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ON A NONHOMOGENEOUS BI-LAYER SHALLOW-WATER PROBLEM: AN EXISTENCE THEOREM

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Abstract. In this paper we prove the existence of a solution for a nonhomogeneous bi-layer shallow-water model in depth-mean velocity formulation. In [7] the homogeneous case was studied. The main difficulties in the nonhomogeneous case arise from the treatment of the boundary terms.

1. Introduction

The problem that motivated our study is the modelling of the dynamics of water masses in the Alboran Sea and the Strait of Gibraltar (the westernmost part of the Mediterranean Sea). In this sea, two layers of water can be distinguished: the surface Atlantic water penetrating into the Mediterranean through the Strait of Gibraltar, and the deeper, denser Mediterranean water flowing into the Atlantic. Observation of this simplified picture shows that, if a bi-dimensional model is used to simulate the flow in this region, it is necessary to consider, at least, a two-layer model.

Here, we propose a model that considers sea water as being composed of two immiscible layers with different constant densities. In such a model, waves appear not only on the surface but also at the interface between the layers. It is assumed that the phenomena to be modelled have wavelengths large enough to make an appropriate shallow-water approximation in each layer. Therefore, the partial differential equations system to be studied is a coupled system of shallow-water equations.

The analysis developed in this paper is based on the techniques used by P. Orenga and F.J. Chatelon, in collaboration with P.L. Lions, in the study of the one-layer model. In [2], these authors presented a theorem for the existence of a solution for the shallow-water problem with nonhomogeneous

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boundary conditions, based on a previous result corresponding to the homogeneous problem [10]. In [3], they also presented some smoothness and uniqueness results for the homogeneous problem. In [7] we applied similar techniques to prove some results for the existence, smoothness, and uniqueness of a solution for the homogeneous bi-layer model. The main difficulty found in that case was the treatment of the terms coupling the equations of both layers. In this paper we prove the existence of a solution for the nonhomogeneous bi-layer problem. The main difficulty in this case arises in the treatment of the boundary terms.

1.1. **Positioning the problem.** Let Ω be a fixed, bounded, and simply connected open domain of $\mathbb{R}^2$ with the boundary Γ sufficiently smooth. In oceanographic applications, Ω is the domain corresponding to the surface of the sea assumed to be at rest. We denote by $x = (x_1, x_2)$ a point in Ω, by n the exterior unit-normal vector to Ω on Γ, and by $t \in [0,T]$ the time during which the flux is studied.

We consider a system composed of two layers of superposed fluids with densities ρ_1 and ρ_2 ($\rho_2 < \rho_1$). In what follows, index 1 refers to the deeper layer, and index 2 to the upper layer of the fluid. Let A_1 and A_2 be the respective coefficients of viscosity for each layer and g the acceleration of gravity.

We denote by u_1 and u_2 the velocity vector fields and by h_1 and h_2 the thickness of the lower and upper layer, respectively.

For $i = 1, 2$, we can split the boundary Γ into three parts: Γ_e, corresponding to the coasts; Γ_i^+, where the flux is going out the i-layer, and Γ_i^-, where the flux is coming into the i-layer. Notice that $u_i \cdot n = 0$ on Γ_e, $u_i \cdot n > 0$ on Γ_i^+, and $u_i \cdot n < 0$ on Γ_i^-.

Then, let $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, $\Sigma_i^+ = \Gamma_i^+ \times (0,T)$, and $\Sigma_i^- = \Gamma_i^- \times (0,T)$, $i = 1, 2$.

If $\varphi = (\varphi_1, \varphi_2)$ is a vector function from Ω into $\mathbb{R}^2$ and q is a scalar function from Ω into $\mathbb{R}$, we define the operators α, Curl, and curl as follows:

$$\alpha(\varphi) = \begin{pmatrix} -\varphi_2 \\ \varphi_1 \end{pmatrix}, \quad \operatorname{Curl} q = \begin{pmatrix} \frac{\partial q}{\partial x_2} \\ -\frac{\partial q}{\partial x_1} \end{pmatrix}, \quad \operatorname{curl} \varphi = \frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2}.$$

The problem we study, which will be referred to as $(\mathcal{P})$ in the sequel, is the following [6]:

$$\frac{\partial u_1}{\partial t} - A_1 \Delta u_1 + \frac{1}{2}\nabla u_1^2 + \operatorname{curl} u_1 \alpha(u_1) + g\nabla h_1 + g\frac{\rho_2}{\rho_1}\nabla h_2 = f_1 \qquad \text{in } Q,$$
$$u_1 \cdot n = G_1 \qquad \text{on } \Sigma,$$

$$\begin{aligned}
\operatorname{curl} u_1 &= 0 && \text{on } \Sigma,\\
u_1(t=0) &= u_{1,0} && \text{in } \Omega,\\
\frac{\partial h_1}{\partial t} + \operatorname{div}(u_1 h_1) &= 0 && \text{in } Q,\\
h_1 &= \mu_1 && \text{on } \Sigma_1^-,\\
h_1(t=0) &= h_{1,0} && \text{in } \Omega,\\
\frac{\partial u_2}{\partial t} - A_2\Delta u_2 + \frac{1}{2}\nabla u_2^2 + \operatorname{curl} u_2\alpha(u_2) + g\nabla h_2 + g\nabla h_1 &= f_2 && \text{in } Q,\\
u_2 \cdot n &= G_2 && \text{on } \Sigma,\\
\operatorname{curl} u_2 &= 0 && \text{on } \Sigma,\\
u_2(t=0) &= u_{2,0} && \text{in } \Omega,\\
\frac{\partial h_2}{\partial t} + \operatorname{div}(u_2 h_2) &= 0 && \text{in } Q,\\
h_2 &= \mu_2 && \text{on } \Sigma_2^-,\\
h_2(t=0) &= h_{2,0} && \text{in } \Omega,
\end{aligned}$$

where $u_{i,0}$ and $h_{i,0} > 0$ are the initial conditions for velocities and depths, respectively, $i = 1, 2$.

As usual, we are going to make a change of variables in order to obtain a homogeneous problem.

Set G_i and $\frac{\partial G_i}{\partial t}$ in $L^2(0,T;H^{1/2}(\Gamma))$, for $i = 1, 2$. Then we solve, for every $t \in [0,T]$, the scalar problem

$$\begin{cases} -\Delta s_i(t) = \tilde{f}_i(t) & \in L^\infty(\Omega),\\ \dfrac{\partial s_i}{\partial n}(t) = G_i(t) & \in H^{1/2}(\Gamma), \end{cases}$$

where $\tilde{f}_i$ is chosen in $H^1(0,T;L^\infty(\Omega))$ and satisfying the condition $\int_\Omega \tilde{f}_i + \int_\Gamma G_i = 0$. We can choose s_i such that $\int_\Omega s_i = 0$. The function $w_i = \nabla s_i$ satisfies $w_i \in H^1(0,T;H^1(\Omega)^2)$, $\operatorname{div} w_i \in H^1(0,T;L^\infty(\Omega))$, $\operatorname{curl} w_i = 0$ in $\overline{\Omega}$, and $w_i \cdot n = G_i$ on Γ, for $i = 1, 2$.

Setting $v_i = u_i - w_i$, $i = 1, 2$, we formulate the problem $(\mathcal{P})$ as the following problem $(\mathcal{P})'$:

$$\begin{aligned}
&\frac{\partial(v_1+w_1)}{\partial t} - A_1\Delta(v_1+w_1) + \frac{1}{2}\nabla(v_1+w_1)^2\\
&+\operatorname{curl}(v_1+w_1)\alpha(v_1+w_1) + g\nabla h_1 + g\frac{\rho_2}{\rho_1}\nabla h_2 = f_1 \qquad \text{in } Q,\\
&v_1 \cdot n = 0 \qquad \text{on } \Sigma,
\end{aligned}$$

$$
\begin{aligned}
\operatorname{curl} v_1 &= 0 && \text{on } \Sigma,\\
v_1(t=0) &= v_{1,0} && \text{in } \Omega,\\
\frac{\partial h_1}{\partial t} + \operatorname{div}(v_1 h_1) + \operatorname{div}(w_1 h_1) &= 0 && \text{in } Q,\\
h_1 &= \mu_1 && \text{on } \Sigma_1^-,\\
h_1(t=0) &= h_{1,0} && \text{in } \Omega,\\
\frac{\partial(v_2+w_2)}{\partial t} - A_2\Delta(v_2+w_2) + \frac{1}{2}\nabla(v_2+w_2)^2 &&&\\
+\operatorname{curl}(v_2+w_2)\alpha(v_2+w_2) + g\nabla h_2 + g\nabla h_1 &= f_2 && \text{in } Q,\\
v_2\cdot n &= 0 && \text{on } \Sigma,\\
\operatorname{curl} v_2 &= 0 && \text{on } \Sigma,\\
v_2(t=0) &= v_{2,0} && \text{in } \Omega,\\
\frac{\partial h_2}{\partial t} + \operatorname{div}(v_2 h_2) + \operatorname{div}(w_2 h_2) &= 0 && \text{in } Q,\\
h_2 &= \mu_2 && \text{on } \Sigma_2^-,\\
h_2(t=0) &= h_{2,0} && \text{in } \Omega,
\end{aligned}
$$

where $v_{i,0} = u_{i,0} - w_i(0)$, $i = 1, 2$.

1.2. **Weak formulation.** We will denote by $(\cdot,\cdot)$ the scalar product of $L^2(\Omega)$ and $L^2(\Omega)^2$ and by $\|\cdot\|_{W^{m,p}}$ the usual norm in $W^{m,p}(\Omega)$ and $W^{m,p}(\Omega)^2$.

Let V be the space

$$V = \{v \in L^2(\Omega)^2,\ \operatorname{div} v \in L^2(\Omega),\ \operatorname{curl} v \in L^2(\Omega),\ v\cdot n = 0 \text{ on } \Gamma\}.$$

As Ω is simply connected, we can consider in V the norm given by

$$\|v\|_V^2 = \|\operatorname{div} v\|_{L^2}^2 + \|\operatorname{curl} v\|_{L^2}^2.$$

Recall that V is algebraically and topologically included in the space $\{v \in H^1(\Omega)^2,\ v\cdot n = 0 \text{ on } \Gamma\}$, and that the bilinear form

$$a(u,v) = (\operatorname{div} u, \operatorname{div} v) + (\operatorname{curl} u, \operatorname{curl} v)$$

is elliptic.

Let us consider the problem $(\mathcal{P})'$ under the following weak formulation: Find (v_1, h_1) and (v_2, h_2) in $[L^\infty(0,T; L^2(\Omega)^2) \cap L^2(0,T;V)] \times [L^\infty(0,T; L^1(\Omega)) \cap L^2(Q)]$ such that $h_1 > 0$, $h_2 > 0$, and the following system, which will be referred as $(\mathcal{V})$, holds:

$$\Big(\frac{\partial v_1}{\partial t}, \varphi\Big) + A_1 a(v_1,\varphi) - \frac{1}{2}(v_1^2, \operatorname{div}\varphi) + (\operatorname{curl} v_1 \alpha(v_1), \varphi) - g(h_1, \operatorname{div}\varphi)$$

$$-g\frac{\rho_2}{\rho_1}(h_2, \operatorname{div}\varphi) - (v_1 w_1, \operatorname{div}\varphi) + (\operatorname{curl} v_1 \alpha(w_1), \varphi) = (f_1, \varphi)$$
$$-\big(\frac{\partial w_1}{\partial t}, \varphi\big) - A_1 a(w_1, \varphi) + \frac{1}{2}(w_1^2, \operatorname{div}\varphi) \quad \forall \varphi \in V,$$
$$\big(\frac{\partial v_2}{\partial t}, \varphi\big) + A_2 a(v_2, \varphi) - \frac{1}{2}(v_2^2, \operatorname{div}\varphi) + (\operatorname{curl} v_2 \alpha(v_2), \varphi) - g(h_2, \operatorname{div}\varphi)$$
$$-g(h_1, \operatorname{div}\varphi) - (v_2 w_2, \operatorname{div}\varphi) + (\operatorname{curl} v_2 \alpha(w_2), \varphi) = (f_2, \varphi)$$
$$-\big(\frac{\partial w_2}{\partial t}, \varphi\big) - A_2 a(w_2, \varphi) + \frac{1}{2}(w_2^2, \operatorname{div}\varphi) \quad \forall \varphi \in V,$$
$$\frac{\partial h_1}{\partial t} + \operatorname{div}(v_1 h_1) + \operatorname{div}(w_1 h_1) = 0 \quad \text{in } L^1(0, T; W^{-1,p}(\Omega)), \quad p < 2,$$
$$\frac{\partial h_2}{\partial t} + \operatorname{div}(v_2 h_2) + \operatorname{div}(w_2 h_2) = 0 \quad \text{in } L^1(0, T; W^{-1,p}(\Omega)), \quad p < 2,$$
$$v_1(t=0) = v_{1,0} \in V, \qquad v_2(t=0) = v_{2,0} \in V,$$
$$h_1 = \mu_1 \text{ on } \Sigma_1^-, \qquad h_2 = \mu_2 \text{ on } \Sigma_2^-,$$
$$h_1(t=0) = h_{1,0} \in L^2(\Omega), \qquad h_2(t=0) = h_{2,0} \in L^2(\Omega).$$

The orthogonal decomposition of $L^2(\Omega)^2$ in a sum of gradient vectors and curl vectors [4, 8], $L^2(\Omega)^2 = \nabla H^1(\Omega) \oplus \operatorname{Curl} H_0^1(\Omega)$, will be used to look for v_1 and v_2 in V under the form

$$v_1 = v_{p,1} + v_{q,1} = \nabla p_1 + \operatorname{Curl} q_1, \quad v_2 = v_{p,2} + v_{q,2} = \nabla p_2 + \operatorname{Curl} q_2,$$

with p_i and q_i solutions of the scalar problems

$$\begin{cases} \Delta p_i = \operatorname{div} v_i & \text{in } \Omega, \\ \dfrac{\partial p_i}{\partial n} = v_i \cdot n = 0 & \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta q_i = \operatorname{curl} v_i & \text{in } \Omega, \\ q_i = 0 & \text{on } \Gamma, \end{cases}$$

$i = 1, 2$. The functions p_i can be chosen such that $\int_\Omega p_i = 0$. We must also remember that $\operatorname{curl} v_{p,i} = 0$ and $\operatorname{div} v_{q,i} = 0$, $i = 1, 2$.

In order to simplify the notation, we consider only the simply connected case. The additional difficulty appearing in the multiply connected domain is solved by taking into account a dissipation condition at the bottom [9]. In this case, some new functions must be added to the decomposition: the functions $\operatorname{curl} r$ with r solutions of the following m problems:

$$-\Delta r_i = 0 \text{ in } \Omega, \quad r_i = 1 \text{ on } \Gamma_i, \quad r_i = 0 \text{ on } \Gamma_j,\ j \neq i,$$

where $i = 1, \dots, m$ and $j = 0, \dots, m$. We have assumed that Γ has a finite number of connected components Γ_i, $i = 0, \dots, m$, Γ_0 indicating the boundary of the infinite connected component of the complement of Ω in $\mathbb{R}^2$.

2. An existence theorem

In this section, we present a global existence result with controlled data.

2.1. **Theorem.** Preliminaries. Let C be the best constant associated with Gagliardo-Nirenberg's inequality:

$$\|u\|_{L^4}^2 \leq C\|u\|_{L^2}\|u\|_V, \quad \forall u \in V,$$

and $\widetilde{C}$ the injection constant of $H^1(\Omega)^2$ into $L^4(\Omega)^2$:

$$\|w\|_{L^4} \leq \widetilde{C}\|w\|_{H^1}, \quad \forall w \in H^1(\Omega)^2.$$

Let C' be the injection constant of $\{\Theta \in W^{1,1}(\Omega) : \int_\Omega \Theta = 0\}$ into $L^2(\Omega)$:

$$\|\Theta\|_{L^2} \leq C'\|\nabla\Theta\|_{L^1}, \quad \forall \Theta \in W^{1,1}(\Omega) : \int_\Omega \Theta = 0.$$

Now set, for $i = 1, 2$, a_i the injection constant of $H^{3/2}(\Gamma_i^-)$ into $L^\infty(\Gamma_i^-)$, and b the constant such that

$$\|p_i\|_{H^2} \leq b\|\operatorname{div} v_i\|_{L^2},$$

with p_i the solution of

$$\begin{cases} \Delta p_i = \operatorname{div} v_i & \text{in } \Omega, \\ \dfrac{\partial p_i}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

We consider the N-function $\Phi(x) = e^{x^2} - 1$ and the associated Orlicz space $L_\Phi(\Omega)$, which is a Banach space with the Orlicz norm, denoted by $\|\cdot\|_{L_\Phi}$. The Sobolev space $H^1(\Omega)$ is embedded in $L_\Phi(\Omega)$ [1]. Let k be the injection constant:

$$\|p\|_{L_\Phi} \leq k\|p\|_{H^1}, \quad \forall p \in H^1(\Omega).$$

It is not possible to give an analytical expression for Ψ, the complementary N-function to Φ. However, it can be shown that Ψ is equivalent to $\tilde{\Psi}$, with

$$\tilde{\Psi}(x) = x\sqrt{\log^+ x}.$$

Let $\|\cdot\|_{L_\Psi}$ and $\|\cdot\|_{L_{\tilde{\Psi}}}$ be the Orlicz norms in the Orlicz spaces $L_\Psi(\Omega)$ and $L_{\tilde{\Psi}}(\Omega)$, respectively. The equivalence relation between the N-functions Ψ and $\tilde{\Psi}$ implies the equivalence between the norms $\|\cdot\|_{L_\Psi}$ and $\|\cdot\|_{L_{\tilde{\Psi}}}$ and allows us to identify the spaces $L_\Psi(\Omega)$ and $L_{\tilde{\Psi}}(\Omega)$. Let k' be the best constant such that

$$\|h\|_{L_\Psi} \leq k'\|h\|_{L_{\tilde{\Psi}}}, \quad \forall h \in L_\Psi(\Omega) = L_{\tilde{\Psi}}(\Omega).$$

Finally, denote by k'' the best constant such that

$$\|p\|_{H^1} \leq k''\|\nabla p\|_{L^2}, \quad \forall p \in H^1(\Omega) : \int_\Omega p = 0.$$

Now we can define $K = kk'k''$. On the other hand, $H^{1/2}(\Gamma)$ is embedded in $L_\Phi(\Gamma)$ [1]. Let κ be the injection constant:

$$\|p\|_{L_\Phi(\Gamma)} \leq \kappa\|p\|_{H^{1/2}(\Gamma)}, \quad \forall p \in H^{1/2}(\Gamma).$$

The equivalence between the N-functions Ψ and $\tilde{\Psi}$ is still valid: let κ' be the best constant such that

$$\|h\|_{L_\Psi(\Gamma)} \leq \kappa'\|h\|_{L_{\tilde{\Psi}}(\Gamma)}, \quad \forall h \in L_\Psi(\Gamma) = L_{\tilde{\Psi}}(\Gamma).$$

Finally, we define $\mathcal{K} = \kappa\kappa' k''$.

Conditions of the theorem. We define the positive[1] constants

$$C_1 = \frac{\|h_{1,0}\|_{L^1} - \int_{\Sigma_1^-} G_1\mu_1}{\text{meas}(\Omega)}, \tag{2.1}$$

$$C_1' = \frac{\|h_{1,0}\|_{L^1} - \int_{\Sigma_1^-} G_1\mu_1}{\text{meas}(\Omega)}\Big(\|w_1\|^2_{L^2(Q)} + A_1\|\text{div}\, w_1\|_{L^1(Q)} + gT\Big(\|h_{1,0}\|_{L^1} - \int_{\Sigma_1^-} G_1\mu_1\Big) + gT\frac{\rho_2}{\rho_1}\Big(\|h_{2,0}\|_{L^1} - \int_{\Sigma_2^-} G_2\mu_2\Big)\Big) \tag{2.2}$$

and

$$C_2 = \frac{\|h_{2,0}\|_{L^1} - \int_{\Sigma_2^-} G_2\mu_2}{\text{meas}(\Omega)}, \tag{2.3}$$

$$C_2' = \frac{\|h_{2,0}\|_{L^1} - \int_{\Sigma_2^-} G_2\mu_2}{\text{meas}(\Omega)}\Big(\|w_2\|^2_{L^2(Q)} + A_2\|\text{div}\, w_2\|_{L^1(Q)} + gT\Big(\|h_{2,0}\|_{L^1} - \int_{\Sigma_2^-} G_2\mu_2\Big) + gT\Big(\|h_{1,0}\|_{L^1} - \int_{\Sigma_1^-} G_1\mu_1\Big)\Big). \tag{2.4}$$

We also define the quantities

$$B_1 = A_1 - \varepsilon - 2\varepsilon' - C_1 - g\frac{\rho_2}{\rho_1}\frac{1}{2\mu} - 2\widetilde{C}\|w_1\|_{L^\infty(0,T;L^4(\Omega)^2)} - \frac{5C'^2}{4\lambda}\|w_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)} \tag{2.5}$$

[1] Notice that $G_i|_{\Sigma_i^-} < 0$, $i = 1, 2$.

and

$$B_2 = A_2 - \varepsilon - 2\varepsilon' - C_2 - g\frac{1}{2\mu} - 2\widetilde{C}\|w_2\|_{L^\infty(0,T;L^4(\Omega)^2)} - \frac{5C'^2}{4\lambda}\|w_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}, \tag{2.6}$$

where ε, ε', λ, and μ are positive numbers,

$$\mathbb{K}_i = \frac{\sqrt{C^2 + 16B_iC_\lambda} - C}{4C_\lambda}, \quad i = 1, 2, \tag{2.7}$$

where

$$C_\lambda = \frac{5(C^2 + 4C'^2)}{16\lambda}, \tag{2.8}$$

and, finally,

$$\begin{aligned}
\mathbb{C}_{\varepsilon,\varepsilon',\lambda} = \sum_{i=1}^{2} \Big[&\frac{1}{2}\|v_{i,0}\|^2_{L^2} + (g + A_i)\int_\Omega h_{i,0}\log h_{i,0} + \frac{A_i}{e}\mathrm{meas}(\Omega) \\
&+ (g + A_i)\|G_i\|_{L^2(0,T;H^{1/2}(\Gamma_i^-))}\|\mu_i \log \mu_i\|_{L^2(0,T;H^{-1/2}(\Gamma_i^-))} \\
&+ \frac{1}{\varepsilon}\Big(\|f_i\|^2_{L^2(0,T;V')} + \|w_{i,t}\|^2_{L^2(Q)} + A_i^2\|w_i\|^2_{L^2(0,T;H^1(\Omega)^2)} + \frac{1}{4}\|w_i\|^4_{L^4(Q)}\Big) \\
&+ C_i' + \int_\Omega p_i(0)h_{i,0} + 2K^2 + \frac{\mathcal{K}^2T}{\varepsilon'} \\
&+ \frac{a_i^2b^2}{4\varepsilon'}\|G_i\|^2_{L^\infty(0,T;H^{1/2}(\Gamma_i^-))}\|\mu_i\|^2_{L^2(0,T;H^{-1/2}(\Gamma_i^-))} + \frac{5k''}{4\lambda}\|w_{i,t}\|^2_{L^2(Q)} \\
&+ \frac{5}{4\lambda}\|f_i\|^2_{L^2(Q)} + \Big(g\|\mathrm{div}\, w_i\|_{L^1(0,T;L^\infty(\Omega))} + \frac{\mathcal{K}^2}{4\varepsilon'} + (g + A_i)\|\log G_i\|_{L^\infty(\Sigma_i^+)}\Big) \\
&\times \Big(\|h_{i,0}\|_{L^1} - \int_{\Sigma_i^-} G_i\mu_i\Big)\Big],
\end{aligned} \tag{2.9}$$

which depends only on the data. Let us assume that the eddy viscosities are great enough to make

$$B_i > 0, \quad i = 1, 2. \tag{2.10}$$

Choose λ and μ such that

$$g > \lambda + g\frac{\mu}{2} \tag{2.11}$$

and ε' such that

$$\varepsilon' < \min_{i=1,2} \frac{g + A_i}{\mathbb{K}_i^2}. \tag{2.12}$$

Let us suppose that the data satisfy the smoothness hypotheses

$$f_i \in L^2(Q), \tag{2.13}$$

$$G_i \in H^1(0,T;H^{1/2}(\Gamma)), \text{ with } \log G_i \in L^\infty(\Sigma_i^+), \tag{2.14}$$

$$\mu_i \in L^2(0,T;H^{-1/2}(\Gamma_i^-)), \text{ with } \mu_i > 0 \text{ and } \mu_i \log \mu_i \in L^2(0,T;H^{-1/2}(\Gamma_i^-)), \tag{2.15}$$

$$v_{i,0} \in V, \tag{2.16}$$

$$h_{i,0} \in L^2(\Omega), \text{ with } h_{i,0} > 0 \text{ and } h_{i,0} \log h_{i,0} \in L^1(\Omega), \tag{2.17}$$

and the compatibility condition

$$u_{i,0} \cdot n = G_i(0), \tag{2.18}$$

for $i = 1, 2$. Finally, we assume the following hypotheses based on small-size data:

$$\|v_{i,0}\|_{L^2} < \mathbb{K}_i, \quad i = 1, 2, \tag{2.19}$$

$$\|h_{i,0}\|_{L^1} - \int_{\Sigma_i^-} G_i \mu_i < \frac{g}{2K^2}, \quad i = 1, 2, \tag{2.20}$$

and

$$\mathbb{C}_{\varepsilon,\varepsilon',\lambda} + \sum_{i=1}^{2} \frac{g + (g + A_i)T}{e} \text{meas}(\Omega) < \frac{1}{4} \min_{i=1,2} \mathbb{K}_i^2. \tag{2.21}$$

Theorem 1. *Let Ω be a simply connected, bounded, smooth, open domain of $\mathbb{R}^2$ with boundary Γ. If* (2.10)–(2.21) *are satisfied, then the weak problem $(\mathcal{V})$ has a solution $\{(v_1, h_1), (v_2, h_2)\}$ that satisfies the following estimate:*

$$\begin{aligned} &\|v_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \|v_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \|v_1\|^2_{L^2(0,T;V)} + \|v_2\|^2_{L^2(0,T;V)} \\ &+ \sup_t \int_\Omega h_1 \log h_1 + \sup_t \int_\Omega h_2 \log h_2 + \int_{\Sigma_1^+} G_1 h_1 \log(G_1 h_1) \\ &+ \int_{\Sigma_2^+} G_2 h_2 \log(G_2 h_2) + \|h_1\|^2_{L^2(Q)} + \|h_2\|^2_{L^2(Q)} \le \mathrm{C}, \end{aligned} \tag{2.22}$$

where $\mathrm{C} > 0$ depends on the initial data.

The proof of this theorem consists in obtaining some a priori estimates, then building a sequence of approximate solutions that satisfy these estimates, and, finally, passing to the limit in the continuity and momentum equations as in [10]. In this case, the main difficulty is to obtain an a priori estimate because of the coupled terms, as in the homogeneous case [7], and because of the boundary terms due to the nonhomogeneous boundary conditions.

2.2. **A priori estimates.**

Lemma 1. *If* $\{(v_1,h_1),(v_2,h_2)\}$ *is a classical solution of the problem* $(\mathcal{V})$ *and if the relations* (2.10)–(2.21) *are satisfied, then we have*

$$h_i > 0, \quad i=1,2, \tag{2.23}$$

$$\frac{d}{dt}\int_\Omega h_i = -\int_\Gamma G_i h_i, \quad i=1,2, \tag{2.24}$$

$$\begin{aligned}
&-\sum_{i=1}^{2}\frac{g+(g+A_i)T}{e} meas(\Omega) \le \frac{1}{4}\|v_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \frac{1}{4}\|v_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\\
&+\Big(B_1 - \frac{C}{2}\|v_1\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda\|v_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\Big)\|v_1\|^2_{L^2(0,T;V)}\\
&+\Big(B_2 - \frac{C}{2}\|v_2\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda\|v_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\Big)\|v_2\|^2_{L^2(0,T;V)}\\
&+\Big(g - 2K^2\Big(\|h_{1,0}\|_{L^1(\Omega)} - \int_{\Sigma_1^-} G_1\mu_1\Big)\Big)\sup_t\int_\Omega h_1\log^+ h_1 - g\sup_t\int_\Omega h_1\log^- h_1\\
&+\Big(g - 2K^2\Big(\|h_{2,0}\|_{L^1(\Omega)} - \int_{\Sigma_2^-} G_2\mu_2\Big)\Big)\sup_t\int_\Omega h_2\log^+ h_2 - g\sup_t\int_\Omega h_2\log^- h_2\\
&+(g+A_1)\int_{\Sigma_1^+} G_1h_1\log^+(G_1h_1) - (g+A_1)\int_{\Sigma_1^+} G_1h_1\log^-(G_1h_1)\\
&+(g+A_2)\int_{\Sigma_2^+} G_2h_2\log^+(G_2h_2) - (g+A_2)\int_{\Sigma_2^+} G_2h_2\log^-(G_2h_2)\\
&+\Big(g-\lambda-g\frac{\mu}{2}\Big)\|h_1\|^2_{L^2(Q)} + \Big(g-\lambda-g\frac{\rho_2}{\rho_1}\frac{\mu}{2})\|h_2\|^2_{L^2(Q)}\\
&\le \varepsilon'\|v_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\int_{\Sigma_1^+} G_1h_1\log^+(G_1h_1)\\
&+\varepsilon'\|v_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\int_{\Sigma_2^+} G_2h_2\log^+(G_2h_2) + \mathbb{C}_{\varepsilon,\varepsilon',\lambda}.
\end{aligned} \tag{2.25}$$

$$B_i - \frac{C}{2}\|v_i\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda\|v_i\|^2_{L^\infty(0,T;L^2(\Omega)^2)} > 0, \quad i=1,2. \tag{2.26}$$

Proof. The inequalities (2.23) are easily deduced from the identity

$$h_i(x,t) = K_i\, e^{-\int_0^t \operatorname{div} u_i(X_i(s),s)\,ds}, \tag{2.27}$$

where $X_i(s)$ is the trajectory of the particle of fluid that is at a point $x \in \Omega$ at instant t; i.e., it is the solution of the problem

$$\frac{dX_i}{dt} = u_i(X_i(t), t), \quad X_i(0) = x,$$

for $i = 1, 2$. If the particle was at $x_0 \in \Omega$ at $t = 0$, then $K_i = h_{i,0}(x_0)$, and if the particle comes into Ω at a point z_0 on boundary Γ_i^- at instant $t = \tau$, then $K_i = \mu_i(z_0, \tau)$.

The relations (2.24) are obtained by integration over Ω of the respective continuity equations:

$$\frac{d}{dt}\int_\Omega h_i + \int_\Gamma G_i h_i = 0.$$

We can obtain an estimate for h_i in $L^\infty(0,T;L^1(\Omega))$, integrating in time the previous equalities:

$$\|h_i\|_{L^\infty(0,T;L^1(\Omega))} + \|G_i h_i\|_{L^1(\Sigma_i^+)} \le \|h_{i,0}\|_{L^1} - \int_{\Sigma_i^-} G_i\mu_i, \quad i = 1, 2. \tag{2.28}$$

The first step to prove the estimate (2.25) is obtaining the energy inequalities, by taking $\varphi = v_1$ in $(\mathcal{V})_1$ and $\varphi = v_2$ in $(\mathcal{V})_2$:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_1\|_{L^2}^2 + A_1\|v_1\|_V^2 - \frac{1}{2}(v_1^2, \operatorname{div} v_1) - g(h_1, \operatorname{div} v_1) \\ &\quad -g\frac{\rho_2}{\rho_1}(h_2, \operatorname{div} v_1) = (v_1 w_1, \operatorname{div} v_1) - (\operatorname{curl} v_1 \alpha(w_1), v_1) \\ &\quad +(f_1, v_1) - \Big(\frac{\partial w_1}{\partial t}, v_1\Big) - A_1 a(w_1, v_1) + \frac{1}{2}(w_1^2, \operatorname{div} v_1), \end{aligned} \tag{2.29}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_2\|_{L^2}^2 + A_2\|v_2\|_V^2 - \frac{1}{2}(v_2^2, \operatorname{div} v_2) - g(h_2, \operatorname{div} v_2) \\ &\quad -g(h_1, \operatorname{div} v_2) = (v_2 w_2, \operatorname{div} v_2) - (\operatorname{curl} v_2 \alpha(w_2), v_2) \\ &\quad +(f_2, v_2) - \Big(\frac{\partial w_2}{\partial t}, v_2\Big) - A_2 a(w_2, v_2) + \frac{1}{2}(w_2^2, \operatorname{div} v_2). \end{aligned} \tag{2.30}$$

The terms on the right-hand side of (2.29) and (2.30) are estimated as follows:

$$|(v_i w_i, \operatorname{div} v_i)| \le \|v_i\|_{L^4}\|w_i\|_{L^4}\|\operatorname{div} v_i\|_{L^2} \le \widetilde{C}\|w_i\|_{L^4}\|v_i\|_V^2,$$

$$|(\operatorname{curl} v_i \alpha(w_i), v_i)| \le \|\operatorname{curl} v_i\|_{L^2}\|w_i\|_{L^4}\|v_i\|_{L^4} \le \widetilde{C}\|w_i\|_{L^4}\|v_i\|_V^2,$$

$$|(f_i, v_i)| \le \|f_i\|_{V'}\|v_i\|_V \le \frac{\varepsilon}{4}\|v_i\|_V^2 + \frac{1}{\varepsilon}\|f_i\|_{V'}^2,$$

$$\Big|\Big(\frac{\partial w_i}{\partial t}, v_i\Big)\Big| \le \|w_{i,t}\|_{L^2}\|v_i\|_{L^2} \le \frac{\varepsilon}{4}\|v_i\|_V^2 + \frac{1}{\varepsilon}\|w_{i,t}\|_{L^2}^2,$$

$$|A_i a(w_i, v_i)| \le A_i \|w_i\|_{H^1}\|v_i\|_V \le \frac{\varepsilon}{4}\|v_i\|_V^2 + \frac{A_i^2}{\varepsilon}\|w_i\|_{H^1}^2,$$

$$\left|\frac{1}{2}(w_i^2, \operatorname{div} v_i)\right| \le \frac{1}{2}\|w_i\|_{L^4}^2\|\operatorname{div} v_i\|_{L^2} \le \frac{\varepsilon}{4}\|v_i\|_V^2 + \frac{1}{4\varepsilon}\|w_i\|_{L^4}^4.$$

To estimate the nonlinear terms (v_1^2,div v_1) and (v_2^2,div v_2), we use Gagliardo-Nirenberg's inequality:

$$(v_i^2, \operatorname{div} v_i) \le \|v_i\|_{L^4}^2\|v_i\|_V \le C\|v_i\|_{L^2}\|v_i\|_V^2, \quad i = 1, 2.$$

Next, we estimate the terms $(h_1, \operatorname{div} v_1)$ and $(h_2, \operatorname{div} v_2)$ by formally writing

$$-(h_i, \operatorname{div} v_i) = (\nabla h_i, v_i) = \Big(\frac{\nabla h_i}{h_i}, v_i h_i\Big) = (\nabla \log h_i, v_i h_i) = -(\log h_i, \operatorname{div}(v_i h_i)).$$

Using the continuity equations and (2.24) we have

$$\begin{aligned}
-(h_i, \operatorname{div} v_i) &= \Big(\log h_i, \frac{\partial h_i}{\partial t}\Big) + (\log h_i, \operatorname{div}(w_i h_i)) \\
&= \frac{d}{dt}(h_i \log h_i - h_i, 1) - (\nabla(\log h_i), w_i h_i) + \int_\Gamma G_i h_i \log h_i \\
&= \frac{d}{dt}(h_i \log h_i, 1) - \frac{d}{dt}\int_\Omega h_i - \Big(\frac{\nabla h_i}{h_i}, w_i h_i\Big) + \int_\Gamma G_i h_i \log h_i \\
&= \frac{d}{dt}(h_i \log h_i, 1) + \int_\Gamma G_i h_i - (\nabla h_i, w_i) + \int_\Gamma G_i h_i \log h_i \\
&= \frac{d}{dt}(h_i \log h_i, 1) + \int_\Gamma G_i h_i + (h_i, \operatorname{div} w_i) - \int_\Gamma G_i h_i + \int_\Gamma G_i h_i \log h_i \\
&= \frac{d}{dt}(h_i \log h_i, 1) + (h_i, \operatorname{div} w_i) + \int_{\Gamma_i^+} G_i h_i \log h_i + \int_{\Gamma_i^-} G_i \mu_i \log \mu_i,
\end{aligned}$$

for $i = 1, 2$. The terms $|(h_i, \operatorname{div} w_i)|$ and $\int_{\Gamma_i^-} |G_i \mu_i \log \mu_i|$ can be easily bounded by virtue of (2.14), (2.15), and (2.28):

$$|(h_i, \operatorname{div} w_i)| \le \|h_i\|_{L^1}\|\operatorname{div} w_i\|_{L^\infty},$$

$$\int_{\Gamma_i^-} |G_i \mu_i \log \mu_i| \le \|G_i\|_{H^{1/2}(\Gamma_i^-)}\|\mu_i \log \mu_i\|_{H^{-1/2}(\Gamma_i^-)}.$$

Then, adding (2.29) and (2.30), and integrating in $(0, t)$, we get

$$\frac{1}{2}\|v_1\|_{L^2}^2 + \frac{1}{2}\|v_2\|_{L^2}^2$$
$$+\Big(A_1 - \varepsilon - 2\widetilde{C}\|w_1\|_{L^\infty(0,T;L^4(\Omega)^2)} - \frac{C}{2}\|v_1\|_{L^\infty(0,T;L^2(\Omega)^2)}\Big)\int_0^t \|v_1\|_V^2$$

$$
\begin{aligned}
&+\Big(A_2-\varepsilon-2\widetilde{C}\|w_2\|_{L^\infty(0,T;L^4(\Omega)^2)}-\frac{C}{2}\|v_2\|_{L^\infty(0,T;L^2(\Omega)^2)}\Big)\int_0^t\|v_2\|_V^2\\
&+g\int_\Omega h_1\log h_1+g\int_\Omega h_2\log h_2\\
&+g\int_0^t\int_{\Gamma_1^+}G_1h_1\log h_1+g\int_0^t\int_{\Gamma_2^+}G_2h_2\log h_2\\
&\le\sum_{i=1}^2\Big[\frac{1}{2}\|v_{i,0}\|_{L^2}^2+g\int_\Omega h_{i,0}\log h_{i,0}\\
&\quad+g\int_0^T\|h_i\|_{L^1}\|\operatorname{div}w_i\|_{L^\infty}+g\int_0^T\|G_i\|_{H^{1/2}(\Gamma_i^-)}\|\mu_i\log\mu_i\|_{H^{-1/2}(\Gamma_i^-)}\\
&\quad+\frac{1}{\varepsilon}\int_0^T\Big(\|f_i\|_{V'}^2+\|w_{i,t}\|_{L^2}^2+A_i^2\|w_i\|_{H^1}^2+\frac{1}{4}\|w_i\|_{L^4}^4\Big)\Big]\\
&\quad+g\frac{\rho_2}{\rho_1}\int_0^T|(h_2,\operatorname{div}v_1)|+g\int_0^T|(h_1,\operatorname{div}v_2)|\,. \qquad (2.31)
\end{aligned}
$$

The second step consists in obtaining estimates for h_1 and h_2 in $L^2(Q)$. To do this, we consider the L^2 projection of the equations $(\mathcal{P})'_1$ and $(\mathcal{P})'_8$ on the gradient vectors field:

$$
\begin{aligned}
&\int_\Omega\Big(\frac{\partial v_1}{\partial t}+\frac{\partial w_1}{\partial t}-A_1\Delta v_1-A_1\Delta w_1+\frac{1}{2}\nabla v_1^2+\nabla(v_1w_1)+\frac{1}{2}\nabla w_1^2\\
&\quad+\operatorname{curl}v_1\alpha(v_1)+\operatorname{curl}v_1\alpha(w_1)+g\nabla h_1+g\frac{\rho_2}{\rho_1}\nabla h_2-f_1\Big)\nabla P\,dx=0,\\
&\int_\Omega\Big(\frac{\partial v_2}{\partial t}+\frac{\partial w_2}{\partial t}-A_2\Delta v_2-A_2\Delta w_2+\frac{1}{2}\nabla v_2^2+\nabla(v_2w_2)+\frac{1}{2}\nabla w_2^2\\
&\quad+\operatorname{curl}v_2\alpha(v_2)+\operatorname{curl}v_2\alpha(w_2)+g\nabla h_2+g\nabla h_1-f_2\Big)\nabla P\,dx=0,
\end{aligned}
$$

where $P\in H^1(\Omega)$. Setting $v_1=v_{p,1}+v_{q,1}$ and $v_2=v_{p,2}+v_{q,2}$, and considering that the projections on the space $\nabla H^1(\Omega)$ of $\frac{\partial v_{q,i}}{\partial t}$ and $\operatorname{Curl}\operatorname{curl}v_{q,i}$ are zero, we have

$$
\begin{aligned}
&\int_\Omega\Big(\frac{\partial v_{p,1}}{\partial t}+\frac{\partial w_1}{\partial t}-A_1\Delta v_{p,1}-A_1\Delta w_1+\frac{1}{2}\nabla v_1^2+\nabla(v_1w_1)+\frac{1}{2}\nabla w_1^2\\
&+\operatorname{curl}v_{q,1}\alpha(v_1)+\operatorname{curl}v_{q,1}\alpha(w_1)+g\nabla h_1+g\frac{\rho_2}{\rho_1}\nabla h_2-f_1\Big)\nabla P\,dx=0,\\
&\int_\Omega\Big(\frac{\partial v_{p,2}}{\partial t}+\frac{\partial w_2}{\partial t}-A_2\Delta v_{p,2}-A_2\Delta w_2+\frac{1}{2}\nabla v_2^2+\nabla(v_2w_2)+\frac{1}{2}\nabla w_2^2
\end{aligned}
$$

$$+\operatorname{curl} v_{q,2}\alpha(v_2)+\operatorname{curl} v_{q,2}\alpha(w_2)+g\nabla h_2+g\nabla h_1-f_2\Big)\nabla P\,dx=0.$$

And recalling that $\operatorname{div} v_{p,i}=\Delta p_i$ and $\operatorname{div} w_i=\Delta s_i$, we arrive at

$$\begin{aligned}&\nabla\Big(\frac{\partial p_1}{\partial t}+\frac{\partial s_1}{\partial t}-A_1\Delta p_1-A_1\Delta s_1+\frac{1}{2}v_1^2+v_1w_1\\&+\frac{1}{2}w_1^2+\Theta_1+\tilde{\Theta}_1+gh_1+g\frac{\rho_2}{\rho_1}h_2-F_1\Big)=0,\\&\nabla\Big(\frac{\partial p_2}{\partial t}+\frac{\partial s_2}{\partial t}-A_2\Delta p_2-A_2\Delta s_2+\frac{1}{2}v_2^2+v_2w_2\\&+\frac{1}{2}w_2^2+\Theta_2+\tilde{\Theta}_2+gh_2+gh_1-F_2\Big)=0,\end{aligned}$$

where $\nabla\Theta_i$ are the projections for the $L^2(\Omega)^2$ scalar product of $\operatorname{curl} v_{q,i}\alpha(v_i)$ on the space $\nabla H^1(\Omega)$, $\nabla\tilde{\Theta}_i$ are the projections of $\operatorname{curl} v_{q,i}\alpha(w_i)$, and ∇F_i the projections of f_i, $i=1,2$. We choose these such that

$$\int_\Omega\Theta_i=\int_\Omega\tilde{\Theta}_i=\int_\Omega F_i=0.$$

Finally, we have

$$\begin{aligned}&\frac{\partial p_1}{\partial t}+\frac{\partial s_1}{\partial t}-A_1\Delta p_1-A_1\Delta s_1+\frac{1}{2}v_1^2+v_1w_1\\&+\frac{1}{2}w_1^2+\Theta_1+\tilde{\Theta}_1+gh_1+g\frac{\rho_2}{\rho_1}h_2-F_1=\zeta_1, &(2.32)\\&\frac{\partial p_2}{\partial t}+\frac{\partial s_2}{\partial t}-A_2\Delta p_2-A_2\Delta s_2+\frac{1}{2}v_2^2+v_2w_2\\&+\frac{1}{2}w_2^2+\Theta_2+\tilde{\Theta}_2+gh_2+gh_1-F_2=\zeta_2, &(2.33)\end{aligned}$$

where ζ_1 and ζ_2 are functions that depend only on time.

Now we define, for $\delta\in[0,T]$, the function

$$\xi_\delta(t)=\begin{cases}1, & 0\le t\le T-\delta,\\ \frac{T-t}{\delta}, & T-\delta\le t\le T.\end{cases}$$

Multiplying (2.32) by $\xi_\delta h_1$ and (2.33) by $\xi_\delta h_2$ and integrating over Q, we obtain

$$\begin{aligned}&g\int_Q\xi_\delta h_1^2-A_1\int_Q\Delta p_1\xi_\delta h_1+g\frac{\rho_2}{\rho_1}\int_Q\xi_\delta h_1h_2\\&\quad=\int_Q\zeta_1\xi_\delta h_1-\int_Q\frac{\partial p_1}{\partial t}\xi_\delta h_1-\int_Q\frac{\partial s_1}{\partial t}\xi_\delta h_1+A_1\int_Q\Delta s_1\xi_\delta h_1\end{aligned}$$

$$-\frac{1}{2}\int_Q v_1^2\xi_\delta h_1 - \int_Q v_1 w_1 \xi_\delta h_1 - \frac{1}{2}\int_Q w_1^2 \xi_\delta h_1$$
$$-\int_Q \Theta_1\xi_\delta h_1 - \int_Q \tilde{\Theta}_1\xi_\delta h_1 + \int_Q F_1\xi_\delta h_1, \tag{2.34}$$

$$g\int_Q \xi_\delta h_2^2 - A_2\int_Q \Delta p_2\xi_\delta h_2 + g\int_Q \xi_\delta h_1 h_2$$
$$= \int_Q \zeta_2\xi_\delta h_2 - \int_Q \frac{\partial p_2}{\partial t}\xi_\delta h_2 - \int_Q \frac{\partial s_2}{\partial t}\xi_\delta h_2 + A_2\int_Q \Delta s_2\xi_\delta h_2$$
$$-\frac{1}{2}\int_Q v_2^2\xi_\delta h_2 - \int_Q v_2 w_2\xi_\delta h_2 - \frac{1}{2}\int_Q w_2^2\xi_\delta h_2$$
$$-\int_Q \Theta_2\xi_\delta h_2 - \int_Q \tilde{\Theta}_2\xi_\delta h_2 + \int_Q F_2\xi_\delta h_2. \tag{2.35}$$

As in the homogeneous case [7], the L^2 estimates for h_1 and h_2 are performed by estimating the terms in these equations and then passing to the limit when $\delta \to 0$. There appear new difficulties due to the boundary terms.

First, notice that $\int_Q \xi_\delta h_1 h_2 \geq 0$ because $h_1, h_2 > 0$. Then, the second terms on the left-hand side are treated by using the equalities

$$-\int_\Omega h_i\Delta p_i = -\int_\Omega h_i \operatorname{div} v_i = \frac{d}{dt}\int_\Omega h_i\log h_i$$
$$+\int_\Omega h_i\operatorname{div} w_i + \int_{\Gamma_i^+} G_i h_i\log h_i + \int_{\Gamma_i^-} G_i\mu_i\log\mu_i$$

as follows

$$-A_i\int_Q \Delta p_i\xi_\delta h_i = -A_i\int_0^T \xi_\delta\int_\Omega h_i\operatorname{div} v_i = A_i\int_0^T \xi_\delta\frac{d}{dt}\int_\Omega h_i\log h_i$$
$$+A_i\int_0^T \xi_\delta\int_\Omega h_i\operatorname{div} w_i + A_i\int_0^T\xi_\delta\int_{\Gamma_i^+} G_i h_i\log h_i + A_i\int_0^T\xi_\delta\int_{\Gamma_i^-} G_i\mu_i\log\mu_i,$$

and

$$A_i\int_0^T \xi_\delta\frac{d}{dt}\int_\Omega h_i\log h_i = A_i\int_0^T\frac{d}{dt}\int_\Omega \xi_\delta h_i\log h_i - A_i\int_0^T\frac{d\xi_\delta}{dt}\int_\Omega h_i\log h_i$$
$$= -A_i\int_\Omega h_{i,0}\log h_{i,0} + \frac{A_i}{\delta}\int_{T-\delta}^T\int_\Omega h_i\log h_i.$$

Using the convexity inequality $h_i \log h_i \geq -\frac{1}{e}$ we have

$$\frac{A_i}{\delta} \int_{T-\delta}^{T} \int_{\Omega} h_i \log h_i \geq -\frac{A_i}{e} \mathrm{meas}(\Omega).$$

Then

$$- A_i \int_Q \Delta p_i \xi_\delta h_i \geq -A_i \int_\Omega h_{i,0} \log h_{i,0} - \frac{A_i}{e} \mathrm{meas}(\Omega)$$
$$+ A_i \int_Q \xi_\delta h_i \mathrm{div}\, w_i + A_i \int_{\Sigma_i^+} \xi_\delta G_i h_i \log h_i + A_i \int_{\Sigma_i^-} \xi_\delta G_i \mu_i \log \mu_i, \quad i = 1, 2.$$

We next consider the second terms on the right-hand side. We first integrate by parts,

$$- \int_Q \frac{\partial p_i}{\partial t} \xi_\delta h_i = - \int_Q \frac{\partial}{\partial t}(p_i \xi_\delta h_i) + \int_Q p_i \frac{\partial}{\partial t}(\xi_\delta h_i)$$
$$= \int_\Omega p_i(0) h_{i,0} + \int_Q p_i \frac{d\xi_\delta}{dt} h_i + \int_Q p_i \xi_\delta \frac{\partial h_i}{\partial t},$$

then we use the continuity equation,

$$-\int_Q \frac{\partial p_i}{\partial t} \xi_\delta h_i = \int_\Omega p_i(0) h_{i,0} + \int_Q p_i \frac{d\xi_\delta}{dt} h_i - \int_Q p_i \xi_\delta \mathrm{div}\,(v_i h_i) - \int_Q p_i \xi_\delta \mathrm{div}\,(w_i h_i)$$
$$= \int_\Omega p_i(0) h_{i,0} + \int_Q p_i \frac{d\xi_\delta}{dt} h_i + \int_Q \nabla p_i \xi_\delta v_i h_i + \int_Q \nabla p_i \xi_\delta w_i h_i - \int_\Sigma \xi_\delta p_i G_i h_i,$$

and finally the decomposition $v_i = v_{p,i} + v_{q,i}$,

$$- \int_Q \frac{\partial p_i}{\partial t} \xi_\delta h_i = \int_\Omega p_i(0) h_{i,0} + \int_Q p_i \frac{d\xi_\delta}{dt} h_i + \int_Q v_{p,i}^2 \xi_\delta h_i$$
$$+ \int_Q v_{p,i} v_{q,i} \xi_\delta h_i + \int_Q v_{p,i} w_i \xi_\delta h_i - \int_{\Sigma_i^+} \xi_\delta p_i G_i h_i - \int_{\Sigma_i^-} \xi_\delta p_i G_i \mu_i, \quad (2.36)$$

for $i = 1, 2$. The terms $-\frac{1}{2} \int_Q v_i^2 \xi_\delta h_i$ and $- \int_Q v_i w_i \xi_\delta h_i$ in (2.34) and (2.35) are split as follows:

$$-\frac{1}{2} \int_Q v_i^2 \xi_\delta h_i = -\frac{1}{2} \int_Q v_{p,i}^2 \xi_\delta h_i - \int_Q v_{p,i} v_{q,i} \xi_\delta h_i - \frac{1}{2} \int_Q v_{q,i}^2 \xi_\delta h_i \quad (2.37)$$

and

$$- \int_Q v_i w_i \xi_\delta h_i = - \int_Q v_{p,i} w_i \xi_\delta h_i - \int_Q v_{q,i} w_i \xi_\delta h_i, \quad (2.38)$$

for $i = 1, 2$. Using the previous results, (2.34) and (2.35) become

$$\begin{aligned}
&g \int_Q \xi_\delta h_1^2 + A_1 \int_{\Sigma_1^+} \xi_\delta G_1 h_1 \log h_1 \\
&\leq A_1 \int_\Omega h_{1,0} \log h_{1,0} + \frac{A_1}{e} \mathrm{meas}(\Omega) - A_1 \int_{\Sigma_1^-} \xi_\delta G_1 \mu_1 \log \mu_1 \\
&+ \int_Q \zeta_1 \xi_\delta h_1 + \int_\Omega p_1(0) h_{1,0} + \int_Q p_1 \frac{d\xi_\delta}{dt} h_1 - \int_{\Sigma_1^+} \xi_\delta p_1 G_1 h_1 - \int_{\Sigma_1^-} \xi_\delta p_1 G_1 \mu_1 \\
&- \int_Q \frac{\partial s_1}{\partial t} \xi_\delta h_1 + \frac{1}{2} \int_Q v_{p,1}^2 \xi_\delta h_1 - \frac{1}{2} \int_Q v_{q,1}^2 \xi_\delta h_1 - \int_Q v_{q,1} w_1 \xi_\delta h_1 - \frac{1}{2} \int_Q w_1^2 \xi_\delta h_1 \\
&- \int_Q \Theta_1 \xi_\delta h_1 - \int_Q \tilde{\Theta}_1 \xi_\delta h_1 + \int_Q F_1 \xi_\delta h_1
\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}
&g \int_Q \xi_\delta h_2^2 + A_2 \int_{\Sigma_2^+} \xi_\delta G_2 h_2 \log h_2 \\
&\leq A_2 \int_\Omega h_{2,0} \log h_{2,0} + \frac{A_2}{e} \mathrm{meas}(\Omega) - A_2 \int_{\Sigma_2^-} \xi_\delta G_2 \mu_2 \log \mu_2 \\
&+ \int_Q \zeta_2 \xi_\delta h_2 + \int_\Omega p_2(0) h_{2,0} + \int_Q p_2 \frac{d\xi_\delta}{dt} h_2 - \int_{\Sigma_2^+} \xi_\delta p_2 G_2 h_2 - \int_{\Sigma_2^-} \xi_\delta p_2 G_2 \mu_2 \\
&- \int_Q \frac{\partial s_2}{\partial t} \xi_\delta h_2 + \frac{1}{2} \int_Q v_{p,2}^2 \xi_\delta h_2 - \frac{1}{2} \int_Q v_{q,2}^2 \xi_\delta h_2 - \int_Q v_{q,2} w_2 \xi_\delta h_2 - \frac{1}{2} \int_Q w_2^2 \xi_\delta h_2 \\
&- \int_Q \Theta_2 \xi_\delta h_2 - \int_Q \tilde{\Theta}_2 \xi_\delta h_2 + \int_Q F_2 \xi_\delta h_2.
\end{aligned} \tag{2.40}$$

To estimate the terms $\int_Q \zeta_i \xi_\delta h_i$ on the right-hand side we look for an expression for ζ_1 and ζ_2, integrating (2.32) and (2.33) over Ω. We use $\int_\Omega p_i = 0$ and $\int_\Omega \Delta p_i = \int_\Omega \mathrm{div}\, v_i = 0$, and $\int_\Omega s_i = 0$ and $\int_\Omega \Delta s_i = \int_\Omega \mathrm{div}\, w_i = \int_\Gamma G_i$. We also have $\int_\Omega \Theta_i = \int_\Omega \tilde{\Theta}_i = \int_\Omega F_i = 0$, $i = 1, 2$. We obtain

$$\begin{aligned}
\zeta_1 &= \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \Big(\frac{1}{2} v_1^2 + v_1 w_1 + \frac{1}{2} w_1^2 - A_1 \mathrm{div}\, w_1 + g h_1 + g \frac{\rho_2}{\rho_1} h_2 \Big) \\
&\leq \frac{1}{\mathrm{meas}(\Omega)} \Big(\|v_1\|_V^2 + \|w_1\|_{L^2}^2 + A_1 \|\mathrm{div}\, w_1\|_{L^1} + g \|h_1\|_{L^1} + g \frac{\rho_2}{\rho_1} \|h_2\|_{L^1} \Big), \\
\zeta_2 &= \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \Big(\frac{1}{2} v_2^2 + v_2 w_2 + \frac{1}{2} w_2^2 - A_2 \mathrm{div}\, w_2 + g h_2 + g h_1 \Big)
\end{aligned}$$

$$\leq \frac{1}{\text{meas}(\Omega)}\Big(\|v_2\|_V^2 + \|w_2\|_{L^2}^2 + A_2\|\text{div}\, w_2\|_{L^1} + g\|h_2\|_{L^1} + g\|h_1\|_{L^1}\Big).$$

Then

$$\int_Q \zeta_i \xi_\delta h_i \leq C_i \|v_i\|^2_{L^2(0,T;V)} + C_i',$$

with C_i and C_i' defined in (2.1)–(2.4), $i = 1, 2$.

The main difficulty is the estimate of the terms

$$\int_Q p_i \frac{d\xi_\delta}{dt} h_i = \int_0^T \frac{d\xi_\delta}{dt}\int_\Omega p_i h_i \quad \text{and} \quad -\int_{\Sigma_i^+} \xi_\delta p_i G_i h_i = -\int_0^T \xi_\delta \int_{\Gamma_i^+} p_i G_i h_i.$$

For this, we must use the theory involving N-functions and Orlicz spaces [1, 5]. The first term appeared in the homogeneous case [7] and the second one is due to the nonhomogeneous boundary conditions.

The first term is bounded as follows:

$$\int_\Omega p_i h_i \leq \|p_i\|_{L_\Phi}\|h_i\|_{L_\Psi} \leq kk'\|p_i\|_{H^1}\|h_i\|_{L_{\tilde\Psi}} \leq kk'k''\|\nabla p_i\|_{L^2}\Big(1 + \int_\Omega \tilde\Psi(h_i)\Big)$$

$$\leq K\|v_i\|_{L^2}\Big(1 + \int_\Omega h_i\sqrt{\log^+ h_i}\Big) = K\|v_i\|_{L^2} + K\|v_i\|_{L^2}\int_\Omega \sqrt{h_i}\sqrt{h_i \log^+ h_i}$$

$$\leq \frac{1}{8}\|v_i\|^2_{L^2} + 2K^2 + \frac{1}{8}\|v_i\|^2_{L^2} + 2K^2\Big(\int_\Omega \sqrt{h_i}\sqrt{h_i\log^+ h_i}\Big)^2$$

$$\leq \frac{1}{4}\|v_i\|^2_{L^2} + 2K^2 + 2K^2\|h_i\|_{L^\infty(0,T;L^1(\Omega))}\int_\Omega h_i \log^+ h_i.$$

Recalling that $\frac{d\xi_\delta}{dt} = 0$ over $(0, T-\delta)$ and $\frac{d\xi_\delta}{dt} = -\frac{1}{\delta}$ over $(T-\delta, T)$ we can conclude that

$$\int_Q \frac{d\xi_\delta}{dt} p_i h_i \leq \frac{1}{4}\|v_i\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + 2K^2 + 2K^2\|h_i\|_{L^\infty(0,T;L^1(\Omega))}\sup_t \int_\Omega h_i \log^+ h_i.$$

Regarding the second term, we start by estimating it in the same way, but we find a new difficulty: before, we used the known estimate for $\|h_i\|_{L^1}$ in $L^\infty(0,T)$; now, for $\|G_i h_i\|_{L^1(\Gamma_i^+)}$, we only know an estimate in $L^1(0,T)$. We proceed as follows:

$$\int_{\Gamma_i^+} p_i G_i h_i \leq \|p_{i|\Gamma_i^+}\|_{L_\Phi(\Gamma_i^+)}\|G_i h_i\|_{L_\Psi(\Gamma_i^+)} \leq \kappa\kappa'\|p_{i|\Gamma_i^+}\|_{H^{1/2}(\Gamma_i^+)}\|G_i h_i\|_{L_{\tilde\Psi}(\Gamma_i^+)}$$

$$\leq \kappa\kappa'\|p_i\|_{H^1}\|G_i h_i\|_{L_{\tilde\Psi}(\Gamma_i^+)} \leq \kappa\kappa' k''\|\nabla p_i\|_{L^2}\Big(1 + \tilde\Psi(G_i h_i)\Big)$$

$$\leq \mathcal{K}\|v_i\|_{L^2}\Big(1 + \int_{\Gamma_i^+} G_i h_i \sqrt{\log^+(G_i h_i)}\Big)$$

$$= \mathcal{K}\|v_i\|_{L^2} + \mathcal{K}\|v_i\|_{L^2}\int_{\Gamma_i^+}\sqrt{G_ih_i}\sqrt{G_ih_i\log^+(G_ih_i)}$$

$$\leq \varepsilon'\|v_i\|_V^2 + \frac{\mathcal{K}^2}{4\varepsilon'} + \varepsilon'\|v_i\|_{L^2}^2\int_{\Gamma_i^+} G_ih_i\log^+(G_ih_i) + \frac{\mathcal{K}^2}{4\varepsilon'}\int_{\Gamma_i^+} G_ih_i.$$

And then

$$-\int_{\Sigma_i^+}\xi_\delta p_iG_ih_i \leq \varepsilon'\|v_i\|_{L^2(0,T;V)}^2 + \frac{\mathcal{K}^2T}{4\varepsilon'} + \varepsilon'\|v_i\|_{L^\infty(0,T;L^2(\Omega)^2)}^2\int_{\Sigma_i^+} G_ih_i\log^+(G_ih_i) + \frac{\mathcal{K}^2}{4\varepsilon'}\|G_ih_i\|_{L^1(\Sigma_i^+)}.$$

The estimate of $-\int_{\Sigma_i^-}\xi_\delta p_iG_i\mu_i$ is less difficult:

$$\begin{aligned}\int_{\Gamma_i^-} p_iG_i\mu_i &\leq \|p_{i|\Gamma_i^-}\|_{L^\infty(\Gamma_i^-)}\|G_i\|_{H^{1/2}(\Gamma_i^-)}\|\mu_i\|_{H^{-1/2}(\Gamma_i^-)}\\ &\leq a_i\|p_{i|\Gamma_i^-}\|_{H^{3/2}(\Gamma_i^-)}\|G_i\|_{H^{1/2}(\Gamma_i^-)}\|\mu_i\|_{H^{-1/2}(\Gamma_i^-)}\\ &\leq a_i\|p_i\|_{H^2}\|G_i\|_{H^{1/2}(\Gamma_i^-)}\|\mu_i\|_{H^{-1/2}(\Gamma_i^-)}\\ &\leq a_ib\|v_i\|_V\|G_i\|_{H^{1/2}(\Gamma_i^-)}\|\mu_i\|_{H^{-1/2}(\Gamma_i^-)}\\ &\leq \varepsilon'\|v_i\|_V^2 + \frac{a_i^2b^2}{4\varepsilon'}\|G_i\|_{H^{1/2}(\Gamma_i^-)}^2\|\mu_i\|_{H^{-1/2}(\Gamma_i^-)}^2,\end{aligned}$$

so

$$-\int_{\Sigma_i^-}\xi_\delta p_iG_i\mu_i \leq \varepsilon'\|v_i\|_{L^2(0,T;V)}^2 + \frac{a_i^2b^2}{4\varepsilon'}\|G_i\|_{L^\infty(0,T;H^{1/2}(\Gamma_i^-))}^2\|\mu_i\|_{L^2(0,T;H^{-1/2}(\Gamma_i^-))}^2.$$

The terms $-\int_Q \frac{\partial s_i}{\partial t}\xi_\delta h_i$ are easily bounded:

$$-\int_Q\frac{\partial s_i}{\partial t}\xi_\delta h_i \leq \frac{\lambda}{5}\int_Q\xi_\delta h_i^2 + \frac{5}{4\lambda}\int_0^T\|s_{i,t}\|_{L^2}^2 \leq \frac{\lambda}{5}\int_Q\xi_\delta h_i^2 + \frac{5}{4\lambda}\int_0^T\|s_{i,t}\|_{H^1}^2$$

$$\leq \frac{\lambda}{5}\int_Q\xi_\delta h_i^2 + \frac{5k''}{4\lambda}\int_0^T\|\nabla s_{i,t}\|_{L^2}^2 = \frac{\lambda}{5}\int_Q\xi_\delta h_i^2 + \frac{5k''}{4\lambda}\int_0^T\|w_{i,t}\|_{L^2}^2.$$

To estimate the terms $\frac{1}{2}\int_Q v_{p,i}^2\xi_\delta h_i$ we use Gagliardo-Nirenberg's inequality:

$$\frac{1}{2}\int_Q v_{p,i}^2\xi_\delta h_i \leq \frac{\lambda}{5}\int_Q\xi_\delta h_i^2 + \frac{5}{16\lambda}\int_0^T\|v_{p,i}\|_{L^4}^4$$

$$\leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5C^2}{16\lambda}\int_0^T \|v_i\|_{L^2}^2\|v_i\|_V^2.$$

Regarding $-\frac{1}{2}\int_Q v_{q,i}^2\xi_\delta h_i$, $-\int_Q v_{q,i}w_i\xi_\delta h_i$, and $-\frac{1}{2}\int_Q w_i^2\xi_\delta h_i$ we simply have to notice that

$$-\frac{1}{2}\int_Q v_{q,i}^2\xi_\delta h_i - \int_Q v_{q,i}w_i\xi_\delta h_i - \frac{1}{2}\int_Q w_i^2\xi_\delta h_i = -\frac{1}{2}\int_Q (v_{q,i}+w_i)^2\xi_\delta h_i$$

is negative. The estimation of the terms on the right-hand side of (2.34) and (2.35) concludes with the estimation of $-\int_Q \Theta_i\xi_\delta h_i$, $-\int_Q \tilde{\Theta}_i\xi_\delta h_i$, and $\int_Q F_i\xi_\delta h_i$:

$$\begin{aligned}
&-\int_Q \Theta_i\xi_\delta h_i \leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5}{4\lambda}\int_0^T \|\Theta_i\|_{L^2}^2 \leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5C'^2}{4\lambda}\int_0^T \|\nabla\Theta_i\|_{L^1}^2\\
&\leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5C'^2}{4\lambda}\int_0^T \|\mathrm{curl}\, v_{q,i}\|_{L^2}^2\|v_i\|_{L^2}^2\\
&\leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5C'^2}{4\lambda}\int_0^T \|v_i\|_{L^2}^2\|v_i\|_V^2,\\
&-\int_Q \tilde{\Theta}_i\xi_\delta h_i \leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5}{4\lambda}\int_0^T \|\tilde{\Theta}_i\|_{L^2}^2 \leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5C'^2}{4\lambda}\int_0^T \|\nabla\tilde{\Theta}_i\|_{L^1}^2\\
&\leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5C'^2}{4\lambda}\int_0^T \|\mathrm{curl}\, v_{q,i}\|_{L^2}^2\|w_i\|_{L^2}^2\\
&\leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5C'^2}{4\lambda}\int_0^T \|w_i\|_{L^2}^2\|v_i\|_V^2
\end{aligned}$$

and

$$\begin{aligned}
&\int_Q F_i\xi_\delta h_i \leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5}{4\lambda}\int_0^T \|F_i\|_{L^2}^2 \leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5}{4\lambda}\int_0^T \|F_i\|_{H^1}^2\\
&\leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5}{4\lambda}\int_0^T \|\nabla F_i\|_{L^2}^2 \leq \frac{\lambda}{5}\int_Q \xi_\delta h_i^2 + \frac{5}{4\lambda}\int_0^T \|f_i\|_{L^2}^2.
\end{aligned}$$

Then, adding (2.39) and (2.40), and making δ tend to zero, we have

$$\begin{aligned}
&(g-\lambda)\int_Q h_1^2 + (g-\lambda)\int_Q h_2^2 + A_1\int_{\Sigma_1^+} G_1 h_1\log h_1 + A_2\int_{\Sigma_2^+} G_2 h_2 \log h_2\\
&\leq \sum_{i=1}^2 \Big[A_i\int_\Omega h_{i,0}\log h_{i,0} + \frac{A_i}{e}\mathrm{meas}(\Omega)
\end{aligned}$$

$$
\begin{aligned}
&+ A_i\|G_i\|_{L^2(0,T;H^{1/2}(\Gamma_i^-))}\|\mu_i \log \mu_i\|_{L^2(0,T;H^{-1/2}(\Gamma_i^-))} + C_i\|v_i\|^2_{L^2(0,T;V)} + C_i' \\
&+ \int_\Omega p_i(0)h_{i,0} + \frac{1}{4}\|v_i\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + 2K^2 \\
&+ 2K^2\|h_i\|_{L^\infty(0,T;L^1(\Omega))}\sup_t \int_\Omega h_i \log^+ h_i + 2\varepsilon'\|v_i\|^2_{L^2(0,T;V)} + \frac{\mathcal{K}^2 T}{4\varepsilon'} \\
&+ \varepsilon'\|v_i\|^2_{L^\infty(0,T;L^2(\Omega)^2)} \int_{\Sigma_i^+} G_i h_i \log^+(G_i h_i) + \frac{\mathcal{K}^2}{4\varepsilon'}\|G_i h_i\|_{L^1(\Sigma_i^+)} \\
&+ \frac{a_i^2 b^2}{4\varepsilon'}\|G_i\|^2_{L^\infty(0,T;H^{1/2}(\Gamma_i^-))}\|\mu_i\|^2_{L^2(0,T;H^{-1/2}(\Gamma_i^-))} \\
&+ \frac{5k''}{4\lambda}\|w_{i,t}\|^2_{L^2(Q)} + C_\lambda\|v_i\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\|v_i\|^2_{L^2(0,T;V)} \\
&+ \frac{5C'^2}{4\lambda}\|w_i\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\|v_i\|^2_{L^2(0,T;V)} + \frac{5}{4\lambda}\|f_i\|^2_{L^2(Q)}\Big].
\end{aligned}
\tag{2.41}
$$

The next step is adding equation (2.31) to equation (2.41). Before this, we are going to make some changes in the terms on the left-hand sides of both equations. Notice that we can split the term $g\int_\Omega h_i \log h_i$ into

$$g\int_\Omega h_i \log h_i = g\int_\Omega h_i \log^+ h_i - g\int_\Omega h_i \log^- h_i.$$

Using the convexity inequality $h_i \log h_i \geq -\frac{1}{e}$, we get a lower bound for the term $-g\int_\Omega h_i \log^- h_i$:

$$
\begin{aligned}
-g\int_\Omega h_i \log^- h_i &= g\int_{\{x\in\Omega:\ h_i(x,t)<1\}} h_i \log h_i \\
&\geq -\frac{g}{e}\text{meas}(\{x \in \Omega:\ h_i(x,t) < 1\}) \geq -\frac{g}{e}\text{meas}(\Omega).
\end{aligned}
$$

We also split the term $g\int_0^t\int_{\Gamma_i^+} G_i h_i \log h_i$ in the form

$$
\begin{aligned}
&g\int_0^t\int_{\Gamma_i^+} G_i h_i \log h_i = g\int_0^t\int_{\Gamma_i^+} G_i h_i \log(G_i h_i) - g\int_0^t\int_{\Gamma_i^+} G_i h_i \log G_i \\
&= g\int_0^t\int_{\Gamma_i^+} G_i h_i \log^+(G_i h_i) - g\int_0^t\int_{\Gamma_i^+} G_i h_i \log^-(G_i h_i) - g\int_0^t\int_{\Gamma_i^+} G_i h_i \log G_i,
\end{aligned}
$$

where

$$-g\int_0^t\int_{\Gamma_i^+} G_i h_i \log^-(G_i h_i) \geq -\frac{gt}{e}\text{meas}(\Omega),$$

and the last term is bounded by

$$g\int_{\Sigma_i^+}|G_i h_i \log G_i| \leq g\|G_i h_i\|_{L^1(\Sigma_i^+)}\|\log G_i\|_{L^\infty(\Sigma_i^+)}.$$

Now we can take the supremum in the terms on the left-hand side of (2.31). And using that

$$g\frac{\rho_2}{\rho_1}\int_0^T|(h_2,\operatorname{div} v_1)| \leq g\frac{\rho_2}{\rho_1}\frac{\mu}{2}\|h_2\|^2_{L^2(Q)} + g\frac{\rho_2}{\rho_1}\frac{1}{2\mu}\|v_1\|^2_{L^2(0,T;V)}$$

and

$$g\int_0^T|(h_1,\operatorname{div} v_2)| \leq g\frac{\mu}{2}\|h_1\|^2_{L^2(Q)} + g\frac{1}{2\mu}\|v_2\|^2_{L^2(0,T;V)}$$

we finally obtain, from the addition of (2.31) to (2.41), the desired inequality (2.25).

To obtain estimates for v_i and h_i we need only[2] to prove that

$$B_i - \frac{C}{2}\|v_i\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda\|v_i\|^2_{L^\infty(0,T;L^2(\Omega)^2)} > 0, \quad i=1,2.$$

This is done using the small-data hypotheses.

Let us assume that v_1 and v_2 are continuous from $[0,T]$ into $L^2(\Omega)^2$. As $\|v_{i,0}\|_{L^2} < \mathbb{K}_i$, there exists t' such that $\|v_i(t)\|_{L^2} < \mathbb{K}_i$ in $[0,t')$, for $i=1,2$. Assume that $\|v_1(t')\|_{L^2} = \mathbb{K}_1$ and $\|v_2(t')\|_{L^2} \leq \mathbb{K}_2$, for instance. Then (2.25), written for $T=t'$, implies

$$\begin{aligned}
&\frac{\mathbb{K}_1^2}{4} + (g+A_1)\int_{\Sigma_1^+}G_1h_1\log^+(G_1h_1) + (g+A_2)\int_{\Sigma_2^+}G_2h_2\log^+(G_2h_2)\\
&\leq \varepsilon'\mathbb{K}_1^2\int_{\Sigma_1^+}G_1h_1\log^+(G_1h_1) + \varepsilon'\mathbb{K}_2^2\int_{\Sigma_2^+}G_2h_2\log^+(G_2h_2)\\
&\quad + \mathbb{C}_{\varepsilon,\varepsilon',\lambda} + \sum_{i=1}^2\frac{g+(g+A_i)T}{e}\operatorname{meas}(\Omega),
\end{aligned}$$

and so[3],

$$\frac{\mathbb{K}_1^2}{4} \leq \mathbb{C}_{\varepsilon,\varepsilon',\lambda} + \sum_{i=1}^2\frac{g+(g+A_i)T}{e}\operatorname{meas}(\Omega),$$

which contradicts (2.21). The same contradiction holds if $\|v_2(t')\|_{L^2} = \mathbb{K}_2$ and $\|v_1(t')\|_{L^2} \leq \mathbb{K}_1$. Therefore, (2.26) is proved.

[2]Notice that condition (2.11) implies $g > \lambda + g\frac{\rho_2}{\rho_1}\frac{\mu}{2}$ because $\rho_1 > \rho_2$.

[3]Recall that $\varepsilon' < \min_{i=1,2}\frac{g+A_i}{\mathbb{K}_i^2}$.

2.3. **Approximate solutions.** Set a sequence $\{w_{i,n}\}$ with $w_{i,n} \in H^1(0,T; H^4(\Omega)^2)$, and another sequence $\{G_{i,n}\}$ with $G_{i,n} = w_{i,n} \cdot n \in H^1(0,T; H^{\frac{7}{2}}(\Gamma))$, $i = 1, 2$.

Let us introduce a basis for V denoted by $\{\nu_1, \dots, \nu_n, \dots\}$, whose elements belong to $H^4(\Omega)^2$. Let V_n be the set of linear combinations of the n first elements of the basis. We consider the following problem:

Find $(v_{1,n}, h_{1,n})$ and $(v_{2,n}, h_{2,n})$ in $[L^\infty(0,T; L^2(\Omega)^2) \cap L^2(0,T; V_n)] \times \mathcal{C}^1(\overline{Q})$ such that the following system, which will be referred to as $(\mathcal{V}_n)$, holds:

$$\begin{aligned}
\Big(\frac{\partial v_{1,n}}{\partial t}, \nu\Big) + A_1 a(v_{1,n}, \nu) - \frac{1}{2}(v_{1,n}^2, \operatorname{div}\nu) + (\operatorname{curl} v_{1,n}\alpha(v_{1,n}), \nu)& \\
-g(h_{1,n}, \operatorname{div}\nu) - g\frac{\rho_2}{\rho_1}(h_{2,n}, \operatorname{div}\nu) - (v_{1,n}w_{1,n}, \operatorname{div}\nu)& \\
+(\operatorname{curl} v_{1,n}\alpha(w_{1,n}), \nu) = (f_1, \nu)& \\
-\Big(\frac{\partial w_{1,n}}{\partial t}, \nu\Big) - A_1 a(w_{1,n}, \nu) + \frac{1}{2}(w_{1,n}^2, \operatorname{div}\nu) \quad \forall \nu \in V_n,& \\
\Big(\frac{\partial v_{2,n}}{\partial t}, \nu\Big) + A_2 a(v_{2,n}, \nu) - \frac{1}{2}(v_{2,n}^2, \operatorname{div}\nu) + (\operatorname{curl} v_{2,n}\alpha(v_{2,n}), \nu)& \\
-g(h_{2,n}, \operatorname{div}\nu) - g(h_{1,n}, \operatorname{div}\nu) - (v_{2,n}w_{2,n}, \operatorname{div}\nu)& \\
+(\operatorname{curl} v_{2,n}\alpha(w_{2,n}), \nu) = (f_2, \nu)& \\
-\Big(\frac{\partial w_{2,n}}{\partial t}, \nu\Big) - A_2 a(w_{2,n}, \nu) + \frac{1}{2}(w_{2,n}^2, \operatorname{div}\nu) \quad \forall \nu \in V_n,& \\
\frac{\partial h_{1,n}}{\partial t} + \operatorname{div}(v_{1,n}h_{1,n}) + \operatorname{div}(w_{1,n}h_{1,n}) = 0,& \\
\frac{\partial h_{2,n}}{\partial t} + \operatorname{div}(v_{2,n}h_{2,n}) + \operatorname{div}(w_{2,n}h_{2,n}) = 0,& \\
v_{1,n}(t=0) = v_{1,0,n} \in V_n, \quad v_{2,n}(t=0) = v_{2,0,n} \in V_n,& \\
h_{1,n} = \mu_{1,n} \in \mathcal{C}_c^1(\Sigma_1^-), \quad h_{2,n} = \mu_{2,n} \in \mathcal{C}_c^1(\Sigma_2^-),& \\
h_{1,n}(t=0) = h_{1,0,n} \in \mathcal{C}_c^1(\Omega), \quad h_{2,n}(t=0) = h_{2,0,n} \in \mathcal{C}_c^1(\Omega),&
\end{aligned}$$

where the data and the constants satisfy the conditions of Theorem 1. Then we have

Lemma 2. *The problem* $(\mathcal{V}_n)$ *has a solution* $\{(v_{1,n}, h_{1,n}), (v_{2,n}, h_{2,n})\}$ *in* $\big[[L^\infty(0,T; L^2(\Omega)^2) \cap L^2(0,T; V_n)] \times \mathcal{C}^1(\overline{Q})\big]^2$, *which satisfies*

$$\begin{aligned}
&\|v_{1,n}\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \|v_{2,n}\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \|v_{1,n}\|^2_{L^2(0,T;V)} \\
&+ \|v_{2,n}\|^2_{L^2(0,T;V)} + \sup_t \int_\Omega h_{1,n} \log h_{1,n} + \sup_t \int_\Omega h_{2,n} \log h_{2,n}
\end{aligned}$$

$$+\int_{\Sigma_1^+} G_{1,n}h_{1,n}\log(G_{1,n}h_{1,n}) + \int_{\Sigma_2^+} G_{2,n}h_{2,n}\log(G_{2,n}h_{2,n})$$
$$+\|h_{1,n}\|^2_{L^2(Q)} + \|h_{2,n}\|^2_{L^2(Q)} \le \mathrm{C}. \tag{2.42}$$

Proof. To prove this lemma we apply the second Schauder fixed-point theorem [11] as in [2]. We obtain approximate solutions that satisfy the a priori estimates. In fact, due to the regularity of the basis, we have $v_{i,n} \in H^1(0,T;H^4(\Omega)^2)$. Therefore, $v_{i,n} \in \mathcal{C}^0([0,T];\mathcal{C}^2(\overline{\Omega})^2)$ and, using (2.27) and the positivity of initial data $h_{i,0}$, we have $h_{i,n} \in \mathcal{C}^1(\overline{Q})$ and $h_{i,n} > 0$, $i = 1, 2$.

2.4. **Passage to the limit.** Now, we present a lemma that is used to pass to the limit in the approximate equations. Its proof is the same as the one of the lemma used to pass to the limit in the homogeneous problem [7].

Lemma 3. *For each $n \in \mathbb{N}$, let*

$$\{(v_{1,n},h_{1,n}),(v_{1,n},h_{2,n})\} \in \left[[L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;V_n)] \times \mathcal{C}^1(\overline{Q})\right]^2$$

be the solution of $(\mathcal{V}_n)$ given by Lemma 2 that satisfies (2.42). Then we have, for $i = 1, 2$,

$$v_{i,n}h_{i,n} \text{ is bounded in } L^2(0,T;L^1(\Omega)^2), \tag{2.43}$$
$$\frac{\partial v_{i,n}}{\partial t} \text{ is bounded in } L^{4/3}(0,T;H^{-1}(\Omega)^2), \tag{2.44}$$

and we can extract from $v_{i,n}$ and $h_{i,n}$ subsequences still denoted $v_{i,n}$ and $h_{i,n}$ such that

$$v_{i,n} \longrightarrow v_i \text{ in } L^2(0,T;V) \text{ weakly}, \tag{2.45}$$
$$v_{i,n} \longrightarrow v_i \text{ in } L^\infty(0,T;L^2(\Omega)^2) \text{ weakly-star}, \tag{2.46}$$
$$h_{i,n} \longrightarrow h_i \text{ in } L^2(Q) \text{ weakly}, \tag{2.47}$$
$$v_{i,n}h_{i,n} \longrightarrow v_ih_i \text{ in } L^{4/3}(Q)^2 \text{ weakly}, \tag{2.48}$$
$$\mathit{curl}\, v_{i,n}\alpha(v_{i,n}) \longrightarrow \mathit{curl}\, v_i\alpha(v_i) \text{ in } L^{4/3}(Q)^2 \text{ weakly}, \tag{2.49}$$
$$\nabla v_{i,n}^2 \longrightarrow \nabla v_i^2 \text{ in } L^{4/3}(Q)^2 \text{ weakly.} \tag{2.50}$$

Passing to the limit in the boundary terms is done as in [2] for the one-layer problem:

Lemma 4. *Let $\{u_{i,n} = v_{i,n} + w_{i,n}\}$ and $\{h_{i,n}\}$ be sequences such that*

$$u_{i,n} \text{ is bounded in } L^2(0,T;H^1(\Omega)^2), \tag{2.51}$$
$$h_{i,n} \text{ is bounded in } L^2(Q), \tag{2.52}$$
$$\frac{\partial h_{i,n}}{\partial t} + \mathit{div}(u_{i,n}h_{i,n}) = 0, \tag{2.53}$$

$i = 1, 2$. *Let* $\{G_{i,n} = u_{i,n} \cdot n\}$ *be a sequence such that*

$$G_{i,n} \longrightarrow G_i \quad \text{in } H^1(0,T;H^{1/2}(\Gamma)) \tag{2.54}$$

and $\{\mu_{i,n} = h_{i,n|\Sigma_i^-}\}$ *such that*

$$\mu_{i,n} \longrightarrow \mu_i \quad \text{in } L^2(0,T;H^{-1/2}(\Gamma_i^-)), \tag{2.55}$$

$i = 1, 2$. *Then*

$$h_i = \mu_i \quad \text{on } \Sigma_i^-, \tag{2.56}$$

for $i = 1, 2$.

2.5. **Proof of the theorem.** Let $\{w_{i,n}\}$ be a sequence with elements $w_{i,n} \in H^1(0,T;H^4(\Omega)^2)$ such that $w_{i,n} \longrightarrow w_i$ in $H^1(0,T;H^1(\Omega)^2)$; then $\{G_{i,n}\}$, with $G_{i,n} = w_{i,n} \cdot n$, satisfies $G_{i,n} \longrightarrow G_i$ in $H^1(0,T;H^{1/2}(\Gamma))$, $i = 1, 2$. Let $v_{i,0}$ and $h_{i,0}$, $i = 1, 2$, be the initial conditions of the problem $(\mathcal{P})'$.

Let $\{v_{i,0,n}\}$, $\{h_{i,0,n}\}$, and $\{\mu_{i,0,n}\}$ be sequences with elements $v_{i,0,n} \in V_n$, $h_{i,0,n} \in \mathcal{C}_c^1(\Omega)$, and $\mu_{i,n} \in \mathcal{C}_c^1(\Sigma_i^-)$ such that $v_{i,0,n} \longrightarrow v_{i,0}$ in V, $h_{i,0,n} \longrightarrow h_{i,0}$ in $L^1(\Omega)$, and $\mu_{i,n} \longrightarrow \mu_i$ in $L^2(0,T;H^{-1/2}(\Gamma_i^-))$, $i = 1, 2$. For each $n \in \mathbb{N}$, set

$$\{(v_{1,n}, h_{1,n}), (v_{2,n}, h_{2,n})\} \in \left[[L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;V_n)] \times \mathcal{C}^1(\overline{Q}) \right]^2$$

a solution of $(\mathcal{V}_n)$ given by Lemma 2, which satisfies the estimate (2.42).

Using Lemma 3, we can extract subsequences to $\{v_{i,n}\}$ and $\{h_{i,n}\}$, also denoted by $\{v_{i,n}\}$ and $\{h_{i,n}\}$, such that

$$v_{i,n} \longrightarrow v_i \quad \text{in } L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;V) \text{ weakly-star}$$

and $h_{i,n} \longrightarrow h_i$ in $L^2(Q)$. Then, $v_{i,n}h_{i,n} \longrightarrow v_i h_i$ in $L^{4/3}(Q)^2$ weakly, $i = 1, 2$.

We can deduce from the previous results that $\operatorname{div}(v_i h_i)$ belongs to $L^{4/3}(0,T;W^{-1,4/3}(\Omega))$ and so $h_{i,t}$. We also have $h_i(t = 0) = h_{i,0}$. The obtaining of the boundary condition $h_i = \mu_i$ on Σ_i^- is done using Lemma 4.

Finally, passing to the limit in all the terms in momentum equations we obtain $v_i(t = 0) = v_{i,0}$. This concludes the proof that $\{(v_1, h_1), (v_2, h_2)\}$ is a solution of the weak problem $(\mathcal{V})$.

3. Concluding remarks

In this work we have studied a bi-layer shallow-water model in depth-mean velocity formulation, with nonhomogeneous boundary conditions. We have proved a theorem for the existence of a solution, under hypotheses based on small-size data. The main difficulty with its proof is related to the presence of the coupling and the boundary terms. In a forthcoming paper, some smoothness and uniqueness results will be studied.

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