GENERALIZED SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. In this paper, we consider the Cauchy problem for linear first-order partial differential equations with discontinuous coefficients. We show that, under suitable assumptions, it has a unique continuous solution. Moreover, this solution is stable under perturbation of the coefficients. We also show that this solution can be expressed explicitly by integrating along generalized characteristics.

1. INTRODUCTION

We consider the following Cauchy problem:

\[ u_t(t, x) + \alpha(t, x)u_x(t, x) = f(t, x), \quad u(0, x) = u_0(x). \tag{1.1} \]

It is well known that when \( \alpha, f, \) and \( u_0 \) are of class \( C^1 \), the above problem admits a unique solution of class \( C^1 \), which is given by integrating along the characteristics

\[ u(t, x) = u_0(X(0, t, x)) + \int_0^t f(s, X(s, t, x)) ds, \tag{1.2} \]

where \( X(s, t, x) \) denotes the unique solution to the characteristic system

\[ \frac{dX}{ds}(s, t, x) = \alpha(s, X(s, t, x)), \quad X(t, t, x) = x. \tag{1.3} \]

When \( \alpha \) and \( f \) are not continuous, one is led to the problem of constructing a continuous solution of the Cauchy problem with discontinuous coefficients. In this case the last ordinary differential equation (the characteristic system) does not have well-defined solutions. Still we shall show that the solution of (1.1) exists and is given by (1.2) whenever for some \( \gamma > 0 \), \( \alpha(t, x) \in [\gamma^{-1}, \gamma] \) almost everywhere in \( \mathbb{R}^2 \), \( \alpha(\cdot, x) \) and \( f(\cdot, x) \) are Lipschitz, and \( f \in L^\infty \).

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This paper was inspired by a result of Da Prato and Sinestrari, who solved the Cauchy problem with discontinuous periodic coefficients in the sense of semigroup theory. They show that under the following hypotheses,

$$\begin{cases}
  a \in C^1([t_0, T]; L^\infty(0, l)) \text{ with } \gamma^{-1} < a(t, x) < \gamma \text{ a.e.} \\
  f \in W^{1,1}([t_0, T]; L^\infty(0, l)) \\
  u_0 \in Lip_*([0, l]) \\
  a(t_0, \cdot)u_0(\cdot) + f(t_0, \cdot) \in C_*([0, l])
\end{cases}$$

(where $F_*[t_0, T] = \{f \in F; f(t_0) = f(T)\}$), there exists a unique periodic solution of the Cauchy problem.

We propose here a different approach based on the approximation of discontinuous data by $C^1$ functions $\alpha_n$ and $f_n$ and an application of the Egorov theorem.

The solution $u_n$ of the approximated Cauchy problem is given by (1.2), (1.3) with $\alpha$ and $f$ replaced by $\alpha_n$ and $f_n$. We show that the $u_n$ converge uniformly on compact sets to a continuous function $u$ and the characteristics $X_n$ of the approximated problems converge (uniformly on compact sets) to a function $X \in W^{1,1}$. Furthermore, both $u$ and $X$ are independent of approximating sequences $\alpha_n$ and $f_n$ and satisfy the equality (1.2). Such a function $X$ can be seen as the generalized characteristic of (1.1).

The outline of the paper is as follows. In the first section, we find a locally Lipschitz solution to the Cauchy problem. Furthermore, we show that all solutions to the approximated problem converge to this solution, which can be represented by integrating along generalized characteristics. In the second section, we show that the Cauchy problem has a unique solution. In the last section, we establish a result of the same type as Da Prato and Sinestrari’s by showing that, under different assumptions, the solution we have found is also a solution in the sense of semigroup theory.

2. Existence theorem

We consider the following equation:

$$u_t(t, x) + \alpha(t, x)u_x(t, x) = f(t, x), \quad u(0, x) = u_0(x),$$

where

$$\begin{cases}
  u_0 \in Lip_{loc}(\mathbb{R}, \mathbb{R}) \\
  \alpha \in L^\infty(\mathbb{R}^2, \mathbb{R}) \text{ and for some } \delta > 0, \delta^{-1} \leq \alpha(t, x) \leq \delta \text{ a.e.} \\
  f \in L^\infty(\mathbb{R}^2, \mathbb{R}) \\
  \forall x \in \mathbb{R}, \alpha(\cdot, x) \text{ and } f(\cdot, x) \text{ are locally Lipschitz uniformly in } x.
\end{cases}$$

(2.1)
Consider $\alpha_n \in C^1, f_n \in C^1$, and $u_0^n \in C^1(R, R)$. We recall that the unique solution $u^n \in C^1(R_+ \times R, R)$ to the partial differential equation
\[
\partial_t u^n(t, x) + \alpha_n(t, x) \partial_x u^n(t, x) = f_n(t, x), \quad u^n(0, x) = u_0^n(x) \tag{2.2}
\]
is given by
\[
u^n(t, x) = u_0^n(X_n(0, t, x)) + \int_0^t f_n(s, X_n(s, t, x)) ds,
\]
where $X_n(s, t, x)$ denotes the unique solution to the characteristic system
\[
\frac{dX_n}{ds}(s, t, x) = \alpha_n(s, X_n(s, t, x)), \quad X_n(t, t, x) = x. \tag{2.3}
\]

**Theorem 2.1.** Assume that (2.1) holds true. Then there exist locally Lipschitz maps $u_0 : R_+ \times R \to R$ and $X : R \times R_+ \times R \to R$ such that
\[
u(t, x) = u_0(X(0, t, x)) + \int_0^t f(s, X(s, t, x)) ds.
\]
Furthermore, $u$ is stable with respect to perturbation of $\alpha$ and $f$ in the following sense: for all sequences $\alpha_n \in C^1(R^2, R), f_n \in C^1(R^2, R)$, and $u_0^n \in C^1(R, R)$ such that
\[
\begin{align*}
i) \quad &\alpha_n \to \alpha \text{ in } L^1_{\text{loc}}, \quad f_n \to f \text{ in } L^1_{\text{loc}} \\
ii) \quad &\exists \gamma > 0 \text{ such that } \gamma^{-1} \leq \alpha_n(t, x) \leq \gamma \text{ a.e. in } R^2 \\
iii) \quad &\sup_n \|f_n\| < \infty \\
iv) \quad &\forall T > 0, \forall T' > 0 \text{ and all compact } K \subset R \text{ we have }
\end{align*}
\]
\[
\begin{align*}
\sup_{x \in K, t \in [-T, T']} \left\| \frac{\partial \alpha_n}{\partial t}(t, x) \right\| &< \infty \\
\sup_{x \in K, t \in [-T, T']} \left\| \frac{\partial f_n}{\partial t}(t, x) \right\| &< \infty \\
u_0^n &\to u_0 \text{ uniformly on } K \text{ and } \sup_{n \geq 1, x \in K} |(u_0^n)'(x)| < \infty
\end{align*}
\]
the solutions $u^n$ of (2.2) converge uniformly on all compact sets to $u$ and $X_n$ of (2.3) converge uniformly on all compact sets to $X$.

**Remark.** Observe that Theorem 2.1 yields stability also with respect to $L^\infty$ perturbations.

**Proof.** Let us first remark that there exist sequences $\alpha_n, f_n$, and $u_0^n$ which satisfy the hypotheses of the theorem. In fact, we can consider the convolution products of $\alpha, f$, and $u_0$ with mollifiers.
Lemma 2.2. Let \( K \subset \mathbb{R} \) be a compact set. Consider \( T > 0, T' > 0 \), and a sequence \( \delta_n : [-T', T] \times \mathbb{R} \rightarrow \mathbb{R} \) of measurable functions such that
\[
\delta_n \longrightarrow \delta_0 \text{ almost everywhere in } [-T', T] \times K
\]
\[
\exists G > 0, \forall t, t' \in [-T', T] \quad \sup_{x \in K, n \geq 0} |\delta_n(t, x) - \delta_n(t', x)| \leq G|t - t'|.
\]
Then for all \( \varepsilon > 0 \) we can find a measurable set \( A \subset K \), with \( m(K \setminus A) \leq \varepsilon \) and
\[
\exists N, \forall k \geq N, \quad \sup_{t \in [-T', T]} |\delta_n(t, x) - \delta_0(t, x)| \leq \frac{\varepsilon}{2} \text{ for all } x \in A.
\]

**Proof.** Set \( \delta = \delta_0 \) and fix \( t \in [-T', T] \). For all \( n \in \mathbb{N} \), consider the set
\[
A^n_t = \left\{ x \in K : \exists s \in \left[t - \frac{1}{n}, t + \frac{1}{n}\right] \cap [-T', T], \lim_{k \rightarrow \infty} \delta_k(s, x) = \delta(s, x) \right\}
\]
having the full measure in \( K \).

The set \( A_t = \bigcap_{n \in \mathbb{N}} A^n_t \), being the countable intersection of sets of full measure, is of full measure in \( K \). Fix \( \varepsilon > 0, x \in A_t \), and \( n > \frac{G}{\varepsilon} \). Consider \( s \in \left[t - \frac{1}{n}, t + \frac{1}{n}\right] \cap [-T', T] \) such that \( \delta_k(s, x) \longrightarrow \delta(s, x) \).

By our assumptions
\[
|\delta(t, x) - \delta(s, x)| \leq G|t - s| \leq \varepsilon \tag{2.5}
\]
and
\[
\forall k \geq 1, \quad |\delta_k(t, x) - \delta_k(s, x)| \leq G|t - s| \leq \varepsilon. \tag{2.6}
\]
Choose \( N \in \mathbb{N} \) in such a way that for \( k \geq N \), \( |\delta_k(s, x) - \delta(s, x)| \leq \varepsilon \). Since
\[
|\delta_k(t, x) - \delta(t, x)| \leq |\delta_k(t, x) - \delta_k(s, x)| + |\delta(t, x) - \delta(s, x)| + |\delta_k(s, x) - \delta(s, x)|
\]
we deduce that
\[
|\delta_k(t, x) - \delta(t, x)| \leq 3\varepsilon.
\]
Consequently,
\[
\forall -T' \leq t \leq T, \forall x \in A_t, \quad \delta_k(t, x) \longrightarrow \delta(t, x).
\]
and therefore $A_t = \{ x \in K : \delta_k(t, x) \rightarrow \delta(t, x) \}$. As $[-T', T] \cap Q$ is countable, the set

$$A_t = \{ x \in K : \forall t \in [-T', T] \cap Q, \delta_k(t, x) \rightarrow \delta(t, x) \}$$

is of full measure in $K$. We define

$$\mu_k(x) = \sup_{t \in [-T', T]} |\delta_k(t, x) - \delta(t, x)|$$

and show that for almost all $x \in K$, $\mu_k(x) \rightarrow 0$. Indeed, let $r \in Q \cap (0, \frac{\varepsilon}{4}]$. We set $t_i = -T' + (1 + 2i)r$; then there exists a finite number $m$ such that $[-T', T] \subset \bigcup_{i=0}^{m} [t_i - r, t_i + r]$. Fix $x \in \cap_{-T', T} A_t$, $t \in [-T', T]$ and observe that for some $0 \leq i \leq m$, $|t_i - t| \leq \frac{\varepsilon}{4}$. Therefore, using (2.5) and (2.6) we obtain

$$|\delta_k(t, x) - \delta(t, x)| \leq |\delta_k(t, x) - \delta_k(t_i, x)| + |\delta_k(t_i, x) - \delta(t_i, x)|$$

$$+ |\delta(t, x) - \delta(t_i, x)| \leq 2\varepsilon + |\delta_k(t_i, x) - \delta(t_i, x)|.$$ 

Furthermore,

$$\forall i, \exists N_i, \forall k \geq N_i, |\delta_k(t_i, x) - \delta(t_i, x)| \leq \varepsilon,$$

and we finally get

$$\forall k \geq \max_{i=0, \ldots, m} N_i, |\delta_k(t, x) - \delta(t, x)| \leq 3\varepsilon.$$ 

Therefore,

$$\forall k \geq \max_{i=0, \ldots, m} N_i, \mu_k(x) \leq 3\varepsilon \text{ for all } x \in \bigcap_{-T', T} A_t.$$ 

Consequently, $\mu_k(x) \rightarrow 0$ for almost all $x \in K$. We apply Egorov’s theorem to the subsequence $\{\mu_k\} : \forall \varepsilon \geq 0, \exists A \subset K$ which is measurable, such that $m(K \setminus A) \leq \varepsilon$ and $\mu_k \rightarrow 0$ uniformly on $A$. In other words,

$$\exists N, \forall k \geq N, \sup_{t \in [-T', T]} |\delta_k(t, x) - \delta(t, x)| \leq \frac{\varepsilon}{2} \text{ for all } x \in A.$$ 

This ends the proof of the lemma.

Step 1. Let $T > 0$ and $a \leq b$, and fix $x \in [a, b]$. We denote by $K$ the interval $[a - \gamma T, b]$ and set

$$S = \sup_{n \in \mathbb{N}} \|f_n\|, \quad M = \sup_{x \in K, t \in [(1-\gamma^2)T, T], n \geq 1} \left\| \frac{\partial \alpha_{n,t}}{\partial t} (t, x) \right\|.$$
for all \( y \)

In particular, this yields

\[ Q = \sup_{x \in K, t \in [(1 - \gamma^2)T, T], n \geq 1} \left\| \frac{\partial f_n}{\partial t}(t, x) \right\|, \quad F = \sup_{x \in K, n \geq 1} \left| (u_0^n)'(x) \right|. \]

Let \( m \) denote the Lebesgue measure in \( K \). We next consider a subsequence \( \alpha_{n_k} \) converging to \( \alpha \) almost everywhere in \([(1 - \gamma^2)T, T] \times K \). By Lemma 2.2 there exists \( A \subset [(1 - \gamma^2)T, T] \) such that \( m((1 - \gamma^2)T, T \setminus A) \leq \varepsilon \) and

\[ \exists N^1, \forall k \geq N^1, \sup_{t \in [(1 - \gamma^2)T, T]} |\alpha_{n_k}(t, x) - \alpha(t, x)| \leq \frac{\varepsilon}{2} \text{ for } x \in A. \quad (2.7) \]

In particular, this yields

\[ \exists N^1, \forall k \geq N^1, \sup_{t \in [(1 - \gamma^2)T, T]} |\alpha_{n_k}(t, x) - \alpha_{n_j}(t, x)| \leq \varepsilon \text{ for } x \in A. \quad (2.8) \]

We fix \( x \in [a, b] \), \( i, j \geq N^1 \), and \( t \leq T \). Since \( \alpha_{n_i} \in C^1(\mathbb{R}^2) \), there exists a unique solution to the characteristic system

\[ \frac{dX_{n_i}}{ds}(s, t, x) = \alpha_{n_i}(s, X_{n_i}(s, t, x)), \quad X_{n_i}(t, t, x) = x. \]

For all \( t \in [0, T] \) and \( s \in [0, t] \),

\[ x - \gamma t \leq X_{n_i}(s, s, t, x) \leq x - \gamma^{-1}(t - s) \leq x. \]

Thus, \( X_{n_i}(s, t, x) \) is a strictly increasing function of \( s \) for all fixed \( t \) and \( x \). Moreover, \( X_{n_i}(s, t, x) \in K \) and \( \lim_{s \to -\infty} X_{n_i}(s, t, x) = -\infty \). Consequently, for all \( y \leq x \) there exists a unique \( s' \leq t \) such that \( y = X_{n_i}(s', t, x) \). We can define \( \Phi : [0, t] \to \mathbb{R} \) such that \( \Phi(t) = t \), and for all \( s \in [0, t] \)

\[ X_{n_i}(s, t, x) = X_{n_j}(\Phi(s), t, x), \]

\( X_{n_j}(\cdot, t, x) \) is invertible, and as \( \frac{\partial}{\partial s} X_{n_j}(s, t, x) \neq 0 \), \( \Phi \) is derivable and satisfies

\[ \Phi'(s) = \frac{\alpha_{n_i}(s, X_{n_i}(s, t, x))}{\alpha_{n_j}(\Phi(s), X_{n_j}(\Phi(s), t, x))} = \frac{\alpha_{n_i}(s, X_{n_i}(s, t, x))}{\alpha_{n_j}(\Phi(s), X_{n_j}(s, t, x))}. \]

As

\[ x - \gamma(t - s) \leq X_{n_i}(s, t, x) \leq x - \gamma^{-1}(t - s) \]

and

\[ x - \gamma(t - \Phi(s)) \leq X_{n_i}(\Phi(s), t, x) \leq x - \gamma^{-1}(t - \Phi(s)) \]

we deduce that

\[ \gamma^{-1}(t - \Phi(s)) \leq \gamma(t - s), \quad \gamma^{-1}(t - s) \leq \gamma(t - \Phi(s)). \]

This leads to

\[ (1 - \gamma^2)t + \gamma^2 s \leq t - \gamma^{-2}(t - s) \]
and \( \Phi(s) \in [(1 - \gamma^2)T, T] \),
\[
|\Phi'(s) - 1| \leq \gamma|\alpha_{n_1}(s, X_{n_1}(s, t, x)) - \alpha_{n_j}(\Phi(s), X_{n_j}(s, t, x))|.
\] (2.9)
Using that
\[
|\alpha_{n_1}(s, X_{n_1}(s, t, x)) - \alpha_{n_j}(\Phi(s), X_{n_j}(s, t, x))| \leq \\
|\alpha_{n_1}(s, X_{n_1}(s, t, x)) - \alpha_{n_j}(s, X_{n_j}(s, t, x))| \\
+ |\alpha_{n_j}(s, X_{n_j}(s, t, x)) - \alpha_{n_j}(\Phi(s), X_{n_j}(s, t, x))|
\]
we finally obtain from (2.8) and (2.9)
\[
|\Phi'(s) - 1| \leq \gamma\varepsilon + M|\Phi(s) - s| \quad \text{whenever} \quad X_{n_i}(s, t, x) \in A. \quad (2.10)
\]
Moreover,
\[
|\Phi'(s) - 1| \leq \gamma^2 \quad \text{whenever} \quad X_{n_i}(s, t, x) \in K \setminus A.
\]

**Lemma 2.3.** Let \( f \in W^{1,1}(\mathbb{R}, \mathbb{R}) \) and \( \Omega \) be a bounded, measurable subset of \( \mathbb{R} \) satisfying \( f'(t) \geq \gamma^{-1} \) almost everywhere in \( \Omega \). Then
\[
m(\{t : f(t) \in \Omega\}) \leq \gamma m(\Omega).
\]

**Proof.** There exits a sequence of open sets \( \Omega_n \) such that \( \Omega \subset \Omega_n \) with \( m(\Omega_n) \to m(\Omega) \). Then
\[
m(\{t : f(t) \in \Omega\}) \leq m(\{t : f(t) \in \Omega_n\}).
\]
Consider disjoint intervals \([a^n_i, a^n_{i+1})\), \( i \in \mathbb{N} \) such that \( \Omega_n = \bigcup_{i \in \mathbb{N}} [a^n_i, a^n_{i+1}) \) and \( m(\Omega_n) = \sum_{i=0}^{\infty} |a^n_{i+1} - a^n_i| \). From our assumptions, \( f \) is a strictly increasing function of \( t \). Let \( f(\alpha) = a^n_i \) and \( f(\beta) = a^n_{i+1} \). Therefore,
\[
m(\{t : f(t) \in [a^n_i, a^n_{i+1})\}) = \beta - \alpha
\]
and
\[
f(\beta) - f(\alpha) \geq \gamma^{-1}(\beta - \alpha).
\]
Then,
\[
\gamma|a^n_{i+1} - a^n_i| \geq m(\{t : f(t) \in [a^n_i, a^n_{i+1})\}).
\]
Consequently,
\[
\gamma \sum_{i \in \mathbb{N}} |a^n_{i+1} - a^n_i| \geq \sum_{i \in \mathbb{N}} m(\{t : f(t) \in [a^n_i, a^n_{i+1})\}).
\]
Hence,
\[
m(\{t : f(t) \in \Omega_n\}) \leq \gamma m(\Omega_n) \quad \text{and} \quad m(\{t : f(t) \in \Omega\}) \leq \gamma m(\Omega). \quad \square
As $X'_{ni}(s, t, x) \geq \gamma^{-1}$, it follows from the previous lemma that
$$m(\{ s : X_{ni}(s, t, x) \in K \setminus A \}) \leq \gamma \varepsilon.$$ 
Integrating the function $\Phi'(s) - 1$ from $s$ to $t$ we obtain
$$|\Phi(s) - s| \leq \gamma \varepsilon (t - s) + M \gamma \int_s^t |\Phi(u) - u|du + \gamma^3 \varepsilon.$$ 
We set
$$\rho = \gamma \left[ (T - s) + \gamma^2 + M \gamma \left( \frac{(T - s)^2}{2} + \gamma(T - s) \right)e^{M \gamma(T - s)} \right].$$ 
From Gronwall’s lemma it follows that
$$|\Phi(s) - s| \leq \rho \varepsilon.$$  \hfill (2.11)
As
$$|X_{ni}(s, t, x) - X_{ni}(\Phi(s), t, x)| \leq \gamma |\Phi(s) - s|$$
from the very definition of $\Phi$ we obtain
$$|X_{ni}(s, t, x) - X_{nj}(s, t, x)| \leq \rho \gamma \varepsilon,$$
where $\rho$ is a constant depending on $T$. We have shown that $X_{nk}(s, t, x)$ is a Cauchy sequence. Thus, it converges uniformly on $[0, t] \times [0, T] \times [a, b]$ to a limit, which we denote by $X(s, t, x)$.

We next consider a subsequence $f_{mk}$ converging to $f$ almost everywhere in $[(1 - \gamma^2)T, T] \times K$. By Lemma 2.2, there exist some $N^3$ and some measurable $B \subset K$ such that $m(K \setminus B) < \varepsilon$ and for all $k, l > N^3$
$$\sup_{t \in [(1 - \gamma^2)T, T]} |f_{mk}(t, x) - f(t, x)| \leq \frac{\varepsilon}{2}\quad \text{for } x \in B,$$  \hfill (2.12)
$$\sup_{t \in [(1 - \gamma^2)T, T]} |f_{mk}(t, x) - f_{ml}(t, x)| \leq \varepsilon\quad \text{for } x \in B.$$  \hfill (2.13)
As $\alpha_{ni} \in C^1(\mathbb{R}^2)$, $f_{mk} \in C^1(\mathbb{R}^2)$, there exists a unique solution $u_{ni,F_{mk}}$ to (2.2) which is defined by
$$u_{ni,F_{mk}}(t, x) = u_{ni,0}(X_{ni}(0, t, x)) + \int_0^t f_{mk}(s, X_{ni}(s, t, x))ds.$$  \hfill (2.14)
First, we remark that for all $(t, x) \in [0, T] \times K$,
$$|u_{ni,0}(X_{ni}(0, t, x)) - u_{ni,0}(X_{nj}(0, t, x))| \leq F \rho \gamma \varepsilon.$$
As $u_{ni,0} \to u_0$ uniformly on $K$,
$$\exists N^2, \forall i, j \geq N^2, |u_{ni,0}(X_{ni}(0, t, x)) - u_{ni,0}(X_{nj}(0, t, x))| \leq \varepsilon.$$
Notice that
\[ \left| \int_0^t f_{mk}(s, X_{ni}(s, t, x)) - f_{ml}(s, X_{nj}(s, t, x)) ds \right| \]
\[ \leq \left| \int_0^t f_{mk}(s, X_{ni}(s, t, x)) - f_{ml}(s, X_{ni}(s, t, x)) ds \right| 
+ \left| \int_0^t f_{ml}(s, X_{nj}(s, t, x)) - f_{ml}(s, X_{nj}(s, t, x)) ds \right|. \]

From (2.13) we know that for all \( k, l \geq N^3 \)
\[ |f_{mk}(s, X_{ni}(s, t, x)) - f_{ml}(s, X_{ni}(s, t, x))| \leq \varepsilon \text{ if } X_{ni}(s, t, x) \in B \]
and
\[ |f_{mk}(s, X_{ni}(s, t, x)) - f_{ml}(s, X_{ni}(s, t, x))| \leq 2S \text{ if } X_{ni}(s, t, x) \in K \setminus B. \]

Therefore,
\[ \left| \int_0^t f_{mk}(s, X_{ni}(s, t, x)) - f_{ml}(s, X_{ni}(s, t, x)) \right| ds \leq \varepsilon t + 2\varepsilon \gamma S. \]

Using the change of variable \( \tau = \Phi^{-1}(s) \) we get
\[ \int_0^t f_{mi}(s, X_{nj}(s, t, s)) ds = \int_{\Phi^{-1}(0)}^{\Phi^{-1}(t)} f_{mi}(\Phi(\tau), X_{ni}(\tau, t, x)) \Phi'(\tau) d\tau \]
and deduce that
\[ \left| \int_0^t f_{mi}(s, X_{ni}(s, t, x)) ds - \int_0^t f_{mi}(s, X_{nj}(s, t, x)) ds \right| \]
\[ \leq \int_0^t |f_{mi}(\tau, X_{ni}(\tau, t, x)) - f_{mi}(\Phi(\tau), X_{ni}(\tau, t, x))| d\tau 
+ \int_0^t |f_{mi}(\Phi(\tau), X_{ni}(\tau, t, x)) [1 - \Phi'(\tau)]| d\tau 
+ \int_0^{\Phi^{-1}(0)} f_{mi}(\Phi(\tau), X_{ni}(\tau, t, x)) \Phi'(\tau) d\tau. \]

By the choice of \( Q \) and (2.11)
\[ \int_0^t |f_{mi}(\tau, X_{ni}(\tau, t, x)) - f_{mi}(\Phi(\tau), X_{ni}(\tau, t, x))| d\tau \leq \int_0^t Q |\Phi(\tau) - \tau| d\tau \leq Q \rho \varepsilon T \]
and
\[ \int_0^t |f_{mi}(\Phi(\tau), X_{ni}(\tau, t, x)) [1 - \Phi'(\tau)]| d\tau \leq S \int_0^t |1 - \Phi'(\tau)| d\tau; \]
Nathalie Caroff

from (2.11) and (2.10) we deduce that
\[ |\Phi'(\tau) - 1| \leq \gamma \varepsilon + M \rho \varepsilon \quad \text{for } X_{n_i}(\tau, t, x) \in A \]

and
\[ |\Phi'(\tau) - 1| \leq \gamma^2 \quad \text{for } X_{n_i}(\tau, t, x) \in K \setminus A. \]

Integrating \( \tau \rightarrow |\Phi'(\tau) - 1| \) we finally obtain
\[
\int_0^t \left| f_{m_i}(\Phi(\tau), X_{n_i}(\tau, t, x)) \right| d\tau \\
\leq S \left[ \gamma^2 \varepsilon + (\gamma + M \rho) \varepsilon T \right] \\
\leq S \gamma^2 \varepsilon.
\]

We can conclude
\[
|u_{\alpha_n, f_{m_k}}(t, x) - u_{\alpha_{n_j}, f_{m_l}}(t, x)| \leq C \varepsilon,
\]
where \( C \) is a constant independent of \( x \). Consequently \( u_{\alpha_n, f_{m_k}}(t, x) \) converges to a limit denoted \( u(t, x) \) when \( i, k \rightarrow \infty \), uniformly on \([0, T] \times [a, b] \). Hence, \( u \) is continuous.

**Step 2.** We show next that \( X(s, t, x) \) is independent of the subsequence \( \alpha_{n_k} \) of the sequence \( \alpha_n \). For this aim it is enough to show that for every sequence \( \beta_n \) which satisfies i) and ii) of hypotheses (2.4) there exists a subsequence \( \beta_{n_k} \) such that \( X_{n_k}(s, t, x) \) converges to \( X(s, t, x) \) defined in Step 1 on \([0, t] \times [0, T] \times [a, b] \). Let \( \beta_n \) be any sequence which satisfies (2.4) i), ii). There exists a subsequence \( \beta_{n_k} \) converging to \( \alpha \) almost everywhere in \(((1 - \gamma^2)T, T] \times K \). From Lemma 2.2 we deduce that for all \( \varepsilon > 0 \) and for some \( N_1' \) and some measurable \( A' \subset K \) such that \( m(K \setminus A') < \varepsilon \) we have
\[
\forall k \geq N_1', \sup_{t \in [(1-\gamma^2)T, T]} |\beta_{n_k}(t, x) - \alpha(t, x)| \leq \frac{\varepsilon}{2} \quad \text{for } x \in A'.
\]

Therefore,
\[
\forall i, j \geq \sup(N_1, N_1'), \sup_{t \in [(1-\gamma^2)T, T]} |\beta_{n_i}(t, x) - \alpha_{n_j}(t, x)| \leq \varepsilon \quad \text{for } x \in A \cap A'
\]

and \( m(K \setminus (A \cup A')) \leq 2\varepsilon \); arguing as in the first step, we deduce that
\[
|X_{n_i}(s, t, x) - X_{n_j}(s, t, x)| \leq \rho' \varepsilon,
\]
where
\[
\rho' = \gamma^2 \left[ (T - s) + 2\gamma^2 + M \gamma \left( \frac{(T - s)^2}{2} + 2\gamma(T - s) \right) e^{M \gamma (T - s)} \right].
\]
Using that \( \varepsilon > 0 \) is arbitrary, we get
\[
\lim_{i \to \infty} X_{n_i}(s, t, x) = X(s, t, x).
\]
We show next that \( u \) is independent of the subsequences \( \alpha_{n_k} \) and \( f_{m_k} \) of the sequences \( \alpha_n \) and \( f_n \). We already know from (2.15) that \( u^{\alpha_{n_i}, f_{m_k}} \) converges to a function \( u \) and
\[
u^{\alpha_{n_i}, f_{m_k}}(t, x) = u_{0i}(X_{n_i}(0, t, x)) + \int_0^t f_{m_k}(s, X_{n_i}(s, t, x))ds.
\]
As \( f_{m_k} \) is a continuous function
\[
\lim_{i \to \infty} u^{\alpha_{n_i}, f_{m_k}}(t, x) = u_0(X(0, t, x)) + \int_0^t f_{m_k}(s, X(s, t, x))ds.
\]
By Lemma 2.3
\[
\forall s \leq t \leq T, X(s, t, x) \in K \text{ and } m(\{s : X(s, t, x) \in K \setminus B\}) \leq \varepsilon \gamma.
\]
Then from (2.12),
\[
\left| \int_0^t f_{m_k}(s, X(s, t, x)) - f(s, X(s, t, x))ds \right| \leq \frac{\varepsilon}{2} T + 2\varepsilon \gamma S.
\]
Hence,
\[
u(t, x) = u_0(X(0, t, x)) + \int_0^t f(s, X(s, t, x))ds.
\]
In particular, this implies that \( u \) does not depend on \( \alpha_{n_k} \) and \( f_{m_k} \). Since \( \{X_n\} \) and \( \{u^{\alpha_{n_i} f_{m_k}}\} \) are relatively compact, we can conclude that
\[
\lim_{n \to \infty} X_n(s, t, x) = X(s, t, x) \text{ and } \lim_{n \to \infty} u^{\alpha_{n_i} f_{m_k}}(t, x) = u(t, x).
\]

**Step 3.** In this part, we will show that \( u \) is locally Lipschitz and satisfies (1.1) almost everywhere. Indeed, we already know that
\[
\forall n_i, m_k \in \mathbb{N}, u^{\alpha_{n_i} f_{m_k}}(t, x) + \alpha_{n_i}(t, x)u^{\alpha_{n_i} f_{m_k}} = f_{m_k}(t, x).
\]
We will first show that \( \frac{d}{dt} X_{n_i}(s, t, x) \) is locally bounded. From
\[
X_{n_i}(s, t, x) = x - \int_s^t \alpha_{n_i}(\tau, X_{n_i}(\tau, t, x))d\tau
\]
we can define \( \Psi_{h,n_i} : [0, t + h] \to \mathbb{R} \) by
\[
X_{n_i}(s, t + h, x) = X_{n_i}(\Psi_{h,n_i}(s), t, x).
\]
The function $\Psi_{h,n_i}$ is derivable and
\[
\Psi'_{h,n_i}(s) = \frac{\alpha_{n_i}(s, X_{n_i}(s, t + h, x))}{\alpha_{n_i}(\Psi_{h,n_i}(s), X_{n_i}(s, t + h, x))};
\]
s $\leq t + h$ implies that $\Psi_{h,n_i}(s) \leq t$. From
\[
x - \gamma(t + h - s) \leq X_{n_i}(s, t + h, x)
\]
and
\[
X_{n_i}(\Psi_{h,n_i}(s), t, x) \leq x - \gamma^{-1}(t - \Psi_{h,n_i}(s))
\]
we deduce that
\[
t - \gamma^2(t + h - s) \leq \Psi_{h,n_i}(s).
\]
Consequently, for $t < T$ and $h$ small enough, $\Psi_{h,n_i}(s) \in [(1 - \gamma^2)T, T]$ and
\[
|\Psi'_{h,n_i}(s) - 1| \leq \gamma M |\Psi_{h,n_i}(s) - s|.
\]
From Gronwall’s lemma it follows that
\[
|\Psi_{h,n_i}(s) - s| \leq e^{M(t+h-s)} |\Psi_{h,n_i}(t + h) - (t + h)|.
\]
As $\Psi_{h,n_i}(t + h) = t$,
\[
|\Psi_{h,n_i}(s) - s| \leq e^{M(t+h-s)} |h|.
\]
Using the change of variable $\tau = \Psi_{h,n_i}(u)$
\[
\int_s^t \alpha_{n_i}(\tau, X_{n_i}(\tau, t, x))d\tau = \int_{\Psi_{h,n_i}^{-1}(s)}^{\Psi_{h,n_i}^{-1}(t)} \alpha_{n_i}(\Psi_{h,n_i}(u), X_{n_i}(\Psi_{h,n_i}(u), t, x))\Psi'_{h,n_i}(u)du
\]
\[
= \int_{\Psi_{h,n_i}^{-1}(s)}^{t+h} \alpha_{n_i}(\Psi_{h,n_i}(u), X_{n_i}(u, t + h, x))\Psi'_{h,n_i}(u)du;
\]
then
\[
X_{n_i}(s, t + h, x) - X_{n_i}(s, t, x) = \int_s^{t+h} \alpha_{n_i}(\tau, X_{n_i}(\tau, t + h, x)) - \alpha_{n_i}(\Psi_{h,n_i}(\tau), X_{n_i}(\tau, t + h, x))d\tau
\]
\[
+ \int_s^{t+h} \alpha_{n_i}(\Psi_{h,n_i}(\tau), X_{n_i}(\tau, t + h, x))(\Psi'_{h,n_i}(\tau) - 1)d\tau
\]
\[
+ \int_{\Psi_{h,n_i}^{-1}(s)}^{\Psi_{h,n_i}^{-1}(t)} \alpha_{n_i}(\Psi_{h,n_i}(u), X_{n_i}(\Psi_{h,n_i}(u), t + h, x))\Psi'_{h,n_i}(u)d\tau = I_1 + I_2 + I_3
\]
and

\[ |I_1| \leq (t+h+s)e^{M(t+h-s)}|h|, \quad |I_2| \leq \alpha M(t+h-s)e^{M(t+h-s)}|h| \]

\[ |I_3| \leq \alpha (1 + Me^{M(t+h)})e^{M(t+h-s)}|h|. \]

Consequently,

\[ |X_{n_i}(s, t+h, x) - X_{n_i}(s, t, x)| \leq C|h|, \]

where \( C \) is a constant depending only on \( K \times [(1-\gamma^2)T, T] \). We can conclude that \( \frac{d}{dt}X_{n_i} \) is locally bounded. Moreover, as the \( X_{n_i} \) satisfy

\[ \frac{d}{dt}X_{n_i}(s, t, x) + \alpha_{n_i}(t, x)\frac{d}{dx}X_{n_i}(s, t, x) = 0 \]

\( \frac{d}{dt}X_{n_i}(s, t, x) \) is also locally bounded. Consequently, \( X \) is locally Lipschitz.

We now will show that \( u_{i}^{\alpha_{n_i}, f_{m_k}} \) is locally bounded. We first remark that

\[ \int_{0}^{t+h} f_{m_k}(s, X_{n_i}(s, t+h, x))ds = \int_{0}^{t+h} f_{m_k}(s, X_{n_i}(\Psi_{h,n_i}(s), t, x))ds \]

and

\[ \int_{0}^{t} f_{m_k}(s, X_{n_i}(s, t, x))ds \]

\[ = \int_{0}^{t+h} f_{m_k}(\Psi_{h,n_i}(u), X_{n_i}(\Psi_{h,n_i}(u), t, x))\Psi'_{h,n_i}(u)du; \]

we deduce that

\[ \int_{0}^{t+h} f_{m_k}(s, X_{n_i}(\Psi_{h,n_i}(s), t, x))ds - \int_{0}^{t} f_{m_k}(s, X_{n_i}(s, t, x))ds \]

\[ = \int_{0}^{t+h} f_{m_k}(u, X_{n_i}(\Psi_{h,n_i}(u), t, x)) - f_{m_k}(\Psi_{h,n_i}(u), X_{n_i}(\Psi_{h,n_i}(u), t, x))du \]

\[ + \int_{0}^{t+h} f_{m_k}(\Psi_{h,n_i}(u), X_{n_i}(\Psi_{h,n_i}(u), t, x))(1 - \Psi'_{h,n_i}(u))du \]

\[ - \int_{\Psi_{h,n_i}^{-1}(0)}^{\Psi_{h,n_i}(0)} f_{m_k}(\Psi_{h,n_i}(u), X_{n_i}(\Psi_{h,n_i}(u), t, x))\Psi'_{h,n_i}(u)du. \]

Then

\[ \left| \int_{0}^{t+h} f_{m_k}(s, X_{n_i}(\Psi_{h,n_i}(s), t, x))ds - \int_{0}^{t} f_{m_k}(s, X_{n_i}(s, t, x))ds \right| \]

\[ \leq \int_{0}^{t+h} Qe^{M(t+h-u)}|h|du + \int_{0}^{t+h} S\gamma Me^{M(t+h-u)}|h|du + \int_{0}^{\Psi_{h,n_i}(0)} Sdu \]
\[ \leq |h|(Q + S\gamma M)(e^{M(t+h)} - 1)(t + h) + Se^{M(t+h)}|h|. \]

Moreover,
\[ \frac{d}{dt}u_n^i(0, t, x) \leq FC. \]

Hence, \( u_{n_i, f} \) is locally bounded, and consequently \( u_{x, f} \) is also locally bounded. We can conclude that \( u \) is locally Lipschitz and \( \{u_{n_k, f}^i\} \) is bounded in \( W^{1,\infty}([0, T] \times [a, b]) \). On the other hand, \( W^{1,\infty}([0, T] \times [a, b]) = W^{1,1}([0, T] \times [a, b])^* \). Thus there exist \( u_{n_i, f} \) converging to \( u \) in the weak * topology; i.e.,
\[ \forall v \in W^{1,1}([0, T] \times [a, b]), \quad \langle u_{n_i, f}^i, v \rangle \to \langle v, v \rangle. \]

In particular, for all \( v \in W^{1,\infty}([0, T] \times [a, b]) \), \( \langle u_{n_i, f}^i, v \rangle \to \langle u, v \rangle. \) Hence, the \( u_{n_i, f}^i \) converge to \( u \) weakly in \( W^{1,1}([0, T] \times [a, b]) \).

Consider the closed, convex sets
\[ A_\varepsilon = \{ w \in W^{1,\infty}([0, T] \times [a, b]) : \|w_t + \alpha(t, x)w_x - f\|_{L^1} \leq \varepsilon \} . \]

Since
\[ \left\| u_t^{n_i, f} + \alpha u_x^{n_i, f} \right\|_{L^1} \leq \|\alpha - \alpha_n\|_{L^1} \left\| u_x^{n_i, f} \right\|_{L^\infty} + \|f - f\|_{L^1}, \]
for all \( i \) large enough \( u_{n_i, f}^i \in A_\varepsilon \). From the Mazur theorem we deduce that \( u \in A_\varepsilon \) for every \( \varepsilon > 0 \). Consequently,
\[ u(t, x) + \alpha(t, x)u_x(t, x) = f(t, x) \quad \text{a.e.} \]

3. **Uniqueness theorem**

**Theorem 3.4.** The problem
\[ \begin{cases} u_t(t, x) + \alpha(t, x)u_x(t, x) = f(t, x) \text{ almost everywhere} \\ u(0, x) = u_0(x) \end{cases} \tag{3.1} \]

has a unique locally Lipschitz continuous solution. Furthermore, it is given by the formula
\[ u(t, x) = u_0(X(0, t, x)) + \int_0^t f(s, X(s, t, x)) ds, \]
where \( X(s, t, x) \) is defined in the previous section.
Proof. Theorem 2.1 proves the existence of a solution to (3.1). Let us consider \( u_1 \) and \( u_2 \), two solution to our problem. We introduce the continuous function \( v = u_1 - u_2 \) satisfying
\[
\begin{cases}
  v_t(t, x) + \alpha(t, x)v_x(t, x) = 0 & \text{almost everywhere} \\
  v(0, x) = 0.
\end{cases}
\]
We suppose that there exists \((T, x)\) such that \( v(T, x) > 0 \). As \( v \) is continuous, there exists \( \delta > 0 \) such that for all \( y \) satisfying \(|y - x| \leq \delta\), \( v(T, y) > 0 \). We denote \( D = (x - \delta, x + \delta) \). Let \( X \) be the locally Lipschitz function from Theorem 2.1; let us consider the locally Lipschitz function \( h(t, y) = v(t, X(t, T, y)) \). \( h_t \) exists almost everywhere on \( D \times [0, T] \) and is equal to 0. Hence,
\[
\int_D \int_0^T h_t(t, y) dt \, dy = \int_D v(T, y) dx > 0.
\]
This leads to a contradiction.

Corollary 3.5. (Stability of solutions) Let \( \alpha_n \in L^1(\mathbb{R}^2) \) such that
\[
\begin{align*}
  i) \quad & \alpha_n \rightarrow \alpha \quad \text{in} \quad L^1_{\text{loc}}, \quad f_n \rightarrow f \quad \text{in} \quad L^1_{\text{loc}} \\
  ii) \quad & \exists \gamma > 0 \quad \text{such that} \quad \gamma^{-1} \leq \alpha_n(t, x) \leq \gamma \quad \text{a.e. in} \quad \mathbb{R}^2 \\
  iii) \quad & \forall T > 0, \; \forall T' > 0 \; \text{and all compact} \; K \subset \mathbb{R} \quad \text{we have} \quad \sup_{x \in K, t \in [-T', T], n \geq 1} \left\| \frac{\partial \alpha_n}{\partial t}(t, x) \right\| < \infty.
\end{align*}
\]
Then the locally Lipschitz maps \( u_n \), solutions of
\[
\begin{cases}
  u_t(t, x) + \alpha_n(t, x)u_x(t, x) = f(t, x) & \text{almost everywhere} \\
  u(0, x) = u_0(x),
\end{cases}
\]
converge uniformly on all compact sets to the solution of problem (3.1).

4. COMPARISON WITH THE RESULT OF DA PRATO AND SINESTRARI

We consider the following first-order partial differential equation:
\[
\begin{cases}
  u_t(t, x) + a(t, x)u_x(t, x) = f(t, x), \; t \in I, \; x \in J \\
  u(t_0, x) = u_0(x), \quad u(t, 0) = u(t, l),
\end{cases}
\]
where \( I = [t_0, T], \; J = [0, l] \). We will denote by \( E \) the Banach space \( L^\infty(J) \) with the norm
\[
\| u \|_\infty = \text{ess sup}_{x \in J} |u(x)|
\]
and by \( D \) the Banach space \( \text{Lip}_*(J) = \{ f \in \text{Lip}(J); f(t_0) = f(T) \} \) with the norm \( \| u \|_\infty + \| u' \|_\infty \). Let \( a : J \rightarrow \mathbb{R}_+^* \) be such that \( a, \; a^{-1} \in L^\infty \) and define
A : D \subset E \rightarrow E by setting \((Au)(x) = -a(x)u'(x), u \in D, x \in J\). We recall that a function \(u \in C(t_0, T, E)\) is called an F-solution in \(L^1\) of
\[
    u'(t) = A(t)u(t) + f(t) \quad t \in [t_0, T], \quad u(t_0) = u_0
\]
if for each \(k \in \mathbb{N}\), there is \(u_k \in W^{1,1}(t_0, T, E) \cap L^1(t_0, T, D)\) such that
\[
    \lim_{k \rightarrow \infty} (\|u_k(0) - u_0\|_E + \|u - u_k\|_{L^1(t_0, T, E)} + \|u' - A(\cdot)u_k - f\|_{L^1(t_0, T, E)}) = 0.
\]

**Proposition 4.6.** Let \(a \in \text{Lip}(I; L^\infty(J))\) be such that for each \(t \in I\) we have
\[
    \gamma^{-1} < a(t, x) < \gamma \quad \text{for a.e. } x \in J
\]
and let \(f \in W^{1,1}(I; L^\infty)\) and \(u_0 \in \text{Lip}_*(J)\). The unique solution to problem (4.1) is an F-solution in \(L^1\).

**Proof.** Let \(\alpha_k, f_k,\) and \(u_0^k\) be any sequences satisfying the hypotheses of Theorem 2.1. We denote by \(u_k(t)\) the solution \(u^{\alpha_k, J_k}(t, \cdot)\) with \(u(t_0) = u_0^k\), and we define \((A_k(t))u(x) = -\alpha_k(t, x)u'(x), u \in D, x \in J\). Then
\[
    u'_k = A_k(\cdot)u_k + f_k;
\]
consequently,
\[
    \lim_{k \rightarrow \infty} \|[u_k(0) - u_0]_E + \|u - u_k\|_{L^1} + \|u'_k - A(\cdot)u_k - f\|_{L^1}
    = \lim_{k \rightarrow \infty} \|[A_k(\cdot) - A(\cdot)]u_k + f_k - f\|_{L^1}.
\]
By definition
\[
    ([A_k(t) - A(t)]u_k)(x) = [-\alpha_k(t, x) + \alpha(t, x)]u'_k(x)(t)
\]
and \(u'_k(x)(t) = (u_k)x(t, x)\). Consequently, \(\|u'_k(x)\|\) is bounded. So
\[
    \lim_{k \rightarrow \infty} \|[A_k(\cdot) - A(\cdot)]u_k + f_k - f\|_{L^1} = 0,
\]
which ends the proof of the proposition. \(\square\)

It is shown in [1] that under different hypothesis, the F-solution is unique. The above Proposition yields the existence of an F-solution. Our results allow therefore a representation formula for F-solutions.

**References**
