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SOLVABILITY OF NONAUTONOMOUS PARABOLIC VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract. We consider nonautonomous parabolic variational inequalities having the strong formulation

$$\langle u'(t) \,,\, v - u(t) \rangle + \langle A(t)u(t) \,,\, v - u(t) \rangle + \Phi(t,v) - \Phi(t,u(t)) \geq 0, \quad \forall\, v \in V^{**},\ a.e.\ t \geq s,$$

where $u(s) = u_s$ for some admissible initial datum, V is a separable Banach space with separable dual V^*, $A(t) : V^{**} \to V^*$ is an appropriate family of monotone operators, and $\Phi(t,\cdot) : V^{**} \to \mathbb{R} \cup \{\infty\}$ is a family of convex, weak* lower-semicontinuous functionals. Well-posedness follows from an explicit construction of the related evolution family $\{U(t,s) : t \geq s\}$. Illustrative applications are given.

1. Introduction

In [5], we established well–posedness results for autonomous parabolic variational inequalities by developing and then exploiting an appropriate version of the Crandall–Liggett theorem ([2]). To extend the results of [5] to problems involving operators and constraint functionals which depend on time, we modify Crandall and Pazy's extension ([3]) of Crandall and Liggett's earlier results. This yields an explicit construction of the related evolution family $\{\, U(t,s) \,:\, t \,\geq\, s \,\}$. We will see below that the function $u(t) := U(t,s)u_s$ is the unique weak solution of the parabolic variational inequality associated to $A(\cdot)$ and $\Phi(\cdot)$; moreover, $u(t)$ is the unique strong solution of this problem if the initial datum u_s is sufficiently smooth.

The following section specifies our assumptions on the operators $A(\cdot)$ and functionals $\Phi(\cdot)$ and defines strong and weak solutions of the associated parabolic variational inequalities. Section 3 investigates two fundamental tools

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in our analysis, the resolvent map and the Yosida approximation. Section 4 establishes the nonautonomous exponential formula, which leads to the definition of the map $U(\cdot, s)$, and Section 5 provides the analysis necessary to conclude that $\{ U(t,s) : t \geq s \}$ is indeed the related evolution family. Finally, Section 6 applies our results to some quasiautonomous problems. Other applications, including obstacle problems for the parabolic p-Laplace operator (with time-dependent obstacles), will be the subject of future work.

2. Preliminaries

We first adapt the structural framework of [5] to nonautonomous problems, and we refer to [5] and the references therein (cf. also [6]) for terminology not defined below. Let V be a separable Banach space whose dual V^* is separable, and let H be a Hilbert space which embeds continuously into V^*; we thus have the pivot space structure $V^{**} \hookrightarrow H \hookrightarrow V^*$. To accommodate operators and functionals which vary with time, let an initial time $s \geq 0$ be given. For each $t \geq s$, let $A(t) : V^{**} \to V^*$ be a bounded, hemicontinuous operator which satisfies

$$\langle A(t)u - A(t)v \, , \, u - v \rangle \geq \omega \|u - v\|_H^2, \ \forall \ u, v \in V^{**}, \tag{2.1}$$

for some monotonicity constant $\omega \geq 0$, and let $\Phi(t, \cdot) : V^{**} \to \mathbb{R} \cup \{\infty\}$ be a proper, convex, and weak* lower-semicontinuous functional. For simplicity, we henceforth write $\Phi(t)$ instead of the more cumbersome $\Phi(t, \cdot)$. We assume that $A(t)$ and $\Phi(t)$ satisfy the coercivity condition

$$\lim_{\|v\|_{V^{**}} \to \infty} \frac{\langle A(t)v \, , \, v - v_0 \rangle + \Phi(t, v)}{\|v - v_0\|_{V^{**}}} = \infty \tag{2.2}$$

for some $v_0 \in D(\Phi(t))$, the effective domain of $\Phi(t)$.

Define the sets

$$\mathfrak{D}(t) := \{ v \in D(\Phi(t)) \ : \ A(t)v \in H \quad \text{and} \quad \partial\Phi(t, v) \cap H \neq \emptyset \}. \tag{2.3}$$

We assume that each $u \in D(\Phi(s))$ is a limit point of $\mathfrak{D}(s)$ with respect to the strong topology of H. Thus, for each $u \in D(\Phi(s))$, there exists a sequence $\{u_n\} \subseteq D(\Phi(s))$ such that

$$u_n \to u \quad \text{in} \quad H, \quad A(s)u_n \in H, \quad \text{and} \quad \partial\Phi(s, u_n) \cap H \neq \emptyset. \tag{2.4}$$

Clearly, then, $\overline{D(\Phi(s))}^{\|\cdot\|_H} = \overline{\mathfrak{D}(s)}^{\|\cdot\|_H}$. Note that we require only that this approximation hypothesis hold for the single set $\mathfrak{D}(s)$.

For a prescribed initial condition $u_s \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$ and a given final time $T > s$, we have the following nonautonomous analogues of the solutions studied in [5]:

Definition 2.1. *A function $u : [s,T] \longrightarrow V^{**}$ is a strong solution of the parabolic variational inequality associated to $A(\cdot)$ and $\Phi(\cdot)$ if*

(i) $u \in L^\infty(s,T;V^{**}) \cap C^{0,1}(s,T;H)$,
(ii) $u(s) = u_s$, *and*
(iii) *u satisfies*

$$\langle\, u'(t)\,,\, v - u(t)\,\rangle + \langle\, A(t)u(t)\,,\, v - u(t)\,\rangle + \Phi(t,v) - \Phi(t,u(t)) \geq 0, \quad \forall\, v \in V^{**},\ a.e.\ t \in (s,T). \tag{2.5}$$

Definition 2.2. *A function $u : [0,T] \to V^{**}$ is a weak solution of the parabolic variational inequality associated to $A(\cdot)$ and $\Phi(\cdot)$ with initial value u_s if*

(i) $u \in C(s,T;H)$,
(ii) $u(s) = u_s$, *and*
(iii) *there exists a sequence u_n of strong solutions of the parabolic variational inequality associated to $A(\cdot)$ and $\Phi(\cdot)$ with initial values $u_{n,s}$ such that $u_{n,s} \to u_s$ in H and $u_n \to u$ in $C(s,T;H)$.*

3. The resolvent and Yosida approximation

Definition 3.1. *For a given time-step $\lambda > 0$ and $t \geq s$, the resolvent*

$$J_\lambda(t) : H \to D(\Phi(t))$$

is defined by $J_\lambda(t)x := u$, where u is the unique solution of the elliptic variational inequality

$$\langle\, \frac{u - x}{\lambda}\,,\, v - u\,\rangle + \langle\, A(t)u\,,\, v - u\,\rangle + \Phi(t,v) - \Phi(t,u) \geq 0,\ \forall\, v \in V^{**}. \tag{3.1}$$

As t is fixed here, the proof of Proposition 3.1 in [5] shows that $J_\lambda(t)$ is well-defined and verifies part (i) of the following lemma. The proof of Lemma 3.2 in [5] yields properties (ii), (iii), and (iv).

Lemma 3.2. *Let $t \geq s$ and $x \in H$ be given.*

(i) *$J_\lambda(t)$ is Lipschitz continuous with respect to the norm of H:*

$$\|J_\lambda(t)x_1 - J_\lambda(t)x_2\|_H \leq \frac{1}{1+\lambda\omega}\, \|x_1 - x_2\|_H.$$

(ii) *If $x \in \mathfrak{D}(t)$, then*

$$\|J_\lambda(t)x - x\|_H \leq \frac{\lambda}{1+\lambda\omega} \left(\|A(t)x\|_H + \inf_{y \in \partial\Phi(t)(x) \cap H} \|y\|_H \right).$$

(iii) *If n is a positive integer, then*

$$\|J_\lambda^n(t)x - x\|_H \leq n\, \|J_\lambda(t)x - x\|_H\,.$$

(iv) *Let $\lambda \geq \mu > 0$. Then the resolvent identity holds:*

$$J_\lambda(t)x = J_\mu(t)\Big(\frac{\lambda-\mu}{\lambda}J_\lambda(t)x + \frac{\mu}{\lambda}x\Big).$$

Definition 3.3. *For $\lambda > 0$ and $t \geq s$, the Yosida approximation $K_\lambda(t) : H \to H$ is defined by*

$$K_\lambda(t) := \frac{I - J_\lambda(t)}{\lambda}.$$

The basic properties of the Yosida approximation follow directly from Lemma 3.2.

Lemma 3.4. *Let $\lambda > 0$, $t \geq s$, and $x, y \in H$ be given.*

(i) *For $0 < \mu \leq \lambda$,*

$$(1+\lambda\omega)\,\|K_\lambda(t)x\|_H \leq (1+\mu\omega)\,\|K_\mu(t)x\|_H.$$

(ii) *$K_\lambda(t)$ is Lipschitz continuous with respect to the norm of H:*

$$\|K_\lambda(t)x - K_\lambda(t)y\|_H \leq \Big(\frac{1}{\lambda} + \frac{1}{\lambda\,(1+\lambda\omega)}\Big)\|x-y\|_H.$$

(iii) *For λ sufficiently small, the monotonicity constant of $K_\lambda(t)$ is $(\frac{1}{\lambda} - \frac{1}{1+\lambda\omega})$.*

(iv) *If $x \in \mathfrak{D}(t)$, then*

$$\|K_\lambda(t)x\|_H \leq \frac{1}{1+\lambda\omega}\Big(\|A(t)x\|_H + \inf_{y \in \partial\Phi(t)(x)\cap H}\|y\|_H\Big).$$

Proof. All of these proofs are straightforward calculations. For simplicity, we omit the argument t, which remains fixed throughout.

For (i), we have

$$\begin{aligned}
\|K_\lambda x\|_H &\leq \frac{1}{\lambda}\|x - J_\mu x\|_H + \frac{1}{\lambda}\|J_\mu x - J_\lambda x\|_H \\
&= \frac{\mu}{\lambda}\|K_\mu x\|_H + \frac{1}{\lambda}\|J_\mu x - J_\mu\Big(\frac{\lambda-\mu}{\lambda}J_\lambda x + \frac{\mu}{\lambda}x\Big)\|_H \\
&\leq \frac{\mu}{\lambda}\|K_\mu x\|_H + \frac{1}{\lambda(1+\mu\omega)}\|x - \Big(\frac{\lambda-\mu}{\lambda}J_\lambda x + \frac{\mu}{\lambda}x\Big)\|_H \\
&= \frac{\mu}{\lambda}\|K_\mu x\|_H + \frac{\lambda-\mu}{\lambda\,(1+\mu\omega)}\|K_\lambda x\|_H.
\end{aligned}$$

Manipulating this last inequality yields (i). We then compute

$$\|K_\lambda x - K_\lambda y\|_H = \|\frac{1}{\lambda}\,(x - J_\lambda x) - \frac{1}{\lambda}\,(y - J_\lambda y)\,\|_H$$

$$\leq \frac{1}{\lambda}\|x-y\|_H + \frac{1}{\lambda}\|J_\lambda x - J_\lambda y\|_H \leq \frac{1}{\lambda}\Big(1+\frac{1}{1+\lambda\omega}\Big)\|x-y\|_H,$$

which yields (ii).

To obtain (iii), first note that

$$\langle\, J_\lambda x_1 - J_\lambda x_2\,,\, x_1-x_2\,\rangle \leq \|J_\lambda x_1 - J_\lambda x_2\|_H\,\|x_1-x_2\|_H \leq \frac{1}{1+\lambda\omega}\|x_1-x_2\|_H^2.$$

We then have

$$\begin{aligned}\langle\, K_\lambda x_1 - K_\lambda x_2\,,\, x_1-x_2\,\rangle &= \frac{1}{\lambda}\|x_1-x_2\|_H^2 + \frac{1}{\lambda}\langle\, J_\lambda x_2 - J_\lambda x_1\,,\, x_1-x_2\,\rangle \\ &\geq \Big(\frac{1}{\lambda}-\frac{1}{1+\lambda\omega}\Big)\|x_1-x_2\|_H^2.\end{aligned}$$

For λ sufficiently small (namely, $\lambda \leq \frac{1}{1-\omega}$), the coefficient $(\frac{1}{\lambda}-\frac{1}{1+\lambda\omega})$ is positive, verifying (iii).

Finally, (iv) follows directly from Lemma 3.2(ii) and the definition of K_λ. ☐

For $x \in H$, Lemma 3.4(i) shows that the sequence $\{\,\|K_\lambda(t)x\|_H\,\}$ increases as λ decreases. This motivates the following definitions.

Definition 3.5. *For $t \geq s$ and $x \in H$,*

$$|K(t)x| := \lim_{\lambda\to 0^+}\|K_\lambda(t)x\|_H.$$

Definition 3.6. *For $x \in H$,*

$$\mathcal{K}(x) := \sup_{t\geq s}|K(t)x|.$$

Lemma 3.7. *Let $\lambda > 0$, $t \geq s$, and $x \in H$.*

(i) $\|K_\lambda(t)x\|_H \leq \frac{1}{1+\lambda\omega}|K(t)x|$.

(ii) *If $x \in \mathfrak{D}(t)$, then* $|K(t)x| \leq \|A(t)x\|_H + \inf_{y\in\partial\Phi(t)(x)\cap H}\|y\|_H$.

Proof. The first estimate results from letting $\mu \to 0^+$ in Lemma 3.4(i), and the second follows from letting $\lambda \to 0^+$ in Lemma 3.4(iv). ☐

Having defined the Yosida approximation $K_\lambda(t)$ and its associated quantities, we now specify our fundamental assumption. For $x \in H$ and $t, \tau \geq s$, we suppose that there exist a continuous function $\beta : [s,\infty) \to H$ of bounded variation and a nondecreasing function $M : [0,\infty) \to [0,\infty)$ such that

$$\|J_\lambda(t)x - J_\lambda(\tau)x\|_H \leq \lambda\,\|\beta(t)-\beta(\tau)\|_H\, M(\|x\|_H)\,\left(\,1+|K(\tau)x|\,\right). \quad (3.2)$$

The following chain of calculations foreshadows the role of inequality (3.2): we have

$$\begin{aligned}
&\|J_\lambda(t)x - J_\lambda(\tau)x\|_H \le \lambda\,\|\beta(t)-\beta(\tau)\|_H\,M(\|x\|_H)\,(1+|K(\tau)x|)\\
\Longrightarrow&\frac{1}{\lambda}\|J_\lambda(t)x - x + x - J_\lambda(\tau)x\|_H \le \|\beta(t)-\beta(\tau)\|_H M(\|x\|_H)\,(1+|K(\tau)x|)\\
\Longrightarrow&\|K_\lambda(t)x\|_H - \|K_\lambda(\tau)x\|_H \le \|\beta(t)-\beta(\tau)\|_H\,M(\|x\|_H)\ (1+|K(\tau)x|)\\
\Longrightarrow&\|K_\lambda(t)x\|_H \le \|K_\lambda(\tau)x\|_H + \|\beta(t)-\beta(\tau)\|_H\,M(\|x\|_H)\ (1+|K(\tau)x|)\\
\Longrightarrow&|K(t)x| \le |K(\tau)x| + \|\beta(t)-\beta(\tau)\|_H\,M(\|x\|_H)\ (1+|K(\tau)x|)\,,
\end{aligned}$$

where the final inequality follows from letting $\lambda \to 0^+$. Letting $\tau = s$, we find that condition (3.2) implies that $\mathcal{K}(x)$ is finite for every $x \in \mathfrak{D}(s)$.

Lemma 3.8. *Let $x \in D(\Phi(s))$, and let $s_i \ge s$ for $i = 1, \ldots, \ell$, for some given integer $\ell \ge 0$. Then*

$$\Big\|\prod_{i=1}^{\ell} J_\lambda(s_i)x - x\Big\|_H \le \ell\,\lambda\,\mathcal{K}(x).$$

Proof. We have

$$\begin{aligned}
\Big\|\prod_{i=1}^{\ell} J_\lambda(s_i)x - x\Big\|_H &= \Big\|\sum_{k=1}^{\ell}\Big(\prod_{i=k}^{\ell} J_\lambda(s_i)x - \prod_{i=k+1}^{\ell} J_\lambda(s_i)x\Big)\Big\|_H\\
&\le \sum_{k=1}^{\ell}\ \Big\|\prod_{i=k}^{\ell} J_\lambda(s_i)x - \prod_{i=k+1}^{\ell} J_\lambda(s_i)x\Big\|_H\\
&= \sum_{k=1}^{\ell}\ \Big\|J_\lambda(s_\ell)\prod_{i=k}^{\ell-1} J_\lambda(s_i)x - J_\lambda(s_\ell)\prod_{i=k+1}^{\ell-1} J_\lambda(s_i)x\Big\|_H\\
&\le \sum_{k=1}^{\ell}\ \Big\|\prod_{i=k}^{\ell-1} J_\lambda(s_i)x - \prod_{i=k+1}^{\ell-1} J_\lambda(s_i)x\Big\|_H
\end{aligned}$$

by Lemma 3.2. Iteration then yields

$$\begin{aligned}
\Big\|\prod_{i=1}^{\ell} J_\lambda(s_i)x - x\Big\|_H &\le \sum_{k=1}^{\ell}\ \|J_\lambda(s_k)x - x\|_H = \lambda\ \sum_{k=1}^{\ell}\ \|K_\lambda(s_k)x\|_H\\
&\le \lambda\ \sum_{k=1}^{\ell}\ |K(s_k)x| \le \ell\,\lambda\,\mathcal{K}(x). \qquad \square
\end{aligned}$$

For convenience, we adopt the following notation:

Definition 3.9. *For $x \in H$ and $\lambda > 0$,*

$$P_\lambda^\ell(s)x := \prod_{i=1}^{\ell} J_\lambda(s+i\lambda)x.$$

Lemma 3.10. *For $x \in H$, $\lambda > 0$, and an integer $\ell \geq 0$,*

$$|K(s+\ell\lambda)P_\lambda^\ell(s)x| \leq \left\| K_\lambda(s+\ell\lambda)P_\lambda^{\ell-1}(s)x \right\|_H .$$

Proof. Let μ be such that $0 < \mu \leq \lambda$. We have

$$\begin{aligned}
&\left\| K_\mu(s+\ell\lambda)P_\lambda^\ell(s)x \right\|_H = \frac{1}{\mu}\left\| P_\lambda^\ell(s)x - J_\mu(s+\ell\lambda)P_\lambda^\ell(s)x \right\|_H \\
=&\frac{1}{\mu} \left\| J_\lambda(s+\ell\lambda)P_\lambda^{\ell-1}(s)x - J_\mu(s+\ell\lambda)P_\lambda^\ell(s)x \right\|_H \\
=&\frac{1}{\mu} \left\| J_\mu(s+\ell\lambda)\left(\frac{\lambda-\mu}{\lambda}P_\lambda^\ell(s)x + \frac{\mu}{\lambda}P_\lambda^{\ell-1}(s)x\right) - J_\mu(s+\ell\lambda)P_\lambda^\ell(s)x \right\|_H \\
\leq&\frac{1}{\lambda} \left\| P_\lambda^{\ell-1}(s)x - P_\lambda^\ell(s)x \right\|_H = \left\| K_\lambda(s+\ell\lambda)P_\lambda^{\ell-1}(s)x \right\|_H ,
\end{aligned}$$

where we have successively used the definition of $K_\mu(s+\ell\lambda)$, the resolvent identity, and Lemma 3.2. Letting $\mu \to 0^+$ proves the lemma. □

Lemma 3.11. *For $x \in \mathfrak{D}(s)$, $\lambda > 0$, and an integer $\ell \geq 0$ such that $\ell\lambda \leq L$ for some positive constant L, there exists a positive constant $C = C(x, L, \beta)$ such that*

$$\mathcal{K}(P_\lambda^\ell(s)x) \leq C.$$

Proof. Let j be an integer such that $0 \leq j \leq \ell$, and define $a_j := |K(s+j\lambda)P_\lambda^j(s)x|$. Using Lemmas 3.10 and 3.7, we have

$$a_j \leq \left\| K_\lambda(s+j\lambda)P_\lambda^{j-1}(s)x \right\|_H \leq |K(s+j\lambda)P_\lambda^{j-1}(s)x| .$$

Applying the calculations following Definition 3.6 to the right-hand side shows that a_j is bounded above by

$$\begin{aligned}
\|\beta(s+j\lambda)-\beta(s+(j-1)\lambda)\|_H M(\|P_\lambda^{j-1}(s)x\|)(1+|K(s+(j-1)\lambda)P_\lambda^{j-1}(s)x|)& \\
+ |K(s+(j-1)\lambda)P_\lambda^{j-1}(s)x| .&
\end{aligned}$$

Lemma 3.8 and the monotonicity of M now yield

$$\begin{aligned}
a_j &\leq a_{j-1} + \|\beta(s+j\lambda) - \beta(s+(j-1)\lambda)\| \, M(\|P_\lambda^{j-1}(s)x\|) \, (1+a_{j-1}) \\
&\leq a_{j-1} + \|\beta(s+j\lambda) - \beta(s+(j-1)\lambda)\| \, M(\|x\| + L\,\mathcal{K}(x)) \, (1+a_{j-1}) ,
\end{aligned}$$

from which we obtain the recursive relation

$$a_j \leq c_j a_{j-1} + b_j, \tag{3.3}$$

where

$$b_j := \|\beta(s + j\lambda) - \beta(s + (j-1)\lambda)\| \, M(\|x\| + L\,\mathcal{K}(x)) \quad \text{and} \quad c_j = 1 + b_j.$$

The estimate

$$a_\ell \leq \Big(\prod_{i=1}^{\ell} c_i\Big) a_0 + \sum_{j=1}^{\ell} \Big(\prod_{i=j+1}^{\ell} c_i\Big) b_j \tag{3.4}$$

now follows directly from (3.3). Combining this bound with the elementary inequalities

$$\prod_{i=j+1}^{\ell} c_i = \prod_{i=j+1}^{\ell} (1 + b_i) \leq \exp\Big(\sum_{i=j+1}^{\ell} b_i\Big) \leq \exp\Big(\sum_{i=1}^{\ell} b_i\Big),$$

we have

$$a_\ell \leq \exp\Big(\sum_{i=1}^{\ell} b_i\Big)\Big(a_0 + \sum_{i=1}^{\ell} b_i\Big).$$

Note that $a_0 \leq \mathcal{K}(x)$ and that

$$\sum_{i=1}^{\ell} b_i \leq V\, M(\|x\| + L\,\mathcal{K}(x)),$$

where V denotes the variation of β. (3.4) thus yields a bound on a_ℓ, and the necessary uniform bound on $|K(t)P_\lambda^\ell(s)x|$ for $t \geq s$ then follows from the calculations following (3.2). $\square$

Finally, we recall the following result from [2]:

Lemma 3.12. *Let $n \geq m > 0$ be integers, and let α and β be positive numbers such that $\alpha + \beta = 1$. Then*

$$\sum_{j=0}^{m} \binom{n}{j} \alpha^j \beta^{n-j} (m - j) \leq \sqrt{(n\alpha - m)^2 + n\alpha\beta}\,, \quad \textit{and}$$

$$\sum_{j=m}^{n} \binom{j-1}{m-1} \alpha^m \beta^{j-m} (n - j) \leq \sqrt{\frac{m\beta}{\alpha^2} + \Big(\frac{m\beta}{\alpha} + m - n\Big)^2}\,.$$

4. The exponential formula

Theorem 4.1. *Let $t \geq s \geq 0$ be given, let x and y be elements of $\mathfrak{D}(s)$, and define $\lambda_n := \frac{t-s}{n}$ for each integer n. Then the limit*

$$\tilde{U}(t,s)x := \lim_{n\to\infty} \prod_{i=1}^{n} J_{\lambda_n}(s+i\lambda_n)x \tag{4.1}$$

exists, relative to the strong topology of H. Moreover,

(i) *$\tilde{U}(\cdot,s)x$ is Lipschitz continuous:*

$$\|\tilde{U}(t,s)x - \tilde{U}(\tau,s)x\|_H \leq c|t-\tau|.$$

(ii) *$\tilde{U}(t,s)$ is a contraction:*

$$\|\tilde{U}(t,s)x - \tilde{U}(t,s)y\|_H \leq e^{\omega(s-t)}\|x-y\|_H.$$

Proof. Let $\lambda \geq \mu > 0$, and define $a_{k,\ell} := \|P_\lambda^k(s)x - P_\mu^\ell(s)x\|_H$ for integers $k,\ell \geq 0$. For $k,\ell \geq 1$, we have

$$\begin{aligned} a_{k,\ell} &= \|J_\lambda(s+k\lambda)P_\lambda^{k-1}(s)x - J_\mu(s+\ell\mu)P_\mu^{\ell-1}(s)x\|_H \\ &\leq \|J_\lambda(s+k\lambda)P_\lambda^{k-1}(s)x - J_\mu(s+k\lambda)P_\mu^{\ell-1}(s)x\|_H \\ &+ \|J_\mu(s+k\lambda)P_\mu^{\ell-1}(s)x - J_\mu(s+\ell\mu)P_\mu^{\ell-1}(s)x\|_H, \end{aligned}$$

by the triangle inequality. Applying the resolvent identity, we see that the first summand on the right equals

$$\begin{aligned} &\Big\|J_\mu(s+k\lambda)\Big(\frac{\lambda-\mu}{\lambda}J_\lambda(s+k\lambda)P_\lambda^{k-1}(s)x + \frac{\mu}{\lambda}P_\lambda^{k-1}(s)x\Big) - J_\mu(s+k\lambda)P_\mu^{\ell-1}(s)x\Big\|_H \\ &\leq \frac{1}{1+\mu\omega}\Big\|\frac{\mu}{\lambda}P_\lambda^{k-1}(s)x + \frac{\lambda-\mu}{\lambda}J_\lambda(s+k\lambda)P_\lambda^{k-1}(s)x - P_\mu^{\ell-1}(s)x\Big\|_H \\ &= \frac{1}{1+\mu\omega}\Big\|\frac{\mu}{\lambda}P_\lambda^{k-1}(s)x + \frac{\lambda-\mu}{\lambda}P_\lambda^{k}(s)x - P_\mu^{\ell-1}(s)x\Big\|_H \\ &= \frac{1}{1+\mu\omega}\Big\|\frac{\mu}{\lambda}\Big(P_\lambda^{k-1}(s)x - P_\mu^{\ell-1}(s)x\Big) + \frac{\lambda-\mu}{\lambda}\Big(P_\lambda^{k}(s)x - P_\mu^{\ell-1}(s)x\Big)\Big\|_H \\ &\leq \frac{1}{1+\mu\omega}\Big(\frac{\mu}{\lambda}a_{k-1,\ell-1} + \frac{\lambda-\mu}{\lambda}a_{k,\ell-1}\Big). \end{aligned}$$

The second line of this chain of results follows from Lemma 3.2. For $k,\ell \geq 1$, we thus have

$$\begin{aligned} a_{k,\ell} \leq \frac{1}{1+\mu\omega}\Big(\frac{\mu}{\lambda}a_{k-1,\ell-1} + \frac{\lambda-\mu}{\lambda}a_{k,\ell-1}\Big) \\ + \Big\|J_\mu(s+\ell\mu)P_\mu^{\ell-1}(s)x - J_\mu(s+k\lambda)P_\mu^{\ell-1}(s)x\Big\|_H, \end{aligned}$$

which we write in the form

$$a_{k,\ell} \le \alpha_1 a_{k-1,\ell-1} + \beta_1 a_{k,\ell-1} + b_{k,\ell}, \tag{4.2}$$

where

$$\alpha_1 := \frac{\mu}{\lambda(1+\mu\omega)}, \quad \beta_1 := \frac{\lambda-\mu}{\lambda(1+\mu\omega)}, \quad \text{and}$$

$$b_{k,\ell} := \left\| J_\mu(s+\ell\mu)P_\mu^{\ell-1}(s)x - J_\mu(s+k\lambda)P_\mu^{\ell-1}(s)x \right\|_H .$$

Crandall and Pazy prove in the appendix of [3] that the recursive relation (4.2) implies the inequality

$$\begin{aligned} a_{m,n} \le & \sum_{i=0}^{(m-1)\wedge n} \binom{n}{i} \beta_1^{n-i}\alpha_1^i a_{m-i,0} + \sum_{i=m}^{n} \binom{i-1}{m-1} \alpha_1^m \beta_1^{i-m} a_{0,n-i} \\ & + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta_1^{j-i}\alpha_1^i b_{m-i,n-j} , \end{aligned}$$

where $\ell \wedge k := \min(\ell, k)$ and $m, n \ge 0$. Since $(1+\mu\omega)^{-1} \le 1$, we obtain the easier inequality

$$\begin{aligned} a_{m,n} \quad \le & \quad \sum_{i=0}^{(m-1)\wedge n} \binom{n}{i} \beta^{n-i}\alpha^i a_{m-i,0} + \sum_{i=m}^{n} \binom{i-1}{m-1} \alpha^m \beta^{i-m} a_{0,n-i} \\ & + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i}\alpha^i b_{m-i,n-j} , \end{aligned}$$

where $\alpha := \frac{\mu}{\lambda}$ and $\beta := \frac{\lambda-\mu}{\lambda}$. By Lemma 3.8, $a_{\ell,0} \le \ell\lambda\mathcal{K}(x)$ and $a_{0,\ell} \le \ell\mu\mathcal{K}(x)$. Using these estimates, the bound on $a_{m,n}$ becomes

$$\begin{aligned} a_{m,n} \le \lambda\,\mathcal{K}(x) & \sum_{i=0}^{(m-1)\wedge n} \binom{n}{i} \beta^{n-i}\alpha^i (m-i) \\ + \mu\mathcal{K}(x) & \sum_{i=m}^{n} \binom{i-1}{m-1} \alpha^m \beta^{i-m} (n-i) + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i}\alpha^i b_{m-i,n-j} , \end{aligned}$$

to which we apply Lemma 3.12 to obtain

$$\lambda\,\mathcal{K}(x)\left[(n\alpha-m)^2+n\alpha\beta\right]^{1/2}+\mu\,\mathcal{K}(x)\left[\frac{m\beta}{\alpha^2}+(\frac{m\beta}{\alpha}+m-n)^2\right]^{1/2}$$

$$+\sum_{j=0}^{n-1}\sum_{i=0}^{(m-1)\wedge j}\binom{j}{i}\beta^{j-i}\alpha^i b_{m-i,n-j}$$

as an upper bound for $a_{m,n}$. Substituting the values $\alpha=\frac{\mu}{\lambda}$ and $\beta=\frac{\lambda-\mu}{\lambda}$, this upper bound simplifies to

$$\mathcal{K}(x)\left(\left[\,(n\mu-m\lambda)^2+n\mu\,(\lambda-\mu)\right]^{1/2}+\left[m\lambda\,(\lambda-\mu)+(m\lambda-n\mu)^2\right]^{1/2}\right)$$
$$+\sum_{j=0}^{n-1}\sum_{i=0}^{(m-1)\wedge j}\binom{j}{i}\beta^{j-i}\alpha^i b_{m-i,n-j}\,. \tag{4.3}$$

It remains to estimate $b_{k,\ell}$ for $k,\ell\geq 0$ in order to verify the convergence of the $a_{m,n}$ as $m,n\to\infty$. By condition (3.2),

$$b_{k,\ell}=\|J_\mu(s+\ell\mu)P_\mu^{\ell-1}(s)x-J_\mu(s+k\lambda)P_\mu^{\ell-1}(s)x\|_H$$
$$\leq\mu\|\beta(s+\ell\mu)-\beta(s+k\lambda)\|_H M(\|P_\mu^{\ell-1}(s)x\|_H)\big(1+|K(s+k\lambda)P_\mu^{\ell-1}(s)x|\big).$$

Suppose that μ and ℓ satisfy $\mu(\ell-1)\leq t-s$, a condition that will hold when we choose μ below. Then Lemma 3.8 yields

$$\|P_\mu^{\ell-1}(s)x\|_H\leq\|x\|_H+(t-s)\mathcal{K}(x),$$

so that

$$M\left(\|P_\mu^{\ell-1}(s)x\|_H\right)\leq M\left(\|x\|_H+\|(t-s)\mathcal{K}(x)\|_H\right),$$

since M is monotone. By definition of $\mathcal{K}(P_\mu^{\ell-1}(s)x)$, Lemma 3.11 shows that

$$|K(s+k\lambda)P_\mu^{\ell-1}(s)x|\leq C$$

for some constant C that depends only on x, β, t, and s. Combining these two observations with the previous bound on $b_{k,\ell}$, we find that

$$b_{k,\ell}\leq C\,\mu\,\|\beta(s+\ell\mu)-\beta(s+k\lambda)\|_H$$

for a (different) constant C. Thus,

$$\sum_{j=0}^{n-1}\sum_{i=0}^{(m-1)\wedge j}\binom{j}{i}\beta^{j-i}\alpha^i b_{m-i,n-j}$$

$$\leq C\mu \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i}\alpha^i \|\beta\,(s+(n-j)\mu) - \beta\,(s+(m-i)\lambda)\,\|_H,$$

which immediately yields

$$\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i}\alpha^i b_{m-i,n-j}$$
$$\leq C\mu \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i}\alpha^i \sigma\,(\,|(n-j)\mu - (m-i)\lambda|\,),$$

where σ denotes the modulus of continuity of β on the interval $[s,t]$, i.e., the function $\sigma : [0, t-s] \to \mathbb{R}^+$ defined by

$$\sigma(r) := \sup\{\,\|\beta(\xi) - \beta(\tau)\|_H \mid s \leq \xi, \tau \leq t \quad \text{and} \quad |\xi - \tau| \leq r\,\}.$$

Since σ is subadditive,

$$\sigma\,(\,|(n-j)\mu - (m-i)\lambda|\,) \leq \sigma\,(\,|n\mu - m\lambda|\,) + \sigma\,(\,|i\lambda - j\mu|\,)\,,$$

where the parameters involved are such that all three terms make sense. This will be the case when we choose λ and μ below. It follows that

$$\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i}\alpha^i b_{m-i,n-j}$$
$$\leq C\mu \Big[n\sigma(|n\mu - m\lambda|) + \sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i}\alpha^i \sigma\,(\,|i\lambda - j\mu|\,)\Big]\,. \quad (4.4)$$

Let $\delta > 0$ be given, and decompose the preceding double sum as follows:

$$\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i}\alpha^i \sigma\,(\,|i\lambda - j\mu|\,) \;=\; \Sigma_1 + \Sigma_2,$$

where Σ_1 is the sum over indices i and j such that $|i\lambda - j\mu| < \delta$, and Σ_2 is the sum over indices i and j such that $|i\lambda - j\mu| \geq \delta$. By definition, an obvious bound on Σ_1 is $\Sigma_1 \leq n\sigma(\delta)$. As for Σ_2, we have

$$\Sigma_2 \leq \sum_{j=0}^{n-1} \sum_{i=0}^{j} \binom{j}{i} \beta^{j-i}\alpha^i \sigma(t-s)$$

$$\leq \sigma(t-s) \sum_{j=0}^{n-1} \sum_{i=0}^{j} \binom{j}{i} \beta^{j-i} \alpha^i \frac{(i\lambda - j\mu)^2}{\delta^2},$$

$$\text{since} \quad \frac{(i\lambda - j\mu)^2}{\delta^2} \geq 1,$$

$$= \frac{\sigma(t-s)}{\delta^2} n(n-1) \left(\lambda\mu - \mu^2\right) \leq \frac{\sigma(t-s)}{\delta^2} n^2 \left(\lambda\mu - \mu^2\right).$$

Combining the estimates on Σ_1 and Σ_2 with inequality (4.4), we have

$$\sum_{j=0}^{n-1} \sum_{i=0}^{(m-1)\wedge j} \binom{j}{i} \beta^{j-i} \alpha^i b_{m-i,n-j}$$

$$\leq C\mu\Big[n\sigma(|n\mu - m\lambda|) + n\sigma(\delta) + \frac{\sigma(t-s)}{\delta^2} n^2 \mu(\lambda - \mu)\Big],$$

and concatenating this bound with inequality (4.3) yields

$$\mathcal{K}(x)\Big(\Big[\,(n\mu - m\lambda)^2 + n\mu\,(\lambda-\mu)\Big]^{1/2} + \Big[m\lambda\,(\lambda-\mu) + (m\lambda - n\mu)^2\Big]^{1/2}\Big)$$
$$+C\mu\Big[n\sigma(|n\mu - m\lambda|) + n\sigma(\delta) + \frac{\sigma(t-s)}{\delta^2} n^2 \mu(\lambda-\mu)\Big] \tag{4.5}$$

as an upper bound on $a_{m,n}$. Setting $\lambda = \frac{t-s}{m}$ and $\mu = \frac{t-s}{n}$, (4.5) becomes

$$a_{m,n} \quad \leq \quad 2\mathcal{K}(x)(t-s)\sqrt{\tfrac{1}{m} - \tfrac{1}{n}} + C(t-s)\left[\sigma(\delta) + \tfrac{(t-s)^2}{\delta^2}(\tfrac{1}{m} - \tfrac{1}{n})\right],$$

where we have changed the constant C to absorb the constant $\sigma(t-s)$ and used the fact that $\sigma(0) = 0$. We now choose $\delta = \sqrt[4]{\frac{1}{m} - \frac{1}{n}}$ to obtain

$$a_{m,n} \leq 2\mathcal{K}(x)(t-s)\sqrt{\frac{1}{m} - \frac{1}{n}} + C(t-s)\Big[\sigma(\sqrt[4]{\frac{1}{m} - \frac{1}{n}}) + (t-s)^2\sqrt{\frac{1}{m} - \frac{1}{n}}\Big],$$

which goes to 0 as $m, n \to \infty$. Consequently, $\{P^n_{\lambda_n}(s)x\}$ is a Cauchy sequence in H, and the limit

$$\tilde{U}(t,s)x := \lim_{n\to\infty} \prod_{i=1}^{n} J_{\lambda_n}(s + i\lambda_n)x$$

exists, relative to the strong topology of H. This proves the exponential formula (4.1).

Lemma 3.2(i) shows that $(1 + \frac{\omega(t-s)}{n})^{-n}$ is the Lipschitz constant for $P^n_{\lambda_n}(s)$, from which it follows that

$$\|\tilde{U}(t,s)x - \tilde{U}(t,s)y\|_H \leq e^{\omega(s-t)}\|x-y\|_H,$$

after letting $n \to \infty$. $\tilde{U}(t,s)$ is therefore a contraction.

Let $t \geq \tau \geq s$ and $\lambda > 0$ be given, and define $n := \lfloor \frac{t}{\lambda} \rfloor$ and $m := \lfloor \frac{\tau}{\lambda} \rfloor$. By Lemma 3.8,

$$\begin{aligned} \|P_\lambda^n(s)x - P_\lambda^m(s)x\|_H &= \|\prod_{i=m+1}^{n} J_\lambda(s+i\lambda)P_\lambda^m(s)x - P_\lambda^m(s)x\|_H \\ &\leq \lambda(n-m)\mathcal{K}(P_\lambda^m(s)x) \leq C\lambda(n-m), \end{aligned}$$

for some constant C by Lemma 3.11. Sending λ to zero shows that

$$\|\tilde{U}(t,s)x - \tilde{U}(\tau,s)x\|_H \leq C|t-\tau|,$$

proving the Lipschitz continuity of $\tilde{U}(\cdot,s)x$. □

The following corollary extends the map $\tilde{U} : [s,\infty) \times \mathfrak{D}(s) \to H$ to $U : [s,\infty) \times \overline{D(\Phi(s))}^{\|\cdot\|_H} \to H$. We will see that $U(t,s)$ is a contraction for each $t \geq s$ and that $U(\cdot,s)x$ is continuous for $x \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$, but $U(\cdot,s)x$ lacks the Lipschitz continuity of $\tilde{U}$. This distinction between $\tilde{U}$ and U is precisely the reason that we obtain strong solutions for smooth initial data but only have weak solutions for arbitrary initial data.

Corollary 4.2. *The map $\tilde{U} : [s,\infty) \times \mathfrak{D}(s) \to H$ constructed in Theorem* 4.1 *extends uniquely to $U : [s,\infty) \times \overline{D(\Phi(s))}^{\|\cdot\|_H} \to H$. This extension has the following properties:*

(i) *For each $x \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$, $U(\cdot,s)x$ is continuous.*

(ii) *For $x, y \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$ and $t \geq s$, $U(t,s)$ is a contraction:*

$$\|U(t,s)x - U(t,s)y\|_H \leq e^{\omega(s-t)}\|x-y\|_H.$$

Proof. Let $x, y \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$ be given, and let $\{x_n\}$ and $\{y_n\}$ be approximating sequences for x and y, respectively, as in (2.4). As $\{x_n\}$ converges strongly in V^*, $\{x_n\}$ is a Cauchy sequence in H. It follows that $\{\tilde{U}(t,s)x_n\}$ is a Cauchy sequence in H, since

$$\|\tilde{U}(t,s)x_n - \tilde{U}(t,s)x_m\|_H \leq \|x_n - x_m\|_H.$$

We thus define $U(t,s)x$ by the strong H-limit

$$U(t,s)x := \lim_{n\to\infty} \tilde{U}(t,s)x_n.$$

This extension is clearly unique.

Let $\varepsilon > 0$ be given. For $t, \tau \geq s$, we have

$$\|U(t,s)x - U(\tau,s)x\|_H \leq \|U(t,s)x - \tilde{U}(t,s)x_n\|_H$$

$$+ \|\tilde{U}(t,s)x_n - \tilde{U}(\tau,s)x_n\|_H \|\tilde{U}(\tau,s)x_n - U(\tau,s)x\|_H$$
$$\le \|U(t,s)x - \tilde{U}(t,s)x_n\|_H + C|t-\tau| + \|\tilde{U}(\tau,s)x_n - U(\tau,s)x\|_H,$$

where C is the Lipschitz constant of $\tilde{U}$. The right-hand side will clearly be less than ε for n sufficiently large and $|t-\tau|$ sufficiently small, verifying the continuity of $U(\cdot,s)x$.

A similar calculation shows that $U(t,s)$ is a contraction for $t \ge s$:

$$\|U(t,s)x - U(t,s)y\|_H \le \|U(t,s)x - \tilde{U}(t,s)x_n\|_H$$
$$+ \|\tilde{U}(t,s)x_n - \tilde{U}(t,s)y_n\|_H + \|\tilde{U}(t,s)y_n - U(t,s)y\|_H$$
$$\le \|U(t,s)x - \tilde{U}(t,s)x_n\|_H + e^{\omega(s-t)}\|x_n - y_n\|_H + \|\tilde{U}(t,s)y_n - U(t,s)y\|_H,$$

since $\tilde{U}(t,s)$ is a contraction, and letting $n \to \infty$ shows that

$$\|U(t,s)x - U(t,s)y\|_H \le e^{\omega(s-t)}\|x-y\|_H.$$

5. Strong and weak solutions

This section develops results which parallel those of Section 4 of [5]. For $u_s \in \mathfrak{D}(s)$, we analyze the approximations $P^n_{\lambda_n}(s)u_s$ and the corresponding difference quotients in order to prove Theorems 5.5 and 5.6 below. These results establish the existence of a unique strong or weak solution, respectively, of the parabolic variational inequality associated to $A(\cdot)$ and $\Phi(\cdot)$, depending on the smoothness of the initial datum u_s. A direct consequence of uniqueness is the fact that $\{U(t,s) : t \ge s\}$ is an evolution family on $D(\Phi(s))$.

Lemma 5.1. *Let $u_s \in \mathfrak{D}(s)$. The function $u : [s,\infty) \to H$ defined by $u(t) := U(t,s)u_s$ is differentiable almost everywhere and satisfies $u' \in L^p(s,T;H)$ for $1 \le p \le \infty$ and any $T > s$.*

Proof. Theorem 4.1 shows that $u(t)$ is Lipschitz continuous in t. The lemma then follows from basic results for vector-valued functions (cf. Section 1.4 of [1]). □

It follows (cf. [1]) that, for almost every $t \ge s$ and $u_s \in \mathfrak{D}(s)$,

$$u(t) = u_s + \int_s^t u'(\tau)\, d\tau\,. \tag{5.1}$$

Lemma 5.2. *Let $u_s \in \mathfrak{D}(s)$. For $t \ge s$, define $u(t) := U(t,s)u_s$ and $\lambda_n := \frac{t-s}{n}$. For almost every $t > s$, the sequence $\{\frac{P^n_{\lambda_n}(s)u_s - P^{n-1}_{\lambda_n}(s)u_s}{\lambda_n}\}$ has a subsequence which converges weakly in H to $u'(t)$.*

Proof. Let t^* be a point where $u(t)$ is differentiable. For each n, define $\lambda_n^* := \frac{t^*-s}{n}$, and partition the interval $[s, 2t^*-s]$ with the points $t_i := s+i\lambda_n^*$, for $i = 0, 1, \ldots, 2n$. Associate to this partition the piecewise-linear function

$$u_n(t) := \begin{cases} P^{i-1}_{\lambda_n^*}(s)u_s + \left(\frac{t-t_{i-1}}{\lambda_n^*}\right)\left(P^{i}_{\lambda_n^*}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\right), \\ \quad \text{for} \quad t_{i-1} \le t \le t_i,\ 1 \le i \le n-1, \\ P^{n-1}_{\lambda_n^*}(s)u_s + \left(\frac{t-t_{n-1}}{\lambda_n^*}\right)\left(P^{n}_{\lambda_n^*}(s)u_s - P^{n-1}_{\lambda_n^*}(s)u_s\right), \\ \quad \text{for} \quad t_{n-1} \le t \le t_{n+1}, \\ P^{i-1}_{\lambda_n^*}(s)u_s + \left(\frac{t-t_{i-1}}{\lambda_n^*}\right)\left(P^{i}_{\lambda_n^*}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\right), \\ \quad \text{for} \quad t_{i-1} \le t \le t_i,\ n+2 \le i \le 2n. \end{cases}$$

Note that t^* is the midpoint of the unique interval of width $2\lambda_n^*$ in the definition of $u_n(t)$, so that

$$u_n'(t^*) = \frac{P^{n}_{\lambda_n^*}(s)u_s - P^{n-1}_{\lambda_n^*}(s)u_s}{\lambda_n^*}.$$

Observe that

$$\begin{aligned} \|P^{i}_{\lambda_n^*}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\|_H &= \|J_{\lambda_n^*}(s+i\lambda_n^*)P^{i-1}_{\lambda_n^*}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\|_H \\ &\le \lambda_n^*\mathcal{K}(P^{i-1}_{\lambda_n^*}(s)u_s), \end{aligned}$$

by Lemma 3.8. Lemma 3.11 (with $L = 2t^*-s$) shows that $\mathcal{K}(P^{i-1}_{\lambda_n^*}(s)u_s)$ is bounded above by some constant C that is independent of i and n, and we conclude that

$$\left\|\frac{P^{i}_{\lambda_n^*}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s}{\lambda_n^*}\right\|_H \le C, \tag{5.2}$$

for all indices i. The sequence $\{u_n'(t)\}$ is therefore bounded for each $t \in [s, 2t^*-s]$, and, after passing to a subsequence and relabeling, we have $u_n'(t) \rightharpoonup w(t)$ as $n \to \infty$, for some $w(t) \in H$. As observed in the proof of Lemma 4.2 of [5], w is integrable.

To show that this sequence $\{u_n\}$ of piecewise-linear functions converges to u in $C(s, 2t^*-s; H)$, we consider several cases. First, suppose that $t \in [t_{i-1}, t_i]$ for some $i \le n-1$, so that $t < t^*$. We have

$$\begin{aligned} &\|u_n(t) - u(t)\|_H \le \|u(t) - P^{n}_{\lambda_n}(s)u_s\|_H \\ &\quad + \left\|P^{n}_{\lambda_n}(s)u_s - \left\{P^{i-1}_{\lambda_n^*}(s)u_s + \left(\frac{t-t_{i-1}}{\lambda_n^*}\right)\left(P^{i}_{\lambda_n^*}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\right)\right\}\right\|_H \\ &\le \|P^{n}_{\lambda_n}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\|_H + \|P^{i}_{\lambda_n^*}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\|_H + \|u(t) - P^{n}_{\lambda_n}(s)u_s\|_H. \end{aligned}$$

Putting $\mu = \lambda_n$, $\lambda = \lambda_n^*$, and $m = i-1$ in (4.5) yields the estimate

$$\begin{aligned}
&\left\| P^n_{\lambda_n}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s \right\|_H \\
&\le \mathcal{K}(u_s)\Big[\Big(t-s-(i-1)(\frac{t^*-s}{n})\Big)^2 + (t-s)(\frac{t^*-t}{n})\Big]^{1/2} \\
&+ C(\frac{t-s}{n})\Big[n\sigma(|t-s-(i-1)(\frac{t^*-s}{n})|) \\
&\qquad + n\sigma(\delta) + \frac{\sigma(t^*-s)}{\delta^2}n^2(\frac{t-s}{n})(\frac{t^*-t}{n})\Big].
\end{aligned}$$

Note that $t-s-(i-1)(\frac{t^*-s}{n}) \le \frac{t^*-s}{n}$, and, since $s \le t \le t^*$, we have

$$(t-s)(\frac{t^*-t}{n}) \le \frac{(t^*-s)^2}{4n}.$$

Using these facts and letting $\delta = \frac{t^*-s}{\sqrt[4]{n}}$, we obtain

$$\begin{aligned}
&\|P^n_{\lambda_n}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\|_H \\
&\le \mathcal{K}(u_s)(t^*-s)\sqrt{\frac{1}{n^2}+\frac{1}{4n}} + C(t-s)\Big[\sigma(\frac{t^*-s}{n}) + \sigma(\frac{t^*-s}{\sqrt[4]{n}}) + \frac{\sigma(t^*-s)}{4\sqrt{n}}\Big] \\
&\le C\Big[\frac{1}{\sqrt{n}} + \sigma(\frac{t^*-s}{\sqrt[4]{n}})\Big],
\end{aligned}$$

for n sufficiently large and a new constant C which is independent of n.

To bound the other two terms, Lemma 3.8 shows that

$$\|P^i_{\lambda_n^*}(s)u_s - P^{i-1}_{\lambda_n^*}(s)u_s\|_H \le \frac{C(t^*-s)}{n} \le \frac{C(t^*-s)}{\sqrt{n}},$$

where C is the constant in (5.2) and n is sufficiently large, and the proof of Theorem 4.1 shows that

$$\|u(t) - P^n_{\lambda_n}(s)u_s\|_H \le C(t^*-s)\Big[\frac{1}{\sqrt{n}} + \sigma(\frac{1}{\sqrt[4]{n}})\Big],$$

for a constant C which does not depend on n. Combining these bounds therefore yields

$$\|u_n(t) - u(t)\|_H \le C_1\Big[\frac{1}{\sqrt{n}} + \sigma(\frac{C_2}{\sqrt[4]{n}})\Big], \quad \text{for} \quad t \in [t_{i-1}, t_i],\ i \le n-1,$$

where C_1 and C_2 are positive constants that do not depend on n or t.

As in the proof of Lemma 4.2 of [5], the cases in which $t \in [t_{i-1}, t_i]$ for $i \ge n-2$ or in which $t \in [t_{n-1}, t_{n+1}]$ proceed in an analogous fashion, with only minor changes. We finally obtain the uniform bound

$$\|u(t) - u_n(t)\|_H \leq C_1\Big[\frac{1}{\sqrt{n}} + \sigma(\frac{C_2}{\sqrt[4]{n}})\Big], \quad \text{for} \quad t \in [s, 2t^* - s],$$

for constants C_1 and C_2 which are independent of n and t. It follows that u_n converges to u in $C(s, 2t^* - s; H)$.

For each n, the identity

$$\langle\, u_n(t)\,,\, v\,\rangle = \langle\, u_s\,,\, v\,\rangle + \int_\tau^t \langle\, u_n'(\tau)\,,\, v\,\rangle\, d\tau \tag{5.3}$$

holds for all $v \in H$. By construction, $\{u_n'\}$ is a sequence of bounded, integrable functions which converges weakly to the integrable function w. Consequently, by dominated convergence, letting $n \to \infty$ in (5.3) yields

$$\langle\, u(t)\,,\, v\,\rangle = \langle\, u_s\,,\, v\,\rangle + \int_\tau^t \langle\, w(\tau)\,,\, v\,\rangle\, d\tau. \tag{5.4}$$

It then follows from (5.1) that $u_n'(\tau) \rightharpoonup u'(\tau)$ for almost every $\tau \in (s, 2t^* - s)$. In particular, this holds for $\tau = t^*$, which proves that

$$\frac{P^n_{\lambda^*_n}(s)u_s - P^{n-1}_{\lambda^*_n}(s)u_s}{\lambda^*_n} \rightharpoonup u'(t^*). \tag{5.5}$$

This completes the proof of the lemma, as (5.5) holds for every point where u is differentiable. □

Let $t \geq s$ and $\lambda_n = \frac{t-s}{n}$ as above. As $\{P^n_{\lambda_n}(s)u_s\}$ is a Cauchy sequence in H, this sequence is bounded in both H and V^*. By definition of the resolvent, this is a sequence in $D(\Phi(t))$. We show next that this sequence is bounded in V^{**}, thus obtaining the following result.

Lemma 5.3. *Let $u_s \in \mathfrak{D}(s)$ and $u(t) = U(t,s)u_s$. For each $t > 0$, the sequence $\{\, P^n_{\lambda_n}(s)u_s\,\}$ converges to $u(t)$ in the weak-* topology of V^{**}.*

Proof. For each integer n, $P^n_{\lambda_n}(s)u_s$ is the unique solution of

$$\frac{1}{\lambda_n}\langle\, P^n_{\lambda_n}(s)u_s - P^{n-1}_{\lambda_n}(s)u_s\,,\, v - P^n_{\lambda_n}(s)u_s\,\rangle + \langle\, AP^n_{\lambda_n}(s)u_s\,,\, v - P^n_{\lambda_n}(s)u_s\,\rangle$$
$$+ \Phi(t)(v) - \Phi(t)(P^n_{\lambda_n}(s)u_s) \geq 0,\ \forall\, v \in V^{**},$$

which we rewrite as

$$\langle\, AP^n_{\lambda_n}(s)u_s\,,\, P^n_{\lambda_n}(s)u_s - v\,\rangle + \Phi(t)(P^n_{\lambda_n}(s)u_s) - \Phi(t)(v)$$
$$\leq \frac{1}{\lambda_n}\langle\, P^n_{\lambda_n}(s)u_s - P^{n-1}_{\lambda_n}(s)u_s\,,\, v - P^n_{\lambda_n}(s)u_s\,\rangle,\ \forall\, v \in V^{**}.$$

From the proof of Lemma 5.2 and the embedding of V^{**} into H, we have

$$\langle AP^n_{\lambda_n}(s)u_s\,,\,P^n_{\lambda_n}(s)u_s - v\,\rangle + \Phi(t)(P^n_{\lambda_n}(s)u_s) - \Phi(t)(v) \le c\|v - P^n_{\lambda_n}(s)u_s\|_{V^{**}},\ \forall\, v \in V^{**},$$

for a constant $c = c(u_s)$. It now follows from coercivity that $\{P^n_{\lambda_n}(s)u_s\}$ is bounded in V^{**}, hence weak-* precompact by the Banach–Alaoglu theorem. Let $\tilde{u}(t) \in V^{**}$ be the weak-* limit of a convergent subsequence $\{P^{n_i}_{\lambda_{n_i}}(s)u_s\}$ of the sequence $\{P^n_{\lambda_n}(s)u_s\}$. We have

$$\langle\, u(t) - \tilde{u}(t)\,,\, u(t) - P^{n_i}_{\lambda_{n_i}}(s)u_s\,\rangle \le \|u(t) - \tilde{u}(t)\|_H \|u(t) - P^{n_i}_{\lambda_{n_i}}(s)u_s\|_H,$$

from which it follows upon letting $n \to \infty$ that

$$\|u(t) - \tilde{u}(t)\|^2_H \le 0,$$

since $u(t)$ is the strong limit in H of the original sequence. Consequently, $\tilde{u}(t) = u(t)$. As this holds for any convergent subsequence, the sequence $\{P^n_{\lambda_n}(s)u_s\}$ converges in the weak-* topology to $u(t)$. □

Since $D(\Phi(t))$ is weak-* closed, it follows that $u(t) = U(t,s)u_s \in D(\Phi(t))$ for $u_s \in \mathfrak{D}(s)$. Furthermore, we conclude from Lemma 5.3 that $U(t,s)u_s$ belongs to $\overline{D(\Phi(s))}^{\|\cdot\|_H}$ for *any* $u_s \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$. Thus, for $t, \tau \ge s$ and $u_s \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$, the expression $U(t,\tau)\,(U(\tau,s)u_s)$ makes sense; we will see in Corollary 5.7 that this expression equals $U(t,s)u_s$.

The next lemma yields the uniqueness of weak solutions (and hence of strong solutions as well). The special case $\tau = s$ reflects the fact that $U(t,s)$ is a contraction, although we have not yet verified that solutions are given by $U(t,s)u_s$.

Lemma 5.4. *Let u_1 and u_2 be weak solutions of the parabolic variational inequality associated to $A(\cdot)$ and $\Phi(\cdot)$ with the prescribed initial values $u_{1,s}$ and $u_{2,s}$ in $\overline{D(\Phi(s))}^{\|\cdot\|_H}$, respectively. Then the inequality*

$$\|u_1(t) - u_2(t)\|_H \le e^{\omega(\tau - t)}\|u_1(\tau) - u_2(\tau)\|_H \tag{5.6}$$

holds for $t \ge \tau \ge s$.

Proof. Suppose first that u_1 and u_2 are strong solutions. For almost every $\xi \in (\tau, t)$, the inequalities

$$\langle\, u_1'(\xi)\,,\, v - u_1(\xi)\,\rangle + \langle\, A(\xi)u_1(\xi)\,,\, v - u_1(\xi)\,\rangle + \Phi(\xi)(v) - \Phi(\xi)(u_1(\xi)) \ge 0,$$

$$\langle\, u_2'(\xi)\,,\, v - u_2(\xi)\,\rangle + \langle\, A(\xi)u_2(\xi)\,,\, v - u_2(\xi)\,\rangle + \Phi(\xi)(v) - \Phi(\xi)(u_2(\xi)) \ge 0$$

therefore hold for all $v \in V^{**}$. Inserting $v = u_2(\xi)$ in the first inequality and $v = u_1(\xi)$ in the second inequality and then adding yields

$$\langle u_2'(\xi) - u_1'(\xi)\, ,\, u_1(\xi) - u_2(\xi) \rangle + \langle A(\xi)u_2(\xi) - A(\xi)u_1(\xi)\, ,\, u_1(\xi) - u_2(\xi) \rangle \geq 0,$$

from which we conclude that

$$\frac{1}{2}\frac{d}{d\xi}\|u_1(\xi) - u_2(\xi)\|_H^2 + \omega\|u_1(\xi) - u_2(\xi)\|_H^2 \leq 0,$$

since $A(\xi)$ is monotone. Multiplying through by $2e^{2\omega\xi}$, we have

$$\frac{d}{d\xi}\left(e^{2\omega\xi}\|u_1(\xi) - u_2(\xi)\|_H^2\right) \leq 0,$$

which we integrate from τ to t to obtain

$$e^{2\omega t}\|u_1(t) - u_2(t)\|_H^2 - e^{2\omega\tau}\|u_1(\tau) - u_2(\tau)\|_H^2 \leq 0.$$

It follows immediately that

$$\|u_1(t) - u_2(t)\|_H \leq e^{\omega(\tau - t)}\|u_1(\tau) - u_2(\tau)\|_H.$$

For the general case in which u_1 and u_2 are weak solutions, let u_1^n and u_2^n be sequences of strong solutions with initial values $u_{1,s}^n$ and $u_{2,s}^n$, respectively, such that

$$u_{i,s}^n \to u_{i,s} \text{ in } H \quad \text{and} \quad u_i^n \to u_i \text{ in } C(s,T;H),\ i = 1,2.$$

We have

$$\begin{aligned}
&\|u_1(t) - u_2(t)\|_H \\
&\leq \|u_1(t) - u_1^n(t)\|_H + \|u_1^n(t) - u_2^n(t)\|_H + \|u_2^n(t) - u_2(t)\|_H \\
&\leq \|u_1 - u_1^n\|_{C(s,T;H)} + e^{\omega(\tau - t)}\|u_1^n(\tau) - u_2^n(\tau)\|_H + \|u_2^n - u_2\|_{C(s,T;H)},
\end{aligned}$$

since (5.6) holds for the strong solutions u_1^n and u_2^n. Letting $n \to \infty$ then yields the desired bound (5.6) in the general case. □

Theorem 5.5. *Let $u_s \in \mathfrak{D}(s)$. The function $u(t) := U(t,s)u_s$ is the unique strong solution of the parabolic variational inequality associated to $A(\cdot)$ and $\Phi(\cdot)$ with the initial value u_s.*

Proof. Uniqueness follows from Lemma 5.4.

Let $t \geq s$ be a point where u is differentiable, and let v be an arbitrary element of V^{**}. By definition of the resolvent, $P_{\lambda_n}^n(s)u_s$ satisfies

$$\begin{aligned}
\frac{1}{\lambda_n}\langle P_{\lambda_n}^n(s)u_s - P_{\lambda_n}^{n-1}(s)u_s\, ,\, v - P_{\lambda_n}^n(s)u_s \rangle + \langle A(t)P_{\lambda_n}^n(s)u_s\, ,\, v - P_{\lambda_n}^n(s)u_s \rangle \\
+ \Phi(t)(v) - \Phi(t)(P_{\lambda_n}^n(s)u_s) \geq 0, \qquad (5.7)
\end{aligned}$$

which we rewrite in the form

$$\begin{aligned}&\langle\, A(t)P^n_{\lambda_n}(s)u_s\,,\, P^n_{\lambda_n}(s)u_s - u(t)\,\rangle\\ &\le \frac{1}{\lambda_n}\langle\, P^n_{\lambda_n}(s)u_s - P^{n-1}_{\lambda_n}(s)u_s\,,\, v - P^n_{\lambda_n}(s)u_s\,\rangle\\ &+ \langle\, A(t)P^n_{\lambda_n}(s)u_s\,,\, v-u(t)\,\rangle + \Phi(t)(v) - \Phi(t)(P^n_{\lambda_n}(s)u_s).\end{aligned}$$

As $\{P^n_{\lambda_n}(s)u_s\}$ is bounded in V^{**}, $\{A(t)P^n_{\lambda_n}(s)u_s\}$ is bounded in V^*. We thus obtain

$$\begin{aligned}&\langle\, A(t)P^n_{\lambda_n}(s)u_s\,,\, P^n_{\lambda_n}(s)u_s - u(t)\,\rangle\\ &\le \frac{1}{\lambda_n}\langle\, P^n_{\lambda_n}(s)u_s - P^{n-1}_{\lambda_n}(s)u_s\,,\, v - P^n_{\lambda_n}(s)u_s\,\rangle\\ &+ c\|v-u(t)\|_{V^{**}} + \Phi(t)(v) - \Phi(t)(P^n_{\lambda_n}(s)u_s),\end{aligned} \tag{5.8}$$

for some constant $c > 0$, and taking the upper limit of both sides of (5.8) yields

$$\begin{aligned}&\limsup_{n\to\infty}\langle\, A(t)P^n_{\lambda_n}(s)u_s\,,\, P^n_{\lambda_n}(s)u_s - u(t)\,\rangle\\ &\le \langle\, u'(t)\,,\, v-u(t)\,\rangle + c\|v-u(t)\|_{V^{**}} + \Phi(t)(v) - \Phi(t)(u(t)),\end{aligned}$$

by the previously established convergence results and the weak-* lower semicontinuity of $\Phi(t)$. We now let $v = u(t)$ to obtain

$$\limsup_{n\to\infty}\langle\, A(t)P^n_{\lambda_n}(s)u_s\,,\, P^n_{\lambda_n}(s)u_s - u(t)\,\rangle \le 0. \tag{5.9}$$

Let v again represent an arbitrary element of V^{**}. Since $P^n_{\lambda_n}(s)u_s \overset{*}{\rightharpoonup} u(t)$ in V^{**}, it follows from (5.9) and the pseudomonotonicity of $A(t)$ that

$$\langle\, A(t)u(t)\,,\, u(t)-v\,\rangle \le \liminf_{n\to\infty}\langle\, A(t)P^n_{\lambda_n}(s)u_s\,,\, P^n_{\lambda_n}(s)u_s - v\,\rangle. \tag{5.10}$$

However, by (5.7), we have

$$\begin{aligned}&\langle\, A(t)P^n_{\lambda_n}(s)u_s\,,\, P^n_{\lambda_n}(s)u_s - v\,\rangle\\ &\le \frac{1}{\lambda_n}\langle\, P^n_{\lambda_n}(s)u_s - P^{n-1}_{\lambda_n}(s)u_s\,,\, v - P^n_{\lambda_n}(s)u_s\,\rangle + \Phi(t)(v) - \Phi(t)(P^n_{\lambda_n}(s)u_s),\end{aligned}$$

from which we see that

$$\limsup_{n\to\infty}\langle\, A(t)P^n_{\lambda_n}(s)u_s\,,\, P^n_{\lambda_n}(s)u_s - v\,\rangle \le \langle\, u'(t)\,,\, v-u(t)\,\rangle + \Phi(v) - \Phi(u(t)). \tag{5.11}$$

Concatenating (5.10) and (5.11), we conclude that

$$\langle\, u'(t)\,,\, v-u(t)\,\rangle + \langle\, A(t)u(t)\,,\, v-u(t)\,\rangle + \Phi(t)(v) - \Phi(t)(u(t)) \ge 0.$$

This holds for almost every $t \geq s$ and for any $v \in V^{**}$, completing the proof. $\square$

The existence and uniqueness of weak solutions corresponding to arbitrary initial data in $\overline{D(\Phi(s))}^{\|\cdot\|_H}$ now follow easily.

Corollary 5.6. *Let* $u_s \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$. *The function* $u(t) := U(t,s)u_s$ *is the unique weak solution of the parabolic variational inequality associated to* $A(\cdot)$ *and* $\Phi(\cdot)$ *with the initial value* u_s.

Proof. Corollary 4.2 shows that u is *a* weak solution, and Lemma 5.4 verifies its uniqueness. $\square$

Corollary 5.6 reveals that $\{U(t,s) : t \geq 0\}$ is an evolution family of (nonlinear) contractions on $\overline{D(\Phi(s))}^{\|\cdot\|_H}$, as intimated earlier.

Corollary 5.7. *For* $u_s \in \overline{D(\Phi(s))}^{\|\cdot\|_H}$ *and all* $t \geq \tau \geq s$, *the evolution identity,* $U(t,s)u_s = U(t,\tau)\left(U(\tau,s)u_s\right)$, *holds.*

Proof. For each $\tau \geq s$, we simply observe that the functions $u_1(t) := U(t,s)u_s$ and $u_2(t) = U(t,\tau)\left(U(\tau,s)u_s\right)$ for $t \geq \tau$, are weak solutions which agree at the initial time $t = \tau$. $\square$

6. Applications

6.1. **Quasiautonomous problems in reflexive spaces.** Let W be a reflexive Banach space which embeds continuously and densely into the Hilbert space H, let $A : W \to W^*$ be a monotone, hemicontinuous, and bounded operator, let $\Phi : W \to \mathbb{R} \cup \{\infty\}$ be convex, lower semicontinuous, and proper, and suppose that A and Φ satisfy the coercivity condition (2.2). In addition, suppose that the approximation condition (2.4) holds. The applications in [5] provide several concrete examples in which these assumptions hold.

Under these hypotheses, Theorem 4.5 of [5] shows that, given $f \in W^*$ and $u_0 \in \mathfrak{D}$, there exists a unique strong solution $u \in L^p(0,T;W) \cap C^{0,1}(0,T;H)$ such that $u(0) = u_0$ and such that the inequality

$$\langle u'(t) + Au(t) \, , \, v - u(t) \rangle + \Phi(v) - \Phi(u(t)) \geq \langle f \, , \, v - u(t) \rangle$$

holds for all $v \in W$ and for almost every $t \in (0,T)$. If $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$, then there exists a unique weak solution of the parabolic variational inequality associated to A and Φ.

This section applies the results of the present paper to problems in which the perturbation f depends on time. To begin, let $f \in W^{1,1}(0,T;H)$ be given, and define the quasiautonomous operators $A(t) : W \to W^*$ for $0 \leq t \leq T$ by

$$A(t)u := Au - f(t). \tag{6.1}$$

We have the following preliminary result:

Proposition 6.1. *If $u_0 \in \mathfrak{D}$, then there exists a unique strong solution of the parabolic variational inequality associated to $A(\cdot)$ and Φ with the initial value u_0. If $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$, then there exists a unique weak solution of the parabolic variational inequality associated to $A(\cdot)$ and Φ with the initial value u_0.*

Proof. We can apply Theorem 5.5 and Corollary 5.6 directly once we verify that the resolvent $J_\lambda(t)$ corresponding to $A(t)$ and Φ satisfies condition (3.2). For $\lambda > 0$, $x \in H$, and times $t, \tau \in (0, T)$, let $u(t) := J_\lambda(t)x$ and $u(\tau) := J_\lambda(\tau)x$. For any $v \in W$, the inequalities

$$\frac{1}{\lambda}\langle\, u(t) - x\,,\, v - u(t)\,\rangle + \langle\, A(t)u(t)\,,\, v - u(t)\,\rangle + \Phi(v) - \Phi(u(t)) \geq 0,$$

$$\frac{1}{\lambda}\langle\, u(\tau)x - x\,,\, v - u(\tau)\,\rangle + \langle\, A(\tau)u(\tau)\,,\, v - u(\tau)\,\rangle + \Phi(v) - \Phi(u(\tau)) \geq 0$$

hold by definition of the resolvent. Substituting $v = u(\tau)$ into the first inequality and $v = u(t)$ into the second and adding, we have

$$\frac{1}{\lambda}\langle\, u(\tau) - u(t)\,,\, u(t) - u(\tau)\,\rangle + \langle\, A(\tau)u(\tau) - A(t)u(t)\,,\, u(t) - u(\tau)\,\rangle \geq 0,$$

which is equivalent to

$$\frac{1}{\lambda}\langle\, u(\tau) - u(t)\,,\, u(t) - u(\tau)\,\rangle + \langle\, Au(\tau) - f(\tau) - Au(t) + f(t)\,,\, u(t) - u(\tau)\,\rangle \geq 0,$$

by definition of $A(\cdot)$. Therefore,

$$\frac{1}{\lambda}\|u(t) - u(\tau)\|_H^2 + \langle\, Au(t) - Au(\tau)\,,\, u(t) - u(\tau)\,\rangle \leq \langle\, f(t) - f(\tau)\,,\, u(t) - u(\tau)\,\rangle,$$

from which it follows that

$$\frac{1}{\lambda}\|u(t) - u(\tau)\|_H^2 \leq \langle\, f(t) - f(\tau)\,,\, u(t) - u(\tau)\,\rangle,$$

by the monotonicity of A. Thus, by the definitions of $u(t)$ and $u(\tau)$,

$$\|J_\lambda(t)x - J_\lambda(\tau)x\|_H \leq \lambda\,\|f(t) - f(\tau)\|_H.$$

Since $f \in W^{1,1}(0, T; H)$, f is a continuous function of bounded variation, and condition (3.2) holds with $\beta = f$ and $M \equiv 1$. Theorem 5.5 and Corollary 5.6 then complete the proof. □

Further results depend on the next lemma, which provides an estimate similar to that of Lemma 5.4.

Lemma 6.2. *Let $u_{1,0}$, $u_{2,0} \in \overline{D(\Phi)}^{\|\cdot\|_H}$ and $f_1, f_2 \in L^q(0,T;W^*)$ be given, and define the operators $A_i(t) : W \to W^*$ by $A_i(t)u = Au - f_i(t)$ for $i = 1, 2$. If u_i is a weak solution of the parabolic variational inequality associated to $A_i(\cdot)$ and Φ with initial value $u_{i,0}$, then the inequality*

$$\|u_1(t) - u_2(t)\|_H^2 \leq \|u_{1,0} - u_{2,0}\|_H^2 + 2\Big(\int_0^t \|f_1(\tau) - f_2(\tau)\|_{W^*}^q \, d\tau\Big)^{1/q} \Big(\int_0^t \|u_1(\tau) - u_2(\tau)\|_W^p \, d\tau\Big)^{1/p}.$$

holds for $0 \leq t \leq T$.

Proof. Suppose first that $u_{1,0}$ and $u_{2,0}$ belong to $\mathfrak{D}$, and let u_1 and u_2 denote corresponding strong solutions with initial values $u_{1,0}$ and $u_{2,0}$, respectively. For almost every $\tau \in (0,t)$, the inequalities

$$\begin{aligned} \langle\, u_1'(\tau)\,,\, v - u_1(\tau)\,\rangle + \langle\, Au_1(\tau)\,,\, v - u_1(\tau)\,\rangle + \Phi(v) - \Phi(u_1(\tau)) \\ \geq \langle\, f_1(\tau)\,,\, v - u_1(\tau)\,\rangle \\ \langle\, u_2'(\tau)\,,\, v - u_2(\tau)\,\rangle + \langle\, Au_2(\tau)\,,\, v - u_2(\tau)\,\rangle + \Phi(v) - \Phi(u_2(\tau)) \\ \geq \langle\, f_2(\tau)\,,\, v - u_2(\tau)\,\rangle \end{aligned}$$

therefore hold for all $v \in W$. Inserting $v = u_2(\tau)$ into the first inequality and $v = u_1(\tau)$ into the second inequality and then adding yields

$$\langle\, u_2'(\tau) - u_1'(\tau)\,,\, u_1(\tau) - u_2(\tau)\,\rangle + \langle\, Au_2(\tau) - Au_1(\tau)\,,\, u_1(\tau) - u_2(\tau)\,\rangle \geq \langle\, f_2(\tau) - f_1(\tau)\,,\, u_1(\tau) - u_2(\tau)\,\rangle,$$

from which we conclude that

$$\frac{1}{2}\frac{d}{d\tau}\|u_1(\tau) - u_2(\tau)\|_H^2 \leq \langle\, f_1(\tau) - f_2(\tau)\,,\, u_1(\tau) - u_2(\tau)\,\rangle$$

by the monotonicity of A. Integrating from 0 to t produces

$$\|u_1(t) - u_2(t)\|_H^2 - \|u_1(0) - u_2(0)\|_H^2 \leq 2\int_0^t \langle\, f_1(\tau) - f_2(\tau)\,,\, u_1(\tau) - u_2(\tau)\,\rangle d\tau,$$

and applying Hölder's inequality yields the desired result for strong solutions,

$$\|u_1(t) - u_2(t)\|_H^2 \leq \|u_{1,0} - u_{2,0}\|_H^2 + 2\left(\int_0^t \|f_1(\tau) - f_2(\tau)\|_{W^*}^q \, d\tau\right)^{1/q} \left(\int_0^t \|u_1(\tau) - u_2(\tau)\|_W^p \, d\tau\right)^{1/p}.$$

For the general case in which the initial data $u_{1,0}$ and $u_{2,0}$ belong to $\overline{D(\Phi)}^{\|\cdot\|_H}$, we apply the approximation argument of Lemma 5.4 to complete the proof. □

Proposition 6.1 and Lemma 6.2 lead to the principal result of this section, Theorem 6.3. For this theorem, we assume that Φ is bounded below; without loss of generality, suppose that $\Phi(v) \geq 0$ for all $v \in W$. Furthermore, we require that there exist positive constants α and κ and an exponent $p > 1$ such that

$$\langle\, Au - Av\,,\, u - v\,\rangle \geq \alpha\|u - v\|_W^p \quad \text{and} \quad \|Av\|_{W^*} \leq \kappa\|v\|_W^{p-1},\ \forall\, u, v \in W. \tag{6.2}$$

Note that these assumptions imply that the monotonicity constant ω of A is *positive.*

Theorem 6.3. *Let $f \in L^q(0,T;W^*)$ be given, and define the operators $A(t) : W \to W^*$ by $A(t)u := Au - f(t)$. If $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$, then there exists a unique weak solution of the parabolic variational inequality associated to $A(\cdot)$ and Φ with initial value u_0.*

Proof. Uniqueness follows from Lemma 6.2.

To prove existence, suppose first that $u_0 \in \mathfrak{D}$ and that $f \in W^{1,1}(0,T;H)$, and let u denote the corresponding strong solution given by Proposition 6.1. Thus, $u \in L^p(0,T;W) \cap C^{0,1}(0,T;H)$, $u(0) = u_0$, and

$$\langle\, u'(t) + Au(t)\,,\, v - u(t)\,\rangle + \Phi(v) - \Phi(u(t)) \geq \langle\, f(t)\,,\, v - u(t)\,\rangle,$$

for all $v \in W$ and almost every $t \in (0,T)$. Let w_0 denote a fixed element of $D(\Phi)$, and define w to be the constant function $w(t) \equiv w_0$. Substituting $v = w(t)$ into the inequality above yields

$$\langle\, u'(t)\,,\, u(t) - w_0\,\rangle + \langle\, Au(t)\,,\, u(t) - w_0\,\rangle + \Phi(u(t)) - \Phi(w_0) \leq \langle\, f(t)\,,\, u(t) - w_0\,\rangle,$$

from which we readily obtain

$$\begin{aligned}\langle\, u'(t) - w_0'(t)\,,\, u(t) - w_0(t)\,\rangle + \langle\, Au(t) - Aw_0(t)\,,\, u(t) - w_0(t)\,\rangle \\ + \langle\, Aw_0(t)\,,\, u(t) - w_0(t)\,\rangle \leq \langle\, f(t)\,,\, u(t) - w_0\,\rangle + \Phi(w_0(t)),\end{aligned}$$

since Φ is bounded below by 0. We thus have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u(t) - w_0\|_H^2 + \alpha\|u(t) - w_0\|_W^p \\ &\leq \langle\, f(t)\,,\, u(t) - w_0\,\rangle + \langle\, Aw_0\,,\, w_0 - u(t)\,\rangle + \Phi(w_0) \\ &\leq \left(\|f(t)\|_{W^*} + \|Aw_0\|_{W^*}\right)\|u(t) - w_0\|_W + \Phi(w_0)\end{aligned}$$

$$\leq \left(\frac{1}{q}\right)\left(\frac{2}{\alpha p}\right)^{q/p}\left(\|f(t)\|_{W^*} + \|Aw_0\|_{W^*}\right)^q + \frac{\alpha}{2}\|u(t) - w_0\|_W^p + \Phi(w_0),$$

by Young's inequality. Rearranging and applying (6.2) produces

$$\frac{d}{dt}\|u(t) - w_0\|_H^2 + \alpha\|u(t) - w_0\|_W^p \leq C\left(\|f(t)\|_{W^*}^q + 1\right) + \Phi(w_0),$$

for some constant C, and integrating this inequality from 0 to t yields

$$\|u(t) - w_0\|_H^2 + \omega\int_0^t \|u(s) - w_0\|_W^p\, ds \leq \|u_0 - w_0\|_H^2 + C\int_0^t \|f(s)\|_{W^*}^q\, ds + \Phi(w_0)\, t. \quad (6.3)$$

Observe that the upper bound on the right side of this inequality depends only on the given data u_0 and f and the fixed element w_0 of $D(\Phi)$; it does not depend on the solution u.

Suppose now that $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$ and that $f \in L^q(0,T;W^*)$. Let u_0^n be an approximating sequence for u_0 as in (2.4), let f_n be a sequence from $W^{1,1}(0,T;H)$ such that $f_n \to f$ in $L^q(0,T;W^*)$, and let u_n denote the strong solution of the parabolic variational inequality with data u_0^n and f_n. The *a priori* bound (6.3) shows that the sequence u_n is bounded in $L^p(0,T;W)$, from which it follows that $u_n \rightharpoonup u$ for some $u \in L^p(0,T;W)$, after passing to a subsequence. Since the sequence u_n is bounded in $L^p(0,T;W)$ and $f_n \to f$ in $L^q(0,T;W^*)$, Lemma 6.2 shows that u_n converges in $C(0,T;H)$, and its limit must be u. Consequently, u is a weak solution of the parabolic variational inequality associated to $A(\cdot)$ and Φ with initial value u_0. □

The following basic existence and uniqueness theorem for parabolic equations involving monotone operators (see [4]) follows as a corollary of Theorem 6.3. Note that, when $\Phi \equiv 0$, $\overline{D(\Phi)}^{\|\cdot\|_H} = H$ since W is dense in H.

Corollary 6.4. *Suppose that the hypotheses of Theorem* 6.3 *hold, and suppose that the functional* Φ *is trivial,* $\Phi \equiv 0$*. For each* $u_0 \in H$*, there exists a unique* $u \in L^p(0,T;W) \cap C(0,T;H)$ *such that* $u(0) = u_0$*,* $u' \in L^q(0,T;W^*)$*, and the equation*

$$u'(t) + Au(t) = f(t) \quad (6.4)$$

holds in W^* *for almost every* $t \in (0,T)$*.*

Proof. Suppose that $u_0 \in \mathfrak{D}$ and that $f \in W^{1,1}(0,T;H)$. Applying Proposition 6.1 in this special case, we see that there exists a unique $u \in L^p(0,T;W) \cap C^{0,1}(0,T;H)$ which satisfies the equation

$$u'(t) + Au(t) = f(t) \tag{6.5}$$

and the initial condition $u(0) = u_0$. This equation holds in W^* for almost every $t \in (0,T)$, and we see that $u' \in L^q(0,T;W^*)$. Moreover, combining the facts that $u'(t) = f(t) - Au(t)$ and $\|Au(t)\|_{W^*} \le \kappa \|u\|_W^{p-1}$ with the *a priori* bound (6.3) yields the following bound on the $L^q(0,T;W^*)$-norm of u':

$$\int_0^T \|u'(t)\|_{W^*}^q \, dt \le C\Big(\|u_0\|_H^2 + \int_0^T \|f(t)\|_{W^*}^q \, dt\Big). \tag{6.6}$$

Suppose now that $u_0 \in H$ and that $f \in L^q(0,T;W^*)$. Let u_0^n be an approximating sequence for u_0 as in (2.4), let f_n be a sequence from $W^{1,1}(0,T;H)$ such that $f_n \to f$ in $L^q(0,T;W^*)$, and let u_n denote the corresponding solution of equation (6.5) with right-hand side f_n and initial condition u_0^n. As in the proof of Theorem 6.3, the *a priori* bound (6.3) shows that $u_n \rightharpoonup u$ for some $u \in L^p(0,T;W)$, and Lemma 6.2 shows that u_n converges in $C(0,T;H)$ to u. In this case, the additional *a priori* bound (6.6) shows that $u_n' \rightharpoonup u'$ in $L^q(0,T;W^*)$. To see that u solves equation (6.4), note that

$$Au_n(t) = f_n(t) - u_n'(t),$$

from which it follows that $Au_n(t) \rightharpoonup f(t) - u'(t)$ in W^* for almost every $t \in (0,T)$. Since A is of type M, we conclude that $Au(t) = f(t) - u'(t)$, finishing the proof. $\square$

6.2. **Quasiautonomous problems in nonreflexive spaces.** We next consider existence results for the case in which the Banach space of interest is not necessarily reflexive. Thus, let V denote a separable Banach space with separable dual V^*, and suppose that the Hilbert space H embeds continuously into V^*. Let the functional $\Phi : V^{**} \to \mathbb{R} \cup \{\infty\}$ be proper, convex, and weak-* lower semicontinuous.

Given $f \in W^{1,1}(0,T;H)$, define the operators $A(t) : V^{**} \to V^*$ by $A(t)u := f(t)$ for $u \in V^{**}$. Observe that reflexivity plays no role in the proof of Proposition 6.1; thus, given $u_0 \in \mathfrak{D}$, there exists a unique strong solution $u \in L^\infty(0,T;V^{**}) \cap C^{0,1}(0,T;H)$ such that $u(0) = u_0$ and such that

$$f(t) - u'(t) \in \partial\Phi(u(t)), \tag{6.7}$$

for almost every $t \in (0,T)$. Given $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$, there exists a unique weak solution $u \in C(0,T;H)$ of the inclusion (6.7) which satisfies the initial condition $u(0) = u_0$. Using this result, we can weaken the smoothness condition on the perturbation f, obtaining the following:

Theorem 6.5. *Let $f \in L^2(0,T;H)$ be given. Given $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$, there exists a unique weak solution $u \in C(0,T;H)$ of the inclusion* (6.7) *which satisfies $u(0) = u_0$.*

Proof. Let $u_{n,0}$ be an approximating sequence for u_0, let $f_n \in W^{1,1}(0,T;H)$ be a sequence such that $f_n \to f$ in $L^2(0,T;H)$, and let u_n denote the strong solution of (6.7) corresponding to the initial value $u_{n,0}$ and the perturbation f_n. As in the proof of Lemma 6.2, we find that

$$\frac{1}{2}\frac{d}{d\tau}\|u_m(\tau) - u_n(\tau)\|_H^2 \leq \langle\, f_m(\tau) - f_n(\tau)\,,\, u_m(\tau) - u_n(\tau)\,\rangle,$$

for almost every $\tau \in (0,T)$. We thus obtain

$$\frac{1}{2}\frac{d}{d\tau}\|u_m(\tau) - u_n(\tau)\|_H^2 \leq \frac{1}{2}\|f_m(\tau) - f_n(\tau)\|_H^2 + \frac{1}{2}\|u_m(\tau) - u_n(\tau)\|_H^2,$$

which is more tractable in the form

$$\frac{d}{d\tau}\|u_m(\tau) - u_n(\tau)\|_H^2 - \|u_m(\tau) - u_n(\tau)\|_H^2 \leq \|f_m(\tau) - f_n(\tau)\|_H^2.$$

Multiplying this inequality by the integrating factor $e^{-\tau}$ and integrating the result from 0 to t yields

$$\|u_m(t) - u_n(t)\|_H^2 \leq e^t\|u_{m,0} - u_{n,0}\|_H^2 + \int_0^t e^{t-\tau}\|f_m(\tau) - f_n(\tau)\|_H^2\,d\tau,$$

from which we obtain the uniform bound

$$\|u_m(t) - u_n(t)\|_H^2 \leq e^T\left(\|u_{m,0} - u_{n,0}\|_H^2 + \|f_m - f_n\|_{L^2(0,T;H)}^2\right).$$

It follows that the sequence u_n converges in $C(0,T;H)$ to some function u, which is the weak solution being sought. Note that the estimate used to prove the existence of u also verifies its uniqueness. □

6.3. **Periodic solutions and asymptotic behavior.** The preceding sections analyze quasiautonomous problems whose solutions satisfy prescribed initial conditions. We now consider quasiautonomous parabolic variational inequalities whose solutions must satisfy a periodicity condition instead. Specifically, let W be a reflexive Banach space which embeds continuously into the Hilbert space H, and let $A : W \to W^*$, $\Phi : W \to \mathbb{R} \cup \{\infty\}$, and $f \in L^q(0,T;W^*)$ be as in Theorem 6.3. Recall that the hypotheses of Theorem 6.3 imply that A has a positive monotonicity constant $\omega > 0$. Under these assumptions, we have the following:

Theorem 6.6. *There exists a unique weak solution $u \in C(0,T;H)$ of the parabolic variational inequality associated to $A(\cdot)$ and Φ such that $u(0) = u(T)$.*

Proof. Given $v_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$, Theorem 6.3 yields the unique weak solution $v \in C(0,T;H)$ of the parabolic variational inequality associated to $A(\cdot)$ and Φ such that $v(0) = v_0$. Using this fact, define the Poincaré map

$$\Pi : \overline{D(\Phi)}^{\|\cdot\|_H} \to \overline{D(\Phi)}^{\|\cdot\|_H}$$

by $\Pi(v_0) := v(T)$. By Lemma 5.4,

$$\|\Pi(v_0) - \Pi(w_0)\|_H \le e^{-\omega T}\|u_0 - v_0\|_H, \ \forall\, v_0, w_0 \in \overline{D(\Phi)}^{\|\cdot\|_H},$$

proving that Π is a strict contraction. The contraction-mapping principle then assures the existence of a unique fixed point $u_0 \in \overline{D(\Phi)}^{\|\cdot\|_H}$ of Π, completing the proof. □

If f is periodic of period T, then the unique weak solution $u \in C(0,T;H)$ of Theorem 6.6 extends periodically for all time. (If $f \in W^{1,1}(0,T;H)$, for example, then it makes sense to say that $f(0) = f(T)$.) This case is especially interesting, as we find that the periodic solution u attracts all other solutions asymptotically: if $\tilde{T} > T$ is given and $v \in C(0,\tilde{T};H)$ is a weak solution of the parabolic variational inequality associated to $A(\cdot)$ and Φ, then

$$\|u(\tilde{T}) - v(\tilde{T})\|_H \le e^{-\omega\tilde{T}}\|u(0) - v(0)\|_H,$$

from which we see that $\|u(\tilde{T}) - v(\tilde{T})\|_H$ goes to 0 as $\tilde{T} \to \infty$.

It may happen that f fails to be periodic but approaches some limit $f_\infty \in W^*$ asymptotically. In this case, weak solutions of the parabolic variational inequality associated to $A(\cdot)$ and Φ tend to the unique solution $u_\infty \in D(\Phi)$ of the elliptic variational inequality

$$\langle\, Au_\infty\,,\, v - u_\infty \,\rangle + \Phi(v) - \Phi(u_\infty) \ge \langle\, f_\infty\,,\, v - u_\infty\,\rangle, \ \forall\, v \in W.$$

The proof of this result follows, as above, from Lemma 5.4.

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