Differential and Integral Equations
Volume 17, Numbers 9-10, Pages 1149–1174
September/October 2004

BOUNDARY-CONTROL PROBLEMS WITH CONVEX COST AND DYNAMIC PROGRAMMING IN INFINITE DIMENSION PART I: THE MAXIMUM PRINCIPLE

Silvia Faggian
Dipartimento di Matematica, Università di Pisa
Via F. Buonarroti, 2, I-56127, Pisa, Italy

(Submitted by: Viorel Barbu)

Abstract. This is the first of two papers on boundary optimal control problems with linear state equation and convex cost arising from boundary control of PDEs and the associated Hamilton–Jacobi–Bellman equation. In this paper we study necessary and sufficient conditions of optimality (Pontryagin maximum principle), and study the properties of a family of approximating problems that will be useful both in this paper and in the sequel. In the second paper we will apply dynamic programming to show that the value function of the problem is a solution of an integral version of the HJB equation, and moreover that it is the pointwise limit of classical solutions of approximating equations.

1. Introduction

This is the first of two papers on linear convex optimal control problems in Hilbert spaces and the associated Hamilton–Jacobi–Bellman equation (briefly, HJB equation) originating from boundary-control problems for PDEs. The main features of the problem are the following:

The state equation is given by

$$\begin{cases} y'(s) = Ay(s) + Bu(s) \qquad s \in [t,T] \\ y(t) = x, \end{cases} \tag{1}$$

where H and U are separable Hilbert spaces with scalar products $\langle \cdot,\cdot \rangle_H$ and $\langle \cdot,\cdot \rangle_U$ respectively, H is the state space and $y : [t,T] \to H$ is the trajectory, U is the control space and $u : [t,T] \to U$ is the control, $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup of linear operators $\{e^{sA}\}_{s \geq 0}$ on H, and the control operator B is linear and *unbounded*.

Accepted for publication: November 2003.
AMS Subject Classifications: 49J15, 49J20, 35B37.

We want to minimize over all controls u in $L^2(t,T;U)$, the cost functional

$$J(t,x,u) = \int_t^T [g(y(s)) + h(u(s))]\, ds + \phi_0(y(T)), \tag{2}$$

where ϕ_0 and g are convex and merely *continuous* functions, and h is convex and superlinear.

The main result of this paper is contained in Theorem 6.4, which gives necessary and sufficient conditions of optimality for our problem (Pontryagin maximum principle). We obtain the result by approximation, and at the same time we start analyzing the properties of the approximating problems that we will need in the sequel [17] of the paper.

In [17] indeed, we will study dynamic programming and show that the value function defined by

$$W(t,x) := \inf_{u \in L^2(t,T;U)} J(t,x,u)$$

satisfies a suitable integral version of the HJB equation,

$$v_t(t,x) + \langle Ax, v_x(t,x)\rangle_H + g(x) - \mathcal{H}(-B^* v_x(t,x)) = 0, \quad v(T,x) = \phi_0(x),$$

where $\mathcal{H}(u) = h^*(u) = \sup_{v\in U}\{(u,v)_U - h(v)\}$, for all $u \in U$. More precisely we will show that for all fixed $x \in D(A)$ and $t \in [0,T]$ there exists $\xi(t,x) \in \partial_x W(t,x)$ such that $t \mapsto \xi(t,x)$ is measurable and the following equation is satisfied:

$$W(r,x) = \phi_0(x) + \int_r^T G(x, \xi(t,x))dt, \quad r \in [0,T],$$

where $G : H \times H \to \mathbb{R}$, $G(x,p) = \langle Ax, p\rangle_H + g(x) - \mathcal{H}(-B^*p)$. In [17] we will prove also that W is the limit of classical solutions of approximating equations, and that the optimal control may be given in feedback form by means of the spatial subgradient of W.

We remark that the main novelty of the results contained in the present paper and in [17] relies on the fact that linear convex control problems with unbounded B arise when we rephrase in abstract terms some *boundary*-control problem for linear PDEs (or, more generally, problems with control on a subdomain).

We note also that a result of the same type as Theorem 6.4 has been obtained by Barbu and Precupanu in [10] (see further Remark 6.5). On the other hand the results on HJB equations are new and, as far as we know, they represent together with those in [18] and [19], the first on verification theorems and existence of optimal feedback maps in the framework of boundary-control problems with general convex cost (i.e., associated to an infinite-dimensional HJB equation that does not reduce to a Riccati equation).

We recall that optimal-control problems for infinite-dimensional systems and the associated HJB equation have been studied in two different frameworks: one is that of classical and strong solutions, and the other is that of viscosity solutions. We recall also that, as far as we know, verification techniques have been performed in infinite dimension just in the classical/strong context, for they require the value function to be regular (at least in the state variable).

Regarding dynamic programming for boundary-control problems only a few results are available. For the case of linear systems and quadratic costs (where HJB reduces to the operator Riccati equation) the reader is referred *e.g.* to the book by Lasiecka and Triggiani [24], to the book by Bensoussan, Da Prato, Delfour, and Mitter [8], and, for the case of nonautonomous systems, to the papers by Acquistapace, Flandoli, and Terreni [1, 2, 3, 4]. For the case of a linear system and a general convex cost, we mention the papers by this author [17, 18, 19]. On the Pontryagin maximum principle for boundary-control problems we mention again the book by Barbu and Precupanu (Chapter 4 in [10]).

For the case of distributed control the literature is indeed richer: we refer the reader to Barbu and Da Prato [5, 6, 7] for some linear convex problems, to Di Blasio [15, 16] for the case of constrained control, to Cannarsa and Di Blasio [11] for the case of state constraints, and to Barbu, Da Prato, and Popa [9] and to Gozzi [20, 21, 22] for semilinear systems.

For viscosity solutions and HJB equations in infinite dimension we mention the series of papers by Crandall and Lions [13], where some boundary-control problems arise also. Moreover, for boundary control we mention the papers by Cannarsa, Gozzi, and Soner [12] and by Cannarsa and Tessitore [14] on existence and uniqueness of viscosity solutions of HJB. We note also that a verification theorem in the case of viscosity solutions has been proved in a finite-dimensional case in the book by Yong and Zhou [25].

Regarding applications, on control on a subdomain (boundary or point control) we refer the reader to the many examples contained in the books by Lasiecka and Triggiani [24], and by Bensoussan *et al.* [8].

The material is organized as follows: in Section 2 we introduce the notation and some preliminary results, and in Section 3 we state our problem and motivate it with an application to a boundary-control problem for parabolic equations with Cauchy-Neumann conditions. In Section 4 we prove the existence of an optimal control, while in Section 5 we introduce the approximating problems and discuss some of their properties. Finally, in Section 6, we prove the Pontryagin maximum principle.

2. Notation and preliminary results

We begin this section by listing some notation.

• If X is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_X$ we denote by $|\cdot|_X$ the induced norm; if $R > 0$ then $\Sigma_R := \{x \in X \ : \ |x|_X \leq R\}$.

• $\chi_{[a,b]}$ indicates the characteristic function of the interval $[a, b]$, that is,

$$\chi_{[a,b]}(s) = \begin{cases} 1 & \text{if } s \in [a,b] \\ 0 & \text{if } s \notin [a,b]. \end{cases}$$

• If $a < b$, $L^p(a, b; X)$:= space of strongly measurable functions $f : [a, b] \to X$ such that $\int_a^b |f(s)|_X^p ds < \infty$ ($1 \leq p < \infty$, obvious modification for $p = \infty$); $\int_a^b f(s)ds \in X$ will indicate the Bochner integral on X; the norm on $L^p(a, b; X)$ will be denoted by $\|\cdot\|_{L^p(a,b;X)}$, or by $|\cdot|_{L^p(a,b;X)}$.

• $C^k([a, b]; X)$:= space of functions $f : [a, b] \to X$ which are k-times continuously differentiable ($k \in \mathbb{N}$).

If X and Y are Banach spaces, we set the following:

• $L(X, Y)$:= space of bounded, linear operators $T : X \to Y$;

• $L(X) := L(X, X)$; the norm on $L(X, Y)$ will be denoted with $\|\cdot\|_{L(X,Y)}$ (respectively $\|\cdot\|_{L(X)}$).

• If $\varphi : X \to \mathbb{R}$ is a continuously differentiable function we write $\varphi \in C^1(X)$; φ' will indicate its Fréchet differential with respect to $x \in X$, and $\nabla\varphi$ the Gâteaux derivative.

• If $\varphi : X \to (-\infty, +\infty]$ is a lower-semicontinuous (*l.s.c.*), convex, and proper function, we denote with $D(\varphi) := \{x \in X : \varphi(x) < +\infty\}$ the domain of φ, with $\partial\varphi : X \to X$ the subdifferential of φ, i.e.,

$$\partial\varphi(x) = \{x^* \in X \ : \ \varphi(y) - \varphi(x) \geq \langle x^*, y - x\rangle_X, \ \forall y \in X\},$$

and we set $D(\partial\varphi) := \{x \in X : \partial\varphi(x) \neq \emptyset\}$. We denote with φ^* the convex conjugate (or Fenchel transform) of φ; that is,

$$\varphi^*(x) = \sup_{y \in X}\{\langle y, x\rangle_X - \varphi(y)\}.$$

• If φ and ψ are both lower semicontinuous, convex, and proper, we indicate with $\varphi \,\Box\, \psi$ the inf-convolution of φ and ψ; that is,

$$\varphi \,\Box\, \psi(x) = \inf_{y \in X}\{\varphi(y) + \psi(x - y)\}.$$

The convex regularization of a convex, *l.s.c.* and proper function $\varphi : X \to (-\infty, +\infty]$ is given by

$$\varphi_n(x) := \varphi \,\square\, \frac{n}{2}|\cdot|(x) = \inf_{y\in X}\{\varphi(y) + \frac{n}{2}|x-y|^2\},\ n \in \mathbb{N}. \tag{3}$$

Finally, we list some useful properties of convex conjugate functions and convex regularization. We recall that

$$x^* \in \partial\varphi(x) \iff x \in \partial\varphi^*(x^*) \iff \varphi(x) + \varphi^*(x^*) = \langle x^*, x\rangle_X, \tag{4}$$

where the last identity will be referred to as the *conjugacy formula*. Moreover, if $\partial\varphi$ is injective, i.e., $\partial\phi(x) \cap \partial\phi(\bar{x}) = \emptyset$ when $x \neq \bar{x}$, the conjugacy formula implies $(\varphi^*)'(x) = (\partial\varphi)^{-1}(x)$ for all $x \in \partial\varphi(X)$.

Proposition 2.1 (Properties of convex regularization). *Let φ and φ_n be defined as above. Then φ_n is convex, everywhere finite on X, and Fréchet differentiable with Lipschitz differential. Moreover, for all x in X the following properties hold:*

(i) $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$*;*

(ii) the infimum in (3) *is attained at a unique point* $J_n^{\varphi}(x) := (I + \frac{1}{n}\partial\varphi)^{-1}(x)$*. Moreover,* $\varphi(J_n^{\varphi}(x)) \leq \varphi_n(x) \leq \varphi(x)$*;*

(iii) $\varphi_n(x) = \frac{1}{2n}|\varphi_n'(x)|^2 + \varphi(J_n^{\varphi}(x))$*;*

(iv) $\varphi_n'(x) = n(x - J_n^{\varphi}(x))$*, and if we set* $\|\!|\varphi x|\!\| := \inf\{|y| \ :\ y \in \partial\varphi(x)\}$*, then* $|\varphi_n'(x)| \leq \|\!|\varphi x|\!\|$*.*

For the proof of these results see [10].

3. Setting of the problem and an example

We hereafter precisely state all the assumptions of our problem. We suppose

$$\begin{cases} A : D(A) \subset H \to H \text{ is the infinitesimal generator of an} \\ \qquad\qquad\qquad\qquad \text{analytic semigroup of operators} \{e^{sA}\}_{s\geq 0}; \\ B : D(B) \subset U \to H \text{ is linear and unbounded.} \end{cases} \tag{H0}$$

We assume also the following crucial hypothesis:

$$\exists \gamma \in [0, 1/2)\ :\ (-A)^{-\gamma}B \in L(U, H), \tag{H1}$$

where $(-A)^{\gamma}$ is a fractional power of $-A$. (Note that for $\gamma = 0$ we have the case of bounded B.) Moreover,

$$\begin{cases} g : H \to [0, +\infty) \text{ and } \ \phi_0 : H \to [0, +\infty) \text{ are continuous and} \\ \text{convex functions, bounded on bounded subsets of } H. \end{cases} \tag{H2}$$

$$\begin{cases} h : U \to [0, +\infty],\ h \not\equiv +\infty,\ l.s.c. \text{ and convex} \\ \exists a,\ b \in \mathbb{R},\ a > 0\ :\quad h(u) \ge a|u|_U^2 + b,\ \forall u \in U. \end{cases} \tag{H3}$$

Remark 3.2. Note that $(H1)$ is well known in the literature: the reader is referred to the book by Lasiecka and Triggiani [24], or to the papers by Acquistapace, Flandoli, and Terreni [1], [3]. The exponent γ measures in some sense the unboundedness of the operator B: roughly speaking, the larger γ is, the worse B is. Assumption $(H1)$ lets us deal with objects such as $e^{sA}Bu$ with $u \in U$, since $e^{sA}(-A)^\gamma \in L(H)$ and of $(-A)^{-\gamma}B \in L(U, H)$.

Remark 3.3. The assumptions above have several immediate consequences, which we list in the sequel.

- First of all note that $R[(-A^*)^{-\gamma}] \subset D(B^*)$, so that $[(-A)^{-\gamma}B]^* = B^*(-A^*)^{-\gamma}$.
- Assumption $(H0)$ implies that there exist some positive constants $M,\ M_\gamma$, and ω such that

$$\|e^{sA}\|_{L(H)} \le Me^{\omega s},\ \|(-A)^\gamma e^{sA}\|_{L(H)} \le \frac{M_\gamma}{s^\gamma},\ \forall s \ge 0. \tag{5}$$

- In addition, $(H2)$ implies that g is subdifferentiable on H, that ∂g is bounded on every bounded subset of H and that g is locally Lipschitz. The same statements hold for ϕ_0. Then we indicate with $Lip_R(g)$ and $Lip_R(\phi_0)$ respectively the smallest positive constants such that

$$|g(x) - g(\bar{x})| \le Lip_R(g)|x - \bar{x}|,\ |\phi_0(x) - \phi_0(\bar{x})| \le Lip_R(\phi_0)|x - \bar{x}|,\ \forall x, \bar{x} \in \Sigma_R.$$

Controlled parabolic equations with Cauchy–Neumann conditions. We devote the last part of the section to an application taken from the book by Lasiecka and Triggiani [24] (the reader may find there some other applications to boundary and point control). Let Ω be an open and bounded subset of $\mathbb{R}^n$, with sufficiently smooth boundary $\partial\Omega$. In Ω, we consider the following Cauchy–Neumann problem for the heat equation, in the unknown $y(s, \xi)$:

$$\begin{cases} y_t(s, \xi) = (\Delta_\xi + c^2)y(s, \xi), & (s, \xi) \in (t, T) \times \Omega, \\ y(t, \xi) = x(\xi), & \xi \in \Omega \\ \frac{\partial y}{\partial \nu}(s, \xi) = u(s, \xi), & (s, \xi) \in (t, T) \times \partial\Omega =: \Sigma, \end{cases}$$

where $c \neq 0$, ν is the unit outward normal to $\partial\Omega$, Δ_ξ is the Laplacian with respect to the space variable ξ, the control u is a function in $L^2(\Sigma)$, and the initial value x is a function in $L^2(\Omega)$. We minimize the cost functional

$$J(t,x,u) := \int_t^T \left[\int_\Omega g_0(y(s,\xi))d\xi + \int_{\partial\Omega} h_0(u(s,\xi))d\sigma(\xi) \right] ds + \\ + \int_\Omega \psi_0(y(T,\xi))d\xi,$$

where g_0, h_0, and ψ_0 are convex functions from $\mathbb{R}^n$ to $\mathbb{R}$.

The abstract setting of the problem is the following:

$H := L^2(\Omega)$, $U := L^2(\partial\Omega)$, and A is the Neumann realization of the Laplace operator; that is,

$$\begin{cases} D(A) = \{y \in H^2(\Omega) : \frac{\partial y}{\partial \nu}_{|\partial\Omega} = 0\} \\ Ay = (\Delta_\xi + c^2)y, \ \forall y \in D(A). \end{cases}$$

Moreover, if N is the Neumann mapping $N : L^2(\partial\Omega) \to L^2(\Omega)$ defined by

$$Nu = w \quad \Longleftrightarrow \quad \begin{cases} (\Delta_\xi + c^2)w = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial \nu}_{|\partial\Omega} = 0 \end{cases}$$

one may show that the abstract state equation is given by

$$y(s,\cdot) = e^{(s-t)A}x(\cdot) + \int_t^s [(-A^*)^{1-\alpha} e^{(s-\sigma)A^*}]^* (-A)^\alpha N u(\sigma,\cdot) d\sigma$$

so that $B = -AN$.

Note that from elliptic theory (see *e.g.* [23]) we know the following facts:

- A is the generator of an analytic semigroup, it is self-adjoint and nonpositive, and it has a bounded inverse on $L^2(\Omega)$, so that fractional powers are well defined, with domains

$$D(-A)^\alpha = \begin{cases} H^{2\alpha}(\Omega) & \text{if } \alpha \in (0, \frac{3}{4}) - \{\frac{1}{2}\} \\ \{y \in H^{2\alpha}(\Omega) : \frac{\partial y}{\partial \nu}_{|\partial\Omega} = 0\} & \text{if } \alpha \in (\frac{3}{4}, 1); \end{cases}$$

- N is bounded and continuous from $L^2(\partial\Omega)$ to $H^{\frac{3}{2}}(\Omega)$, and then

$$N \in \mathcal{L}(L^2(\partial\Omega), D(-A)^\alpha), \ \forall \alpha \in (0, 3/4).$$

Hence,

$$(-A)^{-\gamma}B = (-A)^{1-\gamma}N \in L(U,H) \quad \Longleftrightarrow \quad \gamma = 1 - \alpha \in (0, 1/4)$$

so that assumption $(H1)$ is satisfied for all $\gamma \in (0, \frac{1}{4})$. We refer the reader to [24] and [8] for details and other examples on boundary and point control. In particular, we notice that our setting may fit in problems such as those on point control appearing in [24] (p. 57 and subsequent).

4. Existence of optimal controls

We denote the trajectory of the control system that starts from x at time t and is driven by a control u by $y_{t,x}(\cdot;u)$, or simply by $y(\cdot,u)$ or y when no confusion may arise. By the variation-of-constants formula, we may write system (1) as

$$y_{t,x}(s,u) = e^{(s-t)A}x + \int_t^s e^{(s-\sigma)A}Bu(\sigma)d\sigma,\ s \in [t,T], \qquad (S)_{t,x}$$

provided the integral on the right-hand side is defined. We then precisely state the optimal control problem as follows:

$$\inf\left\{J(t,x,u)\ :\ u \in L^2(t,T;U),\ y_{t,x}(\cdot,u) \text{ satisfies } (S)_{t,x}\right\}. \qquad (P)_{t,x}$$

When the infimum is attained at some $u^* \in L^2(t,T;U)$ we say that u^* is optimal for $(P)_{t,x}$. We denote optimal controls by $u^*_{t,x}$ when it is necessary to point out their dependence on t and x. A pair $(u^*,y^*) \in L^2(t,T;U) \times L^2(t,T;H)$, where u^* is optimal and y^* is the associated trajectory, is called an optimal pair.

Mild solutions are continuous trajectories, as we see next.

Proposition 4.4. *Let $(H0)$ and $(H1)$ be satisfied, and let $u \in L^2(t,T;U)$. The mild solution of system $(S)_{t,x}$ is in $C^0([t,T];H)$.*

Remark 4.5. If we assume $\gamma \in [0,1)$ in $(H1)$ (that is, possibly $\gamma \geq 1/2$) we may show also what follows: $u \in L^2(t,T;U)$ implies that $y \in L^2(t,T;H)$ and that $(-A)^{-\beta}y \in C^0([t,T];H)$ for all $\beta > \gamma - \frac{1}{2}$; $u \in L^p(t,T;U)$, and $\gamma \in [0,\frac{p-1}{p})$ imply $y \in C^0([t,T];H)$.

A proof of Proposition 4.4 and of Remark 4.5 can be found in [8].

Now we prove existence of an optimal control for $(P)_{t,x}$.

Theorem 4.6. Let $(H0)$, $(H1)$, $(H2)$, and $(H3)$ be satisfied, and let $t \in [0,T]$ and $x \in H$ be fixed. Then there exists an optimal control $u^*_{t,x}$ for problem $(P)_{t,x}$.

Proof. Let $x \in H$ and $t \in [0,T]$ be fixed. The functional $J(t,x,u)$ is a convex function of u. Moreover, since g and ϕ_0 are positive functions, from $(H3)$ it follows that

$$J(t,x,u) \geq a\|u\|_{L^2(t,T;U)} + b(T-t)\ \to +\infty, \quad \text{as} \quad \|u\|_{L^2(t,T;U)} \to +\infty.$$

If we show that $J(t,x,u)$ is a *l.s.c.* function of u, then $J(t,x,\cdot)$ has a minimum over $L^2(t,T;U)$.

To this extent, let $\{u_n\} \in L^2(t,T;U)$, with $u_n \to \bar{u}$ in $L^2(t,T;U)$, and let $y_n := y_{t,x}(\cdot\ ;u_n)$ and $\bar{y} := y_{t,x}(\cdot;\bar{u})$ be the associated trajectories. We show first that $y_n \to \bar{y}$ in $C^0([t,T];H)$ and hence in $L^2(t,T;H)$. In fact, for all $s \in [t,T]$ we have, by (5) and Hölder's inequality,

$$\begin{aligned}
&|y_n(s) - \bar{y}(s)|_H \leq \\
&\leq \int_t^s \|e^{(s-\sigma)A}(-A)^\gamma\|_{L(H)}\|(-A)^{-\gamma}B\|_{L(U,H)}|u_n(\sigma) - u(\sigma)|_U \, d\sigma \\
&\leq \|(-A)^{-\gamma}B\|_{L(U,H)} M_\gamma \Big(\int_t^s (s-\sigma)^{-2\gamma} d\sigma\Big)^{1/2} \|u_n - \bar{u}\|_{L^2(t,T;U)} \\
&\leq \|(-A)^{-\gamma}B\|_{L(U,H)} M_\gamma \Big(\frac{T^{1-2\gamma}}{1-2\gamma}\Big)^{\frac{1}{2}} \|u_n - u\|_{L^2(t,T;U)} \\
&\to 0, \; n \to \infty.
\end{aligned}$$

In particular,

$$\phi_0(y_n(T)) \to \phi_0(\bar{y}(T)). \tag{6}$$

Since the functions $L^2(t,T;H) \to \mathbb{R}$, $y \mapsto \int_t^T g(y)ds$, and $L^2(t,T;U) \to \mathbb{R}$, $u \mapsto \int_t^T h(u)ds$ are (*l.s.c.* and hence) weakly *l.s.c.* functions, we have also

$$\liminf_{n\to\infty} \int_t^T (g(y_n(\sigma)) + h(u_n(\sigma)))\, d\sigma \geq \int_t^T \big(g(\bar{y}) + h(\bar{u})\big) d\sigma. \tag{7}$$

Hence, from (6) and (7) we infer

$$\liminf_{n\to\infty} J(t,x,u_n) \geq J(t,x,\bar{u}),$$

and the proof is complete. □

Remark 4.7. It is possible to show existence of optimal controls also in the case $\gamma \geq 1/2$, under the extra assumption $\phi_0 := \tilde{\phi}_0 \cdot (-A)^{-\beta}$ for some $\beta > \gamma - \frac{1}{2}$, where $\tilde{\phi}_0 : H \to \mathbb{R}$ is convex and continuous, in order to give meaning to the term $\phi_0(y(T))$ in (2) when y is merely a function in $L^2(t,T;H)$.

5. The approximating problem

Our techniques make use of a family of problems $(S)_{n,t,x}(P)_{n,t,x}$ where we replace the unbounded operator B with a sequence of approximating bounded operators B_n, defined by

$$B_n := -[-(-A)^\gamma]_n(-A)^{-\gamma}B \equiv n(-A)^\gamma(n + (-A)^\gamma)^{-1}(-A)^{-\gamma}B.$$

Note that $[-(-A)^\gamma]_n$ denotes the Yosida approximation of the operator $[-(-A)^\gamma]$. Indeed, we recall that if $-A$ is of type (ω, M), then $(-A)^\gamma$ is of type $(\gamma\omega, M)$ (see [T]). Note that $B_n u \to Bu$ for all $u \in D(B)$, and also that $(-A)^{-\gamma}B_n$ is a sequence of linear, bounded operators such that

$(-A)^{-\gamma}B_n u \to (-A)^{-\gamma}Bu,\ \forall u \in U$. Hence by the Banach–Steinhaus theorem, there exists a constant $N > 0$ such that

$$\max\{\|(-A)^{-\gamma}B\|_{L(U,H)}, \|(-A)^{-\gamma}B_n\|_{L(U,H)}\} \leq N,\ \forall n \in \mathbb{N}. \tag{8}$$

We now consider the system

$$\begin{cases} y_n'(s) = Ay_n(s) + B_n u(s) & s \in (t,T) \\ y_n(t) = x \end{cases} \qquad (S)_{n,t,x}$$

and the cost functional

$$J_n(t,x,u) := \int_t^T (g_n(y_{n,t,x}(s;u)) + h_n(u(s)))\,ds + \\ + \phi_{0n}(y_{n,t,x}(T;u)) + \frac{1}{2}\|u - u^*_{t,x}\|^2_{L^2(t,T;U)}$$

where $u^*_{t,x}$ is any fixed optimal control for problem $(P)_{t,x}$, $y_{n,t,x}(\cdot;u)$ indicates the trajectory of system $(S)_{n,t,x}$ associated to the control u, and g_n, h_n, and ϕ_{0n} are the convex regularizations of g, h, and ϕ_0 respectively.[1]

The approximating problem is then

$$\inf\left\{J_n(t,x,u)\ :\ u \in L^2(t,T;U),\ y_{n,t,x}(\cdot,u) \text{ satisfies } (S)_{n,t,x}\right\}. \qquad (P)_{n,t,x}$$

In the next lemmata we prove some properties of convergence of sequences of controls and associated trajectories which will be useful in the sequel.

Lemma 5.1. *Let $(H0)$ and $(H1)$ be satisfied. Let $u,\ u_n \in L^2(t,T;U)$, and let $y := y_{t,x}(\cdot,u)$ and $y_n := y_{n,t,x}(\cdot,u_n)$ be the associated trajectories through $(S)_{t,x}$ and $(S)_{n,t,x}$ respectively. We have the following:*

(i) if $u_n \to u$ weakly in $L^2(t,T;U)$, then $y_n(s) \to y(s)$ weakly in H for all $s \in [t,T]$, and $y_n \to y$ weakly in $L^2(t,T;H)$;

(ii) if $u_n \to u$ strongly in $L^2(t,T;U)$, then $y_n \to y$ in $C^0([t,T],H)$;

(iii) there exists a positive constant C independent of n, t, s, and x such that

$$\begin{cases} |y(s)|_H \leq C(|x|_H + \|u\|_{L^2(t,T;U)}), & \forall s \in [t,T] \\ |y_n(s)|_H \leq C(|x|_H + \|u_n\|_{L^2(t,T;U)}), & \forall s \in [t,T]. \end{cases} \tag{9}$$

Remark 5.2. The property (i) holds true also in the case $\gamma \in [1/2, 1)$. Moreover, we may show that $u_n \to u$ strongly in $L^2(t,T;U)$ implies $y_n \to y$ strongly in $L^2(t,T;H)$ and $(-A)^{-\beta}y_n \to (-A)^{-\beta}y$ in $C^0([t,T],H)$ for all $\beta > \gamma - \frac{1}{2}$.

[1]Note that the idea of approximating the data g, ϕ_0, and h by means of convex regularization is due to Barbu and Precupanu [10].

Proof. We set $S_n := (-A)^{-\gamma}B_n$, and $S := (-A)^{-\gamma}B$. We first prove (i). Let $u_n \to u$ weakly in $L^2(t,T;U)$, and let $s \in [t,T]$ be fixed. Then for all $z \in H$ we have

$$\begin{aligned}\langle y_n(s), z\rangle_H &= \langle e^{(s-t)A}x, z\rangle_H + \int_t^s \langle u_n(\sigma), B^* e^{(s-\sigma)A^*} z\rangle_H d\sigma \\ &= \langle e^{(s-t)A}x, z\rangle_H + \langle u_n, S_n^*(-A^*)^\gamma e^{(s-\cdot)A^*} z\rangle_{L^2(t,s;U)} \to \\ &\to \langle e^{(s-t)A}x, z\rangle_H + \langle u, S^*(-A^*)^\gamma e^{(s-\cdot)A^*} z\rangle_{L^2(t,s;U)} = \langle y(s), z\rangle_H.\end{aligned}$$

Let now $\varphi \in L^2(t,T;U)$ be arbitrarily fixed. Then

$$\begin{aligned}&\langle y_n - y, \varphi\rangle_{L^2(t,T;U)} = \\ &\quad = \int_t^T \Big[\int_t^s \langle S_n u_n(\sigma) - Su(\sigma), (-A^*)^\gamma e^{(s-\sigma)A^*}\varphi(s)\rangle_H d\sigma \Big] ds \\ &\quad = \int_t^T \Big\langle S_n u_n(\sigma) - Su(\sigma)\ ,\ \int_\sigma^T (-A^*)^\gamma e^{(s-\sigma)A^*}\varphi(s) ds \Big\rangle d\sigma \\ &\quad = \langle S_n u_n - Su, f\rangle_{L^2(t,T;H)},\end{aligned}$$

where one can easily show that $f(\sigma) := \int_\sigma^T (-A^*)^\gamma e^{(s-\sigma)A^*}\varphi(s) ds$ is a function in $L^2(t,T;U)$. Now

$$\langle S_n u_n - Su, f\rangle_{L^2(t,T;U)} = \langle u_n, S_n^* f\rangle_{L^2(t,T;U)} - \langle u, S^* f\rangle_{L^2(t,T;U)} \to 0,\ n \to \infty;$$

in fact $S_n^* f(\sigma) \to S^* f(\sigma)$ for almost all σ in $[t,T]$ and $|S_n^* f(\sigma) - S^* f(\sigma)|_U \leq 2N|f(\sigma)|_H$ (where N is the constant appearing in (8)), so that $S_n^* f \to S^* f$ strongly in $L^2(t,T;U)$ by dominated convergence. This completes the proof of (i).

Let now $u_n \to u$ strongly in $L^2(t,T;U)$. Then

$$|y(s) - y_n(s)|_H \leq M_\gamma \left(\frac{T^{1-2\gamma}}{1-2\gamma}\right)^{1/2} \|S_n u_n - Su\|_{L^2(t,T;H)}.$$

On the other hand

$$\begin{aligned}\|S_n u_n - Su\|_{L^2(t,T;H)} &\leq \|S_n(u_n - u)\|_{L^2(t,T;H)} + \|(S_n - S)u\|_{L^2(t,T;H)} \\ &\leq N\|u_n - u\|_{L^2(t,T;U)} + \|(S_n - S)u\|_{L^2(t,T;H)}\ \to 0,\ n \to \infty.\end{aligned}$$

This proves (ii).

Finally, for any $s \in [t,T]$ we have

$$\begin{aligned}|y(s)|_H &\le \|e^{(s-t)A}\|_{L(H)}|x|_H + \\ &+ \int_t^s \|e^{(s-\sigma)A}(-A)^{\gamma}\|_{L(H)}\|(-A)^{-\gamma}B\|_{L(U;H)}|u(\sigma)|_U\, d\sigma \\ &\le Me^{\omega T}|x|_H + M_\gamma\|(-A)^{-\gamma}B\|_{L(U,H)}\left(\int_t^s (s-\sigma)^{-2\gamma}d\sigma\right)^{\frac{1}{2}}\|u\|_{L^2(t,T;U)} \\ &\le C\big(|x|_H + \|u\|_{L^2(t,T;U)}\big),\end{aligned}$$

where $C := \max\left\{Me^{\omega T}, M_\gamma\|(-A)^{-\gamma}B\|_{L(U,H)}\left(\frac{T^{1-2\gamma}}{1-2\gamma}\right)^{\frac{1}{2}}\right\}$. The estimate for y_n can be proved in the same way. □

Lemma 5.3. *Let $(H0)$, $(H1)$, $(H2)$, and $(H3)$ be satisfied, and let $r > 0$ be fixed. Then there exist positive constants C_r and C'_r independent of t such that*

$$\begin{cases} J(t,x,u^*_{t,x}) \le C_r \\ J_n(t,x,u^*_{n,t,x}) \le C'_r, \quad \forall t \in [0,T],\ \forall x \in \Sigma_r,\ \forall n \in \mathbb{N}. \end{cases} \tag{10}$$

If $(u^*_{t,x}, y^*_{t,x})$ and $(u^*_{n,t,x}, y^*_{n,t,x})$ are optimal pairs respectively for $(P)_{t,x}$ and $(P)_{n,t,x}$ then there exist some positive constants K_r and K'_r, independent of t, such that

$$\begin{cases} \|u^*_{t,x}\|_{L^2(t,T;U)} \le K_r, \\ \|u^*_{n,t,x}\|_{L^2(t,T;U)} \le K'_r, \quad \forall t \in [0,T],\ \forall x \in \Sigma_r,\ \forall n \in \mathbb{N}. \end{cases} \tag{11}$$

Moreover, there exist $M_r,\ M'_r > 0$ such that

$$\begin{cases} |y^*_{t,x}(s)|_H \le M_r, \\ |y^*_{n,t,x}(s)|_H \le M'_r, \quad \forall t \in [0,T],\ \forall s \in [t,T],\ \forall x \in \Sigma_r. \end{cases} \tag{12}$$

Remark 5.4. Note that (10) and (11) remain true also for $\gamma \in [1/2, 1)$.

Proof. We first prove the estimates regarding problem $(P)_{t,x}$. Let $u_0 \in D(h)$ be fixed, and set $u_0(s) := u_0$, $\forall s \in [0,t]$. Note that for all t in $[0,T]$, for all s in $[t,T]$, and for all x, $|x|_H \le r$, we have

$$|y_{t,x}(s,u_0)|_H \le Me^{\omega T}|x|_H + N|u_0|_U M_\gamma \frac{T^{1-\gamma}}{1-\gamma} \le N_r \tag{13}$$

for some positive constant N_r. Hence, since g and ϕ_0 are bounded on bounded subsets of H,

$$J(t,x,u^*_{t,x}) \le J(t,x,u_0) \le T\big[h(u_0) + \sup_{|y|\le N_r} g(y)\big] + \sup_{|y|\le N_r} \phi_0(y) =: C_r \tag{14}$$

where C_r is finite and does not depend on t. This proves (10).

Inequality (11) follows from $(H3)$. In fact,

$$J(t,x,u^*_{t,x}) \geq \int_t^T h(u^*_{t,x}(s))ds \geq a\|u^*_{t,x}\|^2_{L^2(t,T;U)} + b;$$

hence,

$$\|u^*_{t,x}\|_{L^2(t,T;U)} \leq \frac{1}{a}(C_r - b) =: K_r.$$

Next we prove the estimates regarding problems $(P)_{n,t,x}$. It is easy to show that also the trajectory of the approximating problem satisfies

$$|y_{t,x,n}(s,u_0)|_H \leq N_r, \ \forall t \in [0,T], \ \forall s \in [t,T], \ \forall n \in \mathbb{N}.$$

Moreover, since Proposition 2.1 (ii) implies $g_n \leq g$, $h_n \leq h$, and $\phi_{0n} \leq \phi_0$, we have

$$J_n(t,x,u^*_{n,t,x}) \leq J_n(t,x,u_0) \leq \int_t^T \big[g(y_{n,t,x}(s,u_0)) + h(u_0)\big]ds$$
$$+ \phi_0(y_{n,t,x}(T,u_0)) + \frac{1}{2}\|u^*_{t,x} - u_0\|^2_{L^2(t,T;H)},$$

which coupled with (14) gives

$$J_n(t,x,u^*_{n,t,x}) \leq C_r + \frac{1}{2}\big(K_r + (T-t)|u_0|\big)^2 =: C'_r.$$

The second inequality in (11) is implied by

$$\frac{1}{2}\|u^*_{n,t,x} - u^*_{t,x}\|^2_{L^2(t,T;H)} \leq J_n(t,x,u^*_{n,t,x}) \leq C'_r.$$

The proof of (12) follows from (11) coupled with Lemma 5.1 (iii). □

We next prove some properties of convergence of optimal couples of the approximating problems.

Lemma 5.5. *Let $(H0)$, $(H1)$, $(H2)$, and $(H3)$ be satisfied, and let $t \in [0,T]$ and $x \in H$ be fixed. For all $n \in \mathbb{N}$ the minimum in $(P)_{n,t,x}$ is attained at a unique point $u^*_{n,t,x}$. Moreover, the optimal pair $(u^*_{n,t,x}, y^*_{n,t,x})$ satisfies*

(i) $u^*_{n,t,x} \to u^*_{t,x}$ strongly in $L^2(t,T;U)$;

(ii) $y^*_{n,t,x} \to y^*_{t,x}$ in $C^0([t,T];H)$;

(iii) $\liminf_{n\to\infty} J_n(t,x,u^*_{n,t,x}) \geq J(t,x,u^*)$.

Proof. Let $y^* := y^*_{t,x}$, $y^*_n := y^*_{n,t,x}$, and $J_n(u) := J_n(t,x,u)$. For any fixed $n \in \mathbb{N}$, the function $u \mapsto J_n(u)$ is strictly convex, *l.s.c.,* and coercive; hence, the minimum in $(P)_{n,t,x}$ is actually attained at a unique point u^*_n in $L^2(t,T;U)$. Let us now show that (i) and (ii) are satisfied.

Claim 1. *A subsequence of optimal pairs* $(u^*_{n_k}, y^*_{n_k})$ *satisfies* $u^*_{n_k} \to \tilde{u}$ *weakly in* $L^2(t,T;U)$ *and* $y^*_{n_k} \to \tilde{y}$ *weakly in* $L^2(t,T;H)$*, where* $\tilde{y}(s) := y_{t,x}(s,\tilde{u})$*.*

We have showed in Lemma 5.3 that $\{u^*_n\}$ is bounded in $L^2(t,T;U)$, uniformly with respect to n. Then there is a subsequence of $\{u^*_n\}$, which we keep denoting with $\{u^*_n\}$, that converges weakly in $L^2(t,T;U)$ to a control $\tilde{u}$. Moreover, in view of Lemma 5.1 (i), $y^*_n \to \tilde{y} := y_{t,x}(\cdot,\tilde{u})$ weakly in $L^2(t,T;H)$, and Claim 1 is proved.

Set now

$$G_n(u^*_n) := \int_t^T \big(g_n(y^*_n(s)) + h_n(u^*_n(s))\big)ds + \phi_{0n}(y^*_n(T))$$

$$I_n(u^*) := \int_t^T \big(g(y_n(s,u^*)) + h(u^*(s))\big)ds + \phi_0(y_n(T,u^*)),$$

and observe that $J_n(u^*_n) \le J_n(u^*) \le I_n(u^*)$, which implies

$$\frac{1}{2}\|u^*_n - u^*\|^2_{L^2(t,T;U)} + G_n(u^*_n) \le I_n(u^*). \tag{15}$$

We now wish to show that $u^*_n \to u^*$ strongly in $L^2(t,T;U)$ by passing to limits in the previous inequality.

Claim 2.

$$I_n(u^*) \to J(u^*),\ n \to \infty. \tag{16}$$

Note that (9) implies the existence of a positive constant $M > 0$ independent of n and s such that $|y^*(s)|_H$ and $|y_n(s,u^*)|_H \le M$, and that by Lemma 5.1 (ii), we get $y_n(\cdot,u^*) \to y^*,$ in $C^0([t,T],H)$. Hence, by Lebesgue's theorem,

$$|I_n(u^*) - J(u^*)| \le Lip_M(g)\int_t^T |y_n(s;u^*) - y^*(s)|_H\,ds + \\ + Lip_M(\phi_0)|y_n(T;u^*) - y^*(T)|_H \ \to 0,\ n \to \infty.$$

Claim 3. *The following inequality holds:*

$$\liminf_{n\to\infty} G_n(u^*_n) \ge J(t,x,\tilde{u}). \tag{17}$$

Set $\mathcal{J}^h_n := \big(1 - \frac{1}{n}\partial h\big)^{-1}$, $\mathcal{J}^g_n := \big(1 - \frac{1}{n}\partial g\big)^{-1}$, and $\mathcal{J}^{\phi_0}_n := \big(1 - \frac{1}{n}\partial \phi_0\big)^{-1}$. In view of inequality (ii) in Proposition 2.1,

$$G_n(u^*_n) \ge \int_t^T g(\mathcal{J}^g_n(y^*_n(s))) + h(\mathcal{J}^h_n(u^*_n(s)))ds + \phi_0(\mathcal{J}^{\phi_0}_n(y^*_n(T))). \tag{18}$$

Now, note that since $\partial\phi_0$ is bounded on bounded sets and $\sup_n |y^*_n(T)|_H < +\infty$, then in view of Proposition 1.1 (iv) there exists $0 < K < +\infty$ such that

$$\|\phi_0(y_n^*(T))\| = \inf\{|y|_H : y \in \partial\phi_0(y_n^*(T))\} \le K, \ \forall n \in \mathbb{N}$$

and

$$|\mathcal{J}_n^{\phi_0}(y_n^*(T)) - y_n^*(T)|_H \le K/n.$$

Moreover, Lemma 5.1 (i) implies $y_n^*(T) \to \tilde{y}(T)$ weakly in H, and hence

$$\begin{cases} \mathcal{J}_n^{\phi_0}(y_n^*(T)) \to \tilde{y}(T), \text{ weakly in } H \\ \liminf_{n\to\infty} \phi_0(\mathcal{J}_n^{\phi_0}(y_n^*(T))) \ge \phi_0(\tilde{y}(T)), \end{cases} \tag{19}$$

for ϕ_0 is weakly *l.s.c.* Next we show that

$$\mathcal{J}_n^h(u_n^*) \to \tilde{u} \text{ weakly in } L^2(t,T;U).$$

To this extent, it is enough to show that $\mathcal{J}_n^h(u_n^*(\cdot)) - u_n^* \to 0$ strongly in $L^2(t,T;U)$. Indeed, in view of (i) and (iv) in Proposition 2.1 and (10) we have

$$\frac{1}{2n}\int_t^T |n[u_n^*(s) - \mathcal{J}_n^h(u_n^*(s))]|_U^2 ds = \frac{1}{2n}\int_t^T |h_n'(u_n^*(s))|_U^2 ds =$$
$$\le \int_t^T h_n(u_n^*(s))ds \le J_n(u_n^*) \le C_r'$$

so that

$$\|u_n^* - \mathcal{J}_n^h(u_n^*(\cdot))\|^2_{L^2(t,T;U)} \le \frac{2}{n}C_r'.$$

Hence, since $u \mapsto \int_t^T h(u)ds$ weakly–*l.s.c.* in $L^2(t,T;U)$, we have

$$\liminf_{n\to\infty} \int_t^T h(\mathcal{J}_n^h(u_n^*(s)))ds \ge \int_t^T h(\tilde{u}(s))ds. \tag{20}$$

Arguing similarly we may show also that $\mathcal{J}_n^g(y_n^*) \to \tilde{y}$ weakly in $L^2(t,T;H)$, so that

$$\liminf_{n\to\infty} \int_t^T g(\mathcal{J}_n^g(y_n^*(s)))ds \ge \int_t^T g(\tilde{y}(s))ds. \tag{21}$$

Then Claim 3 is proved by combining (18), (19), (20), and (21).

We are now ready to draw the conclusion, for we can take the liminf in (15) and, using (16) and (17), obtain

$$\liminf_{n\to\infty} \|u_n^* - u^*\|^2_{L^2} \le J(u^*) - J(\tilde{u}) \le 0.$$

This implies, possibly passing to a further subsequence,

$$u_n^* \to u^* \text{ strongly in } L^2(t,T;U) \text{ and } u^* = \tilde{u}. \tag{22}$$

Note that the whole sequence u_n converges to u^* strongly in $L^2(t,T;U)$. Indeed, any subsequence of u_n^* has itself a subsequence weakly converging to u^*. This proves (i).

The proof of (ii) is a straightforward consequence of Lemma 5.1, and (iii) follows from (i) and (17). □

6. The maximum principle

We turn now our attention to finding necessary (and sufficient) conditions for a control u^* to be optimal. To this extent, we consider the dual system associated to $(S)_{t,x}$,

$$\begin{cases} p'(s) \in -A^*p(s) - \partial g(y_{t,x}(s;u)), \quad s \in [t,T] \\ p(T) \in \partial\phi_0(y_{t,x}(T;u)). \end{cases} \tag{23}$$

Let us suppose for a moment that g and ϕ_0 are functions in $C^1(H)$. By applying a variation-of-constants formula to the initial-value problem $\varphi'(s) = A^*\varphi(s)+g'(y_{t,x}(T-s;u)),\ \varphi(0) = \phi_0'(y_{t,x}(T;u))$, and setting $\varphi(s) := p(T-s)$, we find that (23) is a Cauchy problem with a unique mild solution given by

$$p(s) = e^{(T-s)A^*}\phi_0'(y(T)) + \int_s^T e^{(\sigma-s)A^*}g'(y(\sigma))d\sigma,\ s \in [t,T]. \tag{24}$$

Note that p depends on t, x, and u through the trajectory $y = y_{t,x}(\cdot\ ;u)$. In the case of regular data, it is not difficult to show that a control $u^*_{t,x}$ is optimal for problem $(S)_{t,x}(P)_{t,x}$ if and only if it satisfies the following relation:

$$-B^*p_{t,x}(s;u^*) \in \partial h(u^*(s)),\ a.e. \text{ in } [t,T] \tag{25}$$

(see Remark 6.12), which is equivalent to the Pontryagin maximum principle for our problem.

If we do not assume any extra regularity for g and ϕ_0, many problems arise. First of all (23) is a differential inclusion, which we wish to write as

$$p(s) = e^{(T-s)A^*}\xi_T + \int_s^T e^{(\sigma-s)A^*}q(\sigma)d\sigma,\ \forall s \in [t,T], \tag{C}_{t,x}$$

where $q(\sigma) \in \partial g(y(\sigma))$ for almost all $\sigma \in [t,T]$, and $\xi_T \in \partial\phi_0(y(T))$. However, the right-hand side in $(C)_{t,x}$ is defined if and only if q is a measurable selection. Moreover, we may want to choose q and ξ_T in a way such that (25) is still satisfied.

Whenever the right-hand side in $(C)_{t,x}$ is defined we refer to p as the dual variable, or *costate,* associated to the state variable $y_{t,x}$.

We first point out some regularity properties of the dual variable.

Proposition 6.6. *Let (H0), (H1), (H2), and (H3) be satisfied. Let also* $t \in [0,T]$, $x \in H$, $u \in L^2(t,T;U)$, *and* $\xi_T \in \partial\phi_0(y_{t,x}(\cdot;u))$ *be fixed and let* q *be a measurable function on* $[t,T]$ *such that* $q(\sigma) \in \partial g(y(\sigma))$ *almost everywhere in* $[t,T]$. *Then the dual variable* $p_{t,x}(\cdot;u)$ *defined by* $(C)_{t,x}$ *is a function in* $C^0([t,T];H) \cap C^0([t,T), D(-A^*)^\gamma)$.

Remark 6.7. From $D(-A^*)^\gamma \subset D(B^*)$, we have

$$p_{t,x}(\cdot;u) \in C^0([t,T), D(B^*)),$$

which means also that $B^*p^* \in C^0([t,T);H)$.

Proof. Since ∂g is bounded on every bounded subset and y is continuous on $[t,T]$, then q is in $L^\infty(t,T;H)$.

We now wish to show that the function $p_{t,x}(\cdot;u)$ is in $C^0([t,T];H)$. Note that

$$\begin{aligned}|p(s) - p(\bar{s})|_H \le{}& \|e^{(T-s)A^*}\|_{L(H)}|(1 - e^{(s-\bar{s})A^*})\xi_T|_H + \\ &+ \int_{\bar{s}}^{s} |e^{(\sigma-\bar{s})A^*}q(\sigma)|_H d\sigma + \int_s^T |e^{(\sigma-s)A^*}(1 - e^{(s-\bar{s})A^*})q(\sigma)|_H d\sigma \\ \le{}& Me^{\omega T}\left(|(1 - e^{|s-\bar{s}|A^*})\xi_T|_H + \|q\|_\infty|s-\bar{s}| + \int_t^T |(1 - e^{|s-\bar{s}|A^*})q(\sigma)|_H d\sigma\right),\end{aligned}$$

and the same estimate holds reversing the roles of s and $\bar{s}$. Dominated convergence applies to the last term, so the last expression goes to zero when $s \to \bar{s}$.

Next we show that $p([t,T)) \subset D(-A^*)^\gamma$. Fix any $s \in [t,T)$ and note that since $\{e^{sA^*}\}_{s\ge 0}$ is analytic, then $e^{(T-s)A^*}\xi_T \in D(-A^*)^\gamma$, and $e^{(\sigma-s)A^*}q(\sigma) \in D(-A^*)^\gamma$ for almost all $\sigma \in [s,T]$. Note also that

$$\int_s^T |(-A^*)^\gamma e^{(\sigma-s)A^*}q(\sigma)|_H d\sigma \le \int_s^T M_\gamma(\sigma-s)^{-\gamma}\|q\|_\infty d\sigma \le C|T-s|^{1-\gamma},$$

so that

$$\int_s^T e^{(\sigma-s)A^*}q(\sigma)d\sigma = (-A^*)^{-\gamma}\int_s^T (-A^*)^\gamma e^{(\sigma-s)A^*}q(\sigma)d\sigma \in D(-A^*)^\gamma,$$

which implies $p(s) \in D(-A^*)^\gamma$, for all $s \in [t,T)$. Let $s,\ \bar{s} \in [t,T)$ with $\bar{s} \le s$. Then

$$\begin{aligned}|(-A^*)^\gamma(p(s) - p(\bar{s}))|_H \le{}& M_\gamma(T-s)^{-\gamma}|(1 - e^{(s-\bar{s})A^*})\xi_T|_H + \\ &+ \frac{M_\gamma\|\xi\|_\infty}{1-\gamma}|s-\bar{s}|^{1-\gamma} + M_\gamma\int_s^T (\sigma-s)^{-\gamma}|(1 - e^{|s-\bar{s}|A^*})q(\sigma)|_H d\sigma.\end{aligned}$$

Note that by applying Hölder's inequality and dominated convergence, we may show that

$$\int_s^T (\sigma - s)^{-\gamma} |(1 - e^{|s-\bar{s}|A^*})q(\sigma)|_H d\sigma \leq$$
$$\leq \frac{(T-s)^{\frac{1}{2}-\gamma}}{(1-2\gamma)^{\frac{1}{2}}} \left(\int_s^T |(1 - e^{|s-\bar{s}|A^*})q(\sigma)|_H^2 d\sigma \right)^{\frac{1}{2}} \to 0, \text{ as } s \to \bar{s}.$$

A similar proof holds for $s < \bar{s}$. □

Definition 6.8. We say that a pair $(u^*, y^*) \in L^2(t,T;U) \times L^2(t,T;H)$ is extremal for problem $(P)_{t,x}$ if there exist functions $q \in L^1(t,T;H)$ and $p^* : [t,T] \to D(B^*)$ satisfying along with u^* and y^* the equations

$$y^*(s) = e^{(s-t)A}x + \int_t^s e^{(s-\sigma)A} Bu^*(\sigma) d\sigma \; \forall s \in [t,T] \qquad (S)_{t,x}$$

$$\begin{cases} p^*(s) = e^{(T-s)A^*} p^*(T) + \int_s^T e^{(\sigma-s)A^*} q(\sigma) d\sigma, \; \forall s \in [t,T] \\ p^*(T) \in \partial\phi_0(y^*(T)); \; q(\sigma) \in \partial g(y^*(\sigma)), \; a.e. \; in \; [t,T] \end{cases} \qquad (C)_{t,x}$$

$$-B^* p^*(s) \in \partial h(u^*(s)), \; a.e. \text{ in } [t,T]. \qquad (26)$$

In this case, obviously, $y^* = y_{t,x}(\cdot, u^*)$. The corresponding functions p^*, which are the dual variables associated to the state variable y^* (hence to the control u^*), will be denoted by $p^*_{t,x}(\cdot, u^*)$.

Theorem 6.9. *Let $(H0)$, $(H1)$, $(H2)$, and $(H3)$ be satisfied. A pair (u^*, y^*) is optimal if and only if it is extremal.*

Remark 6.10. We borrowed some of our techniques from [5] and [10, Chapter 4]. In particular, as we mention in the introduction, Barbu and Precupanu obtain a result of the same type (see [10, Chapter 4, Theorem 1.1]). Their assumptions are different, and they apply as well to the Cauchy–Neumann problem presented in Section 3. Also, their proof is different, as it makes use of a change of variables, while we obtain our result by approximation.

Sufficiency. Suppose that (u^*, y^*) is an extremal pair, and that p^* is an associated costate. Let $K : L^2(t,T;U) \to \mathbb{R}$ and $G : L^2(t,T;U) \to \mathbb{R}$ be defined by

$$K(u) := \int_t^T h(u(s))ds, \;\; G(u) := \int_t^T g(y_{t,x}(s;u))ds + \phi_0(y_{t,x}(T;u)),$$

so that $J(u) = K(u) + G(u)$. Since $D(G) = L^2(t,T;U)$ and $D(K) \neq \emptyset$

(*e.g.*, $u(s) := u_0 \in D(h)$ is in $D(K)$), from well-known properties of convex functions it follows that

$$\partial J(u) = \partial K(u) + \partial G(u). \tag{27}$$

We now state that

$$\partial K(u) = \{v \in L^2(t,T;U) \ : \ v(s) \in \partial h(u(s)) \ a.e.\ in\ [t,T]\}. \tag{28}$$

In fact, by definition

$$\partial K(u) = \big\{v \in L^2(t,T;U) \ : \\ K(w) - K(u) \geq \langle v, w-u\rangle_{L^2(t,T;U)},\ \forall w \in L^2(t,T;U)\big\};$$

that is, $v \in \partial K(u)$ if and only if

$$\int_t^T \big[h(w(s)) - h(u(s)) - \langle v(s), w(s)-u(s)\rangle_U\big]ds \geq 0,\ \forall w \in L^2(t,T;U).$$

The proof of "$\supset$" in (28) is straightforward. On the other hand, let $v \in \partial K(u)$ and E be any measurable subset of $[t,T]$. We set

$$w(s) := \begin{cases} u(s) & if s \notin E \\ w(s) & if s \in E, \end{cases}$$

where w is any control in $L^2(t,T;U)$. By definition of $\partial K(u)$, we have

$$\int_E \big[h(w(s)) - h(u(s)) - \langle v(s), w(s)-u(s)\rangle_U\big]ds \geq 0,\ \forall w \in L^2(t,T;U);$$

that is, due to the fact that E was arbitrary,

$$h(w(s)) - h(u(s)) \geq (v(s), w(s)-u(s))_U,\ \text{for } a.e.\ s \in [t,T],$$

which implies $v(s) \in \partial h(u(s))$ for almost all s in $[t,T]$. This completes the proof of (28).

Now, since (u^*, y^*) is extremal, (26) yields $-B^*p^* \in \partial K(u^*)$. If one may establish that $B^*p^* \in \partial G(u^*)$, then (27) would imply $\partial J(u^*) \ni 0$; that is, u^* is optimal. This result follows from the following general lemma.

Lemma 6.11. *In the assumptions of Theorem* 6.9, *let* $\xi_T \in \partial\phi_0(y(T))$ *and* $q \in L^1(t,T;H)$, $q(s) \in \partial g(y(s))$ *for almost all* $s \in [t,T]$. *Let also* p *be a dual variable associated to the trajectory* $y = y_{t,x}(\cdot;u)$, *defined by* $(C)_{t,x}$. *Then* $B^*p \in \partial G(u)$.

Proof. Let v be a fixed control in $L^2(t,T;U)$. Then

$$G(v)-G(u) \geq \int_t^T \langle q(s), y(s;v)-y(s;u)\rangle_H \, ds + \langle \xi_T, y(T;v)-y(T;u)\rangle_H \equiv I_1 + I_2.$$

Note that

$$I_1 = \int_t^T \langle q(s), \int_t^s e^{(s-\sigma)A} B[v(\sigma)-u(\sigma)]d\sigma\rangle_H \, ds = \int_t^T \int_\sigma^T \langle B^* e^{(s-\sigma)A^*} q(s), v(\sigma)-u(\sigma)\rangle_U \, ds \, d\sigma.$$

Similarly

$$I_2 = \int_t^T \langle B^* e^{(T-\sigma)A^*} \xi_T, v(\sigma)-u(\sigma)\rangle_U \, d\sigma.$$

Since by Proposition 6.6 $p(\sigma) \in D(B^*)$ for all σ in $[t,T]$, and

$$B^* e^{(T-\sigma)A^*}\xi_T + \int_\sigma^T B^* e^{(s-\sigma)A^*} q(s) ds = B^* p(\sigma),$$

we have

$$G(v)-G(u) \geq \int_t^T \langle B^* p(\sigma), v(\sigma)-u(\sigma)\rangle_U \; d\sigma = \langle B^* p, v-u\rangle_{L^2(t,T;U)},$$

which implies $B^* p \in \partial G(u)$, for v was arbitrary. □

Doing so we completed the proof of sufficiency.

Remark 6.12. If we assume g and ϕ_0 are C^1, it is easy to infer necessity. In fact it is simple to check that G is Gâteaux differentiable and $\partial G(u) = \{B^* p(\cdot, u)\}$. Moreover u^* optimal implies $\partial J(u^*) \ni 0$, which can be restated, in view of (28), as $\partial h(u^*(s)) + B^* p^*(s) \ni 0$ for almost all s; *i.e.*, $-B^* p^*(s) \in \partial h(u^*(s))$ for almost all s in $[t,T]$.

Necessity. We prove the assertion by approximation. Let (u^*, y^*) be optimal and consider the approximating problem settled in Section 5. First of all note that, due to the regularity of g_n and of ϕ_{0n} and to the uniqueness of optimal control $u^*_{n,t,x}$ (Proposition 2.5), there exists a unique dual variable $p^*_{n,t,x}$ associated to the optimal couple $(u^*_{n,t,x}, y^*_{n,t,x})$, and it is given by

$$p^*_{n,t,x}(s) = e^{(T-s)A^*}\phi'_{0n}(y^*_{n,t,x}(T)) + \int_s^T e^{(\sigma-s)A^*} g'_n(y^*_{n,t,x}(\sigma)) d\sigma, \; \forall s \in [t,T]. \tag{$(C)_{n,t,x}$}$$

Moreover, $J_n(u) := J_n(t,x,u)$ is Gâteaux differentiable in u, and hence

$$\langle J'_n \langle u^*_{n,t,x}), v\rangle_{L^2(t,T;U)} = 0, \; \forall v \in L^2(t,T;U).$$

Writing explicitly the preceding relation we have

$$\begin{aligned}0 &= \lim_{\varepsilon\to 0}\frac{J_n(u^*_{n,t,x}+\varepsilon v)-J_n(u^*_{n,t,x})}{\varepsilon}\\
&= \int_t^T \langle g_n'(y^*_{n,t,x}(s)), \int_t^s e^{(s-\sigma)A}B_n v(\sigma)d\sigma\rangle_H\, ds + \langle u^*_{n,t,x}-u^*, v\rangle_{L^2(t,T;U)} +\\
&\quad + \int_t^T \langle h_n'(u^*_{n,t,x}(s)), v(s)\rangle_U\, ds + \langle \phi_{0n}'(y^*_{n,t,x}(T)), \int_t^T e^{(T-\sigma)A}B_n v(\sigma)d\sigma\rangle_H\\
&= \int_t^T \Big\langle \int_\sigma^T B_n^* e^{(s-\sigma)A^*} g_n'(y^*_{n,t,x}(s))ds, v(\sigma)\Big\rangle_U\, d\sigma + \langle u^*_{n,t,x}-u^*, v\rangle_{L^2(t,T;U)} +\\
&\quad + \int_t^T \langle h_n'(u^*_{n,t,x}(s)), v(s)\rangle_U\, ds + \int_t^T \langle B_n^* e^{(T-\sigma)A^*}\phi_{0n}'(y^*_{n,t,x}(T)), v(\sigma)\rangle_U\, d\sigma\\
&= \langle B_n^* p^*_{n,t,x} + h_n'(u^*_{n,t,x}) + u^*_{n,t,x} - u^*, v\rangle_{L^2(t,T;U)},\ \forall v\in L^2(t,T;U),\end{aligned}$$

which implies

$$u^*(s) - u^*_{n,t,x}(s) - B_n^* p^*_{n,t,x}(s) = h_n'(u^*_{n,t,x}(s)),\ a.e. \text{ in } [t,T]. \tag{29}$$

Now we need the following lemma.

Lemma 6.13. *Let the assumptions of Theorem* 6.9 *be satisfied. Let also* (u^*, y^*) *be an optimal pair for problem* $(P)_{t,x}$, *and* $u^*_{n,t,x}$, $y^*_{n,t,x}$, *and* $p^*_{n,t,x}$ *be defined as above. Then there exist* $q \in L^\infty(t,T;H)$, *with* $q(\sigma)\in\partial g(y^*(\sigma))$ *almost everywhere in* $[t,T]$, $\xi_T \in \partial\phi_0(y^*(T))$, *and a subsequence* $k\mapsto n_k$ *such that the following hold:*

(i) $p^*_{n_k,t,x}(s) \to p^*(s)$ *weakly in* H *for all* $s\in[t,T)$, *and* $p^*_{n_k,t,x} \to p^*$, *weakly in* $L^2(t,T;H)$,

(ii) $B_n^* p^*_{n_k} \to B^* p^*$, *weakly in* $L^2(t,T;U)$,

where $p^*(s) = e^{(T-s)A^*}\xi_T + \int_s^T e^{(\sigma-s)A^*} q(\sigma)d\sigma$.

Proof. We set $(u_n^*, y_n^*, p_n^*) := (u_{n,t,x}, y_{n,t,x}, p_{n,t,x})$, and

$$p^*_{n,t,x}(T) := \phi_{0n}'(y^*_{n,t,x}(T)),\ q_{n,t,x}(s) := g_n'(y^*_{n,t,x}(s)),$$

for all $n\in\mathbb{N}$ and for all $s\in[t,T]$. Since $\partial\phi_0$ and ∂g are bounded on bounded subsets, and (12) and Proposition 2.1 (*iv*) hold, then there exists a constant depending only on x,

$$K_x := \sup_{|z|\le M_x'} \inf_{y\in\partial\phi_0(z)} |y|_H < +\infty,$$

such that

$$|p^*_{n,t,x}(T)|_H \le K_x,\ \forall t\in[0,T],\ \forall n\in\mathbb{N} \tag{30}$$

$$|q_{n,t,x}(s)|_H \le K_x,\ \forall t\in[0,T],\ \forall s\in[t,T],\ \forall n\in\mathbb{N}. \tag{31}$$

Then there exists a subsequence $k \mapsto n_k$ such that

$$p^*_{n_k,t,x}(T) \to \xi_T, \text{ weakly in } H$$
$$q_{n_k,t,x} \to q \text{ weakly star in } L^\infty(t,T;H).$$

To simplify the notation we write from now on $y^*_{n_k}$, $p^*_{n_k}$, and q_{n_k}, in place of $y^*_{n_k,t,x}$, $p^*_{n_k,t,x}$, and $q_{n_k,t,x}$, for the dependence on t, and x is not relevant in this context.

$$\langle p^*_{n_k}(s) - p^*(s), z\rangle_H = \langle p^*_{n_k}(T) - \xi_T, e^{(T-s)A}z\rangle_H +$$
$$+ \langle q_{n_k} - q, e^{(\cdot - s)A}\chi_{[s,T]}z\rangle_{L^\infty,L^1} \to 0, \ k \to \infty.$$

Note that (30), (31), and dominated convergence yield also

$$\int_t^T \langle p^*_{n_k}(s) - p^*(s), \varphi(s)\rangle_H \, ds \to 0, \ n \to \infty, \ \forall \varphi \in L^2(t,T;H),$$

which implies (i).
We show next that $\xi_T \in \partial\phi_0(y^*(T))$. Note that according to Proposition 2.1 (ii) and the notation therein, for all $z \in H$ we have

$$\phi_0(\mathcal{J}^{\phi_0}_{n_k}(y^*_{n_k}(T))) - \phi_{0n_k}(z) \leq$$
$$\leq \phi_{0n_k}(y^*_{n_k}(T)) - \phi_{0n_k}(z) \leq \langle p^*_{n_k}(T), y^*_{n_k}(T) - z\rangle_H, \ \forall k \in \mathbb{N}, \ \forall z \in H;$$

hence, recalling Proposition 6.5 (ii),

$$\phi_0(y^*(T)) - \phi_0(z) \leq \liminf_{k\to+\infty} \big(\phi_0(\mathcal{J}^{\phi_0}_{n_k}(y^*_{n_k}(T))) - \phi_{0n_k}(z)\big) \leq$$
$$\leq \lim_{k\to\infty} \langle p^*_{n_k}(T), y^*_{n_k}(T) - z\rangle_H = \langle \xi_T, y^*(T) - z\rangle_H.$$

Similarly,

$$g(\mathcal{J}^g_{n_k}(y^*_{n_k}(s))) - g_{n_k}(z) \leq g_{n_k}(y^*_{n_k}(s)) - g_{n_k}(z) \leq$$
$$\leq \langle q_{n_k}(s), y^*_{n_k}(s) - z\rangle_H, \ \forall k \in \mathbb{N}, \ \forall z \in H, \ a.e. \ s \in [t,T];$$

hence, if $\varphi \in L^\infty(t,T,H)$, one has

$$\int_t^T \big(g(\mathcal{J}^g_{n_k}(y^*_{n_k}(s))) - g_{n_k}(\varphi(s))\big)\, ds \leq \int_t^T \langle q_{n_k}(s), y^*_{n_k}(s) - \varphi(s)\rangle_H \, ds.$$

We now take the liminf as $k \to \infty$, and, arguing as the proof of Claim 3 in Lemma 5.5, we derive

$$\int_t^T (g(y^*(s)) - g(\varphi(s)))\, ds \leq \int_t^T \langle q(s), y^*(s) - \varphi(s)\rangle_H \, ds.$$

Moreover, arguing as in the proof of (28), we may show that

$$q(s) \in \partial g(y^*(s)), \text{ for } \ a.a. \ s \in [t,T].$$

This completes the proof of (i).

To prove (ii) we choose z in H and observe that

$$\langle B^*_{n_k} p^*_{n_k}(s), z\rangle_H = \langle p^*_{n_k}(T), e^{(T-s)A} B_{n_k} z\rangle_H + \int_s^T \langle q_{n_k}(\sigma), e^{(\sigma-s)A} B_{n_k} z\rangle_H \, d\sigma.$$

Note that $p^*_{n_k}(T) \to \xi_T$ weakly in H and $e^{(T-s)A} B_{n_k} z \to e^{(T-s)A} Bz$ strongly in H; hence, $\langle p^*_{n_k}(T), e^{(T-s)A} B_{n_k} z\rangle_H \to \langle \xi_T, e^{(T-s)A} Bz\rangle_H$. It is easy to check also that $e^{(\cdot-s)A}\chi_{[s,T]} B_{n_k} z \to e^{(\cdot-s)A}\chi_{[s,T]} Bz$ strongly in $L^1(t,T;H)$, so that $\langle q_{n_k}, e^{(\cdot-s)A}\chi_{[s,T]} B_{n_k} z\rangle_{L^\infty,L^1} \to \langle q, e^{(\cdot-s)A}\chi_{[s,T]} Bz\rangle_{L^\infty,L^1}$, and hence

$$B^*_{n_k} p^*_{n_k}(s) \to B^* p^*(s), \text{ weakly in } H, \text{ for all } s \in [t,T].$$

Furthermore, for any φ in $L^2(t,T;U)$ one has

$$\begin{aligned}
\int_t^T \langle B^*_{n_k} p^*_{n_k}(s), \varphi(s)\rangle_H ds &= \Big\langle p^*_{n_k}(T), \int_t^T e^{(T-s)A} B_{n_k}\varphi(s) ds\Big\rangle_H \\
&\quad + \int_t^T \Big[\int_s^T \langle q_{n_k}(\sigma), e^{(\sigma-s)A} B_{n_k}\varphi(s)\rangle_H d\sigma\Big] ds \\
&= \Big\langle p^*_{n_k}(T), \int_t^T e^{(T-s)A} B_{n_k}\varphi(s) ds\Big\rangle_H \\
&\quad + \int_t^T \Big\langle q_{n_k}(\sigma), \int_t^\sigma e^{(\sigma-s)A} B_{n_k}\varphi(s) ds\Big\rangle_H d\sigma \equiv I_n + K_n.
\end{aligned}$$

Set now

$$f_n(\sigma) := \int_t^\sigma e^{(\sigma-s)A} B_{n_k}\varphi(s) ds, \text{ and } f(\sigma) := \int_t^\sigma e^{(\sigma-s)A} B\varphi(s) ds \;\forall \sigma \in [t,T].$$

It is simple to check that $f_n(\sigma) \to f(\sigma)$ for all $\sigma \in [t,T]$, and also that

$$f_{n,t,x} \to f, \text{ strongly in } L^1(t,T;H). \tag{32}$$

In particular, from $f_n(T) \to f(T)$ it follows that

$$I_n \to \langle \xi_T, \int_t^T e^{(T-s)A} B\varphi(s) ds\rangle_H$$

and from (32) that

$$K_n \to \int_t^T \langle q(\sigma), \int_t^\sigma e^{(\sigma-s)A} B\varphi(s) ds\rangle_H \, d\sigma.$$

Hence

$$B^*_{n_k} p^*_{n_k} \to B^* p^* \text{ weakly in } L^2(t,T;H). \tag{33}$$

□

We are now ready to complete the proof of necessity. Let $v \in L^2(t,T;U)$ be fixed; since h_n is convex and (29) holds, we have

$$\int_t^T \Big[h_{n_k}(u^*_{n_k}(s)) - h_{n_k}(v(s)) + \\ + \langle B^*_{n_k} p^*_{n_k}(s) + u^*_{n_k}(s) - u^*(s), u^*_{n_k}(s) - v(s)\rangle_U\Big] ds \le 0.$$

We now show that we may pass to limits in the preceding inequality and obtain

$$\int_t^T (h(u^*(s)) - h(v(s)) + \langle B^* p^*(s), u^*(s) - v(s)\rangle_U)\, ds \le 0,$$

so that, arguing as in the proof of (28), we may infer that $-B^*p^*(s) \in \partial h(u^*(s))$ for almost every $s \in [t,T]$. Indeed, note that $h_n \le h$ and $\liminf_{n\to\infty} \int_t^T h_n(u^*_n(s))ds \ge \int_t^T h(u^*(s))ds$, as we showed in the proof of Lemma 5.5, so that

$$\int_t^T (h(u^*(s)) - h(v(s)))\, ds \le \liminf_{k\to\infty} \int_t^T \big(h(u^*_{n_k}(s)) - h(v(s))\big)\, ds \\ \le \liminf_{k\to\infty} \int_t^T \big(h_{n_k}(u^*_{n_k}(s)) - h_{n_k}(v(s))\big)\, ds.$$

Note also that $u^*_n \to u^*$ strongly and (33) imply

$$\langle B^*_{n_k} p^*_{n_k}, u^*_{n_k} - v\rangle_{L^2(t,T;U)} \to \langle B^* p^*, u^* - v\rangle_{L^2(t,T;U)},$$

and the proof is complete.

Remark 6.14. By (4) we have that $-B^*p^*(s) \in \partial h(u^*(s))$ for almost all s in $[t,T]$, if and only if

$$u^*(s) \in \partial\mathcal{H}(-B^*p^*(s)) \ a.e.\ s \in [t,T]. \tag{34}$$

Note also that if

$$D(\partial h) = U,\ \partial h \text{ is injective, and } (\partial h)^{-1} \in C^0(U,U)$$

(where we recall that by "∂h is injective" we mean that $\partial h(x) \ne \emptyset$ for all $u \in U$, and $\partial h(u) \cap \partial h(\bar u) = \emptyset$ when $u \ne \bar u$, so that $(\partial h)^{-1}$ is a function defined on U), then the preceding relations imply that $\mathcal{H}$ is Gâteaux differentiable and $\mathcal{H}' = (\partial h)^{-1}$, so that $u^*(s) = \mathcal{H}'(-B^*p^*(s))$, and then, by Remark 6.7, u^* is continuous on $[t,T)$.

Acknowledgments. This paper is part of my Ph.D. thesis, which was written under the supervision of Paolo Acquistapace and Fausto Gozzi. Many thanks are due to both of them for their invaluable suggestions, remarks, and support. I would like to thank also Viorel Barbu and Piermarco Cannarsa, who revised the first draft of my thesis.

References

[1] P. Acquistapace, F. Flandoli, and B. Terreni, *Initial boundary value problems and optimal control for nonautonomous parabolic systems*, SIAM J. Control Optimiz, 29 (1991), 89–118.

[2] P. Acquistapace and B. Terreni, *Infinite horizon LQR problems for nonautonomous parabolic systems with boundary control*, SIAM J. Control Optimiz, 34 (1996), 1–30.

[3] P. Acquistapace and B. Terreni, *Classical solutions of nonautonomuos Riccati equations arising in parabolic boundary control problems*, Appl. Math. Optim., 39 (1999), 361–409.

[4] P. Acquistapace and B. Terreni, *Classical solutions of nonautonomuos Riccati equations arising in Parabolic Boundary Control problems II*, Appl. Math. Optim., 41 (2000), 199–226.

[5] V. Barbu and G. Da Prato, “Hamilton–Jacobi Equations in Hilbert Spaces,” Pitman, London, 1983.

[6] V. Barbu and G. Da Prato, *Hamilton-Jacobi equations in Hilbert spaces; variational and semigroup approach*, Ann. Mat. Pura Appl., IV, 42 (1985), 303–349.

[7] V. Barbu and G. Da Prato, *A note on a Hamilton-Jacobi equation in Hilbert space*, Nonlinear Anal., 9 (1985), 1337–1345.

[8] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter, “Representation and Control of Infinite Dimensional Systems,” Vols. 1 & 2, Birkäuser, Boston, 1993.

[9] V. Barbu, G. Da Prato, and C. Popa, *Existence and uniqueness of the dynamic programming equation in Hilbert spaces*, Nonlinear Anal., 7 (1983), 283–299.

[10] V. Barbu and Th. Precupanu, “Convexity and Optimization in Banach Spaces,” Editura Academiei, Bucharest, 1986.

[11] P. Cannarsa and G. Di Blasio, *A direct approach to infinite dimensional Hamilton–Jacobi equations and applications to convex control with state constraints*, Diff. Int. Eq., 8 (1995), 225–246.

[12] P. Cannarsa, F. Gozzi, and H. M. Soner *A dynamic programming approach to nonlinear boundary control problems of parabolic type*, J. Funct. Anal., 117 (1993), 25–61.

[13] M.G. Crandall and P.L. Lions, *Hamilton–Jacobi equations in infinite dimensions. Part I: Uniqueness of viscosity solutions*, J. Funct. Anal., 62 (1985), 379–396;
Part II: Existence of viscosity solutions, J. Funct. Anal., 65 (1986), 368–405;
Part III, J. Funct. Anal., 68 (1986), 214–247;
Part IV: Hamiltonians with unbounded linear terms, J. Funct. Anal., 90 (1990), 237–283;
Part V: Unbounded linear terms and B–continuous solutions, J. Funct. Anal., 97 (1991), 417–465;
Part VI: Nonlinear A and Tataru’s method refined, “Evolution Equations, Control Theory, and Biomathematics,” Han sur Lesse, 1991, 51–89, Lecture Notes in Pure and Appl. Math., 155, Dekker, New York, 1994;
Part VII: The HJB equation is not always satisfied, J. Func. Anal., 125 (1994), 111–148.

[14] P. Cannarsa and M.E. Tessitore, *Infinite dimensional Hamilton–Jacobi equations and Dirichlet boundary control problems of parabolic type*, SIAM J. Control Optim., 34 (1996), 1831–1847.

[15] G. Di Blasio, *Global solutions for a class of Hamilton-Jacobi equations in Hilbert spaces*, Numer. Funct. Anal. Optim., 8 (1985/86), 261–300.

[16] G. Di Blasio, *Optimal control with infinite horizon for distributed parameter systems with constrained controls*, SIAM J. Control & Optim., 29 (1991), 909–925.

[17] S. Faggian, *Boundary control problems with convex cost and dynamic programming in infinite dimension. Part II: Existence for the HJB Equation*, to appear.

[18] S. Faggian, *Regular solutions of Hamilton–Jacobi equations arising in economics*, to appear.

[19] S. Faggian, *Applications of dynamic programming to economic problems with vintage capital*, to appear.

[20] F. Gozzi, *Some results for an optimal control problem with a semilinear state equation I*, Rendiconti dell'Accademia Nazionale dei Lincei, 82 (1988), 423–429.

[21] F. Gozzi, *Some results for an optimal control problem with a semilinear state equation II*, Siam J. Control and Optim., 29 (1991), 751–768.

[22] F. Gozzi, *Some results for an infinite horizon control problem governed by a semilinear state equation*, Proceedings Vorau, July 10–16, 1988, editors F. Kappel, K. Kunisch, and W. Schappacher; International Series of Numerical Mathematics, Vol. 91, Birkäuser-Verlag, Basel, 1989, 145–163.

[23] J.L. Lions and E. Magenes, "Problems aux Limites non Homogènes et Applications," Vols. 1 and 2, 1968, Vol. 3, 1969, Dunod, Paris.

[24] I. Lasiecka and R. Triggiani, "Control Theory for Partial Differential Equations: Continuous and Approximation Theory, Vol. I, Abstract Parabolic Systems," Encyclopedia of Mathematics and its Applications, 74, Vol. II, "Abstract Hyperbolic-Like Systems Over a Finite Time Horizon," Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2000, 74–75.

[25] J. Yong and X.Y. Zhou, "Stochastic Controls, Hamiltonian Systems and HJB quations," Applications of Mathematics, 43, Springer-Verlag, New York, 1999.