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ON A CLASS OF DEGENERATE ELLIPTIC EQUATIONS IN WEIGHTED HÖLDER SPACES

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Abstract. We study the Dirichlet problem for the degenerate elliptic equations

$$P_0\Delta\, u+\gamma(\nabla\, P_0,\nabla\, u)=f(x,u,\nabla\, u)\quad \text{in } \Omega,\qquad u=0 \text{ on } \partial\Omega,$$

where $\gamma\geq 0$ is a given parameter, $\Omega\subset\mathbb{R}^n$ is an annular region, the given function $P_0(x)$ is such that $|\nabla\, P_0|+P_0\geq\epsilon>0$ in $\overline{\Omega}$, and $P_0=0$ on the outer boundary of Ω. The equation is degenerate elliptic when $\gamma>0$, while for $\gamma=0$ it transforms into the classical Poisson equation. We introduce the weighted Hölder spaces suitable for the study of the problem throughout the range of the parameter $\gamma\geq 0$. We derive the Schauder-type estimates and prove the existence of a unique classical solution.

It is shown that in the case $\gamma>0$ the solution of the degenerate equation and the given function P_0 possess the same regularity properties. In the case $\gamma=0$ (the Poisson equation) the regularity of u is better than the regularity of P_0. The proof is based on a new method of estimating the derivatives of solutions of the Poisson equation near the boundary of the problem domain which requires neither differentiation of the equation, nor straightening the boundary.

1. Introduction

Let $\Omega\subset\mathbb{R}^n$ be an annular region with the outer boundary Γ and the interior boundary Σ. We study the Dirichlet problem for degenerate elliptic equations of the form

$$P_0\Delta\, u+\gamma(\nabla\, P_0,\nabla\, u)=f(x,u,\nabla\, u)\quad \text{in } \Omega,\qquad u=0 \text{ on } \partial\Omega, \tag{1.1}$$

where the given function $P_0(x)$ is such that $P_0=0$ on Γ, $|\nabla\, P_0|+P_0\geq\epsilon>0$ in $\overline{\Omega}$, and $\gamma\geq 0$ is a given parameter. Our aim is to introduce a scale of weighted Hölder spaces where both equation (1.1) with $\gamma>0$ (an elliptic

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equation which degenerates on the boundary of the problem domain) and equation (1.1) with $\gamma = 0$ (the Poisson equation) admit classical solutions, and to derive the Schauder estimates in these spaces throughout the range of the parameter $\gamma \geq 0$.

Schauder-type estimates for solutions of boundary-value problems of equations similar to (1.1) were derived in [1] (see also the further references therein). This paper deals with the equations

$$L\,u = \sum_{h=0}^{\min\{k,m\}} P_0^{k-h}(x) Q^{m-h}(x, D_x) u,$$

where P_0 is a smooth function satisfying $|\nabla P_0| + P_0 \geq \epsilon > 0$ in $\overline{\Omega}$, $k, m \in \mathbb{N}$, and Q^{m-h} are differential operators with smooth coefficients of orders less than $m-h$. The a priori estimates are established in Hölder spaces defined as the completion of $C^\infty(\overline{\Omega})$ in the norm

$$\|u\|_{C^\mu_\sigma(\overline{\Omega})} = \sum_{h=0}^{\sigma} \|P_0^h u\|_{C^{\mu+h}(\overline{\Omega})}$$

with integer σ and noninteger μ. It is shown that under certain spectral conditions on the operator L every function $u \in C^{\mu+m-k}_{\sigma+k}(\overline{\Omega})$ satisfies the estimate

$$\|u\|_{C^{\mu+m-k}_{\sigma+k}(\overline{\Omega})} \leq C\Big[\|Lu\|_{C^\mu_\sigma(\overline{\Omega})} + \|u\|_{C^{\mu'+m-k}_{\sigma+k}(\overline{\Omega})}\Big], \quad \mu' < \mu.$$

The next result of [1] is that under some additional conditions on the spectrum of L the inclusions $u \in C^{\mu'+m-k}(\overline{\Omega})$, $Lu \in C^\mu_\sigma(\overline{\Omega})$ with $\mu' < \mu$ imply $u \in C^{\mu+m-k}_{\sigma+k}(\overline{\Omega})$. This gives the Schauder estimates in the spaces $C^{\mu+m-k}_{\sigma+k}(\overline{\Omega})$ once global Hölder continuity of solutions to (1.1) is established. By introducing a system of local charts and straightening the boundary the problem is reduced to a system of degenerate elliptic problems posed in the half-planes $\{x_n > 0\}$, which are analyzed by means of the Fourier transform in the directions $x_1, \ldots, x_{n-1}$. The derivation of the Schauder estimates is thus reduced to proving the global Hölder continuity of the solutions to problem (1.1). For the study of local Hölder continuity of solutions to equations (1.1) with $\gamma > 0$, see [3, 14] and further references therein.

It is worth noting here that since the method of the cited paper uses local straightening of the boundary given by the equation $P_0 = 0$, the provable regularity of the solution cannot be better than the regularity of P_0.

The series of papers [8, 9, 10] is devoted to a systematic study of the Neumann problem for the equation $\operatorname{div}(P_0 \nabla u) = f$ and its generalizations. The author constructs the Green's function, derives the Schauder estimates in appropriate weighted Hölder spaces, and proves the existence of classical solutions. The theory of degenerate elliptic operators $\operatorname{div}(x_n^\alpha \nabla u)$ in the half-space $\mathbb{R}^n \cap \{x_n > 0\}$ is developed in [5, 6, 7, 11].

Another approach to the study of regularity of solutions of equations related to (1.1) was developed in [2, 13]. These works deal with the degenerate parabolic equations whose elliptic parts coincide with (1.1) with $\gamma > 0$. It happens to be so that the analysis of the free-boundary problems for the celebrated porous-medium equation

$$u_t = \Delta u^m, \quad m > 1, \qquad \text{(PME)}$$

leads to the model linear degenerate parabolic equation

$$v_t - x_n \Delta v - \gamma D_{x_n} v = f \quad \text{in } (\mathbb{R}^n \cap \{x_n > 0\}) \times (0, T]. \tag{1.2}$$

The arguments of [2, 13] are based on introduction of a new non-Euclidean metric in the half space $\mathbb{R}^n \cap \{x_n > 0\}$. Unfortunately, the methods of [2, 13] do not work if $\gamma = 0$.

A different method of solving the free-boundary problem for PME and the diffusion-absorption equation (including PME as a partial case) also leads to equations related to (1.1) and (1.2), [15, 16]. The free-boundary problem for the equation $u_t = \Delta u^m - u^p$ with the parameters $m > 1$, $p > 0$, $m + p \geq 2$, is reduced to a system of equations consisting of a degenerate elliptic equation (1.1) with $\gamma > 0$ and a degenerate parabolic equation (1.2) with $\gamma = 0$ posed in a time-independent domain.

The principal objectives of the present paper are to introduce the weighted Hölder spaces appropriate for deriving the Schauder-type estimates associated with the operator $P_0 \Delta + \gamma(\nabla P_0, \nabla)$ throughout the range of the parameter $\gamma \geq 0$, and to study the dependence of the regularity of solutions on the regularity of P_0. When $\gamma = 0$, we show that although the surface $\partial\Omega$ is defined by the equation $P_0 = 0$, the regularity of solutions to problem (1.1) is better than the regularity of P_0.

2. Assumptions and results

Let $P_0 \in C^k(\overline{\Omega})$, $k \geq 1$, $P_0 = 0$ on $\Gamma = \partial\Omega$, and $|\nabla P_0| + P_0 \geq \kappa > 0$ in $\overline{\Omega}$. Then the $(n-1)$-dimensional manifold Γ can be parametrized as follows:

(1) given an arbitrary point $x_0 \in \Gamma$ we may introduce local coordinates in $\mathbb{R}^n$ with the origin x_0 so that the axis x_n coincides with the inner normal to Γ at x_0;

(2) there exists $\epsilon > 0$ such that for every $x_0 \in \Gamma$ the set $B_\epsilon(x_0) \cap \Gamma$ is defined by the formulas

$$\begin{cases} x_i = y_i \text{ if } i \neq n, \\ y_n = P_0(y', x_n), \end{cases} \qquad y' = (y_1, \ldots, y_{n-1}) \in B_\epsilon(x_0) \cap \{x_n = 0\}. \tag{2.1}$$

We will assume that there exists a constant $\nu > 0$ such that

$$\forall\, x \in \Omega \qquad \nu \leq \frac{P_0(x)}{\operatorname{dist}(x, \Gamma)} \leq 2\nu. \tag{2.2}$$

Adopt the notation

$$\left| D^k v \right| = \sum_{\beta = (\beta_1, \ldots, \beta_N),\, |\beta| = k} \left| D^\beta v \right|.$$

Given a set $G \subseteq \Omega$ and a function P_0, we define the seminorms and norms $|u|_{0,G} = \sup_G |u|$,

$$\langle\langle u \rangle\rangle_{k,G} = \sum_{|\beta|=0}^{k} \left| d^{|\beta|} D^\beta u \right|_{0,G} + \sum_{|\beta|=k} \left\{ d^{|\beta|} D^\beta u \right\}_{\alpha,G}, \quad d = \operatorname{dist}(x, \Gamma),$$

$$\{u\}_{\alpha,G} = \sup_{x,y \in G,\, x \neq y} \left\{ d^\alpha(x,y) \frac{|u(x) - u(y)|}{|x-y|^\alpha} \right\}, \quad d(x,y) = \min\{P_0(x), P_0(y)\},$$

$$\langle u \rangle_{0,G} = |u|_{0,G} + \{u\}_{\alpha,G},$$

$$\langle u \rangle_{2k+1,G} = \sum_{|\mu|=0}^{k} |D^\mu u|_{0,G} + \sum_{|\mu|=k+1}^{2k+1} \left| d^{|\mu|-k-\beta} D^\mu u \right|_{0,G} + \sum_{|\mu|=2k+1} \{d^{k+1-\beta} D^\mu u\}_{\alpha,G} \quad \text{for } k \geq 0, \quad \beta,\, \alpha \in (0,1).$$

The Banach spaces $V(2k+1, \Omega)$ with $k \geq 0$ are defined as the completion of $C^\infty(\overline{\Omega})$ in the norm $\langle \cdot \rangle_{2k+1,\Omega}$.

We will also assume that the domain Ω is so "thin" that the following condition is fulfilled:

$$-P_0 \sum_{i,j} |D^2_{ij} P_0| + |\nabla\, P_0|^2 \geq \delta > 0 \quad \text{in } \Omega \tag{2.3}$$

with a constant $\delta > 0$.

Theorem 2.1. *Let the following conditions hold:* $P_0 \in V(2k+3, \Omega)$, $f \equiv f(x) \in V(2k+1, \Omega)$ *with* $k \geq 0$, P_0 *satisfies conditions* (2.2) *and* (2.3), *and* $\Sigma \in C^{2k+3}$. *Then for every* $\gamma > 0$ *problem* (1.1) *has a unique solution* $u \in V(2k+3, \Omega)$ *satisfying the estimate*

$$\|u\|_{V(2k+3,\Omega)} \leq C\, \|f\|_{V(2k+1,\Omega)}$$

with a constant C independent of u.

The proof of Theorem 2.1 relies on the standard arguments based on straightening the boundary and introducing the local coordinates by formulas (2.1). A drawback of this method is that we are unable to improve the regularity of the solution in comparison with the regularity of P_0.

Theorem 2.2. *Let under the conditions of Theorem* 2.1 $P_0 \in V(2k+1,\Omega)$ *for some* $k \geq 1$ *and suppose condition* (2.2) *holds. Let us assume that the domain* Ω *satisfies the interior and exterior sphere conditions:*

$$\begin{cases} \text{there exists } R>0 \text{ such that every point } x_0 \in \Gamma \text{ can be touched by balls} \\ B_R(\xi) \subset \Omega \text{ and } B_R(\eta) \not\subset \Omega, \end{cases} \tag{2.4}$$

and is so "thin" that

$$P_0^{\beta} \leq (2\,\nu\,\mathrm{dist}\,(\Gamma,\Sigma))^{\beta} < \frac{1}{8}\beta\,\nu^2 \qquad \text{in } \Omega. \tag{2.5}$$

Then for every $f(x) \in V(2k+1,\Omega)$, $f=0$ *on* Γ, *problem* (1.1) *with* $\gamma = 0$ *has a unique solution* $u \in V(2k+3,\Omega)$ *satisfying the estimate*

$$\|u\|_{V(2k+3,\Omega)} \leq C\,\|f\|_{V(2k+1,\Omega)}$$

with a constant C independent of u and f.

The rest of the article is organized as follows. In Section 3 we study the linear equation (1.1) with $\gamma > 0$. Section 4 is devoted to derivation of special a priori estimates for the solutions of the Dirichlet problem for the Poisson equation. We present a method of estimating the derivatives in the directions tangential to the boundary which neither requires differentiation of the equation nor relies on the possibility of straightening a boundary portion. In Section 5 we apply the results of Section 4 to prove the existence of a unique classical solution of the linear equation (1.1) with $\gamma = 0$ (the "degenerate" Poisson equation). Finally, in Section 6 results of Section 3 are extended to the equation

$$P_0\,\mathrm{div}\,(A\,\nabla\,u) + \gamma\,(\nabla\,P_0,\;A\,\nabla\,u) = f(x,u,\nabla\,u)$$

with a suitable matrix A.

Remark 2.1. For the sake of convenience, throughout the paper we assume that the equation under study degenerates on the exterior boundary Γ of Ω and is uniformly elliptic near the interior boundary Σ. All the results remain true if the equation degenerates on Σ and is uniformly parabolic near Γ.

3. DEGENERATE EQUATION ($\gamma > 0$)

In this section we consider the problem

$$\begin{cases} \mathcal{M}\, u \equiv P_0 \Delta\, u + \gamma(\nabla P_0, \nabla\, u) = f(x) \quad \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega = \Gamma \cup \Sigma, \quad \gamma > 0. \end{cases} \tag{3.1}$$

3.1. **Existence and the maximum principle.** Let $\{\Omega^{(\mu)}\}$ be a family of annular domains with the exterior boundaries $\partial\Omega^{(\mu)} \in C^\infty$ and the interior boundary Σ, chosen so that $\Omega^{(\mu)} \subset \Omega^{(\nu)} \subset \Omega$ for $\mu > \nu > 0$, $\bigcup_{\{\delta>0\}} \Omega^{(\delta)} = \Omega$. A solution to problem (3.1) is obtained as the pointwise limit of the sequence of solutions to regularized problems

$$\mathcal{M} u^{(\mu)} = f(x) \quad \text{in } \Omega^{(\mu)}, \qquad u = 0 \text{ on } \partial\Omega^{(\mu)} \cup \Sigma. \tag{3.2}$$

For every $\mu > 0$ and $f \in C^{k+\alpha}(\overline{\Omega^{(\mu)}})$ problem (3.2) has a classical solution $u^{(\mu)} \in C^{k+2+\alpha}(\overline{\Omega^{(\mu)}})$.

Lemma 3.1. *Let $\mathcal{M}P_0 \geq \delta > 0$ in ω. If $\gamma > 0$, then the solutions of problems* (3.2) *satisfy the estimates*

$$|u^{(\mu)}| \leq L(\delta, \sup P_0) \sup |f| \quad \text{in } \overline{\Omega^{(\mu)}}$$

with a constant L independent of μ.

Proof. Consider the function $q(x) = K - \lambda\, P_0(x)$ depending on positive constants K and λ. The function q satisfies the inequality

$$\mathcal{M}\, q = -\lambda \left(P_0 \Delta\, P_0 + \gamma |\nabla\ P_0|^2\right) \equiv -\lambda \mathcal{M} P_0 \leq -\lambda\delta < f(x) \quad \text{in } \Omega^{(\mu)}$$

provided that $\lambda = 2\delta^{-1} \sup |f|$. Let us take $K = 2\lambda \sup |P_0|$. Then $\mathcal{M}q < h$ in $\Omega^{(\mu)}$ and $q(x) = 2\delta^{-1}(2 \sup P_0 - P_0) \sup |f| > u^{(\mu)}$ on $\partial\Omega^{(\mu)} \cup \Sigma$. By the maximum principle

$$|u^{(\mu)}| \leq 2\delta^{-1}(2 \sup P_0 - P_0) \sup |f| \leq L(\delta, \sup P_0) \sup |f|. \qquad \square$$

It follows from the Schauder estimates for solutions of uniformly elliptic equations that for every $\mu > 0$ the solution $u^{(\mu)}$ of problem (3.2) satisfies the estimate

$$\|u^{(\mu)}\|_{C^{k+2+\alpha}(\overline{\Omega^{(\mu)}})} \leq C \|f\|_{C^{k+\alpha}(\overline{\Omega^{(\mu)}})} \tag{3.3}$$

with a constant C depending on the ellipticity constants $\sup P_0$ and $\inf P_0$ in $\Omega^{(\mu)}$. For every δ we may extract from the sequence $\{u^{(\epsilon)}\}$ a subsequence $\{u^{(\epsilon,\delta)}\}$ which converges in $\Omega^{(\delta)}$ to a function $u_\delta \in C^k(\overline{\Omega^{(\delta)}})$. Organizing the diagonal procedure we see that the subsequence $\{u^{(\epsilon,\epsilon)}\}$ converges to a solution of problem (3.1) $u \in C^k(\overline{\Omega'})$ on every subdomain $\Omega' \subset \Omega$ such that $\operatorname{dist}(\partial\Omega', \Gamma) > 0$.

3.2. **A priori estimates near the boundary Γ.** We have to estimate $\|u\|_{V(2k+1,B_\rho(x_0)\cap\Omega)}$ for all $x_0 \in \Gamma$ with some $\rho > 0$. Let $\zeta(x) \in C^\infty$ be a smooth bump function for the ball $B = B_\rho(x_0)$:

$$\zeta(\xi) = \begin{cases} 0 & \text{if } |x| > \rho, \\ 1 & \text{if } |x| < \rho/2, \end{cases} \qquad |D^s\zeta| \le C\rho^{-s}.$$

The function $w = u\,\zeta$ is a solution of the problem

$$\mathcal{M}\,w = G \quad \text{in } B\cap\Omega, \qquad w = 0 \text{ on } \partial\{B\cap\Omega\}, \tag{3.4}$$

where

$$G = f\,\zeta + 2P_0(\nabla\,\zeta, \nabla\,u) + P_0 u\Delta\,\zeta + \gamma u(\nabla\,P_0, \nabla\,\zeta).$$

We assume that x_0 is the origin of $\mathbb{R}^n$. Let us introduce in $B_\epsilon(0)$ the local coordinates by formulas (2.1) and define the new functions $W(y) \equiv u(x)\zeta(x)$ and $\Phi(y) \equiv G(x)$. The derivatives are calculated by the formulas

$$D_{x_i}p = D_{y_i}W + D_{x_i}P_0\,D_{y_n}W, \quad i \ne n, \qquad D_{x_n}p = D_{x_n}P_0\,D_{y_n}W, \tag{3.5}$$

$$\begin{aligned} \Delta\,p \;&= \Delta_{(n-1)}W + |\nabla\,P_0|^2 D^2_{y_n y_n}W + 2\left(\widetilde{\nabla}\,P_0,\, \widetilde{\nabla}(D_{y_n}W)\right) \\ &+ \left[\Delta_x P_0 + \frac{1}{D_{x_n}P_0}\left(\widetilde{\nabla}\,P_0,\, \widetilde{\nabla}(D_{y_n}P_0)\right)\right] D_{y_n}W, \end{aligned} \tag{3.6}$$

$$(\nabla_x P_0,\, \nabla_x p) = \left(\widetilde{\nabla}\,P_0,\, \widetilde{\nabla}\,W\right) + |\nabla\,P_0|^2\; D_{y_n}W,$$

under the notation $\widetilde{\nabla}\,u = (D_{y_1}u, \ldots, D_{y_{n-1}}u)$, $\Delta_{(n-1)} = \sum_{i=1}^{n-1} D^2_{y_i}$. The supports of Φ and W are contained in the image of $B_\rho(0)$. The problem for w transforms into the following problem for W:

$$\begin{cases} \mathcal{L}W = \Phi \quad \text{in } H = \{|y'| < \epsilon\} \times \{y_n > 0\}, \\ W = 0 \text{ for } y_n = 0 \text{ and } |y'| = \epsilon, \end{cases} \tag{3.7}$$

$$\begin{aligned} \mathcal{L}W \;&\equiv y_n\left(\Delta_{(n-1)}W + 2\left(\widetilde{\nabla}\,P_0,\, \widetilde{\nabla}(D_nW)\right) + |\nabla\,P_0|^2 D^2_{nn}W\right) \\ &+ \left(P_0\,\Delta_x P_0 + \gamma|\nabla\,P_0|^2\right) D_nW + \gamma(\widetilde{\nabla}\,P_0, \widetilde{\nabla}\,W) \\ &+ \frac{P_0}{D_{x_n}P_0}\left(\widetilde{\nabla}\,P_0,\, \widetilde{\nabla}(D_nP_0)\right) D_nW. \end{aligned}$$

The coefficients of $\mathcal{L}$ are bounded in the norm $\langle\cdot\rangle_{0,H}$. Let $a(y)$ be any of these coefficients. Then $a(y) = a(0) + \phi(y)$ with ϕ such that $\phi(0) = 0$ and $\langle\phi\rangle_{0,H} < \infty$. We may assume the images of the balls $B_{\rho/2}(0)$ and $B_\rho(0)$ in the plane ξ are contained in the balls $B_\rho(0)$ and $B_{2\rho}(0)$ in the plane y. Let us take the radial continuations of the coefficients of $\mathcal{L}$:

$$\widehat{a}(\xi) = \begin{cases} a(0) + \phi(\xi) & \text{in } B_\rho(0), \\ a(0) + \phi\left(\rho \dfrac{y}{|y|}\right)\left(2 - \dfrac{|y|}{\rho}\right) & \text{for } y \in \overline{B_{2\rho}(0) \setminus B_\rho(0)}, \\ a(0) & \text{in } B_\epsilon(0) \setminus B_{2\rho}(0) \end{cases} \tag{3.8}$$

so that $\langle \widehat{a} \rangle_{0,H} < \infty$. The functions W and Φ vanish outside $B_\rho(0)$, which is why we may view W as a solution of problem (3.7) with the coefficients continued to the whole of H by formulas (3.8).

Lemma 3.2. *Let $x_0 \in \Omega$ and $d = \operatorname{dist}(x_0, \Gamma)$. Assume that $\Gamma \in C^{s+2+\alpha}$, $s \geq 0$. If $|u| \leq M\, P_0^q$ in $B_{\nu d/2}(x_0)$ with $q = const$, then*

$$\begin{aligned} &P_0^{-q}(x_0)\langle\langle u \rangle\rangle_{s+2, B_{\nu d/4}(x_0)\cap\Omega} \\ &\leq C\left(|P_0^{-q} u|_{0, B_{\nu d/2}(x_0)\cap\Omega} + P_0^{1-q}(x_0)\langle\langle \mathcal{M}u \rangle\rangle_{s, B_{\nu d/2}(x_0)\cap\Omega}\right). \end{aligned}$$

Proof. Let us introduce the new independent variables and the new thought function by the formulas

$$x = x_0 + \frac{1}{2}P_0(x_0)z, \quad |z| < 1, \qquad W(z) = P_0^{-q}(x_0)u(x), \quad h(z) = \mathcal{M}u(x).$$

Observe that since $\nu \in (0,1]$, the ball $B' = \{|x - x_0| < P_0(x_0)\}$ does not touch the boundary Γ. Then

$$\frac{4P_0(x)}{P_0(x_0)}\Delta_z W + 2\gamma(\nabla_x P_0, \nabla_z W) = P_0^{1-q}(x_0)\, h \qquad \text{in } \{|z| < 1\}.$$

It follows from the Schauder interior estimates for solutions of uniformly elliptic equations [4, Chapter 6] that $W \in C_z^{s+2+\alpha}(\{|z| < 1/2\})$ and

$$\|W\|_{C^{s+2+\alpha}(\{|z|<1/2\})} \leq K\Big\{\sup_{\{|z|<1\}} |W| + P_0^{1-q}(x_0)\|h\|_{C^{s+\alpha}(\{|z|<1\})}\Big\}.$$

Reverting to the original variables x we obtain the needed estimate. □

Corollary 3.1. *Under the conditions of Lemma 3.1, for every $k \geq 0$*

$$\langle\langle \pi \rangle\rangle_{2k+3,\Omega} \leq C\,\nu\,(1 + \operatorname{dist}(\Gamma, \Sigma))\langle\langle \mathcal{M}\pi \rangle\rangle_{2k+1,\Omega}$$

with a constant C independent of v and ν.

Corollary 3.2. *The same argument shows that for every $s \geq 0$ and every $z \in H$*

$$z_n^{-q}\langle\langle w \rangle\rangle_{s+2, B_{z_n/4}(z)\cap H} \leq C\left(|z_n^{-q} w|_{0, B_{z_n/2}(z)\cap H} + z_n^{1-q}\langle\langle \mathcal{L}w \rangle\rangle_{s, B_{z_n/2}(z)\cap H}\right).$$

3.3. **A model problem.** Let us define the operator with constant coefficients

$$\mathcal{L}_0 v \equiv y_n \left(\Delta_{(n-1)} v + \mu^2 D^2_{nn} v\right) + \gamma D_n v$$

with $\Delta_{(n-1)} = \sum_{i \neq n} D^2_{ii}$ and consider the problem

$$\begin{cases} \mathcal{L}_0 w = \Phi \quad \text{in } H, \\ w = 0 \text{ for } y_n = 0 \text{ and for } |y'| = 8\rho, \quad \gamma > 0,\ \mu > 0 \text{ are given constants.} \end{cases} \tag{3.9}$$

Let us consider problem (3.9) with a special right-hand side. Let $\Phi = f(y_n)\psi(y')$, where ψ is the first eigenfunction of the Dirichlet problem for the Laplace operator in the ball $\{|y'| < \epsilon\}$:

$$\Delta_{(n-1)}\psi + \lambda\,\psi = 0 \quad \text{in } \{|y'| < \epsilon\}, \qquad \psi = 0 \text{ for } |y'| = \epsilon.$$

Lemma 3.3. *Let $\Phi = f(y_n)\psi$, where f is a smooth function vanishing for $|y| \geq \epsilon$. Assume that $\gamma > 0$. Then problem (3.9) has a unique solution W bounded in H and such that $|W| \leq K y_n \sup |\Phi|$ in H, $|W| \to 0$ when $y_n \to \infty$.*

Proof. It is convenient to assume that $\sup |f| = 1$. We search for a solution in the form $W = w(y_n)\psi$ with w defined from the conditions

$$\begin{cases} y_n w'' + \dfrac{\gamma}{\mu^2} w' - y_n \dfrac{\lambda}{\mu^2}\, w = \dfrac{f}{\mu^2} \quad \text{for } y_n > 0, \\ w(0) = 0, \quad w \to 0 \text{ when } y_n \to \infty. \end{cases} \tag{3.10}$$

Since the point $y_n = 0$ is a regular singular point for equation (3.10), there always exists a smooth solution w. The solution of the homogeneous equation (3.10) can be explicitly constructed [12]. If $\nu = \frac{1-\gamma/\mu^2}{2}$ is not an integer, the solution has the form

$$w_0^{\pm}(z) = z^{\nu}\, J_{\pm\nu}\left(z\frac{\imath\sqrt{\lambda}}{\mu}\right), \quad \text{where} \quad J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\dfrac{z}{2}\right)^{\nu+2m}}{m!\,\Gamma(m+\nu+1)}$$

are Bessel's functions. The known asymptotic formulas for Bessel's functions show that for small $|z|$

$$w_0(z) = \left[C_1 \frac{z^{2\nu}}{2^{\nu}\Gamma(1+\nu)} + C_2 \frac{1}{2^{-\nu}\Gamma(1-\nu)}\right]\left(1 + \mathcal{O}(|z|^2)\right),$$

while for large $|z|$

$$J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left[\cos\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)(1 + \mathcal{O}(|z|^{-2})) + \frac{1}{|z|}\sin\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)(1 + \mathcal{O}(|z|^{-2}))\right].$$

If ν is an integer, the solutions of the homogeneous equation (3.10) are given by the formula

$$w_0(z) = z^\nu\Big(C_1\, J_\nu(\imath\sqrt{\lambda}z/\mu) \\ + C_2 \lim_{\epsilon\to 0}\epsilon^{-1}[J_{\nu+\epsilon}(\imath\sqrt{\lambda}z/\mu) + (-1)^\nu J_{-\nu-\epsilon}(\imath\sqrt{\lambda}z/\mu)]\Big).$$

It follows that for $\gamma > 0$, (i.e., $2\nu < 1$), the homogeneous equation has no solutions which vanish and have bounded derivative when $y_n \to 0$. The solution of the nonhomogeneous equation is given by the formula of variation of constants:

$$w(x) = -\frac{\pi x^{2\nu}}{2\sin\pi\nu}\sum_{m=0}^{\infty}\frac{(x\sqrt{\lambda}/2)^{2m}}{m!\Gamma(m+\nu+1)}\int_x^\infty f(s)s^{-2\nu}\sum_{m=0}^{\infty}\frac{(s\sqrt{\lambda}/2)^{2m}}{m!\Gamma(m-\nu+1)}\,ds \\ -\frac{\pi}{2\,\sin\,\pi\nu}\sum_{m=0}^{\infty}\frac{(x\sqrt{\lambda}/2)^{2m}}{m!\,\Gamma(m-\nu+1)}\int_0^x f(s)\sum_{m=0}^{\infty}\frac{(s\sqrt{\lambda}/2)^{2m}}{m!\,\Gamma(m+\nu+1)}\,ds.$$

It follows that w is bounded, $|w| \le Ky_n$ for small y_n, and $|w| \to 0$ when $y_n \to \infty$ (if $\operatorname{supp} f$ is finite in $\mathbb{R}_+$).

The general case is reduced to the considered one by passing to the scaled right-hand side $\phi = f/\sup|f|$: there exists a unique solution w such that $|w| + |w/y_n| \le K$ with an absolute constant K independent of ϕ. Reverting to the function f we get the needed estimate. □

Corollary 3.3. *Let in problem* (3.9) $\gamma = 0$. *Then* $2\nu = 1$, *and the assertion of Lemma* 3.3 *is no longer valid unless we impose an additional condition on* f. *The following assertion holds. Problem* (3.9) *with* $\gamma = 0$ *has a solution* w *such that*

$$|w| \le K\begin{cases} x^\beta & \text{if } |f(s)| \le Ks^{\beta-1} \text{ with } \beta\in(0,1),\\ x\,|\ln x|^2 & \text{if } |f(s)| \le K|\ln s|,\\ x\,|\ln x| & \text{if } |f(s)| \le Ks^{\beta-1} \text{ with } \beta = 1,\\ x & \text{if } |f(s)| \le Ks^{\beta-1} \text{ with } \beta > 1.\end{cases}$$

Corollary 3.4. *Let under the conditions of Lemma* 3.3 $|\Phi| \le My_n^{\beta-1}$ *with* $\beta\in(0,1)$. *Then problem* (3.9) *has a unique solution vanishing when* $y_n\to\infty$ *and such that* $|W| \le Ky_n^\beta \sup|\Phi y_n^{1-\beta}|$ *in* H.

3.4. **Equation with variable coefficients.**

Lemma 3.4. *Let* W *be a solution of problem* (3.7) *with the right-hand side* Φ *such that* $\operatorname{supp} W, \operatorname{supp}\Phi \subset H\cap\{|y| < R\}$ *for some* $\epsilon > 2R > 0$. *Let* $x_0 = 0$ *and*

$$\gamma\mu^2 = \lim_{x\to 0}\left[P_0\,\Delta_x P_0 + \gamma|\nabla P_0|^2\right] > 0, \qquad \mu^2 = \lim_{x\to 0}|\nabla P_0|^2. \tag{3.11}$$

Then there exists $\rho^* \equiv \rho^*(\epsilon) > 0$ such that for all $\rho \in (0, \rho^*]$, $|W| \leq C(\rho) y_n \langle \Phi \rangle_{0,H}$ in H.

Proof. The coefficients of the operator $\mathcal{L}$ depend on ρ and are defined by formulas (3.8). Decreasing ρ we may always assume that $\{|y| < 4R, y_n > 0\} \subset H$. Let us compare W with a solution U of the model problem (3.9) with the coefficients γ and μ^2 given in (3.11) and a smooth right-hand side Ψ that vanishes for $|y'| = \epsilon$ and $y_n > 2R$. The coefficients of $\mathcal{L}_0$ are the limits of the coefficients of $\mathcal{L}$ when $y \to 0$. Moreover, $\mathcal{L} - \mathcal{L}_0 \equiv 0$ in $H \cap \{|y| > 2\rho\}$ (see (3.8)). The function $V = W - U$ is a solution of the equation

$$\mathcal{L}V = \Phi - \Psi + (\mathcal{L}_0 - \mathcal{L})\, U \quad \text{in } H,$$

$V = 0$ on the plane $y_n = 0$, $V = 0$ for $|y'| = \epsilon$, and $|V| \to 0$ when $|y| \to \infty$. According to Corollary 3.2

$$|\,(\mathcal{L}_0 - \mathcal{L})\, U(y)| \leq K(\rho) \langle U \rangle_{2,H} \leq K(\rho) \langle \Psi \rangle_{0,H} \quad \text{in } H \cap \{|y| < R\},$$

where $K(\rho) \to 0$ when $\rho \to 0$ and $(\mathcal{L}_0 - \mathcal{L})\, U(y) \equiv 0$ for $y_n > R$. Let us take $\Psi = f(y_n)\psi(y')$ with

$$f(s) = L \begin{cases} 1 & \text{if } s \in [0, R], \\ \exp\left(\dfrac{1}{R} - \dfrac{1}{2R - s}\right) & \text{if } s \in [R, 2R], \end{cases} \qquad L = const < 0, \quad 2R \in (0, \epsilon].$$

Then $\operatorname{supp} f \subset [0, 2R]$, $\langle \Psi \rangle_{0,H} \leq M(\epsilon)\, |L|$ and

$$\Phi - \Psi + (\mathcal{L}_0 - \mathcal{L})\, U \geq -\sup |\Phi| - \Psi - K(\rho) M(R, \epsilon)\, |L|.$$

Taking ρ so small that $K(\rho) M(\epsilon) \leq 1/2$, we choose L from the condition $\Phi - \Psi + (\mathcal{L}_0 - \mathcal{L})\, U > 0$ in H. By the maximum principle $V \leq 0$ in H, which means that $W \leq U \leq K y_n$ in H. The lower estimate on W follows in the same way. □

Corollary 3.5. *If under the conditions of Lemma 3.4 $\langle y_n^{1-\beta} \Phi \rangle_{0,H} < \infty$ with $\beta \in (0, 1)$, then*

$$|W| \leq C(\rho) y_n^{\beta} \langle y_n^{1-\beta} \Phi \rangle_{0,H}.$$

The assertion follows via comparison of W with the solution U of problem (3.9) with the right-hand side $\Psi = y_n^{\beta-1} f(s) \psi(y')$ satisfying the condition

$$\Phi - \Psi + (\mathcal{L}_0 - \mathcal{L})\, U > 0, \quad \langle y_n^{1-\beta} \Psi \rangle_{0,H} \leq M(R, \epsilon)\, |L|.$$

By Corollary 3.4 and Lemma 3.2, for every point $\overline{y} \in H$

$$|(\mathcal{L}_0 - \mathcal{L}) U|(\overline{y}) \leq K(\rho)(\overline{y}_n)^{\beta-1} \langle y_n^{1-\beta} U \rangle_{2, B_{\overline{y}_n/4}(\overline{y})} \leq K(\rho)(\overline{y}_n)^{\beta-1} \langle y_n^{1-\beta} \Psi \rangle_{0,H}.$$

Lemma 3.5. *Let conditions* (3.11) *be fulfilled,* $P_0 \in V(3,\Omega)$, *and* $G \in V(1, B\cap\Omega)$. *Then there exists* $\rho^* > 0$ *such that for every* $\rho \in (0,\rho^*)$ *the solution of problem* (3.4) *satisfies the estimate*

$$\|w\|_{V(3,B\cap\Omega)} \le C\,\|G\|_{V(1,B\cap\Omega)}$$

with a constant C *depending on* ρ *and* $\|P_0\|$.

Proof. It is sufficient to consider the solution $W(y)$ of problem (3.7). By Lemma 3.4 $|W| \le C y_n \langle\Phi\rangle_{0,H}$. By Corollary 3.2 with $q=1$

$$y_n^{-1}\langle\langle W\rangle\rangle_{2,B_{y_n/4}(y)\cap H} \le C\left(|y_n^{-1}W|_{0,B_{y_n/2}(y)\cap H} + \langle\langle\Phi\rangle\rangle_{0,B_{y_n/2}(y)\cap H}\right),$$

whence

$$|\nabla W| + y_n|D^2 W| \le C. \tag{3.12}$$

Differentiating equation (3.7) in the tangential direction y_j we arrive at the equation for the function $Z_j = D_{y_j}W$:

$$\mathcal{L}Z_j = \Phi^{(j)} \quad \text{in } H, \qquad Z = 0 \text{ for } y_n = 0. \tag{3.13}$$

Since $\operatorname{supp} Z_j$ is compact in H, we also have that $Z_j = 0$ for $|y'| = \epsilon$. The right-hand side $\Phi^{(j)}$ has the form

$$\begin{aligned}\Phi^{(j)} &= D_{y_j}\Phi - 2y_n\left(D_{y_j}(\widetilde{\nabla}P_0),\,\widetilde{\nabla}(D_nW)\right) - 2y_n(\nabla P_0, \nabla(D_{y_j}P_0))D^2_{nn}W\\ &\quad - D_{y_j}\left(P_0\,\Delta_\xi P_0 + \gamma|\nabla P_0|^2\right)D_nW - \gamma(D_{y_j}(\widetilde{\nabla}P_0), \widetilde{\nabla}W)\\ &\quad + D_{y_j}\left[\frac{P_0}{D_{x_n}P_0}\left(\widetilde{\nabla}P_0,\,\widetilde{\nabla}(D_nP_0)\right)\right]D_nW.\end{aligned}$$

The norm $\langle y_n^{1-\beta}\Phi^{(j)}\rangle_{0,H}$ is bounded by the assumption and due to (3.12). By Corollary 3.5 $|Z_j| \le K(\rho)y_n^{\beta}\langle y_n^{1-\beta}\Phi^j\rangle_{0,H}$. Applying to Z_j Lemma 3.2 with $q=\beta$ and using the obtained estimate on Z_j, we have that for all $j \ne n$

$$|D_{y_j}W| + y_n|D^2_{y_iy_j}W| + y_n^2|D^3_{y_iy_ky_j}W| \le Cy_n^{\beta}.$$

We have estimated all second-order derivatives except $D^2_{nn}W$, which cannot be estimate in the same way. Let us write equation (3.7) in the form

$$y_nD^2_{nn}W + \gamma D_nW = \Psi, \tag{3.14}$$

where the right-hand side Ψ depends on the data and on combinations of the already-estimated functions ∇W and $\nabla(D_\tau W)$. (By $\tau \ne n$ we denote an arbitrary tangential direction.) Notice that by virtue of the above estimates and the assumptions on P_0 and G we have $|D_n\Psi| \le Cy_n^{\beta-1}$. Differentiating (3.14) in y_n and multiplying by y_n^{γ}, we may write

$$D_n\left(y_n^{\gamma+1}D^2_{nn}W\right) = y_n^{\gamma}D_n\Psi. \tag{3.15}$$

According to (3.12), $\lim_{y_n\to 0} y_n^{1+\gamma} D^2_{nn}W = 0$ for every $\gamma > 0$. Integrating (3.15) over the interval $y_n \in (0, s)$ we have now $|D^2_{nn}W| \le C y_n^{\beta-1}$. Moreover, by virtue of (3.15) we also have that $|D^3_{nnn}W| \le C y_n^{\beta-2}$. Gathering the above estimates we have that

$$\sum_{0\le|\sigma|\le 1} |D^\sigma W| + \sum_{2\le\sigma\le 3} |y_n^{|\sigma|-1-\beta} D^\sigma W| \le C.$$

The last step is to estimate Hölder's quotients of $|y_n^{2-\beta} D^3 W|$. For $D^2(D_\tau W)$ they are estimated by Lemma 3.2 applied to the functions $D_\tau W$. To estimate the Hölder quotient of $y_n^{2-\beta} D^3_{nnn}W$ we use (3.14):

$$\langle y_n^{2-\beta} D^3_{nnn}W\rangle_{0,H} \le C\left(\langle y_n^{1-\beta} D_n\Psi\rangle_{0,H} + (1+\gamma)\langle y_n^{1-\beta} D^2_{nn}W\rangle_{0,H}\right).$$

□

Lemma 3.6. *Let, under the conditions of Lemma* 3.5, $G \in V(2k+1, B\cap\Omega)$ *and* $P_0 \in V(2k+3,\Omega)$ *with* $k \ge 0$. *Then the solution of problem* (3.4) *satisfies the estimate*

$$\|w\|_{V(2k+3,B\cap\Omega)} \le C\|G\|_{V(2k+1,B\cap\Omega)}$$

with a constant C *depending on* ρ *and* $\|P_0\|$.

Proof. We argue by induction. For $k = 0$ the assertion is proved in Lemma 3.5. Let us assume that it is true for all $s = 0, 1, \ldots, k-1$, and that $G \in V(2k+1, B\cap\Omega)$, $P_0 \in V(2k+3,\Omega)$. Let $\{y\}$ be the system of local coordinates in B and $W(y) = w(x)$. Take an arbitrary multi-index $\kappa = (\kappa_1, \ldots, \kappa_{n-1}, 0)$, $|\kappa| = s \le k$, and consider the function $D^\kappa_y W(y)$. These functions solve the problems obtained from (3.13) by differentiation. By the induction conjecture

$$\sum_{|\delta|=0}^{k} |D^\delta W| + \sum_{s=0}^{k}\sum_{|\kappa|=s} \|D^\kappa W\|_{V(2(k-s)+3,H)} \le C\|\Phi\|_{V(2k+1,H)}. \tag{3.16}$$

Let us estimate $|D^{k+1}_n W|$. We write equation (3.7) in the form (3.14) and differentiate it k times in y_n:

$$y_n D_n(D^{k+1}_n W) + (k+\gamma) D^{k+1}_n W = D^k_n\Psi.$$

It follows that

$$D_n\left(y_n^{k+\gamma} D^{k+1}_n W\right) = y_n^{k+\gamma-1} D^k_n\Psi.$$

By the conjecture $y_n^{k+\gamma} D^{k+1}_n W \to 0$ when $y_n \to 0$, and $|D^k_n\Psi| \le C$. Integrating in y_n we obtain the inequality $|D^{k+1}_n W| \le C$, whence, by virtue of (3.14), $|D^{k+2}_n W| \le C y_n^{-1}$. Let us repeat these arguments differentiating equation (3.14) $k+1$ times in y_n. Multiplying the resulting equations by $y_n^{k+\gamma}$, we get

$$D_n\left(y_n^{k+\gamma+1}D_n^{k+2}W\right) = y_n^{k+\gamma}D_n^{k+1}\Psi.$$

From the above estimate $y_n^{k+\gamma+1}D_n^{k+2}W \to 0$ as $y_n \to 0$, and $|D_n^{k+1}\Psi| \leq Cy_n^{\beta-1}$. Integrating in y_n we have

$$|D_n^{k+2}W| \leq Cy_n^{\beta-1}.$$

The proof is completed by applying Lemma 3.2 with $q=\beta-1$ to the functions $D_n^{k+2}W$ and $D_n^{k+1}(D_\tau W)$. $\square$

3.5. **Global estimates.**

Theorem 3.1. *Let $P_0 \in V(2k+3,\Omega)$ and $f \in V(2k+1,\Omega)$ with $k \geq 0$, and suppose condition* (2.2) *holds with the constant $\nu \in (0,1]$. Assume that condition* (2.3) *is fulfilled. Then problem* (3.1) *with $\gamma > 0$ has a unique solution $u \in V(2k+3,\Omega)$ satisfying the estimate*

$$\|u\|_{V(2k+3,\Omega)} \leq C\,\|f\|_{V(2k+1,\Omega)}$$

with the constant C depending on Ω, $\|P_0\|$, and the constants δ in (2.3), *ϵ in* (2.1), *and ν in* (2.2).

Proof. Let $\{B_i\}_{i=1}^K$ be a finite covering of the layer $\Omega_\rho = \{x \in \Omega : \text{dist}\,(x,\Gamma) \in (0,\rho/8)\}$ by the balls of radius $\rho/8$ centered at points ξ_i such that $|\xi_i - \xi_j| \geq \rho/64$, and let $\{\xi_i\}_{i=1}^K$ be the partition of unity associated with this covering. The estimates of Lemma 3.6 give

$$\|u\|_{V(2k+3,\Omega_\rho)} \leq \sum_{i=1}^K \|u\|_{V(2k+3,\Omega\cap B_i)}$$
$$\leq C\sum_{i=1}^K \left(\|f\|_{V(2k+1,\Omega\cap B_i)} + \|u\|_{V(2k+1,\Omega\cap B_i)} + \|P_0\nabla\, u\|_{V(2k+1,\Omega\cap B_i)}\right).$$

The norms of u on the right-hand side of this inequality are estimated by combining the standard interpolation inequalities in Hölder spaces in Ω_ρ, the global estimate of Lemma 3.1, and estimates (3.3) for $\|u\|_{V(2k+3,\Omega\setminus\Omega_\rho)}$. Gathering the result with estimate (3.3) in the domain $\Omega\cap\{\text{dist}\,(x,\Gamma) > \rho/8\}$ leads to the global estimate

$$\|u\|_{V(2k+3,\Omega)} \leq C\,\|f\|_{V(2k+1,\Omega)}$$

with the constant C depending on Ω, $\|P_0\|$, and the constants ϵ in condition (2.1) and ν in (2.2). $\square$

4. The Poisson equation

Let $u(x)$ be a solution of the Poisson equation

$$\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{4.1}$$

Fix a point $x_0 \in \Gamma$, and assume that at the point $x_0 \in \Gamma$ the interior and exterior sphere condition (2.4) is fulfilled. Let us place at x_0 the origin of the coordinate system in $\mathbb{R}^n$ and choose the system so that the axes x_i, $i \neq n$, belong to the tangential plane to Γ at x_0 and x_n coincides with the direction of the inner normal to Γ at x_0.

Throughout the text we use the notation

$$x' = (x_1, \dots, x_{n-1}), \quad |x'|^2 = \sum_{i=1}^{n-1} x_i^2, \quad \overline{x} = (x_2, \dots, x_n),$$

$$B = \left\{x \in \Omega : |x'|^2 + (x_n - \delta)^2 < \delta^2\right\} \quad \text{(the ball)},$$

$$\mathcal{E} = \{x \in B : 2|x'|^2 + (x_n - \delta)^2 < \delta^2\}$$

(the co-axial ellipsoid inscribed in B).

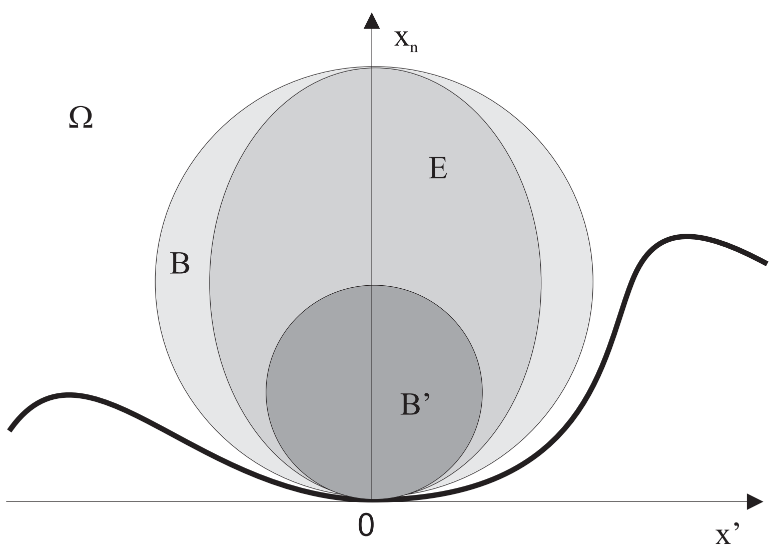

Figure 1. The inscribed balls B, B' and ellipsoid E

Lemma 4.1. *Let $u \in C(\overline{B}) \cap C^2(B)$ satisfy equation* (4.1) *and f be bounded in $\overline{B}$. If the domain Ω satisfies condition* (2.4) *and* $|u(x) - u(0)| \leq L\,|x|^\alpha$

with some $\alpha \in (0,1]$*, then for every direction* $\tau = x_i$*,* $i \neq n$*,* $|D_\tau u| \leq C\, d^{\alpha-1}$ *in* $\mathcal{E} \cap \{x_n = d\}$ *with a finite constant* C *depending only on* $\max_{\overline{B}} |f|$*,* L*,* n*, and* α*.*

Lemma 4.2. *If under the conditions of Lemma* 4.1 *the solution* $u(x)$ *satisfies the estimate* $|u(x) - u(0)| \leq K\, x_n^\alpha$ *in* $\overline{B}$ *with some* $\alpha \in (0,1]$*, then*

$$\begin{cases} |D_\tau u| \leq C\, d\, |\ln d| & \text{in } \mathcal{E} \cap \{x_n = d\} \text{ if } \alpha = 1, \\ |D_\tau u| \leq C d^{2\alpha-1} & \text{in } \mathcal{E} \cap \{x_n = d\} \text{ if } \alpha < 1. \end{cases}$$

Proof. [Proof of Lemma 4.1.] It is sufficient to consider the case $\tau = x_1$ assuming that $u(0) = 0$. Introduce the function $w(x_1, \zeta, \overline{x}) = u(x_1, \overline{x}) - u(\zeta, \overline{x})$ depending on $N+1$ independent variables. Subtracting the equation for $u(\zeta, \overline{x})$ from the equation for $u(x_1, \overline{x})$ we obtain the following equation for w:

$$\widetilde{\Delta}\, w = \Phi(x_1, \zeta, \overline{x}, t), \qquad \widetilde{\Delta} \equiv \frac{\partial^2}{\partial \zeta^2} + \Delta_x,$$

where $\Phi = f(x_1, \overline{x}) - f(\zeta, \overline{x})$, so that $|\Phi| \leq C$. The domain of definition of w is the ball

$$D = \left\{ (\zeta, x_1, x') \in \mathbb{R}^{n+1} : \, x_n \in (0, 2\delta),\, \zeta^2 + \left|x'\right|^2 + (x_n - \delta)^2 < \delta^2 \right\}.$$

The equation for w yields the elliptic differential inequality

$$\widetilde{\Delta}\, w + C \geq 0 \qquad \text{in } D \tag{4.2}$$

with the constant C from the estimate on Φ.

In view of the Hölder continuity of u and the definition of B, there is a constant L such that

$$w(x_1, \zeta, \overline{x}) \leq L |x_n|^{\alpha/2} \text{ on the boundary of } D^+ = D \cap \{x_1 > \zeta\}. \tag{4.3}$$

Let us accept the notation $B^+ = B \cap \{x_1 > 0\}$, $S = \partial B^+ \cap \{x_1 > 0\}$ and consider the auxiliary problem in the space of n independent variables:

$$\Delta\, W + C = 0 \quad \text{in } B^+, \qquad W(0, x') = 0, \qquad W \geq L|x_n|^{\alpha/2} \text{ on } S. \tag{4.4}$$

By virtue of (4.3), and the continuity of u in $\overline{B}$, there exists a function continuous and bounded on $\overline{S}$, $\psi(x)$, $\psi(0, \overline{x}) = 0$, such that the boundary restriction on W in (4.4) is fulfilled with $W|_S = \psi(x)|x'|^\alpha|_S$.

Let us search for W in the form $W = W_0 - \dfrac{C}{2} x_1^2$ with W_0 defined from the conditions

$$\Delta\, W_0 = 0 \quad \text{in } B^+, \qquad W_0(0, \overline{x}) = 0, \qquad W_0|_S = W|_S + \frac{C x_1^2}{2}.$$

W_0 is then obtained as the restriction to B^+ of the solution U to the problem

$$\Delta U = 0 \quad \text{in } B, \quad U|_{\partial B} = \begin{cases} W_0(x) & \text{if } x_1 > 0, \\ -W_0(-x_1, \overline{x}) & \text{if } x_1 < 0 \end{cases}$$

which is given by Poisson's integral,

$$U(x) = \frac{\mu}{\delta}(\delta^2 - |x-\theta_0|^2) \int_{\partial B} \frac{U(\eta)}{|(x-\theta_0)-\eta|^n}\, dS_\eta,$$

$\mu \equiv \mu(n), \quad \theta_0 = (0, \ldots, 0, \delta).$

Lemma 4.3. *For $x = (0, \ldots, 0, d)$, we have $|D_\tau W(x)| \le L(n,\delta)\, d^{\alpha-1}$.*

Proof. Let $n = 2$. Differentiating in x_1 the representation for U we have

$$|D_{x_1}U(x)| \le \frac{4\mu}{\delta}(\delta^2 - |x-\theta_0|^2) \int_{|\eta|=\delta,\, \eta_2<0} \frac{\max\, |\psi||\eta_1|^{1+\alpha}}{\left(\eta_1^2 + (\eta_2+\delta-d)^2\right)^2}\, dS_\eta$$
$$+ \frac{4\mu}{\delta}(\delta^2 - |x-\theta_0|^2) \int_{|\eta|=\delta,\, \eta_2\ge 0} \frac{\max\, |\psi||\eta_1|^{1+\alpha}}{\left(\eta_1^2 + (\eta_2+\delta-d)^2\right)^2}\, dS_\eta \equiv I_1 + I_2.$$

I_2 can be immediately estimated:

$$I_2 \le M\, \frac{4\mu}{\delta}(\delta^2 - (\delta-d)^2) \int_{|\eta|=\delta,\, \eta_2\ge 0} \frac{2\delta \max\, |\psi|}{(\delta/2)^4} \le K\, d$$

with a constant K depending only on δ, α, and $\max|\psi|$. Let $x = (r,\phi)$ be the polar coordinates in $\mathbb{R}^2$ with the origin at θ_0. Denote $\varepsilon = d/\delta$. Passing in I_1 to polar coordinates we have

$$I_1 \le \max\, |\psi| \frac{4\mu}{\delta}(\delta^2 - (\delta-d)^2) \int_{|\eta|=\delta,\, \eta_2<0} \frac{|\eta_1|^{1+\alpha}}{\left(\eta_1^2 + (\eta_2+\delta-d)^2\right)^2}\, dS_\eta$$
$$= K\, \epsilon \int_\pi^{2\pi} \frac{d\phi}{\left(2(1-\epsilon)(1+\sin\,\phi) + \epsilon^2\right)^{(3-\alpha)/2}}\ .$$

Writing

$$\int_\pi^{2\pi} \frac{d\phi}{\left(1+\sin\,\phi+\gamma^2\right)^{(3-\alpha)/2}} = 2\int_{-\pi/2}^0 \frac{d\phi}{\left(1+\sin\,\phi+\gamma^2\right)^{(3-\alpha)/2}},$$

$\gamma = \frac{\epsilon}{\sqrt{2(1-\epsilon)}}$, and then substituting $1 + \sin\,\phi = \lambda$ and simplifying we have

$$\int_\pi^{2\pi} \frac{d\phi}{\left(1+\sin\,\phi+\gamma^2\right)^{(3-\alpha)/2}} \le 4\int_0^1 \frac{d\lambda}{\left(\gamma^2+\lambda^2\right)^{(3-\alpha)/2}}$$
$$\le 2^{2+(3-\alpha)/2} \int_0^1 \frac{d\lambda}{\left(\gamma+\lambda\right)^{3-\alpha}} \le M(\delta,\alpha)\, d^{\alpha-2}.$$

Thus, $|D_{x_1}U(x)| \le C\, d^{\alpha-1}$. The case $n \ge 3$ is studied in the same way. The only difference is that now the integral I_1 over the semisphere $\{|\eta| = \delta,\ \eta_n < 0\}$ which becomes singular when d goes to zero and, thus, requires special estimating, has the form

$$\begin{aligned}&\frac{4\mu}{\delta}(\delta^2 - |x-\theta_0|^2)\int_{|\eta|=\delta,\,\eta_n<0} \frac{|\psi||\eta_1||\eta'|^\alpha\, dS_\eta}{(|\eta'|^2 + (\eta_n + \delta - d)^2)^{(n+2)/2}}\\ &\le \frac{4\mu}{\delta}(\delta^2 - |x-\theta_0|^2)\int_{|\eta|=\delta,\,\eta_n<0} \frac{|\psi|\, dS_\eta}{(|\eta'|^2 + (\eta_n + \delta - d)^2)^{(n+1-\alpha)/2}} \equiv J.\end{aligned}$$

Passing to the spherical coordinates in $\mathbb{R}^n$ with the origin θ_0 and $\eta_n = \delta \sin\theta$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we estimate J as follows:

$$\begin{aligned}J &\le \frac{4\mu}{\delta}(\delta^2 - |x-\theta_0|^2)\,\omega_{n-1}\int_{-\pi/2}^{0} \frac{(\cos\theta)^{n-2}\, d\theta}{[\gamma^2 + (1+\sin\theta)]^{(n+1-\alpha)/2}}\\ &\le M(n,\delta)\, d^{\alpha-2},\end{aligned}$$

where ω_{n-1} is the surface of the unit sphere in $\mathbb{R}^{n-1}$. □

Let us consider now the function $\widetilde{W}(x_1,\zeta,\overline{x}) \equiv W((x_1-\zeta)/\sqrt{2},\overline{x})$. It satisfies the equation $\widetilde{\Delta}\,\widetilde{W} + C = 0$ in $D^+ = D \cap \{x_1 - \zeta > 0\}$. By the definition of W, $\widetilde{W} \ge w$ on the boundary of D^+. Set $H(\zeta, x_1, \overline{x}) \equiv w - \widetilde{W}$. We have $\widetilde{\Delta}\,H \ge 0$ in D^+, $H \le 0$ on the boundary of D^+, which by the maximum principle yields the inequality $H \le 0$ in $\overline{D^+}$; i.e.,

$$\begin{aligned}w(x_1,\zeta,\overline{x}) = u(x) - u(\zeta,\overline{x}) &\le K\widetilde{W}(x_1,\zeta,\overline{x})\\ &\equiv K[W((x_1-\zeta)/\sqrt{2},\overline{x}) - W(0,\overline{x})] \quad \text{in } \overline{D^+}.\end{aligned}$$

Dividing the both parts by $|x_1 - \zeta|$, taking $\overline{x} = (0,\dots,0,d)$, and then letting $\zeta \to x_1-$, we have $|D^+_{x_1}u(x_1,\overline{x})| \le K|D_{x_1}W(x_1,\overline{x})| \le L\, d^{\alpha-1}$. Since we assumed $u \in C^1(B)$, this estimate on the right derivative continues to hold for $D_{x_1}u$. □

Proof. [Proof of Lemma 4.2.] The proof of Lemma 4.2 follows the same steps. Under the assumptions of the theorem the boundary restriction (4.3) takes on the form $w(x_1,\zeta,\overline{x}) \le L|x_n|^\alpha$ on the boundary of the semiball $D^+ = D \cap \{x_1 > \zeta\}$, which allows us to take for the function W_0 the solution of the problem

$$\Delta W_0 = 0 \quad \text{in } B^+, \qquad W_0(0,\overline{x}) = 0, \qquad W_0|_S = M\, x_n^\alpha + \frac{Cx_1^2}{2}.$$

The choice of W_1 does not change. Since $|x'|^2 = x_n(2\delta - x_n)$ on ∂B, the only difference with the proof of Lemma 4.1 is that now the integral I_1 is estimated as follows:

$$\begin{aligned} I_1 &= \frac{4\mu}{\delta}(\delta^2 - |x-\theta_0|^2) \int_{|\eta|=\delta,\, \eta_n<0} \frac{\max |\psi||\eta_1||\eta'|^{2\alpha}}{(|\eta'|^2 + (\eta_n+\delta-d)^2)^{(n+2)2}}\, dS_\eta \\ &\le \frac{4\mu}{\delta}(\delta^2 - |x-\theta_0|^2) \int_{|\eta|=\delta,\, \eta_n<0} \frac{|\psi|\, dS_\eta}{(|\eta'|^2 + (\eta_n+\delta-d)^2)^{(n+1-2\alpha)/2}}. \end{aligned}$$

Further estimating is straightforward. □

Let us claim now that $|f| \le K$ and

$$|f(x_1,\overline{x}) - f(y_1,\overline{x})| \le C x_n^{\beta-2}|x_1-y_1| \quad \text{for } (x_1,\overline{x}),\, (y_1,\overline{x}) \in B', \tag{4.5}$$

$B' = \{x \in B : |x'|^2 + (x_n - \delta/2)^2 < (\delta/2)^2\} \subset \mathcal{E}'$, with some $\beta > 0$.

Lemma 4.4. *Let under the conditions of Lemma* 4.2 *the function f satisfy* (4.5). *Then*

$$|D_\tau u| \le C \max\left\{d^\beta,\, d^{2\alpha-1}\right\} \quad \text{in } \mathcal{E}'' \cap \{x_n = d\} \text{ if } \alpha \in (0,1) \text{ and } \beta \in (0,1),$$

$$|D_\tau u| \le C d^\beta \quad \text{in } \mathcal{E}'' \cap \{x_n = d\} \text{ if } \alpha = 1 \text{ and } \beta \in (0,1),$$

$$|D_\tau u| \le C\, d \quad \text{in } \mathcal{E}'' \cap \{x_n = d\} \text{ if } \alpha = 1 \text{ and } \beta \ge 1,$$

$$|D_\tau u| \le C \max\left\{d,\, d^{2(2\alpha-1)}\right\} \quad \text{in } \mathcal{E}'' \cap \{x_n = d\} \text{ if } \alpha \in (0,1) \text{ and } \beta \ge 1,$$

where $\mathcal{E}'' = \{x \in B' :\, 4|x'|^2 + (x_n - \delta/2)^2 < (\delta/2)^2\}$.

Proof. Since the conditions of Lemma 4.2 are fulfilled, we modify the proof of Lemma 4.1 passing to the smaller ball $B' \subset \mathcal{E}$ where $|D_\tau u|$ is already estimated:

$$|w| = \left|[u]|_\zeta^{x_1}\right| \le K \begin{cases} |D_\tau u||x_1-\zeta| \le C\, d^{2\alpha-1}|x_1-\zeta| & \text{if } \alpha < 1\,, \\ |D_\tau u||x_1-\zeta| \le C\, d\,|\ln d||x_1-\zeta| & \text{if } \alpha = 1 \end{cases} \quad \text{on } \partial D'.$$

If $\beta < 1$ we take for the barrier functions $W_1(x) = \frac{C}{\beta(1-\beta)} x_n^\beta x_1$ and W_0 defined from the conditions

$$\Delta\, W_0 = 0 \text{ in } B'^+, \quad W_0(0,\overline{x}) = 0,$$

$$W_0 \ge K \begin{cases} x_n^{2\alpha-1} x_1 & \text{if } \alpha < 1\,, \\ x_n\,|\ln x_n| x_1 & \text{if } \alpha = 1 \text{ on } \partial B'^+. \end{cases}$$

If $\beta > 1$, then $W_1(x) = \frac{C}{\beta(1-\beta)} x_n^\beta x_1 \le 0$ and the function W_0 has to be chosen from the conditions $\Delta\, W_0 = 0$ in B'^+, $W_0(0,\overline{x}) = 0$,

$$W_0 \ge -\frac{C}{\beta(1-\beta)} x_n^\beta x_1 + K \begin{cases} x_n^{2\alpha-1} x_1 & \text{if } \alpha < 1\,, \\ x_n\,|\ln x_n| x_1 & \text{if } \alpha = 1 \end{cases} \qquad \text{on } \partial B'^+.$$

For $\beta = 1$ we set $W_1 = -Cx_1x_n\,|\ln x_n|$ and

$$W_0 \geq K \begin{cases} x_n^{2\alpha-1}x_1 & \text{if } \alpha < 1\ , \\ x_n\,|\ln x_n|x_1 & \text{if } \alpha = 1 \end{cases} \qquad \text{on } \partial B'^+.$$

To satisfy the boundary condition for W_0, it suffices to take

$$W_0 = 2K \begin{cases} x_n^{2\alpha-1}x_1 & \text{if } \alpha < 1\ , \\ x_n\,|\ln x_n|x_1 & \text{if } \alpha = 1\ . \end{cases}$$

Lemma 4.5. *Let $|u(x)-u(0)| \leq Kx_n$ in $\overline{B}$ and $f(x)$ satisfy condition* (4.5) *with some $\beta > 0$ for all tangential directions. Then*

$$|D_\tau u| \leq C \begin{cases} d & \text{in } \mathcal{E} \cap \{x_n = d\} & \text{if } \beta \geq 1, \\ d^\beta & \text{in } \mathcal{E} \cap \{x_n = d\} & \text{if } \beta \in (0,1). \end{cases}$$

Proof. Let us consider the case $\beta \geq 1$. We choose

$$W_1(x) = K \begin{cases} x_1\,x_n|\ln x_n| & \text{if } \beta = 1, \\ x_1\,x_n^\beta & \text{if } \beta > 1 \end{cases}$$

and define W_0 from the conditions

$$\Delta\,W_0 = 0 \text{ in } B'^+, \quad W_0(0,\overline{x}) = 0, \quad W_0 = Cx_n \text{ on } \partial B^+. \tag{4.6}$$

This gives the estimate $|D_\tau v| \leq Kd\,|\ln d|$ in $\mathcal{E} \cap \{x_n = d\}$. Passing to the smaller ball $B' \subset \mathcal{E}$ we repeat estimating under the stronger condition

$$|\ [u]|_{x_1}^{\zeta}\ | \leq K\,d\,|\ln d||x_1 - \zeta| \quad \text{on } \partial B^+.$$

Choosing W_1 as before, for W_0 we take the solution of the problem (4.6) with the boundary condition

$$W_0(x) = \max\{K\,x_n\,|\ln x_n||x_1 - \zeta|;\, W_1(x)\} \quad \text{on } \partial B^+.$$

In the case $\beta \in (0,1)$ the proof is similar. □

5. The degenerate Poisson equation

Let us consider problem (3.1) with $\gamma = 0$:

$$P_0\Delta\,u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \Gamma, \tag{5.1}$$

with the right-hand side $f = 0$ on Γ.

5.1. **Existence. The maximum principle.** A classical solution of problem (5.1) can be constructed by means of the same regularization procedure that was used in the case $\gamma > 0$. Notice that the prospective solution satisfies the estimate $|u| \leq C(\delta)$ on every set $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \Gamma) < \delta\}$.

Lemma 5.1. *Let $P_0 \in V(3, \Omega)$ and $f \in V(1, \Omega)$, $f = 0$ on Γ. Then the classical solution of problem* (5.1) *satisfies the estimate*

$$|u| \leq C P_0 \qquad \text{in } \Omega$$

with a constant C depending on $\|P_0\|_{V(3,\Omega)}$, β, and λ.

Proof. Notice that if $f = 0$ on Γ, the inclusion $f \in V(1, \Omega)$ implies $|f| \leq \lambda P_0^\beta$ in $\overline{\Omega}$. Let us consider the function

$$U(x) = K\, P_0 - M\, P_0^{1+\beta}, \qquad K,\, M = const > 0.$$

It is easy to calculate that

$$\begin{aligned}\mathcal{M}\, U &= K\, P_0 \Delta\, P_0 - M(1+\beta) P_0^{1+\beta} \Delta\, P_0 - M\beta(1+\beta) P_0^\beta |\nabla\, P_0|^2 \\ &\leq P_0^\beta \left(K\, \|P_0\|_{V(3,\Omega)} + (1+\beta) M P_0^\beta - \beta(1+\beta) M |\nabla\, P_0|^2 \right).\end{aligned}$$

According to conditions (2.2) and (2.5) $|\nabla\, P_0| \geq \nu/2$ and $P_0^\beta < \beta\nu^2/8$ in Ω, so that

$$\mathcal{M}\, U < \frac{1}{8} \nu^2 \beta(1+\beta) P_0^\beta \left(\frac{8K}{\nu^2 \beta(1+\beta)} \|P_0\|_{V(3,\Omega)} - M \right).$$

Letting

$$M = K \left(1 + \frac{8K}{\nu^2 \beta(1+\beta)} \|P_0\|_{V(3,\Omega)} \right),$$

we obtain

$$\mathcal{M}\, U < -\frac{K}{8} \nu^2 \beta(1+\beta) P_0^\beta < -\lambda P_0^\beta \leq f \quad \text{in } \Omega$$

and $U \geq u$ on $\Gamma \cup \Sigma$ provided that K is taken sufficiently large. □

Notice that given any small $\delta > 0$, the classical solution of problem (5.1) satisfies the estimate

$$\|u\|_{C^{k+2+\alpha}(\Omega_\delta)} \leq C(\delta)\, \|f\|_{C^{k+\alpha}(\Omega_\delta)}. \tag{5.2}$$

5.2. **Estimates near the boundary Γ.** The estimates near the boundary are easy to obtain imitating the proofs given in the case $\gamma > 0$. However, as long as our arguments are based on the idea of straightening a boundary portion given by the equation $P_0 = 0$, the provable regularity of the solution cannot be better than the regularity of P_0. The special form of equation (5.1) makes it possible to overcome this inconvenience. Applying the results of the previous section we may estimate the derivatives in the tangential directions to Γ without differentiating equation (5.1), which allows us to show that unlike the case $\gamma > 0$ the regularity of the solution is better than the regularity of P_0.

Lemma 5.2. *Let P_0, $f \in V(3,\Omega)$, $P_0 = f = 0$ on Γ. Assume that at every point of Γ the interior and exterior sphere conditions are fulfilled. For every point $x_0 \in \Gamma$ there is a ball $B = B_\rho(y_0) \subset \Omega$, touching Γ at x_0, such that for every direction $\tau \neq n$*

$$|\nabla u| \le C, \quad |D_\tau u| \le C\, d, \quad |D^2_{ij} u| \le C \quad \text{in } \mathcal{E},$$

where $\mathcal{E}$ is the ellipsoid co-axial with B.

Proof. Take a point $x_0 \in \Gamma$ and assume that $x_0 = 0$ and that x_0 is the origin of $\mathbb{R}^n$. The axes are chosen so that x_n coincides with the interior normal to Γ at x_0. By Lemma 5.1 $|u| \le CP_0$ near Γ. It follows from Lemma 3.2 that $|\nabla u| \le C$. Let us write equation (5.1) in the form

$$\Delta u = F \equiv \frac{f}{P_0} \quad \text{in } \Omega.$$

The right-hand side of this equation satisfies

$$\begin{aligned} |F(\xi,\overline{x}) - F(\zeta,\overline{x})| &\le \Big(\Big|\frac{D_{x_1} f(\eta,\overline{x})}{P_0(\eta,\overline{x})}\Big| + \Big|\frac{f(\eta,\overline{x})\, D_{x_1} P_0(\eta,\overline{x})}{P_0^2(\eta,\overline{x})}\Big|\Big)|\xi - \zeta| \\ &\le C x_n^{-1} |\xi - \zeta|. \end{aligned}$$

By Lemma 4.5 with $\alpha = \beta = 1$

$$|D_\tau u| \le C\, d \qquad \text{in } \mathcal{E} \cap \{x_n = d\}.$$

Let us differentiate equation (5.1) in $\tau \neq n$ in $\mathcal{E}$: $v = D_\tau u$,

$$P_0 \Delta v = D_\tau f - D_\tau P_0\, \Delta u \qquad \text{in } \mathcal{E}. \tag{5.3}$$

Introduce the function $w = P_0 \Delta u$. It satisfies the conditions

$$P_0 \Delta w = P_0 \Delta f \quad \text{in } \Omega, \qquad w = 0 \text{ on } \Gamma.$$

Since $f \in V(3,\Omega)$, $|P_0 \Delta f| \le C\, d^\beta$. It follows from Lemma 5.1 that $|w| \le C\, P_0$ and, thus, $|\Delta u| \le C$ in $\overline{\Omega}$. This means that the right-hand side in (5.3) is bounded by Cd^β. Applying Lemma 3.2 we have that $|\nabla(D_\tau u)| \le C$. This estimate gathered with $|\Delta u| \le C$ gives a bound for $D^2_{nn}u$. □

Lemma 5.3. *Under the conditions of Lemma* 5.2

$$|D^3 u| \le C\, d^{\beta-1} \quad \text{in } \mathcal{E}.$$

Proof. Consider the function $v = D_\tau u$ in $B' \subset \mathcal{E}$. Let us write equation (5.3) in the form

$$\Delta\, v = H \equiv \frac{D_\tau f}{P_0} - \frac{D_\tau P_0}{P_0} \Delta\, u \quad \text{in } B'.$$

Since $f \in V(3, \Omega)$, we may apply to u Lemma 3.2, which gives $x_n|\nabla(\Delta\, u)| \le C$. Notice that since for every $i \neq n$ and $\widehat{x} = (0, \dots, 0, x_n)$

$$|D^2_{ni} f(\widehat{x})| + |D^2_{ni} P_0(\widehat{x})| \le C' x_n^{\beta-1}$$

and $D_i P_0(0) = D_i f(0) = 0$, we have the estimate $|D_i f(\widehat{x})| + |D_i P_0(\widehat{x})| \le C x_n^\beta$. It follows that for any tangential direction τ

$$|D_\tau f(\widehat{x})| + |D_\tau P_0(\widehat{x})| \le C P_0^\beta \quad \text{in } B'.$$

By the Lagrange theorem we have then

$$|H(\xi, \overline{x}) - H(\zeta, \overline{x})| \le C x_n^{\beta-2} |\xi - \zeta| \quad \text{in } B'.$$

Since $|v| \le C x_n$ by Lemma 5.2, it follows from Lemma 4.5 that for every tangential direction $\theta \neq n$

$$|D^2_{\tau\theta} u| = |D_\theta v| \le C\, d^\beta \quad \text{in } \mathcal{E}'.$$

By Lemma 3.2 we conclude that

$$d|\nabla(D^2_{\tau\theta} u)| + d^2 |D^2_{ij}(D^2_{\tau\theta} u)| \le C d^\beta. \tag{5.4}$$

It remains to show that $|D^3_{nnn} u| + |D^2_{nn}(D_\tau u)| \le C\, d^{\beta-1}$. Let us consider the function $w = P_0 \Delta\, u$. This function satisfies the conditions

$$\Delta\, w = \Delta\, f \quad \text{in } B', \qquad |w| \le C P_0.$$

Since $f \in V(3, \Omega)$, then $d|\Delta\, f| + d^2 |\nabla(\Delta\, f)| \le C\, d^\beta$ in B', and by Lemma 4.5 we conclude that $|D_\tau w| \le C\, d^\beta$. According to (5.4), the last inequality yields

$$|\Delta(D_\tau u)| \le C\, d^{\beta-1},$$

whence

$$|D^3_{nn\tau} u| \le C\, d^{\beta-1} \quad \text{for every } \tau \neq n.$$

Let us estimate $D^3_{nnn}u$. To this end we differentiate twice equation (5.1) in the direction x_n and write the result as

$$P_0 D^2_{nn}(D^2_{nn}u)+2D_nP_0\,D_n(D^2_{nn}u) = D^2_{nn}f - D^2_{nn}P_0\,D^2_{nn}u - D^2_{nn}(P_0\sum_{i\neq n} D^2_{ii}u).$$

It follows from the above estimates and the assumptions on P_0 and f that the right-hand side of this equation is bounded by $C\,d^{\beta-1}$. Multiplying by P_0 we have

$$|D_n\left(P_0^2 D^3_{nnn}u\right)| \leq C\,x_n^{\beta} \quad \text{for } x=(0,\ldots,0,x_n).$$

Integrating along the ray x_n and simplifying we finally have $|D^3_{nnn}u| \leq C\,d^{\beta-1}$. □

5.3. **Global estimate.**

Lemma 5.4. *Under the conditions of Lemma* 5.2 *the solution of problem* (5.1) *satisfies the estimate*

$$\|u\|_{V(5,\Omega)} \leq C\,\|f\|_{V(3,\Omega)}.$$

The proof follows by gathering the estimates of Lemmas 5.2, 5.3, and applying Lemma 3.2 to estimate the higher-order derivatives.

Arguing by induction we extend the estimate to the case when P_0, $f \in V(2k+1,\Omega)$ with arbitrary natural $k\geq 1$: $\|u\|_{V(2k+3,\Omega)} \leq C\,\|f\|_{V(2k+1,\Omega)}$. Plugging into this estimate (5.2), we prove the estimate of Theorem 2.2.

6. EXTENSIONS

Let us consider the problem

$$L_1 u \equiv P_0\,\mathrm{div}(A\nabla u) + \gamma(\nabla P_0,\, A\nabla u) = f(x) \quad \text{in } \Omega, \quad u=0 \text{ on } \partial\Omega \tag{6.1}$$

with the parameter $\gamma > 0$, under the above assumptions on the function $f(x)$. Here $A(x) = I + B(x)$, where I is the identity matrix, and $B(x)$ is a symmetrical matrix with the entries

$$B_{ij}(x) = D^2_{ij}v(x) \text{ for some } v \in V(2k+3,\Omega),\ k\geq 0. \tag{6.2}$$

It is assumed that the matrix $A(x)$ is strictly positive defined on $\overline{\Omega}$,

$$\forall x \in \overline{\Omega} \qquad (A(x)\xi,\,\xi) \geq \kappa\,|\xi|^2, \quad |\xi| \neq 0. \tag{6.3}$$

We prove the existence of a solution to problem (6.1) using the method of continuity [4, Chapter 5]. Since the problem with $B\equiv 0$ is already solved, it suffices to derive suitable a priori estimates for the solutions of (6.1) in the spaces $V(2k+3,\Omega)$. We revise the derivation of the a priori estimates for the solutions of problem (3.1) given in Section 3.

Let us denote $D = \text{dist}\,(\Gamma, \Sigma)$. According to (2.2), we have $P_0 \leq 2\nu\, D$. Assuming that condition (2.5) is fulfilled, and $B_{ij}(x) = D^2_{ij} v$ with $v \in V(3, \Omega)$, we may estimate

$$\begin{aligned} L_1 P_0 &= P_0 \text{div}(A\,\nabla\, P_0) + \gamma\,(\nabla\, P_0,\; A\,\nabla\, P_0) \\ &\geq \gamma\kappa\delta - 2\,n^2\,\|v\|_{V(3,\Omega)}\|P_0\|_{V(3,\Omega)}\,(2\nu D)^\beta \geq \mu > 0 \end{aligned} \tag{6.4}$$

in Ω, $\mu = \mathit{const} > 0$, provided that $\text{dist}\,(\Gamma, \Sigma)$ is small. It follows from the maximum principle that for any bounded function f and appropriately chosen constant C the solution of problem (6.1) satisfies the estimate $|u| \leq C(\sup\, P_0 -\, P_0)\, \sup\, |f|$ in $\overline{\Omega}$ (see the proof of Lemma 3.1).

Further estimating is based on the study of the model problem with constant coefficients. We place the origin at a boundary point $x_0 \in \Gamma$, introduce the new coordinates according to formulas (2.1), and then "freeze" the coefficients at $x_0 = 0$. The operators of the resulting equations coincide with $\mathcal{L}$ and $\mathcal{L}_0$, which allows us to imitate all the arguments of Section 3 without any substantial changes.

Theorem 6.1. *Let $P_0 \in V(2k+3, \Omega)$, $f \in V(2k+1, \Omega)$ with $k \geq 0$, and the domain Ω be such that conditions* (2.3) *and* (2.5) *are fulfilled. Let us assume that conditions* (6.2) *and* (6.3) *hold, and the domain Ω is so thin that condition* (6.4) *is fulfilled. Then problem* (6.1) *has a unique solution satisfying the estimate*

$$\|u\|_{V(2k+3,\Omega)} \leq C\|f\|_{V(2k+1,\Omega)}.$$

Let us consider now the problem

$$\text{div}\;(P_0^\gamma\, A(x)\,\nabla\, u) = P_0^{\gamma-1} f(x, u, \nabla\, u) \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega \tag{6.5}$$

with $\gamma > 0$ and under the previous assumptions on the matrix $A(x)$. We will assume that

$$f(x, u, \nabla\, u) = f_0(x) + q(x)u + (\mathbf{p}(x), \nabla\, u) \tag{6.6}$$

where q and $\mathbf{p}$ are given functions and that there exists a scalar function g such that

$$\begin{aligned} \|f(x, s, \nabla\, r)\|_{V(2k+1,\Omega)} &\leq K\|f_0\|_{V(2k+1,\Omega)} \\ &+ K\|g\|_{V(2k+1,\Omega)}\left(\|s\|_{V(2k+1,\Omega)} + \|\nabla\, r\|_{V(2k+1,\Omega)}\right). \end{aligned} \tag{6.7}$$

Theorem 6.2. *Let the conditions of Theorem* 6.1 *be fulfilled, the function $f(x, s, \mathbf{r})$ satisfy conditions* (6.6) *and* (6.7)*, and the following condition hold:*

$$\begin{aligned} &\gamma\kappa\nu - 2\,n^2\,\|v\|_{V(3,\Omega)}\|P_0\|_{V(3,\Omega)}\,(2\nu\, D)^\beta - 2\sup\,|q|\,\nu\, D \\ &\quad - \|P_0\|_{V(3,\Omega)} \sum_i \sup\,|\mathbf{p}_i| \geq \mu > 0 \quad \textit{in } \Omega,\ \mu = \mathit{const} > 0. \end{aligned} \tag{6.8}$$

Then for every $f_0 \in V(2k+1,\Omega)$ problem (6.5) *has a unique solution u satisfying the estimate*

$$\|u\|_{V(2k+3,\Omega)} \leq C \, \|f_0\|_{V(2k+1,\Omega)}$$

with a constant C independent of u and f_0.

The proof also uses the method of continuity. Condition (6.8) provides the validity of the estimate $\sup |u| \leq C \sup |f_0|$, while conditions (6.6) and (6.7) allow one to derive the a priori estimate in the norm of $V(2k+3,\Omega)$.

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