Differential and Integral Equations September/October 2004
Volume 17, Numbers 9-10, Pages 1079–1092

OPTIMAL WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR EVOLUTION EQUATIONS WITH C^N COEFFICIENTS

Massimo Cicognani
Dipartimento di Matematica
Piazza di Porta S. Donato, 5, 40127 Bologna, Italy

Ferruccio Colombini
Dipartimento di Matematica
Via Buonarroti, 2, 56127 Pisa, Italy

(Submitted by: Reza Aftabizadeh)

Abstract. We deal with the Cauchy problem for a 2-evolution operator of Schrödinger type with C^N coefficients in the time variable, $N > 2$. We find the Levi conditions for well-posedness in Gevrey classes of index $1/2 + N/4$, which is the best possible, as we show by means of counterexamples.

1. Introduction

Let us consider the Cauchy problem in $[0,T] \times \mathbf{R}^n$,

$$\partial_t^2 u + A(t, D_x)u + \sum_{k=0}^{3} B_k(t,x,D_x)u = 0, \quad t > 0 \tag{1.1}$$
$$u(0,x) = u_0,\ \partial_t u(0,x) = u_1,$$

$D_x = -i\nabla_x$, where A is of order 4 and each B_k of order k

$$A(t,D_x) = \sum_{|\alpha|=4} a_\alpha(t) D_x^\alpha, \quad B_k(t,x,D_x) = \sum_{|\alpha|=k} b_{\alpha,k}(t,x) D_x^\alpha. \tag{1.2}$$

Let us assume that A and B_3 are real and that

$$A(t,\xi) \geq 0, \quad t \in [0,T],\ \xi \in \mathbf{R}^n, \tag{1.3}$$

so we are dealing with a 2-evolution equation with real (Schrödinger-type) roots.

For a space X of functions $v(x)$ in $\mathbf{R}^n$, we say that the Problem (1.1) is well posed in X if for every $u_0, u_1 \in X$ there is a unique solution $u \in C^1([0,T]; X)$.

Accepted for publication: March 2004.
AMS Subject Classifications: 35G10, 35J10, 35B30.

The natural spaces X of well-posedness will be in the scale of Sobolev-Gevrey type $H^{\infty,\lambda,s}(\mathbf{R}^n) = \bigcap_\mu H^{\mu,\lambda,s}(\mathbf{R}^n)$, $\lambda > 0$, $s > 1$, with $H^{\mu,\lambda,s}(\mathbf{R}^n) = e^{-\lambda\langle D_x\rangle^{1/s}} H^\mu(\mathbf{R}^n)$, $\langle D_x\rangle = (1 + |D_x|^2)^{1/2}$, H^μ the usual Sobolev space.

For every $t \in [0,T]$ we need well-defined operators $B_k(t) : X \to X$, so in (1.2) we assume $b_{\alpha,k}(t,\cdot) \in \gamma^s(\mathbf{R}^n)$, where $\gamma^s(\mathbf{R}^n)$ denotes the Gevrey space of all functions $v(x)$ in $\mathbf{R}^n$ such that

$$|\partial_x^\alpha v(x)| \le C K^{|\alpha|} \alpha!^s$$

for some $K, C > 0$ and every α and x. Fixing K one obtains a Banach space taking the best constant C as the norm of v. Then γ^s has the natural topology of limit space as K tends to ∞.

In this paper we are concerned with the problem of determining the sharp C^N regularity of the coefficients a_α and the related Levi conditions on the B_k's in order to have a well-posed problem (1.1). Here $N > 2$ is a real number, and for $N = k + \chi$, k an integer and $0 < \chi \le 1$, C^N is a short notation for the space $C^{k,\chi}$ of all functions $a(t)$ such that $\partial_t^k a$ is Hölder continuous of exponent χ in $[0,T]$.

We focus on the case $A(t,\xi) = 0$ for some $t \in [0,T]$ and $\xi \ne 0$ in (1.3). Under the stronger assumption

$$A(t,\xi) \ge c|\xi|^4, \quad c > 0$$

one can allow less than C^1 coefficients even for H^∞ well-posedness, $H^\infty = \bigcap_\mu H^\mu$, as we have shown in [3] (see also [2]).

Testing the equation

$$\partial_t^2 u + A(t, D_x)u = 0 \tag{1.4}$$

with $a_\alpha \in C^N([0,T])$ we find well-posedness in $H^{\infty,\lambda,s}$ for

$$s < s_0 := 1/2 + N/4. \tag{1.5}$$

For more general evolution equations, $p \ge 2$,

$$\partial_t^2 u + A(t, D_x)u = 0, \quad A(t,\xi) = \sum_{|\alpha|=2p} a_\alpha(t)\xi^\alpha \ge 0, \quad p \ge 1,$$

we prove that well-posedness holds for

$$s < 1/p + N/2p, \quad N > 2(p-1), \tag{1.6}$$

which is in line with the weakly hyperbolic case $p = 1$ of [4], and we give a counterexample to show that this bound for the Gevrey index s is optimal. In particular one can not expect H^∞ well-posedness with less than C^∞ coefficients.

Coming back to the case $p = 2$, adding lower-order terms

$$\sum_{k=0}^{3} B_k(t, D_x)u$$

with coefficients depending only on t in (1.4), it is sufficient to assume $b_{\alpha,3} \in C^1([0, T])$ and $b_{\alpha,k} \in C^0([0, T])$, $k \leq 2$, but we need Levi conditions on B_k, $1 \leq k \leq 3$,

$$|B_k(t, \xi)| \leq C(A(t, \xi))^{\delta_k} |\xi|^{k-4\delta_k}, \quad \delta_k \geq k/4 - (4 - k)/2N \tag{1.7}$$

and

$$|\partial_t B_3(t, \xi)| \leq C(A(t, \xi))^{\eta} |\xi|^{3-4\eta}, \quad \eta \geq 3/4 - 3/2N \tag{1.8}$$

in order to maintain the sharp bound (1.5) for the Gevrey index of well-posedness.

In (1.7) for $k = 1, 2$ and (1.8) one can replace $A(t, \xi)$ with $|\partial_t A(t, \xi)|$ taking suitable larger δ_k and η.

All our results hold true in dimension of space $n = 1$ if we let the coefficients $b_{\alpha,k}$ in (1.2) depend also on the variable x.

This paper is organized as follows: in Section 2 we consider p-evolution equations, $p \geq 2$, with coefficients depending only on time, and we show the way condition (1.6) arises by Fourier transform. We show also that this bound for s is optimal by means of counterexamples. In Section 3 we take $p = 2$ and add lower-order terms under the Levi conditions (1.7) and (1.8). In Section 4 we let the coefficients of B_k, $k \leq 3$, depend also on a space variable $x \in \mathbf{R}$ using pseudo-differential calculus.

2. Sharp C^N regularity in homogeneous equations of any order

For an integer $p \geq 1$, let us consider the operator L_0 in $[0, T] \times \mathbf{R}^n$:

$$L_0 = \partial_t^2 + A(t, D_x), \quad A(t, D_x) = \sum_{|\alpha|=2p} a_\alpha(t) D_x^\alpha. \tag{2.1}$$

We assume

$$a_\alpha \in C^N([0, T]; \mathbf{R}), \ N > 2, \quad A(t, \xi) \geq 0, \quad t \in [0, T], \ \xi \in \mathbf{R}^n \tag{2.2}$$

and study the Cauchy problem

$$L_0 u = 0, \quad t > 0, \quad u(0, x) = u_0, \ \partial_t u(0, x) = u_1 \tag{2.3}$$

in Sobolev-Gevrey spaces

$$H^{\mu,\lambda,s}(\mathbf{R}^n) = e^{-\lambda \langle D_x \rangle^{1/s}} H^\mu(\mathbf{R}^n), \ \lambda > 0, \ s > 1,$$

with norm

$$\|u\|_{\mu,\lambda,s} = \|e^{\lambda\langle D_x\rangle^{1/s}}u\|_\mu.$$

Let $u(t,x)$ be a solution of the equation $L_0u=0$. For the Fourier transform $v(t,\xi)$ of u with respect to x, we define the energy

$$E_\varepsilon = |\partial_t v(t,\xi)|^2 + |\xi|^{2p}(a(t,\xi)+\varepsilon)|v(t,\xi)|^2, \quad a(t,\xi) = A(t,\xi)|\xi|^{-2p} \tag{2.4}$$

with $\varepsilon = |\xi|^{-\sigma}$, $\sigma > 0$ to be chosen later. From

$$\partial_t^2 v(t,\xi) + |\xi|^{2p}a(t,\xi)v(t,\xi) = 0$$

we have

$$\partial_t E_\varepsilon \le (|\partial_t a|/(a+\varepsilon) + \varepsilon^{1/2}|\xi|^p)E_\varepsilon, \tag{2.5}$$

which gives

$$E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi)\exp\{t\varepsilon^{1/2}|\xi|^p + \int_0^t |\partial_\tau a(\tau,\xi)|/(a(\tau,\xi)+\varepsilon)d\tau\} \tag{2.6}$$

by Gronwall's inequality. From Lemma 1 of [4] we know that $a^{1/N}$ is an absolutely continuous function of the variable t, so

$$t\varepsilon^{1/2}|\xi|^p + \int_0^t |\partial_\tau a(\tau,\xi)|/(a(\tau,\xi)+\varepsilon)d\tau$$
$$\le t\varepsilon^{\frac12}|\xi|^p + \varepsilon^{-\frac1N}\int_0^t |\partial_\tau a^{\frac1N}(\tau,\xi)|d\tau = t|\xi|^{p-\sigma/2} + |\xi|^{\sigma/N}\int_0^t |\partial_\tau a^{\frac1N}(\tau,\xi)|d\tau.$$

The best choice $\sigma = 2pN/(N+2)$ of σ and (2.6) yield

$$\begin{aligned} &E_\varepsilon(t,\xi) \le E_\varepsilon(0,\xi)e^{w(t,\xi)}, \\ &w(t,\xi) = |\xi|^{2p/(N+2)}\Big(t + \int_0^t |\partial_\tau a^{1/N}(\tau,\xi)|d\tau\Big). \end{aligned} \tag{2.7}$$

This allows us to solve the Cauchy problem for L_0 in Gevrey spaces provided that the Gevrey index s is related to the C^N regularity of A by the assumption

$$1 < s \le s_0 := 1/p + N/2p. \tag{2.8}$$

In fact, we have proved the following:

Theorem 2.1. *Let $w(t,\xi)$ be as in* (2.7), *and let s be as in* (2.8). *Then for every $u \in \bigcap_{j=0}^2 C^j([0,T]; H^{\mu+p(2-j),\lambda,s})$, $\mu \in \mathbf{R}$, $\lambda > 0$, we have*

$$\begin{aligned} &\|e^{-w(t,D_x)}u(t)\|^2_{\mu+2p/(N+2),\lambda,s} + \|e^{-w(t,D_x)}\partial_t u(t)\|^2_{\mu,\lambda,s} \\ &\le C\Big(\|u(0)\|^2_{\mu+p,\lambda,s} + \|\partial_t u(0)\|^2_{\mu,\lambda,s} + \int_0^t \|e^{-w(\tau,D_x)}L_0u(\tau)\|^2_{\mu,\lambda,s}d\tau\Big), \end{aligned} \tag{2.9}$$

$C > 0$, $0 \le t \le T$.

Remark 2.2. For $s < s_0$ the energy estimate (2.9) implies the well-posedness of the Cauchy problem (2.3) in the space $\bigcup_{\lambda>0} H^{\infty,\lambda,s}$. In particular for every Cauchy data $u_0, u_1 \in \gamma^s$ with compact support there is a solution $u \in C^1([0,T];\gamma^s)$.

Also for $s = s_0$ the Cauchy problem can be uniquely solved but the lifespan of the solution as a Gevrey function depends on the space of Cauchy data. For $u_0, u_1 \in H^{\infty,\lambda^*,s_0}$ there is a unique solution $u \in C^1\left([0,T^*];\bigcup_{\lambda>0} H^{\infty,\lambda,s_0}\right)$ with $w(T^*,\xi) \le \lambda^*\langle\xi\rangle^{1/s_0}$.

Next we show that condition (2.8) is sharp. Let us consider the Cauchy problem

$$\begin{aligned} &L_0 u(t,x) = 0,\ t\in[0,1],\ x\in\mathbf{T}\\ &u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \end{aligned} \tag{2.10}$$

taking now

$$L_0 = \partial_t^2 + a(t)D_x^{2p} \tag{2.11}$$

in $[0,1]\times\mathbf{T}$, $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, the torus of dimension 1.

Theorem 2.3. *Let us take $N > 2(p-1)$ and let $s_0 = 1/p + N/2p$. There are a function $a(t)$*

$$a \in C^N([0,1]),\ a(t) \ge 0 \tag{2.12}$$

and two Cauchy data $u_0, u_1 \in \gamma^s(\mathbf{T})$ for every $s > s_0$ such that the Cauchy problem (2.10) does not have any solution in $C^1([0,1];\mathcal{D}'_s(\mathbf{T}))$ for any $s > s_0$, $\mathcal{D}'_s$ the space of all Gevrey ultradistributions of index s.

Proof. Our construction is inspired by the examples in [5] and in [4]. Let us take a real, nonnegative, 2π-periodic C^∞ function φ such that $\varphi(\tau) = 0$ for τ in a neighborhood of $\tau = 0$ and

$$\int_0^{2\pi} \varphi(\tau)\cos^2\tau\ d\tau = \pi.$$

Then, for every $\tau \in \mathbf{R}$ we define

$$\alpha(\tau) = 1 + 4\varepsilon\varphi(\tau)\sin 2\tau - 2\varepsilon\ \varphi'(\tau)\cos^2\tau - 4\varepsilon^2\varphi^2(\tau)\cos^4\tau,$$

where ε is fixed in such a way that $1/2 \le \alpha(\tau) \le 3/2$,

$$\tilde{w}(\tau) = \cos\tau\exp\Big(-\varepsilon\tau + 2\varepsilon\int_0^\tau \varphi(s)\cos^2 s\ ds\Big),\quad w(\tau) = \tilde{w}(\tau)e^{\varepsilon\tau}.$$

So, $\alpha(\tau)$ and $\tilde{w}(\tau)$ are 2π-periodic C^∞ functions; let us now denote

$$M = \|\alpha'\|_{L^\infty}.$$

Furthermore, w is the solution of the Cauchy problem

$$w''(\tau) + \alpha(\tau)w(\tau) = 0, \quad w(0) = 1, \ w'(0) = 0. \tag{2.13}$$

Let now $\beta(\tau)$ be a nonincreasing C^∞ function such that $\beta(\tau) = 1$ for $\tau \le 0$, $\beta(\tau) = 0$ for $\tau \ge 1$. We use also four positive monotone sequences $\{\delta_k\}$, $\{\varrho_k\}$, $\{\nu_k\}$, and $\{h_k\}$ such that

$$\begin{aligned} &h_k \to +\infty, \ \nu_k \to +\infty, \ \delta_k \to 0, \ \varrho_k \to 0; \ \nu_k, h_k \in \mathbf{N}, \\ &\delta_1 \le 1, \ 2\sum_{k=1}^{\infty} \varrho_k = T < 1. \end{aligned} \tag{2.14}$$

Finally, let us define two families of intervals I_k and J_k, $k \ge 1$, by setting

$$\begin{aligned} &I_k = [t_k - \varrho_k/2, t_k + \varrho_k/2], \quad J_k = [t_k + \varrho_k/2, t_k + 3\varrho_k/2] \\ &t_k = \varrho_k/2 + 2\sum_{j=1}^{k-1} \varrho_j \quad (t_1 = \varrho_1/2). \end{aligned} \tag{2.15}$$

Now we are ready to construct the coefficient $a(t)$ for $t \in [0,1]$ as follows:

$$a(t) = \begin{cases} \delta_k \alpha(4\pi\nu_k(t - t_k)/\varrho_k) & \text{for } t \in I_k \\ \delta_{k+1} + (\delta_k - \delta_{k+1})\beta((t - t_k)/\varrho_k - 1/2) & \text{for } t \in J_k \\ 0 & \text{for } t \ge T. \end{cases} \tag{2.16}$$

It is easy to see that $a \in C^\infty([0,T))$. Moreover, requiring

$$\delta_k \left(\frac{4\pi\nu_k}{\varrho_k} \right)^N = 1 \tag{2.17}$$

we obtain $a \in C^N([0,1])$.

Now we define a solution $u \in C^\infty([0,T); \gamma^s(\mathbf{T}))$ for any $s > s_0$ of $Lu = 0$ and take $u_0 = u(0,x)$ and $u_1 = \partial_t u(0,x)$ as Cauchy data in (2.10). Let us set

$$u(t,x) = \sum_{k=1}^{\infty} v_k(t) e^{ih_k x}. \tag{2.18}$$

We have

$$v_k''(t) + h_k^{2p} a(t) v_k(t) = 0; \tag{2.19}$$

hence, if we impose $v_k(t_k) = 1$ and $v_k'(t_k) = 0$ we have, thanks to (2.13),

$$v_k(t) = w(4\pi\nu_k(t - t_k)/\varrho_k), \ t \in I_k, \tag{2.20}$$

provided that

$$h_k^{2p} = (4\pi\nu_k/\varrho_k)^2 \delta_k^{-1}. \tag{2.21}$$

In particular

$$v_k(t_k - \varrho_k/2) = e^{-2\pi\varepsilon\nu_k}, \quad v_k'(t_k - \varrho_k/2) = 0, \tag{2.22}$$

$$v_k(t_k + \varrho_k/2) = e^{2\pi\varepsilon\nu_k}, \quad v_k'(t_k + \varrho_k/2) = 0. \tag{2.23}$$

For

$$E_k(t) = |v_k'(t)|^2 + h_k^{2p} a(t)|v_k(t)|^2$$

from (2.22), for $t \le t_k - \varrho_k/2$ and taking (2.21) into account, we obtain

$$E_k(t) \le E_k(t_k - \varrho_k/2) \exp\Big[\int_0^{t_k-\varrho_k/2} |a'(t)|/a(t)dt\Big]$$

$$= h_k^{2p}\delta_k \exp\Big[-4\pi\varepsilon\nu_k + \sum_{j=1}^{k-1}\int_{I_j}|a'(t)|/a(t)dt + \sum_{j=1}^{k-1}\int_{J_j}|a'(t)|/a(t)dt\Big].$$

But

$$\int_{I_j}|a'(t)|/a(t)dt \le 8\pi M\nu_j,$$

$$\int_{J_j}|a'(t)|/a(t)dt = \log(1/\delta_{j+1}) - \log(1/\delta_j),$$

so, finally, for $t \le t_k - \varrho_k/2$, taking (2.17) into account, we obtain

$$E_k(t) \le C\exp\Big[-4\pi\varepsilon\nu_k + 8\pi M\sum_{j=1}^{k-1}\nu_j + (N+2)\log(\nu_k/\varrho_k)\Big]. \tag{2.24}$$

Now we choose $\varrho_k = (k + k_0)^{-2}$ in such a way that $\sum_{k=1}^{\infty}\varrho_k < 1/2$ and $\nu_k = \mu^k$ with μ an integer so large that

$$4\pi\varepsilon\nu_k > 8\pi M\sum_{j=1}^{k-1}\nu_j + (N+2)\log(\nu_k/\varrho_k) + \varepsilon\nu_k. \tag{2.25}$$

From (2.24), (2.17), and (2.25), we obtain

$$E_k(t)\exp(h_k^{1/s}) \le C\exp\big[-\varepsilon\nu_k + (4\pi\nu_k/\varrho_k)^{(N+2)/2ps}\big],$$

and this expression goes to 0 for $k \to \infty$, for any $s > 1/p + N/2p$.

So, for u defined by (2.18), $u(0,x)$ and $\partial_t u(0,x)$ are in $\gamma^s(\mathbf{T}))$ for any $s > s_0$. On the other hand, from (2.23) immediately it follows that $u(t,\cdot)$ is not bounded in $\mathcal{D}_s'(\mathbf{T})$ as $t \to T$, for any $s > s_0$. □

3. Levi conditions

Let us now consider the Cauchy problem

$$\begin{aligned} &Lu = 0, \quad t > 0 \\ &u(0,x) = u_0, \ \partial_t u(0,x) = u_1, \end{aligned} \tag{3.1}$$

for an operator L in $[0,T] \times \mathbf{R}^n$,

$$\begin{aligned} &L = \partial_t^2 + A(t,D_x) + \textstyle\sum_{k=0}^3 B_k(t,D_x) \\ &A(t,D_x) = \textstyle\sum_{|\alpha|=4} a_\alpha(t) D_x^\alpha, \quad B_k(t,D_x) = \textstyle\sum_{|\alpha|=k} b_{\alpha,k}(t) D_x^\alpha. \end{aligned} \tag{3.2}$$

In this section we take the evolution degree $p = 2$ only for simplicity's sake. We assume

$$\begin{aligned} &a_\alpha \in C^N([0,T];\mathbf{R}), \ N > 2, \quad A(t,\xi) \geq 0, \quad t \in [0,T], \ \xi \in \mathbf{R}^n \\ &b_{\alpha,3} \in C^1([0,T];\mathbf{R}), \quad b_{\alpha,k} \in C^0([0,T];\mathbf{C}), \ k \leq 2. \end{aligned} \tag{3.3}$$

We have proved in Section 2 that $s_0 = 1/2 + N/4$ is the largest possible index of Gevrey well-posedness for the principal part $L_0 = \partial_t^2 + A(t,D_x)$. In order to reach it also for the full operator L, we assume that for $\delta_k, \eta \geq 0$, $a(t,\xi) = |\xi|^{-4} A(t,\xi)$, and $b_k(t,\xi) = |\xi|^{-k} B_k(t,\xi)$, the lower-order terms satisfy the Levi conditions

$$|b_3(t,\xi)| \leq C a(t,\xi)^{\delta_3}, \quad \delta_3 > 3/4 - 1/2N, \tag{3.4}$$

$$|\partial_t b_3(t,\xi)| \leq C a(t,\xi)^{\eta}, \quad \eta \geq 3/4 - 3/2N, \tag{3.5}$$

$$|b_k(t,\xi)| \leq C a(t,\xi)^{\delta_k}, \quad \delta_k \geq k/4 - (4-K)/2N, \ k = 1,2. \tag{3.6}$$

In particular we do not need any condition on B_1 for $s_0 \leq 2$ since we can take $\delta_1 = 0$ in this case.

The Gevrey well-posedness of the Cauchy problem (3.1) for $s < s_0$ is obtained by the following:

Theorem 3.1. *Let the operator L defined by* (3.2) *fulfill conditions* (3.3), (3.4), (3.5), *and* (3.6), *and let $w(t,\xi)$ be as in* (2.7) *taking there $p = 2$. Let $s \leq 1/2 + N/4$.*

Then there is $\varrho > 0$ such that for every $u \in \bigcap_{j=0}^2 C^j([0,T]; H^{\mu+4-2j,\lambda,s})$, $\mu \in \mathbf{R}$, $\lambda > 0$, we have

$$\begin{aligned} &\|e^{-\varrho w(t,D_x)} u(t)\|^2_{\mu+4/(N+2),\lambda,s} + \|e^{-\varrho w(t,D_x)} \partial_t u(t)\|^2_{\mu,\lambda,s} \\ &\leq C\Big(\|u(0)\|^2_{\mu+2,\lambda,s} + \|\partial_t u(0)\|^2_{\mu,\lambda,s} + \int_0^t \|e^{-\varrho w(\tau,D_x)} Lu(\tau)\|^2_{\mu,\lambda,s} d\tau \Big), \end{aligned} \tag{3.7}$$

$C > 0$, $0 \leq t \leq T$.

Proof. Only to have a simpler notation let us give the proof for u such that $Lu = 0$. Denoting by $v(t, \xi)$ the Fourier transform of $u(t, x)$ with respect to the space variable x, let us define

$$E(t, \xi) = |\partial_t v|^2 + (a(t, \xi) + \varepsilon)|\xi|^4|v|^2 + b_3(t, \xi)|\xi|^3|v|^2, \quad \varepsilon = |\xi|^{-4N/(N+2)}. \tag{3.8}$$

From (3.4) we have

$$E(t, \xi) \geq |\partial_t v|^2 + |\xi|^{8/(N+2)}|v|^2/2 \tag{3.9}$$

for large $|\xi|$. Since

$$\partial_t^2 v + |\xi|^4 a(t, \xi)v + \sum_{k=0}^{3} |\xi|^k b_k(t, \xi)v = 0, \tag{3.10}$$

the function E satisfies

$$\partial_t E \leq \Big(|\partial_t a + |\xi|^{-1}\partial_t b_3|/(a + \varepsilon) + \varepsilon^{1/2}|\xi|^2 + (a + \varepsilon)^{-1/2} \sum_{k=0}^{2} |b_k||\xi|^{k-2}\Big)E, \tag{3.11}$$

so from (3.5) and (3.6) we have

$$\partial_t E \leq \left(|\partial_t a|/(a + \varepsilon) + 2\varrho|\xi|^{1/s_0}\right) E \tag{3.12}$$

for some $\varrho > 1$ and large $|\xi|$, which gives

$$E_\varepsilon(t, \xi) \leq E_\varepsilon(0, \xi)e^{2\varrho w(t,\xi)} \tag{3.13}$$

by Gronwall's inequality as in (2.7), taking there $p = 2$. The energy estimate (3.7) follows from (3.13) and (3.9). □

Remark 3.2. In (3.5) and (3.6) one can replace a with $|\partial_t a|$ taking now $\eta, \delta_k \in [0, 1)$ such that

$$\eta(N - 1)/N \geq 3/4 - 3/2N, \quad \delta_k(N - 1)/N \geq k/4 - (4 - K)/2N, \;\; k = 1, 2. \tag{3.14}$$

Using the Hölder inequality and Lemma 1 of [4] in

$$\int_0^t |\partial_\tau a(\tau, \xi)|^\eta/(a(\tau, \xi) + \varepsilon)d\tau$$

and

$$\int_0^t |\partial_\tau a(\tau, \xi)|^{\delta_k}/(a(\tau, \xi) + \varepsilon)^{1/2}d\tau,$$

the energy estimate (3.7) holds with

$$
\begin{aligned}
&w(t,\xi) = |\xi|^{1/s_0}(t + t^{1-\eta}h(t,\xi)^{\eta} + \sum_{k=1}^{2} t^{1-\delta_k}h(t,\xi)^{\delta_k}),\\
&h(t,\xi) = \textstyle\int_0^t |\partial_\tau a^{1/N}(\tau,\xi)|d\tau.
\end{aligned}
\tag{3.15}
$$

Remark 3.3. The conditions (3.4), (3.5), and (3.6) can be slightly weakened in several directions.

For instance, in (3.5) and (3.6) one can replace the constant C by a positive function $\lambda(t,\xi)$ such that

$$\sup_{|\xi|=1} \int_0^T \lambda(t,\xi)dt < +\infty.$$

Furthermore, in the case $a(0,\xi) = 0$ for some $\xi \neq 0$, we obtain the weaker conditions

$$
\begin{aligned}
&|t^{2(1-q)}b_3(t,\xi)| \le C(t^{2(1-q)}a(t,\xi))^{\delta_3},\\
&|t^{2(1-q)}\partial_t b_3(t,\xi)| \le \lambda(t,\xi)(t^{2(1-q)}a(t,\xi))^{\eta},\\
&|t^{1-q}b_k(t,\xi)| \le \lambda(t,\xi)(t^{2(1-q)}a(t,\xi))^{\delta_k}, \quad k = 1,2
\end{aligned}
$$

for any $q \in (0,1)$ by choosing

$$
\begin{aligned}
&\varepsilon(t,\xi) = |\xi|^{-4N/(N+2)}t^{-2(1-q)}, \quad t > |\xi|^{-2N(N+2)^{-1}(1-q)^{-1}},\\
&\varepsilon(t,\xi) = 1, \quad t < |\xi|^{-2N(N+2)^{-1}(1-q)^{-1}}
\end{aligned}
$$

in (3.8).

The following example shows that the exponents δ_k in (3.4) and (3.6) are optimal:

Example. For a fixed k, $1 \le k \le 3$, and postive integers h and ℓ, let us consider the operator

$$L = \partial_t^2 + t^{2h}D_x^4 + t^{\ell}D_x^k, \quad (h+1)/(\ell+2) > 2/k$$

in $[0,T] \times \mathbf{R}$. From the above Remark 3.3 we obtain Gevrey well-posedness for

$$s < s_0 = \frac{2h - \ell}{k(h+1) - 2(\ell+2)},$$

taking there $\lambda(t,\xi) = t^{q-1}$. In [1] it is shown that the Cauchy problem for L is not well posed for $s > s_0$. There the author generalizes the famous weakly hyperbolic example

$$L = \partial_t^2 + t^{2h}D_x^2 + t^{\ell}D_x$$

treated in [6].

4. Space variable

In this section, using pseudo-differential calculus, we prove an energy inequality in spaces $H^{\mu,\lambda,s}$ in order to solve the Cauchy problem for an operator L in $[0,T]\times\mathbf{R}$ with coefficients of lower-order terms depending also on a space variable x of dimension 1.

Let S^m denote the usual space of symbols $p(x,\xi)$ of order m such that

$$|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|\le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|},\quad |p|_{m,\ell}:=\sup_{|\alpha|+|\beta|\le\ell}\sup_{x,\xi}|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|\langle\xi\rangle^{-m+|\alpha|}, \tag{4.1}$$

and for $s>1$, $A>0$, let $S_A^{m,s}$ denote the subspace of all symbols $p\in S^m$ such that

$$\begin{aligned}&|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|\le C_\alpha A^{|\beta|}\beta!^s\langle\xi\rangle^{m-|\alpha|},\\&|p|_{m,\ell,A,s}:=\sup_{|\alpha|\le\ell,\beta\ge0}\sup_{x,\xi}|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)|A^{-|\beta|}\beta!^{-s}\langle\xi\rangle^{-m+|\alpha|}.\end{aligned} \tag{4.2}$$

Furthermore, let $S^{m,s}$ denote the limit space $S^{m,s}=\bigcup_{A>0}S_A^{m,s}$. Operators with symbol in $S^{m,s}$ are continuous in spaces $H^{\mu,\lambda,s}$. In fact, denoting

$$p_\Lambda(x,D_x)=e^\Lambda p(x,D_x)e^{-\Lambda},\quad \Lambda(D_x)=\lambda\langle D_x\rangle^{1/s},$$

we have

Proposition 4.1. [7]. *Let $p\in S_A^{m,s}$. There is $\lambda_0=\lambda_0(A,s)>0$ such that for every $\lambda\le\lambda_0$ the symbol of p_Λ has the asymptotic expansion*

$$p_\Lambda=\sum_{|\alpha|\le M}\alpha!^{-1}(\partial_\xi^\alpha e^\Lambda)D_x^\alpha p(x,\xi)e^{-\Lambda}+r_M^{(\Lambda)},\quad r_M^{(\Lambda)}\in S^{m-(M+1)(1-1/s),s}, \tag{4.3}$$

$M\ge0$, and for every ℓ there is ℓ' such that

$$|p_\Lambda|_{m,\ell}\le C|p|_{m,\ell',A,s} \tag{4.4}$$

with $C>0$ independent of p and $\lambda\in(0,\lambda_0]$.

Now let us consider the Cauchy problem in $[0,T]\times\mathbf{R}$,

$$\begin{aligned}&Lu=0,\quad t>0\\&u(0,x)=u_0,\ \partial_t u(0,x)=u_1,\end{aligned} \tag{4.5}$$

for an operator L,

$$L=\partial_t^2+a(t)D_x^4+\sum_{k=0}^{3}b_k(t,x)D_x^k,\quad t\in[0,T],\ x\in\mathbf{R}, \tag{4.6}$$

with $a(t)$ and $b_3(t,x)$ real valued and such that

$$a \in C^N([0,T]),\ N > 2,\quad a(t) \geq 0, \tag{4.7}$$

$$b_3 \in C^1([0,T];\gamma^s),\quad b_k \in C^0([0,T];\gamma^s),\ k \leq 2,\ s \leq s_0 = 1/2 + N/4, \tag{4.8}$$

$$|\partial_x^\beta b_3(t,x)| \leq C^{|\beta|+1}\beta a(t)^{\delta_3},\quad \delta_3 > 3/4 - 1/2N, \tag{4.9}$$

$$|\partial_x^\beta \partial_t b_3(t,x)| \leq C^{|\beta|+1}\beta a(t)^{\eta},\quad \eta \geq 3/4 - 3/2N, \tag{4.10}$$

$$|\partial_x^\beta b_k(t,x)| \leq C^{|\beta|+1}\beta a(t)^{\delta_k},\quad \delta_k \geq k/4 - (4-K)/2N,\ k = 1,2 \tag{4.11}$$

for $\eta, \delta_k \in [0,1)$.

From the above assumptions on the coefficients of L we have the factorization

$$\begin{aligned}
&L = (\partial_t - i\omega D_x^2 - i\gamma D_x)(\partial_t + i\omega D_x^2 + i\gamma D_x) + r\omega\langle D_x\rangle^2,\\
&\omega(t,\xi) = (a(t) + \langle\xi\rangle^{-\sigma})^{1/2},\quad \sigma = 4N/(N+2),\quad \xi \in \mathbf{R},\\
&\gamma(t,x,\xi) = 2^{-1}b_3(t,x)\omega^{-1}(t,\xi) \in C^0([0,T];S^{0,s}),\\
&r(t,x,\xi) = p(t,x,\xi)a'(t)\omega^{-2}(t,\xi) + q(t,x,\xi)\langle\xi\rangle^{1/s_0},\\
&p, q \in C^0([0,T];S^{0,s}).
\end{aligned} \tag{4.12}$$

Notice that ω is of order 0 but $\partial_\xi\omega$ is of order $-\sigma/2 - 1$. Furthermore, we can take an elliptic symbol $m(t,x,\xi) \in C^0([0,T];S^{0,s})$ by (4.9) such that

$$m(t,x,\xi) = \left(2 + b_3(t,x)\xi^{-1}\omega^{-2}(t,\xi)\right)^{-1},\quad |\xi| \text{ large} \tag{4.13}$$

so for a given scalar function $u(t,x)$ we can define the vector

$$U =^t (u_1, u_2)$$

by

$$u_1 = (\partial_t + i\omega D_x^2 + i\gamma D_x)u,\quad u_2 = \omega\langle D_x\rangle^2 u + imu_1 \tag{4.14}$$

in order to make the scalar equation $Lu = 0$ equivalent to a 2×2 system $SU = 0$ with

$$\begin{aligned}
&S = \partial_t + K(t,x,D_x),\\
&K(t,x,\xi) = D(t,x,\xi) + P(t,x,\xi)a'(t)\omega^{-2}(t,\xi) + Q(t,x,\xi)\langle\xi\rangle^{1/s_0},\\
&D(t,x,\xi) = \begin{pmatrix} -i\omega(t,\xi)\xi^2 - i\gamma(t,x,\xi)\xi & 0 \\ 0 & i\omega(t,\xi)\xi^2 + i\gamma(t,x,\xi)\xi \end{pmatrix},\\
&P = (p_{jk})_{1\leq j,k\leq 2},\quad Q = (q_{jk})_{1\leq j,k\leq 2},\quad p_{jk}, q_{jk} \in C^0([0,T];S^{0,s}).
\end{aligned} \tag{4.15}$$

Since $a(t) = 0$ implies $a'(t) = 0$ for $t \in (0,T)$, we have

$$|a'(t)|\omega^{-2} = |a'(t)|(a(t) + \langle\xi\rangle^{-\sigma})^{-1} \le \alpha(t)\langle\xi\rangle^{1/s_0}, \quad t \in (0,T),$$
$$\alpha(t) = 0 \quad \text{for} \quad a(t) = 0, \quad \alpha(t) = |a'(t)|a(t)^{1/N-1} \quad \text{for} \quad a(t) \neq 0, \tag{4.16}$$

where $\alpha \in L^1(0,T)$ by Lemma 1 of [4]. Let us define

$$w(t,\xi) = \langle\xi\rangle^{1/s}\left(t + \int_0^t \alpha(\tau)d\tau\right). \tag{4.17}$$

We have

Theorem 4.2. *Let S be as in* (4.15), *α as in* (4.16), *and w as in* (4.17). *There are $\lambda_0, \varrho > 0$ such that for every $U \in C^0([0,T]; H^{\mu+2,\lambda,s}) \cap C^1([0,T]; H^{\mu,\lambda,s})$, $\mu \in \mathbf{R}$, $\lambda \in (0,\lambda_0]$, we have*

$$\|e^{-\varrho w(t,D_x)}U(t)\|^2_{\mu,\lambda,s} \le C\Big(\|U(0)\|^2_{\mu,\lambda,s} + \int_0^t \|e^{-\varrho w(\tau,D_x)}SU(\tau)\|^2_{\mu,\lambda,s}d\tau\Big),$$
$$C > 0, \ 0 \le t \le T^*, \tag{4.18}$$

with $T^* + \int_0^{T^*} \alpha(\tau)d\tau \le \lambda/\varrho$.

Proof. Let us denote $\Lambda(t,\xi) = \lambda\langle\xi\rangle^{1/s} - \varrho w(t,\xi)$, $V = e^{\Lambda}U$, and $F = e^{\Lambda}SU$. We have to prove

$$\|V(t)\|^2_\mu \le C\Big(\|V(0)\|^2_\mu + \int_0^t \|F(\tau)\|^2_\mu d\tau\Big), \quad 0 \le t \le T^*. \tag{4.19}$$

Let $(\cdot,\cdot)$ denote the scalar product in H^μ. From Proposition 4.1, (4.15), and (4.16), for $\lambda \le \lambda_0$ and almost every $t \in [0,T^*]$, the absolutely continuous function $\|V(t)\|^2_\mu$ satisfies

$$d/dt\|V(t)\|^2_\mu =$$
$$-2\Re\left(DV(t) + \varrho(1+\alpha(t))\langle D_x\rangle^{1/s}V(t) + P_0V(t) + Q_0V(t) - F(t), V(t)\right),$$
$$P_0 \in L^1([0,T]; S^{1/s_0}), \quad |P_0(t)|_{1/s_0,\ell} \le C_\ell\alpha(t),$$
$$Q_0 \in C^0([0,T]; S^{1/s}), \quad |Q_0(t)|_{1/s,\ell} \le C_\ell \tag{4.20}$$

with $C_\ell > 0$ independent of ϱ so that we can fix ϱ in order to make

$$\varrho(1+\alpha(t))\langle D_x\rangle^{1/s} + P_0 + Q_0$$

a positive operator. Since the symbol of $D + D^*$ belongs to $C^0([0,T]; S^0)$, now (4.19) follows by Gronwall's method. □

Coming back to the scalar equation, from (4.18) and (4.14), we obtain

Theorem 4.3. *Let L in (4.6) satisfy the conditions from (4.7) to (4.11), and let α be as in (4.16), w as in (4.17). There are $\lambda_0, \varrho > 0$ such that for every $u \in \bigcap_{j=0}^{2} C^j([0,T]; H^{\mu+4-2j,\lambda,s})$, $\mu \in \mathbf{R}$, $\lambda \in (0, \lambda_0]$, we have*

$$\|e^{-\varrho w(t,D_x)}u(t)\|^2_{\mu+4/(N+2),\lambda,s} + \|e^{-\varrho w(t,D_x)}\partial_t u(t)\|^2_{\mu,\lambda,s} \leq$$

$$C\Big(\|u(0)\|^2_{\mu+2,\lambda,s} + \|\partial_t u(0)\|^2_{\mu,\lambda,s} + \int_0^t \|e^{-\varrho w(\tau,D_x)}Lu(\tau)\|^2_{\mu,\lambda,s}d\tau\Big), \tag{4.21}$$

$$C > 0,\ 0 \leq t \leq T^*,$$

with $T^* + \int_0^{T^*} \alpha(\tau)d\tau \leq \lambda/\varrho$.

Remark 4.4. From (4.21) it follows that for $s \leq s_0$ and $u_0, u_1 \in H^{\infty,\lambda^*,s}$ there is a unique solution $u \in C^1\left([0,T^*]; \bigcup_{\lambda>0} H^{\infty,\lambda,s}\right)$ of the Cauchy problem (4.5) with $T^* + \int_0^{T^*} \alpha(\tau)d\tau \leq \lambda^*/\varrho$. In particular for every $u_0, u_1 \in \gamma^{s-\varepsilon}$, $\varepsilon > 0$, with compact support there is a global solution $u \in C^1([0,T];\gamma^s)$ since u_0 and u_1 belong to $H^{\infty,\lambda,s}$ for every $\lambda > 0$.

References

[1] R. Agliardi, *A generalization of an example by V. Ya. Ivriĭ,* Ann. Univ. Ferrara, 39 (1993), 93–109.

[2] M. Cicognani and R. Agliardi, *Operators of p-evolution with non regular coefficients in the time variable,* to appear in J. Differential Equations.

[3] M. Cicognani and F. Colombini, *Sharp regularity of the coefficients in the Cauchy problem for a class of evolution equations,* Differential and Integral Equations, 16 (2003), 1321–1344.

[4] F. Colombini, E. Jannelli, and S. Spagnolo, *Well-posedness in the Gevrey classes of the Cauchy problem for a nonstrictly hyperbolic equation with coefficients depending on time,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., 10 (1983), 291–312.

[5] F. Colombini and S. Spagnolo, *An example of a weakly hyperbolic Cauchy problem not well posed in C^∞,* Acta Math., 148 (1982), 243–253.

[6] V. Ja. Ivriĭ, *Conditions for correctness in Gevrey classes of the Cauchy problem for hyperbolic operators with characteristics of variable multiplicity,* (Russian) Sibirsk. Mat. Ž., 17 (1976), 1256–1270.

[7] K. Kajitani and S. Wakabayashi, *Microhyperbolic operators in Gevrey classes,* Publ. Res. Inst. Math. Sci., 25 (1989), 169–221.