Differential and Integral Equations Volume 17, Numbers 3–4, March/April 2004, Pages 369–390

# MULTIPLICITY FOR A NONLINEAR FOURTH-ORDER ELLIPTIC EQUATION IN MAXWELL-CHERN-SIMONS VORTEX THEORY

Tonia Ricciardi
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II
Via Cintia, 80126 Naples, Italy

(Submitted by: Haim Brezis)

**Abstract.** We prove the existence of at least two solutions for a fourth-order equation, which includes the vortex equations for the $U(1)$ and $CP(1)$ self-dual Maxwell-Chern-Simons models as special cases. Our method is variational, and it relies on an "asymptotic maximum principle" property for a special class of supersolutions to this fourth-order equation.

## 0. Introduction

Vortex solutions for self-dual Maxwell-Chern-Simons models may be generally reduced to systems of two nonlinear elliptic equations of the second order, defined on two-dimensional Riemannian manifolds. See [14, 15] and the monographs [9, 11, 22]. These systems are also equivalent to scalar nonlinear elliptic equations of the fourth order. The existence of *multiple* solutions for such fourth-order equations, in the case of compact manifolds, is the main question addressed in this note.

We were motivated to consider this problem by our previous joint work with Tarantello [19] concerning the $U(1)$ Maxwell-Chern-Simons model introduced in [15], and by the results of Chae and Nam [6] concerning the $CP(1)$ Maxwell-Chern-Simons model introduced in [14]. By variational methods, it is shown in [19] that the $U(1)$ system admits in general at least two distinct vortex solutions. On the other hand, the method employed in [6] allows the authors to obtain only one vortex solution for the $CP(1)$ system.

Our main result (see Theorem 0.1 below) will yield multiple solutions for a *general* system containing the $U(1)$ system and the $CP(1)$ system as special

Accepted for publication: June 2003.
AMS Subject Classifications: 35J60.
Partially supported by MIUR, National Project "Variational Methods and Nonlinear Differential Equations".

cases. As in the $U(1)$ case, we shall reduce the system to an elliptic fourth-order equation admitting a convenient variational structure. We shall obtain two solutions corresponding to a local minimum and a mountain pass. Due to our abstract formulation, we cannot use the *ad hoc* methods employed in [19], based on minimization with integral constraints. Instead, we shall exploit an "asymptotic maximum principle" property for a special class of supersolutions of this fourth-order equation, which we believe is of interest on its own.

More precisely, we denote by $M$ a compact Riemannian two-manifold, and we fix $n > 0$ points $p_1, \ldots, p_n \in M$ (we already showed in [18] that the case $n = 0$ admits only the trivial solution $(e^{\widetilde{u}}, v) = (f^{-1}(s), s)$). We consider (distributional) solutions $(\widetilde{u}, v)$ for the following system:

$$-\Delta\widetilde{u} = \varepsilon^{-1}\lambda(v - f(e^{\widetilde{u}})) - 4\pi\sum_{j=1}^{n}\delta_{p_j} \qquad \text{on } M \tag{0.1}$$

$$-\Delta v = \varepsilon^{-1}\left[\lambda f'(e^{\widetilde{u}})e^{\widetilde{u}}(s - v) - \varepsilon^{-1}(v - f(e^{\widetilde{u}}))\right] \qquad \text{on } M. \tag{0.2}$$

Here $s \in \mathbb{R}$, $\varepsilon > 0$, and $\lambda > 0$ are constants, $\delta_{p_j}$, $j = 1, \ldots, n$ is the Dirac measure centered at $p_j$, and $f : [0, +\infty) \to \mathbb{R}$ is smooth and *strictly increasing*, i.e., $f'(t) > 0$ for all $t \in [0, +\infty)$. Some further technical assumptions on $f$ will be made below. System (0.1)–(0.2) was introduced in [18]. It contains the system for $U(1)$ Maxwell-Chern-Simons vortices introduced in [15] and analyzed in [4, 17, 19], as well as the system for $CP(1)$ Maxwell-Chern-Simons vortices (in the "single-signed case") introduced in [14] and analyzed in [6] as special cases. Indeed, the $U(1)$ system [15] is given by

$$\Delta\widetilde{u} = 2q^2 e^{\widetilde{u}} - 2\mu N + 4\pi\sum_{j=1}^{n}\delta_{p_j} \qquad \text{on } M$$

$$\Delta N = (\mu^2 + 2q^2 e^{\widetilde{u}})N - q^2(\mu + \frac{2q^2}{\mu})e^{\widetilde{u}} \qquad \text{on } M.$$

Setting $\lambda = 2q^2/\mu$, $\varepsilon = 1/\mu$, and $v := \mu/q^2\, N$, the above system takes the form

$$-\Delta\widetilde{u} = \varepsilon^{-1}\lambda(v - e^{\widetilde{u}}) - 4\pi\sum_{j=1}^{n}\delta_{p_j} \qquad \text{on } M \tag{0.3}$$

$$-\Delta v = \varepsilon^{-1}\{\lambda e^{\widetilde{u}}(1 - v) - \varepsilon^{-1}(v - e^{\widetilde{u}})\} \qquad \text{on } M, \tag{0.4}$$

which corresponds to (0.1)–(0.2) with $f(t) = t$ and $s = 1$. On the other hand, the $CP(1)$ system [14] is given by

$$\Delta\widetilde{u} = 2q\Big( - N + S - \frac{1 - e^{\widetilde{u}}}{1 + e^{\widetilde{u}}}\Big) + 4\pi \sum_{j=1}^{n} \delta_{p_j} \qquad \text{on } M$$

$$\Delta N = -\kappa^2 q^2\Big( - N + S - \frac{1 - e^{\widetilde{u}}}{1 + e^{\widetilde{u}}}\Big) + q\frac{4e^{\widetilde{u}}}{(1 + e^{\widetilde{u}})^2}N \qquad \text{on } M.$$

Setting $v = N - S$, $s = -S$, $\lambda = 2/\kappa$, and $\varepsilon = 1/(\kappa q)$, the system above takes the form

$$-\Delta\widetilde{u} = \varepsilon^{-1}\lambda\Big(v - \frac{e^{\widetilde{u}} - 1}{e^{\widetilde{u}} + 1}\Big) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \qquad \text{on } M \tag{0.5}$$

$$-\Delta v = \varepsilon^{-1}\Big\{\lambda\frac{2e^{\widetilde{u}}}{(1 + e^{\widetilde{u}})^2}(s - v) - \varepsilon^{-1}\Big(v - \frac{e^{\widetilde{u}} - 1}{e^{\widetilde{u}} + 1}\Big)\Big\} \qquad \text{on } M, \tag{0.6}$$

which corresponds to (0.1)–(0.2) with $f(t) = (t - 1)/(t + 1)$.

In turn, system (0.1)–(0.2) is equivalent to the following *fourth-order* equation (see Section 1 for the details):

$$\begin{aligned}\varepsilon^2\Delta^2 u - \Delta u = {} & -\varepsilon\lambda[f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})]e^{\sigma+u}|\nabla(\sigma + u)|^2 \\ & + 2\varepsilon\lambda\Delta f(e^{\sigma+u}) + \lambda^2 f'(e^{\sigma+u})e^{\sigma+u}(s - f(e^{\sigma+u})) - \frac{4\pi n}{|M|} \qquad \text{on } M,\end{aligned} \tag{0.7}$$

where $\sigma$ is the Green's function uniquely defined by $-\Delta\sigma = 4\pi(n/|M| - \sum_{j=1}^{n} \delta_{p_j})$ on $M$ and $\int_M \sigma = 0$, with $|M|$ the volume of $M$.

We make the following

**Assumptions on $f$:**

- $(f0)$ $f : [0, +\infty)$ is smooth and $f'(t) > 0$ for all $t \in [0, +\infty)$
- $(f1)$ $f(0) < s < \sup_{t>0} f(t)$
- $(f2)$ $f$, $f'$, and $f''$ have at most polynomial growth
- $(f3)$ $f$ satisfies one of the following conditions:
  - (a) $f''(t)t + f'(t) \geq 0$ and $\sup_{t>0} |f(t)|/[f'(t)t] < +\infty$
  - (b) $\sup_{t>0} f'(t)t(|\log t| + |f(t)|) < +\infty$.

The aim of this note is to establish the following result for (0.7):

**Theorem 0.1.** *Suppose $f$ satisfies assumptions $(f0)$, $(f1)$, $(f2)$, and $(f3)$. Then there exists $\lambda_0 > 0$ with the property that for every $\lambda \geq \lambda_0$ there exists $\varepsilon_\lambda > 0$ such that the fourth-order equation (0.7) admits at least* two *solutions for all $0 < \varepsilon < \varepsilon_\lambda$.*

We note that assumption $(f3)$-(a) allows $f(t) = t^\alpha$, for every $\alpha > 0$, and therefore it includes the $U(1)$ case $f(t) = t$. On the other hand, assumption $(f3)$-(b) is satisfied by the $CP(1)$ case $f(t) = (t-1)/(t+1)$. It follows that the existence result stated in Theorem 0.1 includes indeed the $U(1)$ system and the $CP(1)$ system as special cases, as well as all power growths for $f$.

As already mentioned, we shall prove Theorem 0.1 variationally. Indeed, in Section 1 we show that solutions to (0.7) correspond to critical points for the functional

$$I_\varepsilon(u) = \frac{\varepsilon^2}{2}\int(\Delta u)^2 + \frac{1}{2}\int|\nabla u|^2 \\ + \varepsilon\lambda\int f'(e^{\sigma+u})e^{\sigma+u}|\nabla(\sigma+u)|^2 + \frac{\lambda^2}{2}\int(f(e^{\sigma+u}) - s)^2 + \frac{4\pi n}{|M|}\int u,$$

defined on the Sobolev space $H^2(M)$ (we choose to emphasize the dependence on $\varepsilon$ only, since $\lambda$ will be fixed in Section 2). In Section 2 we show the existence of a subsolution $\underline{u}_\varepsilon$, for $\varepsilon$ and $\lambda$ as in Theorem 0.1. This subsolution will be employed in Section 3 to obtain a critical point $u_\varepsilon$ satisfying $I_\varepsilon(u_\varepsilon) = \min\{I_\varepsilon(u) : u \in H^2(M), u \geq \underline{u}_\varepsilon\}$, in the spirit of some results of Brezis and Nirenberg [3] concerning second-order equations. Since our equation is of the fourth order, and thus the Hopf maximum principle is not directly applicable, the main technical difficulty will be to show that $u_\varepsilon > \underline{u}_\varepsilon$, pointwise on $M$. Nevertheless, by exploiting the decomposition $\varepsilon^2\Delta^2 - \Delta = (-\varepsilon^2\Delta + 1)(-\Delta)$, we shall derive a kind of "strong maximum principle" property for a special class of supersolutions of (0.7), for small values of $\varepsilon$. Finally, in Section 4 we show that under the assumptions of Theorem 0.1, the functional $I_\varepsilon$ satisfies the Palais-Smale condition. Therefore, the existence of a second critical point of the "mountain-pass" type will follow by the Ambrosetti and Rabinowitz theorem [1].

We showed in our previous note [18] that under assumptions $(f0)$ and $(f1)$, solutions for system (1.3)–(1.4) (equivalently solutions to (0.7)) tend to a solution for

$$-\Delta u = \lambda^2 f'(e^{\sigma+u})e^{\sigma+u}(s - f(e^{\sigma+u})) - \frac{4\pi n}{|M|} \qquad \text{on } M, \tag{0.8}$$

as $\varepsilon \to 0$, in any relevant topology. Equation (0.8) is a generalization of the $U(1)$ "pure" Chern-Simons equation derived in [10, 12] and thoroughly analyzed in [7, 21, 16, 8] in the case of compact manifolds (see references therein for the noncompact case), and of the $CP(1)$ "pure" Chern-Simons equation derived in [13] and analyzed in [5]. We note that solutions for (0.8) correspond to critical points for the functional $I_0$ defined for $u \in H^1(M)$ by

$$I_0(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{\lambda^2}{2}\int (f(e^{\sigma+u}) - s)^2 + \frac{4\pi n}{|M|}\int u.$$

A multiplicity result may be obtained for (0.8) variationally, which implies that the multiplicity results for the particular "pure" Chern-Simons equations obtained in [5, 21] are in fact a *general property* of equations of the form (0.8). Since the proof may be obtained from the proof of Theorem 0.1 by setting $\varepsilon = 0$, we omit the details.

**Notation.** Henceforth, unless otherwise specified, all equations are defined on $M$. All integrals are taken over $M$ with respect to Lebesgue measure. All functional spaces are defined on $M$ in the usual way. In particular, we denote by $L^p$, $1 \le p \le +\infty$, the Lebesgue spaces and by $H^k$, $1 \le k \le +\infty$, the Sobolev spaces. We denote by $C > 0$ a general constant, independent of certain parameters that will be specified in the sequel, and whose actual value may vary from line to line.

## 1. Variational setting

Our aim in this section is to provide a suitable variational formulation for the generalized Maxwell-Chern-Simons system (0.1)–(0.2), by reducing it to the fourth-order equation (0.7). We note that the resulting formulation is new even for the special case $f(t) = (t-1)/(t+1)$, corresponding to the $CP(1)$ model.

In order to work in Sobolev spaces, it is standard (see [11, 22]) to subtract from $\widetilde{u}$ its "singular part," which we denote by $\sigma$. Namely, we denote by $\sigma$ the unique solution for the problem

$$-\Delta\sigma = A - 4\pi\sum_{j=1}^{n}\delta_{p_j}, \qquad \int \sigma = 0,$$

where $A = 4\pi n/|M| > 0$. Since $\sigma(x) \approx \log|x - p_j|^2$ near $p_j$, $j = 1, \dots, n$, it follows that $e^\sigma$ and $e^\sigma\nabla\sigma = \nabla e^\sigma$ are *smooth* on $M$. Furthermore, $e^\sigma|\nabla\sigma|^2$ and $e^\sigma\Delta\sigma$ are also smooth. Indeed, it is easy to check that

$$e^\sigma\Delta\sigma = -Ae^\sigma \tag{1.1}$$

$$e^\sigma|\nabla\sigma|^2 = \Delta e^\sigma + Ae^\sigma, \tag{1.2}$$

in the sense of distributions.

Setting $\widetilde{u} = \sigma + u$, we obtain from (0.1)–(0.2) the equivalent system for $(u, v) \in H^1(M) \times H^1(M)$:

$$-\Delta u = \varepsilon^{-1}\lambda\left(v - f(e^{\sigma+u})\right) - A \tag{1.3}$$

$$-\Delta v = \varepsilon^{-1}\left[\lambda f'(e^{\sigma+u})e^{\sigma+u}(s-v) - \varepsilon^{-1}\left(v - f(e^{\sigma+u})\right)\right]. \tag{1.4}$$

System (1.3)–(1.4) is equivalent to a fourth-order equation:

**Lemma 1.1.** *$(u,v)\in H^1\times H^1$ is a weak solution for (1.3)–(1.4) if and only $u\in H^2$ is a weak solution for the fourth-order equation*

$$\begin{aligned}\varepsilon^2\Delta^2 u - \Delta u = &-\varepsilon\lambda[f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})]e^{\sigma+u}|\nabla(\sigma+u)|^2\\ &+ 2\varepsilon\lambda\Delta f(e^{\sigma+u}) + \lambda^2 f'(e^{\sigma+u})e^{\sigma+u}(s - f(e^{\sigma+u})) - A\end{aligned} \tag{1.5}$$

*and $v$ is defined by*

$$v = -\varepsilon\lambda^{-1}\Delta u + \varepsilon\lambda^{-1}A + f(e^{\sigma+u}). \tag{1.6}$$

**Proof.** By elliptic regularity, weak solutions $(u,v)\in H^1\times H^1$ for (1.3)–(1.4) are smooth. Clearly, (1.6) is equivalent to (1.3). Inserting (1.6) into (1.4), we obtain

$$\begin{aligned}\varepsilon\lambda^{-1}\Delta^2 u - \Delta f(e^{\sigma+u}) = &\,\varepsilon^{-1}\lambda f'(e^{\sigma+u})e^{\sigma+u}(s + \varepsilon\lambda^{-1}\Delta u - \varepsilon\lambda^{-1}A - f(e^{\sigma+u}))\\ &- \varepsilon^{-2}(-\varepsilon\lambda^{-1}\Delta u + \varepsilon\lambda^{-1}A).\end{aligned}$$

Equivalently, multiplying by $\varepsilon\lambda$,

$$\begin{aligned}\varepsilon^2\Delta^2 u - \Delta u = &\,\varepsilon\lambda\Delta f(e^{\sigma+u}) + \varepsilon\lambda f'(e^{\sigma+u})e^{\sigma+u}(\Delta u - A)\\ &+ \lambda^2 f'(e^{\sigma+u})e^{\sigma+u}(s - f(e^{\sigma+u})) - A.\end{aligned} \tag{1.7}$$

By (1.1) we have

$$f'(e^{\sigma+u})e^{\sigma+u}(\Delta u - A) = f'(e^{\sigma+u})e^{\sigma+u}\Delta(\sigma+u).$$

Furthermore, we have

$$\begin{aligned}&\Delta f(e^{\sigma+u})\\ &= \{f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})\}e^{\sigma+u}|\nabla(\sigma+u)|^2 + f'(e^{\sigma+u})e^{\sigma+u}\Delta(\sigma+u).\end{aligned} \tag{1.8}$$

Therefore,

$$\begin{aligned}\Delta f(e^{\sigma+u}) + f'(e^{\sigma+u})e^{\sigma+u}(\Delta u - A) &= \Delta f(e^{\sigma+u}) + f'(e^{\sigma+u})e^{\sigma+u}\Delta(\sigma+u)\\ &= 2\Delta f(e^{\sigma+u}) - \{f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})\}e^{\sigma+u}|\nabla(\sigma+u)|^2,\end{aligned}$$

and (1.7) reduces to (1.5). The converse follows similarly. □

Now we obtain a variational formulation for (1.5):

**Lemma 1.2.** *$u\in H^2$ is a weak solution for (1.5) if and only if it is a critical point for the functional $I_\varepsilon$ defined on $H^2$ by*

$$I_\varepsilon(u) = \frac{\varepsilon^2}{2}\int(\Delta u)^2 + \frac{1}{2}\int|\nabla u|^2$$

$$+\varepsilon\lambda\int f'(e^{\sigma+u})e^{\sigma+u}|\nabla(\sigma+u)|^2+\frac{\lambda^2}{2}\int(f(e^{\sigma+u})-s)^2+A\int u.$$

**Proof.** First of all, we note that $I_\varepsilon$ is well-defined on $H^2$ and smooth. Indeed, if $u\in H^2$, then by Sobolev embeddings we have $|\nabla u|\in L^p$ for all $p\geq 1$ and $u\in L^\infty$. We can write, in view of (1.2),

$$\begin{aligned}\int f'(e^{\sigma+u})e^{\sigma+u}|\nabla(\sigma+u)|^2 & \qquad (1.9)\\ &=\int f'(e^{\sigma+u})e^{\sigma+u}(|\nabla\sigma|^2+2\nabla\sigma\cdot\nabla u+|\nabla u|^2)\\ &=\int f'(e^{\sigma+u})e^{u}(\Delta e^\sigma+Ae^\sigma)+2\int f'(e^{\sigma+u})e^u\nabla e^\sigma\cdot\nabla u\\ &\quad+\int f'(e^{\sigma+u})e^{\sigma+u}|\nabla u|^2,\end{aligned}$$

and therefore the third term in $I_\varepsilon$ is well-defined on $H^2$. Clearly, all other terms in $I_\varepsilon$ are well-defined, and thus $I_\varepsilon$ is well-defined on $H^2$. Smoothness of $I_\varepsilon$ is checked similarly. We check that solutions in $H^2$ to (1.5) correspond to critical points for $I_\varepsilon$. We compute, for any $\phi\in H^2$,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\int f'(e^{\sigma+u+t\phi})e^{\sigma+u+t\phi}|\nabla(\sigma+u+t\phi)|^2 &\\ =\int\left[f''(e^{\sigma+u})e^{\sigma+u}+f'(e^{\sigma+u})\right]e^{\sigma+u}|\nabla(\sigma+u)|^2\phi &\\ +2\int f'(e^{\sigma+u})e^{\sigma+u}\nabla(\sigma+u)\cdot\nabla\phi, &\end{aligned}$$

and therefore

$$\begin{aligned}\langle I'_\varepsilon(u),\phi\rangle=\varepsilon^2\int\Delta u\Delta\phi+\int\nabla u\cdot\nabla\phi & \qquad (1.10)\\ +\varepsilon\lambda\int\left[f''(e^{\sigma+u})e^{\sigma+u}+f'(e^{\sigma+u})\right]e^{\sigma+u}|\nabla(\sigma+u)|^2\phi &\\ +2\varepsilon\lambda\int f'(e^{\sigma+u})e^{\sigma+u}\nabla(\sigma+u)\cdot\nabla\phi &\\ +\lambda^2\int f'(e^{\sigma+u})e^{\sigma+u}(f(e^{\sigma+u})-s)\phi+A\int\phi. &\end{aligned}$$

Since

$$\int f'(e^{\sigma+u})e^{\sigma+u}\nabla(\sigma+u)\cdot\nabla\phi=\int\nabla f(e^{\sigma+u})\cdot\nabla\phi=-\int\Delta f(e^{\sigma+u})\phi,$$

it follows that critical points of $I_\varepsilon$ correspond to solutions to (1.5), as asserted. □

We say that $u \in H^2$ is a subsolution (supersolution) for (1.5) if $u$ satisfies $\langle I'_\varepsilon(u), \phi\rangle \le 0$ ($\langle I'_\varepsilon(u), \phi\rangle \ge 0$), for all $\phi \in H^2$ such that $\phi \ge 0$ on $M$.

## 2. Existence of a subsolution

In this section we show that for suitable values of $\lambda$ and $\varepsilon$, the fourth-order equation (1.5) admits a subsolution. Namely, we show

**Proposition 2.1.** *Suppose $f$ satisfies $(f0)$ and $(f1)$. Then there exists $\lambda_0 > 0$ such that for every fixed $\lambda \ge \lambda_0$ there exists $\varepsilon_\lambda > 0$ with the property that for every $0 < \varepsilon < \varepsilon_\lambda$ equation (1.5) admits a subsolution $\underline{u}_\varepsilon$. Furthermore, $\underline{u}_\varepsilon \to \underline{u}_0$ in $H^2$ as $\varepsilon \to 0$, with $\underline{u}_0$ a subsolution for (0.8).*

We begin by proving some properties of the Green's function $G_\varepsilon$ for the operator $-\varepsilon^2\Delta + 1$:

**Lemma 2.1.** *Let $G_\varepsilon = G_\varepsilon(x,y)$ be the Green's function defined by*

$$(-\varepsilon^2\Delta_x + 1)G_\varepsilon(x,y) = \delta_y \qquad \text{on } M.$$

*Then*

(i) *$G_\varepsilon > 0$ on $M \times M$, and for every fixed $y \in M$ we have $G_\varepsilon \rightharpoonup \delta_y$ as $\varepsilon \to 0$, weakly in the sense of measures;*
(ii) *$\|G_\varepsilon * h\|_q \le \|h\|_q$ for all $1 \le q \le +\infty$;*
(iii) *If $\Delta h \in L^q$ for some $1 < q < +\infty$, then $\|G_\varepsilon * h - h\|_q \le \varepsilon^2\|\Delta h\|_q$.*

**Proof.** Proof of (i). Note that since $-\varepsilon^2\Delta + 1$ is coercive, $G_\varepsilon$ is well defined (e.g., by Stampacchia's duality argument [20]). By the maximum principle, $G_\varepsilon > 0$ on $M \times M$. Integrating over $M$ with respect to $x$, we have

$$\int G_\varepsilon(x,y)\,\mathrm{d}x = \int |G_\varepsilon(x,y)|\,\mathrm{d}x = 1,$$

and therefore there exists a Radon measure $\mu$ such that $G_\varepsilon(\cdot,y) \rightharpoonup \mu$ as $\varepsilon \to 0$, weakly in the sense of measures. For $\varphi \in C^\infty$ we compute

$$\varphi(y) = \varepsilon^2 \int G_\varepsilon(x,y)(-\Delta\varphi)(x)\,\mathrm{d}x + \int G(x,y)\varphi(x)\,\mathrm{d}x \to \int \varphi\,\mathrm{d}\mu$$

as $\varepsilon \to 0$. By density of $C^\infty$ in $C$, we conclude that $\mu = \delta_y$. Proof of (ii). For $q = 1$, we have

$$\|G_\varepsilon * h\|_1 = \int |(G_\varepsilon * h)(x)|\,\mathrm{d}x \le \int \mathrm{d}y|h(y)| \int G_\varepsilon(x,y)\,\mathrm{d}x = \int |h| = \|h\|_1.$$

For $q = \infty$ we have, for any $x \in M$,

$$|G_\varepsilon * h(x)| \leq \|h\|_\infty \int G_\varepsilon(x,y)\,\mathrm{d}y = \|h\|_\infty \int G_\varepsilon(x,y)\,\mathrm{d}x = \|h\|_\infty,$$

and therefore $\|G_\varepsilon * h\|_\infty \leq \|h\|_\infty$. The general case follows by interpolation. Proof of (iii). Let $U_\varepsilon = G_\varepsilon * h$. Then we can write

$$-\varepsilon^2\Delta(U_\varepsilon - h) + (U_\varepsilon - h) = \varepsilon^2\Delta h.$$

Multiplying by $|U_\varepsilon - h|^{q-2}(U_\varepsilon - h)$ and integrating, we obtain

$$\varepsilon^2(q-1)\int |U_\varepsilon - h|^{q-2}|\nabla(U_\varepsilon - h)|^2 + \int |U_\varepsilon - h|^q = \varepsilon^2 \int \Delta h |U_\varepsilon - h|^{q-2}(U_\varepsilon - h).$$

By positivity of the first term above and Hölder's inequality,

$$\int |U_\varepsilon - h|^q \leq \varepsilon^2 \int |\Delta h||U_\varepsilon - h|^{q-1} \leq \varepsilon^2 \|\Delta h\|_q \|U_\varepsilon - h\|_q^{q-1}.$$

Hence $\|U_\varepsilon - h\|_q \leq \varepsilon^2\|\Delta h\|_q$, and (iii) follows recalling the definition of $U_\varepsilon$. □

Now we can prove Proposition 2.1.

**Proof of Proposition 2.1.** Equation (1.5) is of the form

$$\varepsilon^2\Delta^2 u - \Delta u = \varepsilon\lambda\, a(u) + \lambda^2 f'(e^{\sigma+u})e^{\sigma+u}(s - f(e^{\sigma+u})) - A, \tag{2.1}$$

where $a$ is the operator defined by

$$a(u) := -[f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})]e^{\sigma+u}|\nabla(\sigma+u)|^2 + 2\Delta f(e^{\sigma+u}). \tag{2.2}$$

By (1.8), we can also write

$$a(u) = [f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})]e^{\sigma+u}|\nabla(\sigma+u)|^2 + 2f'(e^{\sigma+u})e^{\sigma+u}\Delta(\sigma+u), \tag{2.3}$$

and therefore, recalling (1.1) and (1.2) we can estimate

$$\|a(u)\|_\infty \leq \Phi(\|\Delta u\|_\infty + \|u\|_{C^1}), \tag{2.4}$$

for some continuous function $\Phi : [0, +\infty) \to \mathbb{R}$. We denote by $\varphi$ a smooth function defined on $M$ with the following properties:

$$\varphi = \begin{cases} -A - 1 & \text{in } \cup_{j=1}^n B_\delta(p_j) \\ \varphi_0 & \text{in } M \setminus \cup_{j=1}^n B_{2\delta}(p_j) \\ -A - 1 \leq \varphi \leq \varphi_0 & \text{on } M \\ \int \varphi = 0 \end{cases},$$

where $\varphi_0$ is a suitable constant, and $\delta > 0$ is sufficiently small so that $B_{2\delta}(p_j) \cap B_{2\delta}(p_k) = \emptyset$ for all $j, k = 1, \ldots, n$ with $j \neq k$. We denote by $\widetilde{u}_\varepsilon$ the unique solution for the problem

$$\begin{aligned} &\varepsilon^2 \Delta^2 \widetilde{u}_\varepsilon - \Delta \widetilde{u}_\varepsilon = \varphi \qquad \text{on } M \\ &\int \widetilde{u}_\varepsilon = 0. \end{aligned} \tag{2.5}$$

Note that $\widetilde{u}_\varepsilon$ is well-defined. Indeed, since $\int \varphi = 0$, we have $\int G_\varepsilon * \varphi = 0$. Therefore there exists a unique solution $\widetilde{u}_\varepsilon$ for the problem $-\Delta \widetilde{u}_\varepsilon = G_\varepsilon * \varphi$ satisfying $\int \widetilde{u}_\varepsilon = 0$. Writing $\varepsilon^2\Delta^2 - \Delta = (-\varepsilon^2\Delta + 1)(-\Delta)$, we see that $\widetilde{u}_\varepsilon$ is the desired unique solution for (2.5). By Lemma 2.1-(ii) we have $\|\Delta \widetilde{u}_\varepsilon\|_\infty = \|G_\varepsilon * \varphi\|_\infty \leq \|\varphi\|_\infty$. By elliptic regularity we have in turn

$$\|\widetilde{u}_\varepsilon\|_{C^{1,\alpha}} \leq C_1 \|\varphi\|_\infty. \tag{2.6}$$

We set $\underline{u}_\varepsilon := \widetilde{u}_\varepsilon - k$, where $k$ is defined by

$$e^k = \frac{e^{\max_M \sigma + C_1 \|\varphi\|_\infty}}{f^{-1}(\frac{s+f(0)}{2})},$$

and where $C_1$ is the constant in (2.6). In view of $(f0)$ and $(f1)$, such a choice of $k$ implies that $s - f(e^{\sigma + \underline{u}_\varepsilon}) \geq (s - f(0))/2 > 0$. Indeed, since $f$ is strictly increasing, we have

$$\begin{aligned} f(e^{\sigma+\underline{u}_\varepsilon}) = f(e^{\sigma+\widetilde{u}_\varepsilon - k}) \leq f(e^{\max_M \sigma + \|\widetilde{u}_\varepsilon\|_\infty - k}) \leq& f(e^{\max_M \sigma + C_1\|\varphi\|_\infty - k}) \\ =& \frac{s+f(0)}{2}. \end{aligned}$$

Now we check that for $\lambda \geq \lambda_0$ and $\varepsilon \leq \varepsilon_\lambda$, for suitable $\lambda_0$ and $\varepsilon_\lambda$, the function $\underline{u}_\varepsilon$ is indeed a subsolution for (1.5).

**Claim:** There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ and for all $0 < \varepsilon < 1$, $\underline{u}_\varepsilon$ is a subsolution for (1.5) in $M \setminus \cup_{j=1}^n B_\delta(p_j)$. Namely,

$$\varphi \leq \varepsilon\lambda\, a(\underline{u}_\varepsilon) + \lambda^2 f'(e^{\sigma+\underline{u}_\varepsilon}) e^{\sigma+\underline{u}_\varepsilon}(s - f(e^{\sigma+\underline{u}_\varepsilon})) - A \quad \text{in } M \setminus \cup_{j=1}^n B_\delta(p_j). \tag{2.7}$$

Proof of (2.7). By (2.4) and (2.6) there exists a constant $C_0$ such that $\|a(\underline{u}_\varepsilon)\|_\infty + \|\underline{u}_\varepsilon\|_\infty + \max_M \sigma \leq C_0$. Let $c_0 = \min\{f'(t) : t \in [0, C_0]\} > 0$ and $\mu_0 = \min_{M \setminus \cup_{j=1}^n B_\delta(p_j)} \sigma$. It suffices to check that

$$\varphi_0 \leq -\lambda C_0 + \lambda^2 c_0 e^{\mu_0 - C_0} \frac{s - f(0)}{2} - A.$$

The above inequality is clearly achieved for all $\lambda \geq \lambda_0$, for sufficiently large $\lambda_0$. Hence, (2.7) is established.

Now we fix $\lambda \geq \lambda_0$.

**Claim:** For every fixed $\lambda \geq \lambda_0$, there exists $\varepsilon_\lambda > 0$ such that $\underline{u}_\varepsilon$ is a subsolution for (1.5) in $\cup_{j=1}^n B_\delta(p_j)$, for all $0 < \varepsilon < \varepsilon_\lambda$. Namely,

$$\varphi \leq \varepsilon\lambda\, a(\underline{u}_\varepsilon) + \lambda^2 f'(e^{\sigma+\underline{u}_\varepsilon})e^{\sigma+\underline{u}_\varepsilon}(s - f(e^{\sigma+\underline{u}_\varepsilon})) - A \quad \text{in } \cup_{j=1}^n B_\delta(p_j). \quad (2.8)$$

Proof of (2.8). It suffices to prove the following condition:

$$-A - 1 \leq -\varepsilon\lambda C_0 - A,$$

which is clearly satisfied for all $0 < \varepsilon \leq \varepsilon_\lambda$, with $\varepsilon_\lambda > 0$ such that $\varepsilon_\lambda \lambda C_0 \leq 1$. Hence, (2.8) is also established. Consequently, for $\lambda \geq \lambda_0$ and for $0 < \varepsilon \leq \varepsilon_\lambda$, $\underline{u}_\varepsilon$ is a subsolution for (1.5), as asserted.

We are left to analyze the asymptotic behavior of $\underline{u}_\varepsilon$ as $\varepsilon \to 0$. We denote by $\widetilde{u}_0$ the unique solution for $-\Delta\widetilde{u}_0 = \varphi$ satisfying $\int \widetilde{u}_0 = 0$. Then $\widetilde{u}_\varepsilon - \widetilde{u}_0$ satisfies

$$-\Delta(\widetilde{u}_\varepsilon - \widetilde{u}_0) = G_\varepsilon * \varphi - \varphi, \qquad \int(\widetilde{u}_\varepsilon - \widetilde{u}_0) = 0.$$

By Lemma 2.1-(iii) we have $\|G_\varepsilon * \varphi - \varphi\|_2 \leq \varepsilon^2\|\Delta\varphi\|_2 \to 0$, as $\varepsilon \to 0$. Therefore, $\|\widetilde{u}_\varepsilon - \widetilde{u}_0\|_{H^2} \leq C\varepsilon^2 \to 0$. It is simple to check that $\underline{u}_0 := \widetilde{u}_0 - k$ is a subsolution for (0.8). Clearly, $\|\underline{u}_\varepsilon - \underline{u}_0\|_{H^2} = \|\widetilde{u}_\varepsilon - \widetilde{u}_0\|_{H^2} \to 0$. □

Henceforth, $\lambda$ denotes a *fixed* constant satisfying $\lambda \geq \lambda_0$.

## 3. Existence of a local minimum

We take $\lambda_0$ and $\varepsilon_\lambda$ as in Proposition 2.1. In this section we show

**Proposition 3.1.** *Suppose $f$ satisfies* $(f0)$, $(f1)$, *and* $(f2)$. *Then* (*possibly taking a smaller* $\varepsilon_\lambda$), *for every fixed* $\lambda \geq \lambda_0$ *and for every* $0 < \varepsilon < \varepsilon_\lambda$ *there exists a solution* $u_\varepsilon$ *for* (1.5), *corresponding to a local minimum for* $I_\varepsilon$.

We define $\mathcal{A}_\varepsilon := \{u \in H^2 : u \geq \underline{u}_\varepsilon\}$. $\mathcal{A}_\varepsilon$ is a closed, convex subset of $H^2$, and therefore it is weakly closed. It is readily checked that $I_\varepsilon$ attains its minimum on $\mathcal{A}_\varepsilon$; i.e., there exists $u_\varepsilon$ such that

$$I_\varepsilon(u_\varepsilon) = \min_{\mathcal{A}_\varepsilon} I_\varepsilon.$$

The remaining part of this section is devoted to showing that $u_\varepsilon$ is a solution for (1.5) corresponding to a local minimum for $I_\varepsilon$. The main issue is to show that $u_\varepsilon$ belongs to the *interior* of $\mathcal{A}_\varepsilon$ (in the sense of $H^2$), and thus it is a critical point for $I_\varepsilon$. It is readily checked that $u_\varepsilon$ is a supersolution for (1.5). Indeed, for all $\phi \in H^2(M)$ such that $\phi \geq 0$ and for all $t > 0$ we have $u_\varepsilon + t\phi \in \mathcal{A}_\varepsilon$; therefore,

$$\frac{I_\varepsilon(u_\varepsilon + t\phi) - I_\varepsilon(u_\varepsilon)}{t} \geq 0.$$

Consequently, taking into account (1.10), we obtain

$$\begin{aligned} 0 \le \langle I'_\varepsilon(u_\varepsilon), \phi\rangle &= \varepsilon^2 \int \Delta u_\varepsilon \Delta\phi + \int \nabla u_\varepsilon \cdot \nabla \phi \qquad (3.1)\\ &+ \varepsilon\lambda \int \left[f''(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon} + f'(e^{\sigma+u_\varepsilon})\right] e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2\phi \\ &+ 2\varepsilon\lambda \int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}\nabla(\sigma+u_\varepsilon)\cdot\nabla\phi \\ &+ \lambda^2 \int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}(f(e^{\sigma+u_\varepsilon}) - s)\phi + A\int \phi, \end{aligned}$$

for all $\phi \in H^2$, $\phi \ge 0$. Hence, $u_\varepsilon$ is a supersolution for (1.5). We define $\mathcal{A}_0 = \{u \in H^1 : u \ge \underline{u}_0 \text{ almost everywhere}\}$. (Note that $\mathcal{A}_0$ is a subset of $H^1$, while $\mathcal{A}_\varepsilon$ is a subset of $H^2$.) The next lemma provides estimates for $u_\varepsilon$, independent of $\varepsilon \to 0$. Throughout this section, we denote by $C > 0$ a general constant independent of $\varepsilon$. Recall that $I_0$ is the functional defined at the end of Section 1.

**Lemma 3.1.** *There exists a solution $u_0 \in H^1$ for (0.8) such that $u_\varepsilon \to u_0$ strongly in $H^1$. Furthermore,*

(i) $\lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) = \inf_{\mathcal{A}_0} I_0$
(ii) $\lim_{\varepsilon\to 0} \varepsilon\|\Delta u_\varepsilon\|_2 = 0$
(iii) $\lim_{\varepsilon\to 0} \varepsilon \int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2 = 0.$

**Proof.** Since $I_\varepsilon(u_\varepsilon) \le I_\varepsilon(\underline{u}_\varepsilon) \le C$ and since $\int u_\varepsilon \ge \int \underline{u}_\varepsilon \ge -C$, we readily have the following estimates:

$$\begin{aligned} &\varepsilon\|\Delta u_\varepsilon\|_2 + \|\nabla u_\varepsilon\|_2 \le C \qquad (3.2)\\ &\varepsilon \int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2 \le C, \qquad \left|\int u_\varepsilon\right| \le C. \end{aligned}$$

In particular, $\|u_\varepsilon\|_{H^1} \le C$, and therefore we may assume that for some $u_0 \in H^1$ we have $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1$, strongly in $L^p$ for all $1 \le p < +\infty$ and almost everywhere.

Proof of (i). We can write

$$I_\varepsilon(u) = \frac{\varepsilon^2}{2}\|\Delta u\|_2^2 + \varepsilon\lambda \int f'(e^{\sigma+u})e^{\sigma+u}|\nabla(\sigma+u)|^2 + I_0(u)$$

for all $u \in H^2$. In particular, $I_\varepsilon(u) \ge I_0(u)$ for all $u \in H^2$. Since $\underline{u}_\varepsilon \to \underline{u}_0$ in $H^2$, we have

$$\inf_{\mathcal{A}_0} I_0 = \inf_{\mathcal{A}_\varepsilon} I_0 + \circ_\varepsilon(1); \qquad (3.3)$$

hence,

$$I_\varepsilon(u_\varepsilon) = \inf_{\mathcal{A}_\varepsilon} I_\varepsilon \geq \inf_{\mathcal{A}_\varepsilon} I_0 = \inf_{\mathcal{A}_0} I_0 + \circ_\varepsilon(1).$$

It follows that

$$\liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geq \inf_{\mathcal{A}_0} I_0. \tag{3.4}$$

In order to obtain the inverse inequality, we fix $\eta > 0$ and we select $u_\eta \in \mathcal{A}_\varepsilon$ such that

$$I_0(u_\eta) \leq \inf_{\mathcal{A}_\varepsilon} I_0 + \eta.$$

(Note that $\mathcal{A}_\varepsilon$ in not closed in $H^1$.) We have

$$I_\varepsilon(u_\varepsilon) \leq I_\varepsilon(u_\eta) \leq I_0(u_\eta) + \circ_\varepsilon(1) \leq \inf_{\mathcal{A}_\varepsilon} I_0 + \eta + \circ_\varepsilon(1) = \inf_{\mathcal{A}_0} I_0 + \eta + \circ_\varepsilon(1).$$

Therefore,

$$\limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \leq \inf_{\mathcal{A}_0} I_0 + \eta.$$

Since $\eta$ is arbitrary, we derive

$$\limsup_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \leq \inf_{\mathcal{A}_0} I_0. \tag{3.5}$$

Now by (3.4) and (3.5) the asserted equality (i) is established.

Proof of (ii) and (iii). By weak $H^1$ convergence and assumption $(f2)$,

$$\liminf_{\varepsilon\to 0} I_0(u_\varepsilon) \geq I_0(u_0).$$

Therefore, we have

$$\begin{aligned}
\inf_{\mathcal{A}_0} I_0 &= \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon)\\
&= \lim_{\varepsilon\to 0}\{\frac{\varepsilon^2}{2}\|\Delta u_\varepsilon\|_2^2 + \varepsilon\lambda\int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2 + I_0(u_\varepsilon)\}\\
&\geq \lim_{\varepsilon\to 0}\{\frac{\varepsilon^2}{2}\|\Delta u_\varepsilon\|_2^2 + \varepsilon\lambda\int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2\} + I_0(u_0)\\
&\geq \lim_{\varepsilon\to 0}\{\frac{\varepsilon^2}{2}\|\Delta u_\varepsilon\|_2^2 + \varepsilon\lambda\int f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2\} + \inf_{\mathcal{A}_0} I_0.
\end{aligned}$$

Therefore we obtain (ii) and (iii). Furthermore, we find that $\lim_{\varepsilon\to 0} I_0(u_\varepsilon) = I_0(u_0)$, which implies that $u_\varepsilon \to u_0$ strongly in $H^1$, and that $I_0(u_0) = \inf_{\mathcal{A}_0} I_0$. Arguing similarly as for $u_\varepsilon$, we see that $u_0$ is a supersolution for (0.8). By the Hopf maximum principle, $u_0 > \underline{u}_0$. Therefore, $u_0$ is a local minimum for $I_0$ in the $C^1$ topology. By the Brezis-Nirenberg argument [3], $u_0$ is a local minimum for $I_0$ in the $H^1$ topology, and thus it is in fact a solution for (0.8). By elliptic regularity, $u_0$ is smooth. □

The next lemma shows that the strong-maximum-principle property for $u_0$ and $\underline{u}_0$ carries over to $u_\varepsilon$ and $\underline{u}_\varepsilon$, for small values of $\varepsilon$:

**Lemma 3.2.** *Suppose $f$ satisfies* $(f0)$, $(f1)$, *and* $(f2)$. *Then, for all $\varepsilon$ sufficiently small, $u_\varepsilon > \underline{u}_\varepsilon$, pointwise on $M$.*

**Proof.** We define

$$F_\varepsilon = \varepsilon\lambda\, a(u_\varepsilon) + \lambda^2 f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}(s - f(e^{\sigma+u_\varepsilon})) - A,$$

where $a$ is the operator defined in (2.2). Then $u_\varepsilon$ satisfies

$$\varepsilon^2\Delta^2 u_\varepsilon - \Delta u_\varepsilon \geq F_\varepsilon.$$

Since the Green's function $G_\varepsilon$ for $-\varepsilon^2\Delta + 1$ is positive (see Lemma 2.1), the above yields

$$-\Delta u_\varepsilon \geq G_\varepsilon * F_\varepsilon. \tag{3.6}$$

We define

$$F_0 = f'(e^{\sigma+u_0})e^{\sigma+u_0}(s - f(e^{\sigma+u_0})) - A.$$

**Claim:** There exists some $q > 1$ such that

$$\|F_\varepsilon - F_0\|_q \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.7}$$

Proof of (3.7). We show that $\varepsilon a(u_\varepsilon) \to 0$. In view of (2.3), it suffices to show that

$$\varepsilon\|[f''(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon} + f'(e^{\sigma+u_\varepsilon})]e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2\|_q \to 0$$

and

$$\varepsilon\|f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}\Delta(\sigma+u_\varepsilon)\|_q \to 0.$$

By (1.9), the fact $\|u_\varepsilon\|_{H^1} \leq C$, and Sobolev embeddings we have

$$\|[f''(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon} + f'(e^{\sigma+u_\varepsilon})]e^{\sigma+u_\varepsilon}|\nabla(\sigma+u_\varepsilon)|^2\|_q \leq C$$

for some $q > 1$, and therefore the first limit follows easily. In order to prove the second limit, we write, using (1.1),

$$\varepsilon f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}\Delta(\sigma+u_\varepsilon) = -\varepsilon A f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon} + \varepsilon f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}\Delta u_\varepsilon.$$

By similar arguments as above, the $L^q$ norm of the first term on the right-hand side above vanishes as $\varepsilon \to 0$. In order to estimate the second term, we write for $r > 2$ such that $1/r + 1/2 = 1/q$

$$\|\varepsilon f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}\Delta u_\varepsilon\|_q \leq \|f'(e^{\sigma+u_\varepsilon})e^{\sigma+u_\varepsilon}\|_r\|\varepsilon\Delta u_\varepsilon\|_2 \leq C\|\varepsilon\Delta u_\varepsilon\|_2 \to 0,$$

where we used Lemma 3.1-(ii) to derive the last step. Hence, (3.7) is established. By (3.7) and by Lemma 2.1-(ii)-(iii), it follows that

$$\begin{aligned}\|G_\varepsilon * F_\varepsilon - F_0\|_q &\le \|G_\varepsilon * (F_\varepsilon - F_0)\|_q + \|G_\varepsilon * F_0 - F_0\|_q \\ &\le (\|F_\varepsilon - F_0\|_q + \varepsilon^2 \|\Delta F_0\|_q) \to 0,\end{aligned}$$

as $\varepsilon \to 0$. We define $w_\varepsilon$ as the unique solution for

$$(-\Delta + 1) w_\varepsilon = G_\varepsilon * F_\varepsilon + u_\varepsilon.$$

Then, by (3.6),

$$(-\Delta + 1)(u_\varepsilon - w_\varepsilon) \ge 0,$$

and therefore, by the maximum principle,

$$u_\varepsilon \ge w_\varepsilon.$$

Since $u_0$ satisfies

$$-\Delta u_0 = F_0$$

we have

$$(-\Delta + 1)(w_\varepsilon - u_0) = G_\varepsilon * F_\varepsilon - F_0 + u_\varepsilon - u_0,$$

and by standard elliptic estimates

$$\|w_\varepsilon - u_0\|_{C^\alpha} \le C(\|G_\varepsilon * F_\varepsilon - F_0\|_q + \|u_\varepsilon - u_0\|_q) \to 0.$$

In conclusion, we have $\underline{u}_\varepsilon \to \underline{u}_0$ in $H^2$ and in particular uniformly, $w_\varepsilon \to u_0$ uniformly, and $u_0 > \underline{u}_0$. It follows that $w_\varepsilon > \underline{u}_\varepsilon$ for small $\varepsilon$. Consequently, $u_\varepsilon > \underline{u}_\varepsilon$ for small $\varepsilon$, as asserted. □

**Proof of Proposition 3.1.** By Lemma (3.2) and the Sobolev embedding $\|\phi\|_\infty \le C\|\phi\|_{H^2}$, $u_\varepsilon$ belongs to the interior of $\mathcal{A}_\varepsilon$, in the sense of $H^2$. Therefore $u_\varepsilon$ is a critical point for $I_\varepsilon$ corresponding to a local minimum, as asserted. □

## 4. The Palais-Smale condition

The main result of this section is

**Proposition 4.1.** *Suppose $f$ satisfies $(f0)$, $(f1)$, and $(f3)$. Then $I_\varepsilon$ satisfies the Palais-Smale condition.*

We denote by $(u_j)$, $u_j \in H^2$ a Palais-Smale sequence. That is, we assume that $I_\varepsilon(u_j) \to \alpha$ for some $\alpha \in \mathbb{R}$ and $\|I'_\varepsilon(u_j)\|_{H^{-1}} \to 0$ as $j \to +\infty$. In order to prove Proposition 4.1 we have to show that $(u_j)$ admits a subsequence which converges strongly in $H^2$. By standard compactness arguments, it suffices to show that $(u_j)$ is bounded in $H^2$. It will be convenient to decompose

$$u_j = u_j' + c_j, \qquad \int u_j' = 0, \qquad c_j \in \mathbb{R}.$$

Unless otherwise stated, throughout this section we denote by $C > 0$ a general constant independent of $j \to +\infty$, whose actual value may vary from line to line. The Palais-Smale assumption for $(u_j)$ implies in particular the following facts:

$$\begin{aligned} I_\varepsilon(u_j) = \frac{\varepsilon^2}{2}\int(\Delta u_j)^2 + \frac{1}{2}\int|\nabla u_j|^2 + \varepsilon\lambda\int f'(e^{\sigma+u_j})e^{\sigma+u_j}|\nabla(\sigma+u_j)|^2 \\ + \frac{\lambda^2}{2}\int(f(e^{\sigma+u_j})-s)^2 + A|M|c_j \to \alpha, \end{aligned} \tag{4.1}$$

and (see (1.10))

$$\begin{aligned} \langle I_\varepsilon'(u_j), 1\rangle =& \varepsilon\lambda\int[f''(e^{\sigma+u_j})e^{\sigma+u_j} + f'(e^{\sigma+u_j})]e^{\sigma+u_j}|\nabla(\sigma+u_j)|^2 \\ &+ \lambda^2\int f'(e^{\sigma+u_j})e^{\sigma+u_j}(f(e^{\sigma+u_j})-s) + A|M| \le C, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \langle I_\varepsilon'(u_j), u_j\rangle = \varepsilon^2\int(\Delta u_j)^2 + \int|\nabla u_j|^2 \\ + \varepsilon\lambda\int[f''(e^{\sigma+u_j})e^{\sigma+u_j} + f'(e^{\sigma+u_j})]e^{\sigma+u_j}|\nabla(\sigma+u_j)|^2 u_j \\ + 2\varepsilon\lambda\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma+u_j)\cdot\nabla u_j \\ + \lambda^2\int f'(e^{\sigma+u_j})e^{\sigma+u_j}(f(e^{\sigma+u_j})-s)u_j + A|M|c_j \\ \le \circ_j(1)(\|\Delta u_j\|_2 + |c_j|), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \langle I_\varepsilon'(u_j), u_j'\rangle = \varepsilon^2\int(\Delta u_j)^2 + \int|\nabla u_j|^2 \\ + \varepsilon\lambda\int[f''(e^{\sigma+u_j})e^{\sigma+u_j} + f'(e^{\sigma+u_j})]e^{\sigma+u_j}|\nabla(\sigma+u_j)|^2 u_j' \\ + 2\varepsilon\lambda\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma+u_j)\cdot\nabla u_j \\ + \lambda^2\int f'(e^{\sigma+u_j})e^{\sigma+u_j}(f(e^{\sigma+u_j})-s)u_j' \le \circ_j(1)\|\Delta u_j\|_2. \end{aligned} \tag{4.4}$$

**Lemma 4.1.** *If either $c_j \geq -C$ or $\|\Delta u_j\|_2 \leq C$, then $(u_j)$ is bounded in $H^2$.*

**Proof.** Suppose $c_j \geq -C$. Condition (4.1) implies that $c_j \leq C$. Since $c_j \geq -C$, (4.1) yields $\|\Delta u_j\|_2 \leq C$. Hence $(u_j)$ is bounded in this case.

Now suppose $\|\Delta u_j\|_2 \leq C$. Then, by Sobolev embeddings, $\|\nabla u_j\|_p \leq C_p$ for all $p \geq 1$ and $\|u_j'\|_\infty \leq C$. It follows that $\|e^{u_j}\|_\infty = e^{c_j}\|e^{u_j'}\|_\infty \leq C$. Hence, by (1.9), we obtain $\int f'(e^{\sigma+u_j})e^{\sigma+u_j}|\nabla(\sigma + u_j)|^2 \leq C$. Inserting into (4.1), we find that $A|M|c_j + O_j(1) \to \alpha$. Consequently, $c_j \geq -C$, and therefore $(u_j)$ is bounded also in this case. □

In view of Lemma 4.1, we assume henceforth without loss of generality

$$\|\Delta u_j\|_2 \to +\infty \qquad \text{as } j \to +\infty \tag{4.5}$$

and

$$c_j \to -\infty \qquad \text{as } j \to +\infty. \tag{4.6}$$

**Proof of Proposition 4.1 under assumption ($f$3)-(a).** By ($f$1) there exists $t_0 > 0$ such that $f(t) - s > 0$ for all $t \geq t_0$, and consequently $f'(t)t(f(t) - s) \geq -C$ for some $C > 0$ independent of $t$. We may further assume that $f'(t)t|f(t) - s| \leq f'(t)t(f(t) - s) + C$, and in view of ($f$3)-(a), we may assume that $f^2(t) \leq f'(t)t|f(t)| \leq Cf'(t)t(f(t) - s)$ for all $t \geq t_0$. Therefore, we derive from (4.2) that

$$\int [f''(e^{\sigma+u_j})e^{\sigma+u_j} + f'(e^{\sigma+u_j})]e^{\sigma+u_j}|\nabla(\sigma + u_j)|^2 \leq C \tag{4.7}$$

$$\int f'(e^{\sigma+u_j})e^{\sigma+u_j}|f(e^{\sigma+u_j}) - s| \leq C \tag{4.8}$$

$$\|f(e^{\sigma+u_j})\|_2 \leq C. \tag{4.9}$$

Consequently, we may easily estimate the terms in the right-hand side of (4.4). Indeed, using (4.7), we have

$$\begin{aligned} &\Big|\int [f''(e^{\sigma+u_j})e^{\sigma+u_j} + f'(e^{\sigma+u_j})]e^{\sigma+u_j}|\nabla(\sigma + u_j)|^2 u_j'\Big| \\ &\qquad \leq \int [f''(e^{\sigma+u_j})e^{\sigma+u_j} + f'(e^{\sigma+u_j})]e^{\sigma+u_j}|\nabla(\sigma + u_j)|^2 \, \|u_j'\|_\infty \leq C\|\Delta u_j\|_2. \end{aligned}$$

Using (4.8), we have

$$\begin{aligned} &\Big|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}(f(e^{\sigma+u_j}) - s)u_j'\Big| \\ &\qquad \leq \int f'(e^{\sigma+u_j})e^{\sigma+u_j}|f(e^{\sigma+u_j}) - s| \, \|u_j'\|_\infty \leq C\|\Delta u_j\|_2. \end{aligned}$$

Finally, integrating by parts, we have

$$\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma+u_j)\cdot\nabla u_j = \int \nabla f(e^{\sigma+u_j})\cdot\nabla u_j = -\int f(e^{\sigma+u_j})\Delta u_j.$$

Hence, using (4.9) we have

$$\Big|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma+u_j)\cdot\nabla u_j\Big| = \Big|\int f(e^{\sigma+u_j})\Delta u_j\Big| \\ \le \|f(e^{\sigma+u_j})\|_2\|\Delta u_j\|_2 \le C\|\Delta u_j\|_2.$$

Inserting into (4.4) we find $\|\Delta u_j\|_2 \le C$, which is in contradiction with (4.5). □

In order to prove Proposition 4.1 in the remaining case ($f$3)-(b), we first establish an identity:

**Lemma 4.2.** *For all $u \in H^2$ the following identity holds:*

$$\int [f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})]e^{\sigma+u}|\nabla(\sigma+u)|^2 u \\ + 2\int f'(e^{\sigma+u})e^{\sigma+u}\nabla(\sigma+u)\cdot\nabla u \\ = \int f'(e^{\sigma+u})e^{\sigma+u}\nabla(\sigma+u)\cdot\nabla u - \int f'(e^{\sigma+u})e^{\sigma+u}\Delta(\sigma+u)u.$$

**Proof.** Integrating by parts, we have

$$\int [f''(e^{\sigma+u})e^{\sigma+u} + f'(e^{\sigma+u})]e^{\sigma+u}|\nabla(\sigma+u)|^2 u \\ = \int \nabla[f'(e^{\sigma+u})e^{\sigma+u}]\cdot\nabla(\sigma+u)u \\ = -\int f'(e^{\sigma+u})e^{\sigma+u}\Delta(\sigma+u)u - \int f'(e^{\sigma+u})e^{\sigma+u}\nabla(\sigma+u)\cdot\nabla u.$$

The asserted identity follows. □

**Proof of Proposition 4.1 under assumption ($f$3)-(b).** By Lemma 4.2 with $u = u_j$, condition (4.3) may be equivalently written in the form

$$\langle I'_\varepsilon(u_j), u_j\rangle = \varepsilon^2\int(\Delta u_j)^2 + \int|\nabla u_j|^2 \\ + \varepsilon\lambda\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma+u_j)\cdot\nabla u_j \\ - \varepsilon\lambda\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta(\sigma+u_j)u_j$$

$$+ \lambda^2 \int f'(e^{\sigma+u_j})e^{\sigma+u_j}(f(e^{\sigma+u_j}) - s)u_j + A|M|c_j$$
$$\leq \circ_j(1)(\|\Delta u_j\|_2 + |c_j|),$$

and consequently, we have

$$\varepsilon^2 \int (\Delta u_j)^2 + \int |\nabla u_j|^2 \leq -\varepsilon\lambda \int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma + u_j) \cdot \nabla u_j \tag{4.10}$$
$$+ \varepsilon\lambda \int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta(\sigma + u_j)u_j$$
$$- \lambda^2 \int f'(e^{\sigma+u_j})e^{\sigma+u_j}(f(e^{\sigma+u_j}) - s)u_j - A|M|c_j + \circ_j(1)(\|\Delta u_j\|_2 + |c_j|).$$

We estimate term by term the right-hand side in (4.10).

**Claim:** There holds

$$-\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma + u_j) \cdot \nabla u_j \leq C\|\Delta u_j\|_2. \tag{4.11}$$

Proof of (4.11). Since $f'(t)t \leq C$ we have, for any $1 < p < 2$,

$$-\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma + u_j) \cdot \nabla u_j$$
$$= -\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla\sigma \cdot \nabla u_j - \int f'(e^{\sigma+u_j})e^{\sigma+u_j}|\nabla u_j|^2$$
$$\leq \left|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla\sigma \cdot \nabla u_j\right| \leq C\|\nabla\sigma\|_p\|\nabla u_j\|_{p'} \leq C\|\Delta u_j\|_2,$$

and thus (4.11) is established.

**Claim:** There holds

$$\left|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta(\sigma + u_j)u_j\right| \leq C\|\Delta u_j\|_2. \tag{4.12}$$

Proof of (4.12). By the assumption $\sup_{t>0} f'(t)t|\log t| \leq +\infty$, we readily derive

$$\left|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta u_j u_j\right|$$
$$\leq \left|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}(\sigma + u_j)\Delta u_j\right| + \left|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\sigma\Delta u_j\right|$$
$$\leq C\|\Delta u_j\|_2 + C\|\sigma\|_2\|\Delta u_j\|_2 \leq C\|\Delta u_j\|_2.$$

On the other hand, recalling (1.1) we have

$$\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta\sigma\, u_j = -A \int f'(e^{\sigma+u_j})e^{\sigma+u_j}u_j$$

$$= - A\int f'(e^{\sigma+u_j})e^{\sigma+u_j}(\sigma+u_j) + A\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\sigma.$$

Hence, in view of $(f3)$-(b) we derive

$$|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta\sigma\, u_j| \le C.$$

Now (4.12) follows (recall that by assumption $\|\Delta u_j\|_2 \to +\infty$).

Finally, we easily estimate using $(f3)$-(b)

$$|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}(f(e^{\sigma+u_j}-s))u_j| \le C\int |u_j| \le C(\|\Delta u_j\|_2 + |c_j|).$$

Inserting (4.11)–(4.12) and the above estimate into (4.10), we finally obtain

$$\varepsilon^2\|\Delta u_j\|_2^2 \le C(\|\Delta u_j\|_2 + |c_j|). \tag{4.13}$$

In turn, (4.13) yields

$$\|\Delta u_j\|_2 \le C|c_j|^{1/2}. \tag{4.14}$$

Using (4.14), we show

**Claim:** There holds $\|e^{u_j}\|_\infty \to 0$. More precisely, the following estimate holds, for some $\gamma > 0$:

$$\|e^{u_j}\|_\infty \le e^{-\gamma|c_j|}. \tag{4.15}$$

Proof of (4.15). By (4.14) and Sobolev embeddings, we have

$$\|e^{u_j}\|_\infty \le e^{c_j + \|u_j'\|_\infty} \le e^{c_j + C\|\Delta u_j\|_2} \le e^{-|c_j| + C|c_j|^{1/2}},$$

and (4.15) follows. Finally, we show

**Claim:** There holds

$$|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta(\sigma+u_j)u_j'| = \circ_j(1)\|\Delta u_j\|_2. \tag{4.16}$$

Proof of (4.16). By (4.15) and the fact $\|u_j'\|_\infty \le C\|\Delta u_j\|_2 \le C|c_j|^{1/2}$, we have

$$\begin{aligned}|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta u_j u_j'| &\le \|f'(e^{\sigma+u_j})e^{\sigma+u_j}\|_\infty\|u_j'\|_\infty\|\Delta u_j\|_2\\ &\le C\|e^{u_j}\|_\infty\|u_j'\|_\infty\|\Delta u_j\|_2 \le Ce^{-\gamma|c_j|}|c_j|^{1/2}\|\Delta u_j\|_2 = \circ_j(1)\|\Delta u_j\|_2.\end{aligned}$$

On the other hand, recalling (1.1), we have by arguments similar to those above

$$|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta\sigma u_j'| = A|\int f'(e^{\sigma+u_j})e^{\sigma+u_j}u_j'| \le Ce^{-\gamma|c_j|}|c_j|^{1/2} \to 0.$$

Hence (4.16) is established.

Now, using Lemma 4.2 with $u = u_j'$, we rewrite (4.4) in the form

$$\begin{aligned}\varepsilon^2 \int (\Delta u_j)^2 + \int |\nabla u_j|^2 \le -\varepsilon\lambda \int f'(e^{\sigma+u_j})e^{\sigma+u_j}\nabla(\sigma+u_j)\cdot\nabla u_j \\ + \varepsilon\lambda \int f'(e^{\sigma+u_j})e^{\sigma+u_j}\Delta(\sigma+u_j)u_j' \\ - \lambda^2 \int f'(e^{\sigma+u_j})e^{\sigma+u_j}(f(e^{\sigma+u_j})-s)u_j' + \circ_j(1)\|\Delta u_j\|_2.\end{aligned}$$

In view of (4.11), (4.16), and assumption $(f3)$-b, we derive from the above that $\|\Delta u_j\|_2 \le C$. This is in contradiction with (4.5). □

Now we can finally prove our main result:

**Proof of Theorem 0.1.** By Proposition 3.1, the functional $I_\varepsilon$ admits a critical point corresponding to a local minimum. By Proposition 4.1, $I_\varepsilon$ satisfies the Palais-Smale condition. If $u_\varepsilon$ is not a strict local minimum, it is known that $I_\varepsilon$ has a continuum of critical points (see, e.g., [21]). In particular, $I_\varepsilon$ has at least two critical points. If $u_\varepsilon$ is a strict local minimum, we note that for $c \in \mathbb{R}$, $c \to -\infty$, we have $I_\varepsilon(c) \to -\infty$. Therefore $I_\varepsilon$ admits a mountain-pass structure in the sense of Ambrosetti and Rabinowitz [1]. Hence by the "mountain-pass theorem" [1] we obtain the existence of a second critical point for $I_\varepsilon$. In either case, we conclude that the fourth-order equation (1.5) (equivalently, system (0.1)–(0.2)) admits at least two solutions, as asserted. □

**Acknowledgments.** In am grateful to Professor Gabriella Tarantello for interesting and stimulating discussions.

## References

[1] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Funct. Anal., 14 (1973), 349–381.

[2] Th. Aubin, "Nonlinear Analysis on Manifolds: Monge-Ampère Equations," Springer-Verlag, New York, Berlin, 1982.

[3] H. Brezis and L. Nirenberg, *$H^1$ versus $C^1$ local minimizers,* C.R. Acad. Sci. Paris, Série I, 317 (1993), 465–472.

[4] D. Chae and N. Kim, *Vortex condensates in the relativistic self-dual Maxwell-Chern-Simons-Higgs system,* preprint.

[5] D. Chae and H.S. Nam, *Multiple existence of the multivortex solutions of the self-dual Chern-Simons $CP(1)$ model on a doubly periodic domain,* Lett. Math. Phys., 49, 297–315.

[6] D. Chae and H.S. Nam, *On the condensate multivortex solutions of the self-dual Maxwell-Chern-Simons $CP(1)$ model,* Ann. H. Poincaré, 2 (2001), 887–906.

[7] L.A. Caffarelli and Y. Yang, *Vortex condensation in the Chern-Simons-Higgs model: An existence theorem,* Commun. Math. Phys., 168 (1995), 321–336.
[8] W. Ding, J. Jost, J. Li, and G. Wang, *An analysis of the two-vortex case in the Chern-Simons Higgs model,* Calc. Var. and PDE, 7 (1998), 87–97.
[9] G. Dunne, "Self-Dual Chern-Simons Theories," Lecture Notes in Physics, vol. M36, Springer-Verlag, Berlin, New York, 1995.
[10] J. Hong, Y. Kim, and P.Y. Pac, *Multivortex solutions of the Abelian Chern-Simons theory,* Phys. Rev. Lett., 64 (1990), 2230–2233.
[11] A. Jaffe and C. Taubes, "Vortices and Monopoles," Birkhäuser, Boston, 1980.
[12] R. Jackiw and E. Weinberg, *Self-dual Chern–Simons vortices,* Phys. Rev. Lett., 64 (1990), 2234–2237.
[13] K. Kimm, K. Lee, and T. Lee, *The self-dual Chern-Simons $CP(N)$ models,* Phys. Lett. B, 380 (1996), 303–307.
[14] K. Kimm, K. Lee, and T. Lee, *Anyonic Bogomol'nyi solitons in a gauged $O(3)$ sigma model,* Phys. Rev. D, 53 (1996), 4436–4440.
[15] C. Lee, K. Lee, and H. Min, *Self-Dual Maxwell-Chern-Simons solitons,* Phys. Lett. B, 252 (1990), 79–83.
[16] M. Nolasco and G. Tarantello, *Double vortex condensates in the Chern-Simon-Higgs theory,* Cal. Var. and PDE, 9 (1999), 31–94.
[17] T. Ricciardi, *Asymptotics for Maxwell-Chern-Simons multivortices,* Nonlin. Anal. T.M.A., 50 (2002), 1093–1106.
[18] T. Ricciardi, *On a nonlinear elliptic system from Maxwell-Chern-Simons vortex theory,* Asympt. Anal., 35 (2003), 113–126.
[19] T. Ricciardi and G. Tarantello, *Vortices in the Maxwell-Chern-Simons theory,* Comm. Pure Appl. Math., 53 (2000), 811–851.
[20] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus,* Ann. Inst. Fourier, Grenoble, 15 (1965), 189–258.
[21] G. Tarantello, *Multiple condensate solutions for the Chern-Simons-Higgs theory,* J. Math. Phys., 37 (1996), 3769–3796.
[22] Y. Yang, "Solitons in Field Theory and Nonlinear Analysis," Springer Monographs in Mathematics, Springer-Verlag, New York, 2001.