max\{u_\lambda(x) : x \in \mathbb{R}^N\}$, we find that $f'(x, \cdot)$ is uniformly continuous on $[0, 2M]$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f'(x, t_1) - f'(x, t_2)| \leq \varepsilon$ as $t_1, t_2 \in [0, 2M]$ and $|t_1 - t_2| < \delta$.

By (f1), (f2), and (f5), we obtain that $f(x, u_\lambda + t) - f(x, u_\lambda) - f'(x, u_\lambda)t \leq C_\varepsilon t^p$ as $t \geq \delta$. □

Lemma 4.4. *If (f1)–(f5) hold, then there exist small $\rho > 0$ and $\alpha > 0$ such that $J(v) \geq \alpha > 0$ for all $v \in S_\rho = \{u \in H^1(\mathbb{R}^N) : ||u|| = \rho\}$.*

Proof. By Lemma 4.3, the definition of μ, and Sobolev embeddings, we gave

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2) - \lambda\int_{\mathbb{R}^N}(F(x, u_\lambda + v^+) - F(x, u_\lambda) - f(x, u_\lambda)v^+)$$
$$= \frac{1}{2}||v||^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N} f'(x, u_\lambda)|v^+|^2$$
$$- \lambda\int_{\mathbb{R}^N}\int_0^{v^+}\big(f(x, u_\lambda + t) - f(x, u_\lambda) - f'(x, u_\lambda)t\big)$$
$$\geq \frac{1}{2}||v||^2 - \frac{\lambda}{2}\int_{\mathbb{R}^N} f'(x, u_\lambda)|v^+|^2 - \frac{1}{2}\lambda\varepsilon\int_{\mathbb{R}^N}|v^+|^2 - \frac{1}{p+1}\lambda C_\varepsilon\int_{\mathbb{R}^N}|v^+|^{p+1}$$
$$\geq \frac{1}{2}||v||^2 - \frac{\lambda}{2}\mu^{-1}||v||^2 - \frac{1}{2}\lambda\varepsilon||v||^2 - \lambda C_\varepsilon||v||^{p+1}$$
$$= \frac{1}{2}\mu^{-1}(\mu - \lambda - \lambda\mu\varepsilon)||v||^2 - \lambda C_\varepsilon||v||^{p+1}.$$

Since $\mu > \lambda$, we may choose $\varepsilon > 0$ small enough such that $\mu - \lambda - \lambda\mu\varepsilon > 0$. If we fix $\varepsilon = (\mu - \lambda)/(2\lambda\mu)$, then

$$J(v) \geq \frac{1}{4}\mu^{-1}(\mu - \lambda)||v||^2 - \lambda C||v||^{p+1}.$$

Hence, there exist small $\rho > 0$ and $\alpha > 0$ such that $J(v) \geq \alpha > 0$ for all $v \in S_\rho = \{u \in H^1(\mathbb{R}^N) : ||u|| = \rho\}$. □

Similar to Proposition 2.3, as to J, we have the following:

Proposition 4.5. *Assume (f1)–(f4) hold. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a (PS) sequence for J. Then there exist a subsequence (still denoted $\{v_n\}$) for which the following holds: there exist an integer $m \geq 0$, a sequence $\{x_n^i\} \subset \mathbb{R}^N$ for $1 \leq i \leq m$, a solution $\bar{v}$ of equation (4.1), and solutions v^i, for $1 \leq i \leq m$, of equation (2.2) such that as $n \to \infty$,*

$$v_n \rightharpoonup \bar{v} \text{ weakly in } H^1(\mathbb{R}^N),$$
$$v_n - \bar{v} - \textstyle\sum_{i=1}^m v^i(x - x_n^i) \to 0 \text{ strongly in } H^1(\mathbb{R}^N),$$
$$J(v_n) \to J(\bar{v}) + \textstyle\sum_{i=1}^m I^\infty(v^i),$$
$$|x_n^i| \to \infty,\ |x_n^i - x_n^j| \to \infty, \text{ for } 1 \leq i \neq j \leq m,$$

where we agree that in the case $m = 0$ the above holds without v^i and x_n^i.

Let w be a ground-state solution of equation (2.2), $S^\infty = I^\infty(w) = \sup_{t\geq 0} I^\infty(tw)$.

Lemma 4.6. *If (f1)–(f5) hold, then*

(i) *there exists $s_0 > 0$ such that $J(sw) < 0$ for all $s \geq s_0$;*
(ii) $\sup_{s\geq 0} J(sw) < S^\infty$.

Proof. (i) By (f3) and (f5), for all $x \in \mathbb{R}^N$ and $t_1, t_2 \geq 0$,

$$\begin{aligned} f(x, t_1 + t_2) &\geq f(x, t_1) + f(x, t_2), \\ f(x, t_1 + t_2) &\not\equiv f(x, t_1) + f(x, t_2). \end{aligned} \tag{4.2}$$

By (f4) and (4.2), we have

$$\begin{aligned} J(sw) &= \frac{s^2}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) - \lambda \int_{\mathbb{R}^N} \int_0^{sw} [f(x, u_\lambda + t) - f(x, u_\lambda)] \\ &= I^\infty(sw) - \lambda \int_{\mathbb{R}^N} \int_0^{sw} [f(x, u_\lambda + t) - f(x, u_\lambda) - \bar{f}(t)] \\ &\leq I^\infty(sw) - \lambda \int_{\mathbb{R}^N} \int_0^{sw} [f(x, u_\lambda + t) - f(x, u_\lambda) - f(x, t)] \\ &\leq I^\infty(sw). \end{aligned} \tag{4.3}$$

By (f4), $I^\infty(sw) \to -\infty$ as $s \to \infty$. Hence (i) holds.

(ii) From (i) and Lemma 4.4 we know that

$$\sup_{s \geq 0} J(sw) = \sup_{s \leq s_0} J(sw).$$

By the continuity of $J(sw)$ as a function of $s \geq 0$, along with the fact that $\sup_{s \geq 0} I^\infty(sw) = I^\infty(w) = S^\infty$ and (4.4), there exists $s_1 \in (0, s_0)$ such that

$$\sup_{0 \leq s \leq s_1} J(sw) < S^\infty.$$

Thus, we need only to show that

$$\sup_{s_1 \leq s \leq s_0} J(sw) < S^\infty.$$

To this end, by (4.3), we have

$$\begin{aligned} \sup_{s_1 \leq s \leq s_0} J(sw) &\leq S^\infty - \lambda \inf_{s_1 \leq s \leq s_0} \int_{\mathbb{R}^N} \int_0^{sw} [f(x, u_\lambda + t) - f(x, u_\lambda) - f(x, t)] \\ &< S^\infty. \end{aligned}$$

Hence (ii) holds. □

Theorem 4.7. *If (f1)–(f5) hold, then equation* (4.1) *has a positive solution as* $\lambda \in (0, \lambda^*)$.

Proof. We set $\gamma = \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t))$, where

$$\Gamma = \{g \in C([0, 1], H^1(\mathbb{R}^N)) : g(0) = 0,\, g(1) = s_0 w\}.$$

By Lemma 4.4 and Lemma 4.6, we deduce that $0 < \alpha \leq \gamma < S^\infty$. The mountain-pass lemma insures the existence of a (PS) sequence $\{v_n\}$ for J at level γ. By Proposition 4.6, we deduce that

$$\gamma = \lim_{n\to\infty} J(v_n) = J(\bar{v}) + \sum\nolimits_{i=1}^{m} I^{\infty}(v^i),$$

for some $\bar{v}$ and v^i satisfying $J'(\bar{v}) = 0$ and $I^{\infty\prime}(v^i)$, for $1 \le i \le m$.

By the strong maximum principle, we need only to prove $\bar{v} \not\equiv 0$. In fact, we have

$$\gamma = J(\bar{v}) \ge \alpha > 0 \text{ if } m = 0;\ S^{\infty} > \gamma \ge J(\bar{v}) + S^{\infty} \text{ if } m \ge 1.$$

This implies $\bar{v} \not\equiv 0$. □

Remark 4.8. To the author's knowledge, for the general nonlinearity $f(x, u)$ it is still unknown whether λ^* is bounded or infinite.

5. Exactly two positive solutions

Denote by $Q = \{(\lambda, u) : u \text{ solves equation (1.1)}\}$, the set of solutions of equation (1.1), $\lambda \in (0, \lambda^*)$. For each $(\lambda, u) \in Q$, let $\mu_\lambda(u)$ denote the number defined by

$$\mu_\lambda(u) = \inf\left\{\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) : v \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} f'(x, u)v^2 = 1\right\},$$

which is the smallest eigenvalue of the following problem:

$$\begin{cases} -\Delta v + v = \mu_\lambda(u) f'(x, u)v \text{ in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N),\ v > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{5.1}$$

Lemma 5.1. *Assume (f1)–(f5) hold, let $(\lambda, u) \in Q$, and suppose v is a nonnegative supersolution of equation* (1.1); *then*

$$v \ge u \textit{ if } \mu_\lambda(u) > \lambda,\ v \le u \textit{ if } \mu_\lambda(u) < \lambda,\ \textit{and } v = u \textit{ if } \mu_\lambda(u) = \lambda.$$

Proof. Let $\psi \ge 0$, $\psi \in H^1(\mathbb{R}^N)$. Then by (f5), we obtain

$$\int_{\mathbb{R}^N} \nabla(u - v)\cdot\nabla\psi + \int_{\mathbb{R}^N} (u - v)\psi \le \lambda \int_{\mathbb{R}^N} (f(x, u) - f(x, v))\psi\, dx \tag{5.2}$$

$$= \lambda \int_{\mathbb{R}^N} \int_v^u f'(x, t)dt\, \psi\, dx \le \lambda \int_{\mathbb{R}^N} f'(x, u)(u - v)\psi\, dx. \tag{5.3}$$

Consider the case $\mu_\lambda(u) > \lambda$. Set $\psi = (u - v)^+$. If $\psi \not\equiv 0$, then by (5.3) and the definition of $\mu_\lambda(u)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla\psi|^2 + |\psi|^2) \le \lambda \int_{\mathbb{R}^N} f'(x, u)\psi^2 dx \\ &< \mu_\lambda(u) \int_{\mathbb{R}^N} f'(x, u)\psi^2 dx \le \int_{\mathbb{R}^N} (|\nabla\psi|^2 + |\psi|^2), \end{aligned}$$

which is impossible. Hence $\psi \equiv 0$; i.e., $u \le v$ in $\mathbb{R}^N$.

Consider the case $\mu_\lambda(u) < \lambda$. We show then that we must have that the set $G = \{x \in \mathbb{R}^N : v(x) > u(x)\}$ is a proper subset of $\mathbb{R}^N$; otherwise, let $\psi = \omega_\lambda$ be a minimizer of $\mu_\lambda(u)$, and by (5.3),

$$\begin{aligned}\mu_\lambda(u)\int_{\mathbb{R}^N} f'(x,u)\omega_\lambda(v-u) &= \int_{\mathbb{R}^N} \nabla\omega_\lambda\cdot\nabla(v-u) + \int_{\mathbb{R}^N}\omega_\lambda(v-u)\\ &\ge \lambda\int_{\mathbb{R}^N} f'(x,u)(v-u)\omega_\lambda,\end{aligned}$$

which contradicts the case $\mu_\lambda(u) < \lambda$. Next we show that G is empty. If not, set $\psi = (v-u)^+$, and for each $n = 1, 2, \ldots$, let $\Omega_n = G \cup \{x \in \mathbb{R}^N : |x| \le n\}$. Consider the minimization problems

$$\mu_{\lambda,n}(u) = \inf\left\{\int_{\Omega_n}(|\nabla v|^2 + v^2) : v \in H_0^1(\Omega_n),\ \int_{\Omega_n} f'(x,u)v^2 = 1\right\}.$$

Since $\Omega_n \subseteq \Omega_{n+1} \subseteq \mathbb{R}^N$ and $\cup_{n=1}^\infty \Omega_n = \mathbb{R}^N$, we have $\mu_\lambda(u) \le \mu_{\lambda,n+1}(u) \le \mu_{\lambda,n}(u)$ and $\lim_{n\to\infty}\mu_{\lambda,n}(u) = \mu_\lambda(u)$. Hence for $\delta = (\lambda - \mu_\lambda(u))/2$, there exists n_0 such that $\mu_{\lambda,n}(u) \le \mu_\lambda(u) + \delta$ for $n \ge n_0$. Since

$$\begin{gathered}-\Delta\psi + \psi \ge \lambda f'(x,u)\psi \text{ in } \Omega_n,\\ -\Delta\omega_{\lambda,n} + \omega_{\lambda,n} = \mu_{\lambda,n}(u)f'(x,u)\omega_{\lambda,n} \text{ in } \Omega_n,\end{gathered}$$

where $\omega_{\lambda,n}$ is a minimizer of $\mu_{\lambda,n}(u)$ for $n = 1, 2, \ldots$, we have

$$\begin{aligned}\mu_{\lambda,n}(u)\int_{\Omega_n} f'(x,u)\omega_{\lambda,n}\psi &= \int_{\Omega_n}\nabla\omega_{\lambda,n}\cdot\nabla\psi + \int_{\Omega_n}\omega_{\lambda,n}\psi\\ \ge \lambda\int_{\Omega_n} f'(x,u)\psi\omega_{\lambda,n};\end{aligned}$$

then $\mu_{\lambda,n}(u) \ge \lambda,$ for $n = 1, 2, \ldots$. However, for $n \ge n_0$, $\mu_{\lambda,n}(u) \le \mu_\lambda(u) + \delta$; therefore, $2\delta = \lambda - \mu_\lambda(u) \le \delta$, a contradiction.

Consider the case $\mu_\lambda(u) = \lambda$. Assume to the contrary that $v - u \not\equiv 0$ in $\mathbb{R}^N$. First we will claim that

$$-\Delta(v-u) + (v-u) = \lambda f'(x,u)(v-u) \text{ in } \mathbb{R}^N. \tag{5.4}$$

Otherwise, by (5.3),

$$-\Delta(v-u) + (v-u) > \lambda f'(x,u)(v-u) \text{ in } \mathbb{R}^N;$$

let $\psi = \omega_\lambda$ be a minimizer of $\mu_\lambda(u)$,

$$\int_{\mathbb{R}^N}\nabla(v-u)\cdot\nabla\omega_\lambda + \int_{\mathbb{R}^N}(v-u)\omega_\lambda > \lambda\int_{\mathbb{R}^N} f'(x,u)(v-u)\omega_\lambda$$

$$= \mu_\lambda(u) \int_{\mathbb{R}^N} f'(x,u)(v-u)\omega_\lambda = \int_{\mathbb{R}^N} \nabla\omega_\lambda \cdot \nabla(v-u) + \int_{\mathbb{R}^N} \omega_\lambda(v-u),$$

a contradiction. Hence equation (5.4) holds.

From equation (5.4), we may assume that $v-u$ has a fixed sign in $\mathbb{R}^N$.

If $v(x)-u(x) \geq 0$ in $\mathbb{R}^N$, by (5.4) and (5.2),

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} f'(x,u)(v-u)^2 = \int_{\mathbb{R}^N} |\nabla(v-u)|^2 + (v-u)^2 \\
&\geq \lambda \int_{\mathbb{R}^N} (f(x,v) - f(x,u))(v-u) \\
&= \lambda \int_{\mathbb{R}^N} f'(x,u)(v-u)^2 + \lambda \int_{\mathbb{R}^N} (f(x,v) - f(x,u) - f'(x,u)(v-u))(v-u) \\
&= \lambda \int_{\mathbb{R}^N} f'(x,u)(v-u)^2 + \lambda \int_{\mathbb{R}^N} \int_u^v (f'(x,t) - f'(x,u))(v-u);
\end{aligned}$$

by (f5), we obtain $v \equiv u$.

Similarly, if $v(x)-u(x) \leq 0$ in $\mathbb{R}^N$, by (5.4) and (5.2),

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} f'(x,u)(u-v)^2 = \int_{\mathbb{R}^N} |\nabla(u-v)|^2 + (u-v)^2 \\
&\leq \lambda \int_{\mathbb{R}^N} (f(x,u) - f(x,v))(u-v) \\
&= \lambda \int_{\mathbb{R}^N} f'(x,u)(u-v)^2 + \lambda \int_{\mathbb{R}^N} (f(x,u) - f(x,v) - f'(x,u)(u-v))(u-v) \\
&= \lambda \int_{\mathbb{R}^N} f'(x,u)(u-v)^2 + \lambda \int_{\mathbb{R}^N} \int_v^u (f'(x,t) - f'(x,u))(u-v);
\end{aligned}$$

by (f5), we obtain $v \equiv u$. □

Theorem 5.2. *Let $(\lambda, u) \in Q$, $0 < \lambda < \lambda^*$. Then*

(i) *$\mu_\lambda(u) > \lambda$ if and only if $u = u_\lambda$;*

(ii) *$\mu_\lambda(u) < \lambda$ if and only if $u = U_\lambda = u_\lambda + v_\lambda$, where v_λ is any positive solution of equation* (4.1).

Proof. (i) By Lemma 5.1, for each fixed $\lambda > 0$, solutions of equation (1.1) satisfying $\mu_\lambda(u) > \lambda$ (or $\mu_\lambda(u) < \lambda$) are unique, and then by Lemma 3.6, $\mu_\lambda(u_\lambda) > \lambda$, so (i) holds.

(ii) If $u = U_\lambda = u_\lambda + v_\lambda$, where v_λ is any positive solution of equation (4.1), then $(\lambda, U_\lambda) \in Q$. Since $U_\lambda > u_\lambda$, then by Lemma 5.1, $\mu_\lambda(U_\lambda) < \lambda$. Since solutions of equation (1.1) satisfying $\mu_\lambda(u) < \lambda$ are unique, we denote by U_λ the unique solution of equation (1.1) satisfying $\mu_\lambda(u) < \lambda$, and from that we know equation (4.1) has a unique solution v_λ for each $\lambda \in (0, \lambda^*)$. □

REFERENCES

[1] R.A. Adams, "Sobolev Space," Academic Press, New York, 1975.

[2] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces,* SIAM. Review, 18 (1976), 620–709.

[3] A. Ambrosetti and P.H. Rabinowitz, *Dual variational method in critical point theory and applications,* J. Funct. Anal., 14 (1973), 349–381.

[4] A. Bahri and P.L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains,* Ann. Inst. H. Poincare Anal. Non Lineaire, 14 (1997), 365–413.

[5] V. Benci and G. Cerami, *Positive solutions of some nonlinear elliptic problems in unbounded domains,* Arch. Rational Mech. Anal., 99 (1987), 283–300.

[6] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations, I, Existence of a ground state; II, Existence of infinitely many solutions,* Arch. Rational Mech. Anal., 82 (1983), 313–375.

[7] H. Brezis and T. Kato, *Remarks on the Schrodinger operator with singular complex potentials,* J. Math. Pures Appl., 58 (1979), 137–151.

[8] R. Dautray and J.L. Lions, *Analyse mathematique et calaul numerique pour les sciences et les techniques,* Vol. 1, Masson, Pairs, 1984.

[9] W.Y. Ding and W.M. Ni, On the existence of positive entire solution of a semilinear elliptic equation, Arch. Rational Mech. Anal., 91 (1986), 283–308.

[10] I. Ekeland, *Non-convex minimization problems,* Bull. Amer. Mat. Soc., 1 (1979), 443–474.

[11] M.J. Esteban and P.L. Lions, *Existence and nonexistence results for semilinear elliptic problems in unbounded domains,* Proc. Roy. Soc. Edin., 93 (1982), 1–14.

[12] P.L. Lions, *On positive solutions of semilinear elliptic equations in unbounded domains,* in "Nonlinear Diffusion Equations and Their Equilibrium States," eds. W.M. Ni, L.A. Peletier, and J. Serrin, Springer, New York/Berlin, 1988.

[13] W. Strauss, *Existence of solitary waves in higher dimensions,* Comm. Math. Phys., 55 (1977), 149–162.

[14] C.A. Stuart, *Bifurcation in $L^p(\mathbb{R}^N)$ for a semilinear elliptic equation,* Proc. London Math. Soc., 57 (1988), 511–541.

[15] X.P. Zhu, *A perturbation result on positive entire solutions of a semilinear elliptic equation,* J. Diff. Equ., 92 (1991), 163–178.

[16] X.P. Zhu and D.M. Cao, *The concentration-compactness principle in nonlinear elliptic equation,* Acta. Math. Sci., 9 (1989).

[17] X.P. Zhu and H.S. Zhou, *Existence of multiple positive solutions of inhomogeneous semilinear elliptic problems in unbounded domains,* Proc. Roy. Soc. Edin., 115A (1990), 301–318.