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GLOBAL AND ALMOST-GLOBAL EXISTENCE FOR SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH DIFFERENT PROPAGATION SPEEDS

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Abstract. We consider a system of nonlinear wave equations

$$(\partial_t^2 - c_i^2\Delta_x)u_i = F_i(u, \partial u, \partial_x\partial u) \text{ in } (0,\infty)\times\mathbb{R}^3$$

for $i = 1, \dots, m$, where $F = (F_1, \dots, F_m)$ is a smooth function satisfying

$$F(u, \partial u, \partial_x\partial u) = O(|u|^3 + |\partial u|^2 + |\partial_x\partial u|^2) \quad \text{near the origin,}$$

$u = (u_1, \dots, u_m)$, while ∂u and $\partial_x\partial u$ represent the first and second derivatives of u, respectively. We assume $0 < c_1 \le c_2 \le \cdots \le c_m$.

In this paper, we show global existence of classical solutions to the above system with small initial data under the "null condition" for systems with different propagation speeds. We also show "almost-global" existence for the above system for the case where the null condition is not satisfied.

1. Introduction

In this paper, we consider the Cauchy problem for systems of nonlinear wave equations of the form

$$\Box_i u_i = F_i(u, \partial u, \partial_x\partial u) \quad \text{in } (0,\infty)\times\mathbb{R}^n \ (i = 1, 2, \dots, m) \tag{1.1}$$

with initial data

$$u(0,x) = \varepsilon f(x),\ u_t(0,x) = \varepsilon g(x), \tag{1.2}$$

where $\Box_i = \partial_t^2 - c_i^2\Delta_x = \partial_t^2 - c_i^2\sum_{k=1}^n \partial_k^2$ $(i = 1, \dots, m)$, $u = (u_j)_{j=1,\dots,m}$, $\partial u = (\partial_a u_j)_{\substack{j=1,\dots,m\\ a=0,\dots,n}}$, $\partial_x\partial u = (\partial_k\partial_a u_j)_{\substack{j=1,\dots,m\\ k=1,\dots,n\\ a=0,\dots,n}}$. Here we have used the notation $\partial_0 = \partial_t = \partial/\partial t$ and $\partial_k = \partial/\partial x_k$ for $k = 1, \dots, n$. We assume

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$0 < c_1 \le c_2 \le \cdots \le c_m$. We suppose that $F = (F_i)_{i=1,\dots,m}$ is a smooth function of degree p around the origin, namely,

$$F(u,v,w) = F\Big((u_j)_{j=1,\dots,m},\ (v_{j,a})_{\substack{j=1,\dots,m,\\ a=0,\dots,n}},\ (w_{j,ka})_{\substack{j=1,\dots,m\\ k=1,\dots,n\\ a=0,\dots,n}}\Big)$$
$$= O\big(|u|^p + |v|^p + |w|^p\big) \tag{1.3}$$

around the origin in $\mathbb{R}^m \times \mathbb{R}^{m(n+1)} \times \mathbb{R}^{mn(n+1)}$, with some integer $p (\ge 2)$. Here $v_{j,a}$ and $w_{j,ka}$ represent variables for which $\partial_a u_j$ and $\partial_k \partial_a u_j$ are substituted, respectively. ε in the initial condition (1.2) is a positive parameter, which is always assumed to be sufficiently small. To ensure the hyperbolicity of the system (1.1), throughout this paper we always assume

$$c^{ij}_{ka}(u,v,w) = c^{ji}_{ka}(u,v,w)\ \big(i,j \in \{1,\dots,m\}, k \in \{1,\dots,n\}, a \in \{0,\dots,n\}\big) \tag{1.4}$$

for any (u,v,w) in some neighborhood of the origin in $\mathbb{R}^m \times \mathbb{R}^{m(n+1)} \times \mathbb{R}^{mn(n+1)}$, where c^{ij}_{ka} is given by

$$c^{ij}_{ka}(u,v,w) = \frac{\partial F_i}{\partial w_{j,ka}}(u,v,w) \tag{1.5}$$

for i, $j \in \{1,\dots,m\}$, $k \in \{1,\dots,n\}$, and $a \in \{0,\dots,n\}$. Because we consider only classical solutions, without loss of generality we may also assume that

$$c^{ij}_{kl} = c^{ij}_{lk} \quad \big(i,j \in \{1,\dots,m\},\ k,l \in \{1,\dots,n\}\big). \tag{1.6}$$

We note that the local existence of classical solutions to the system (1.1) is ensured under the conditions (1.4) and (1.6), because we can reduce (1.1) to a symmetric hyperbolic system of Friedrichs (see Kubota-Yokoyama [22] for instance).

Our interest lies in the condition to guarantee existence of global solutions for the system (1.1) with small data. In the following, we say that **(GE) holds** when for any f, $g \in C^\infty_0(\mathbb{R}^n)$, there exists a positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$, the Cauchy problem (1.1)–(1.2) admits a unique solution $u \in C^\infty\big([0,\infty) \times \mathbb{R}^n; \mathbb{R}^m\big)$.

Now we recall some known results quickly, restricting our attention to the cases $n = 3$ and $n = 2$. Since there are examples of quadratic (respectively cubic) nonlinearity in three (respectively two) space dimensions for which blowing up of the solution in finite time occurs, we need some special condition to ensure (GE) for these cases.

To state the condition, we introduce a family $I(i)$ $(i \in \{1,\dots,m\})$ of sets of indices by $I(i) = \big\{j \in \{1,\dots,m\}; c_j = c_i\big\}$, and also a family Y^m_i of subspaces in $\mathbb{R}^m$ by $Y^m_i = \{y = (y_1,\dots,y_m) \in \mathbb{R}^m; y_j = 0 \text{ if } j \notin I(i)\}$.

For any given function $G = G(u, v, w)$ and a natural number q, we define $G^{(q)}(u, v, w)$ by the q-th degree term of Taylor expansion of G around the origin; i.e.,

$$G^{(q)}(u,v,w) = \sum_{|\alpha|+|\beta|+|\gamma|=q} \frac{\partial_u^\alpha \partial_v^\beta \partial_w^\gamma G(0,0,0)}{\alpha!\beta!\gamma!} u^\alpha v^\beta w^\gamma. \tag{1.7}$$

Here α, β, and γ are multi-indices, and we have used the standard notation for multi-indices. Then, the known conditions for (GE) can be unified like the following, which we call the "null condition":

Definition 1.1 (the null condition)**.** *We say the system* (1.1) *satisfies the null condition* (*of degree* p) *when the following two conditions are fulfilled:*
(i) *For each* $i \in \{1, \dots, m\}$,

$$F_i^{(p)}\big(U, V(\mu, X), W(\nu, X)\big) \equiv 0 \tag{1.8}$$

holds for any $U, \mu, \nu \in Y_i^m$ *and* $X = (X_0, X_1, \dots, X_n) \in \mathbb{R}^{n+1}$ *with* $X_0^2 - c_i^2 \sum_{k=1}^n X_k^2 = 0$, *where* $V(\mu, X) = \big(V_{j,a}(\mu, X)\big)_{j,a} := (\mu_j X_a)_{j,a}$ *and* $W(\nu, X) = (W_{j,ka}(\nu, X))_{j,k,a} := (\nu_j X_k X_a)_{j,k,a}$. *Here the index* j *runs from* 1 *to* m, *the index* k *from* 1 *to* n, *and the index* a *from* 0 *to* n.
(ii) *For each* $i \in \{1, \dots, m\}$, $F_i^{(p)}(u, 0, 0) = 0$ *holds for any* $u \in \mathbb{R}^m$.

Remark. If $c_1 = \cdots = c_m$, we have $Y_i^m = \mathbb{R}^m$. Therefore, we do not need the condition (ii) in the above definition when $c_1 = \cdots = c_m$, because it is just a special case of the condition (i).

For the single-speed case (i.e., $c_1 = c_2 = \cdots = c_m$), Klainerman ([18]) and Christodoulou ([5]) proved (GE) under the null condition for $(n,p) = (3,2)$ independently by different methods (see also John [11]; refer to Keel-Smith-Sogge [16] for the mixed problems), and (GE) under the null condition for $(n,p) = (2,3)$ was proved by the author [14] (see also Godin [6], Hoshiga [8], and the author [13]). The extremely difficult case $(n,p) = (2,2)$ was investigated by Alinhac ([2], [3]), and he proved that if $F = F(\partial u, \partial_x \partial u)$ and the null conditions of degree 2 and 3 are satisfied, then (GE) holds.

From now on, we consider the case where the speeds c_i do not necessarily coincide with each other. If we assume F is independent of u itself, that is, $F = F(\partial u, \partial_x \partial u)$, (GE) under the null condition was proved by Yokoyama ([26]) for $(n,p) = (3,2)$ and Hoshiga-Kubo ([9]) for $(n,p) = (2,3)$ (see also Kovalyov [21], Agemi-Yokoyama [1] and Sideris-Tu [24]). The case where F depends both on u and its derivatives is rather complicated. This case is treated in Kubota-Yokoyama [22] for $(n,p) = (3,2)$, but unfortunately they needed to add an additional condition to the null condition, in order to obtain (GE). First of all, they assumed the following condition:

$$F(u, v, w) = O(|u|^3 + |v|^2 + |w|^2) \text{ near } (u, v, w) = (0, 0, 0). \tag{1.9}$$

Note that the condition (1.9) follows naturally from the null condition (of degree 2) when we consider the single-speed case $c_1 = \cdots = c_m$.

In Kubota-Yokoyama [22], to prove (GE) for $(n, p) = (3, 2)$, they also assumed that one of three conditions $(\mathrm{A})_1$–$(\mathrm{A})_3$ holds in addition to the null condition (of degree 2) and the condition (1.9). Here we just mention what kind of condition each additional condition is, and omit the details. The condition $(\mathrm{A})_1$ is the further restriction on the quadratic part of F, which makes the decay estimates better. The condition $(\mathrm{A})_2$ is a restriction on all parts of F, which assures that $\operatorname{supp} f \cup \operatorname{supp} g \subset \{|x| \le R\}$ implies $\operatorname{supp} u_i(t, \cdot) \subset \{|x| \le c_i t + R\}$ for all $i \in \{1, \ldots, m\}$. The condition $(\mathrm{A})_3$ is a restriction on the cubic part of F, concerning the dependence on u. Here we remark that the assumption that the data are compactly supported was used essentially in the proof of Kubota-Yokoyama [22] in connection with the additional assumptions $(\mathrm{A})_2$ and $(\mathrm{A})_3$, though the support condition is not needed for all the other results in three space dimensions mentioned above. In fact, we can obtain the results other than [22] just by assuming that the data are in the Schwartz class $\mathfrak{S}$ (the class of rapidly decreasing functions) for example.

Our purpose in this paper is to show that neither $(\mathrm{A})_1$, $(\mathrm{A})_2$, nor $(\mathrm{A})_3$ is necessary for (GE). We also show that the support condition on data is not needed for global existence of solutions either. Our main result is the following theorem.

Theorem 1.1. *Let* $(n, p) = (3, 2)$. *Suppose that* (1.4) *and* (1.6) *hold. If the condition* (1.9) *is fulfilled and* F *satisfies the null condition of degree* 2*, then for any* f, $g \in \mathfrak{S}(\mathbb{R}^3)$*, there exists a positive constant* ε_0 *such that the Cauchy problem* (1.1)–(1.2) *admits a unique global solution* $u \in C^\infty\big([0, \infty) \times \mathbb{R}^3; \mathbb{R}^m\big)$ *for any* $\varepsilon \in (0, \varepsilon_0]$.

Remark. In [15], the author obtained some examples of nonlinearity, which satisfy all the conditions in Theorem 1.1 but the condition (1.9), and for which we have global existence of small solutions. This fact suggests that the condition (1.9) might be able to be somewhat relaxed.

Since local existence of classical solutions is already known, what we need for establishing the global existence result is some *a priori* estimate. Theorem 1.1 can be proved by following almost the same arguments and estimates as in Kubota-Yokoyama [22]. The only difference lies in the treatment of L^2 norms of solutions. They carefully avoided getting control of L^2 norms of

u explicitly, and this is the reason why they needed additional assumptions $(A)_1$–$(A)_3$, and also the support condition on data to estimate the energy. In our proof, we will get a bound for L^2 norms of u explicitly by making use of a rather classical inequality for linear wave equations (see Lemma 2.2 below), along with an elementary inequality (see Proposition 3.4 below). This makes it possible to estimate the energy without any further assumptions than (1.9) and the null condition.

Now, let us consider the case where the null condition is not necessarily satisfied. For this case, we have "almost-global" existence of solutions.

Theorem 1.2. *Let* $(n, p) = (3, 2)$*. Suppose that* (1.4) *and* (1.6) *hold. If the condition* (1.9) *is fulfilled, then for any* f, $g \in \mathfrak{S}(\mathbb{R}^3)$*, there exists two positive constants* ε_0 *and* C *such that the Cauchy problem* (1.1)–(1.2) *admits a unique solution* $u \in C^\infty\big([0, T_\varepsilon) \times \mathbb{R}^3; \mathbb{R}^m\big)$ *for any* $\varepsilon \in (0, \varepsilon_0]$*, where the lifespan* T_ε *satisfies the estimate* $T_\varepsilon \geq \exp(C\varepsilon^{-1})$.

Almost-global existence results in three space dimensions for the single-speed case and multiple-speed case with $F = F(\partial u, \partial_x \partial u)$ were treated in Klainerman [17], John [12], and Klainerman-Sideris [19] (see also Kovalyov [20] for the corresponding result in two space dimensions). Our result can be regarded as an extension of these results to the case where $F = F(u, \partial u, \partial_x \partial u)$. We remark that, for single equations with $F = F(u, \partial u, \partial_x \partial u)$, a better result than our Theorem 1.2 was obtained by Lindblad ([23]). His condition is $F(u, 0, 0) = O(|u|^3)$, instead of the condition (1.9), and hence some quadratic terms like $u(\partial_a u)$ are allowed to be included (see (2.27) of Theorem 2.3 in [23]). Unfortunately, his method requires the data to be compactly supported and, more importantly, is not directly applicable to our system, even if the speeds coincide with each other. Nevertheless the author conjectures Lindblad's result is true also for our system.

The strategy in this paper is as follows. In Section 2, we describe some known estimates for linear wave equations, including L^∞–L^∞ decay estimates by Kubota-Yokoyama [22], the L^2 estimate, and the classical energy inequality for symmetric hyperbolic systems. In Section 3, we present some technical lemmas to treat nonlinearity. Also the estimate for null forms due to Yokoyama [26] will be given in that section. In Section 4, we will give a proof for Theorem 1.1 by deriving *a priori* estimates. Although most of the estimates are essentially the same as those in Kubota-Yokoyama [22], we will give a full proof for the sake of completeness. Theorem 1.2 will be proved in Section 5.

Before concluding this section, we present some notation which will be used throughout this paper.

Notation. Let f and g be functions in $\mathfrak{S}$, where $\mathfrak{S}$ denotes the Schwartz class (the class of rapidly decreasing functions), and let the speeds c_i ($i = 1, \dots, m$) be given positive constants. For $i = 1, \dots, m$, we define a mapping $U_i^*[f,g]$ by

$$U_i^*[f,g](t,x) = u(t,x) \text{ for } t > 0 \text{ and } x \in \mathbb{R}^3, \tag{1.10a}$$

where u is the unique classical solution to

$$\begin{cases} \Box_i u(t,x) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ u(0,x) = f(x),\ (\partial_t u)(0,x) = g(x) & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{1.10b}$$

For a given function $\phi = \phi(t,x)$, we define another mapping $U_i[\phi]$ by

$$U_i[\phi](t,x) = v(t,x) \text{ for } (t,x) \in (0,\infty) \times \mathbb{R}^3\ , \tag{1.11a}$$

where v is the unique classical solution to

$$\begin{cases} \Box_i v(t,x) = \phi(t,x) & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^3, \\ v(0,x) = (\partial_t v)(0,x) = 0 & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{1.11b}$$

For the sake of simplicity of exposition, when a function f and a family of functions $\{g_\alpha, \alpha \in \Lambda\}$ are given, we write $f(t,x) = \sum'_{\alpha\in\Lambda} g_\alpha(t,x)$, if there exist appropriate constants C_α ($\alpha \in \Lambda$) such that $f(t,x) = \sum_{\alpha\in\Lambda} C_\alpha g_\alpha(t,x)$. Finally, we introduce vector fields Γ_a ($0 \le a \le 7$) by

$$\Gamma_0 = t\partial_t + \sum_{k=1}^{3} x_k \partial_k,\ \Gamma_1 = \partial_t,\ \Gamma_2 = \partial_1,\ \Gamma_3 = \partial_2,\ \Gamma_4 = \partial_3, \tag{1.12}$$
$$\Gamma_5 = \Omega_{12},\ \Gamma_6 = \Omega_{13},\ \Gamma_7 = \Omega_{23},$$

where $\Omega_{ij} = x_i\partial_j - x_j\partial_i$ ($1 \le i,j \le 3$). We write Γ^α for $\Gamma_0^{\alpha_0}\Gamma_1^{\alpha_1}\cdots\Gamma_7^{\alpha_7}$ using a multi-index α. It is easy to check that for any smooth function ϕ we have

$$\Gamma^\alpha\Gamma^\beta\phi(t,x) = \Gamma^{\alpha+\beta}\phi(t,x) + \sum_{|\gamma|\le|\alpha|+|\beta|-1}{}' \Gamma^\gamma\phi(t,x), \tag{1.13}$$

$$\begin{cases} \Gamma^\alpha(\partial_a\phi)(t,x) = \partial_a(\Gamma^\alpha\phi)(t,x) + \displaystyle\sum_{\substack{b\in\{0,1,2,3\}\\ |\beta|\le|\alpha|-1}}{}' \partial_b(\Gamma^\beta\phi)(t,x), \\ \partial_a(\Gamma^\alpha\phi)(t,x) = \Gamma^\alpha(\partial_a\phi)(t,x) + \displaystyle\sum_{\substack{b\in\{0,1,2,3\}\\ |\beta|\le|\alpha|-1}}{}' \Gamma^\beta(\partial_b\phi)(t,x). \end{cases} \tag{1.14}$$

We also have

$$\Box_i\Gamma^\alpha\phi(t,x) = \Gamma^\alpha\big(\Box_i\phi(t,x)\big) + \sum_{|\beta|\le|\alpha|-1}{}' \Gamma^\beta\big(\Box_i\phi(t,x)\big), \tag{1.15}$$

where $\Box_i = \partial_t^2 - c_i^2 \Delta_x$.

For a nonnegative integer s and a function v for which the following definitions make sense, we define

$$|v(t,x)|_s = \sum_{|\alpha|\le s} |\Gamma^\alpha v(t,x)|, \tag{1.16}$$

$$\|v(t,\cdot)\|_{s,p} = \big\||v(t,\cdot)|_s\big\|_{L^p(\mathbb{R}^3)} \quad (1 \le p \le \infty). \tag{1.17}$$

In what follows, C stands for various positive constants, which may change line by line.

2. Estimates for linear wave equations

We start this section with the well-known energy inequality for hyperbolic systems.

Lemma 2.1 (the energy inequality)**.** *Let $v = (v_1, \dots, v_m)$ be a solution to*

$$\partial_t^2 v_i(t,x) - \sum_{j=1}^m \sum_{\substack{1\le k\le 3\\ 0\le a\le 3}} S_{k,a}^{i,j}(t,x)\big(\partial_k\partial_a v_j\big)(t,x) = \Phi_i(t,x) \text{ in } (0,\infty)\times\mathbb{R}^3$$

for $i = 1, \dots, m$, with initial data $v = f$ and $v_t = g$ at $t = 0$. Assume that $S_{k,a}^{i,j} = S_{k,a}^{j,i}$ and $S_{k,l}^{i,j} = S_{l,k}^{i,j}$ hold for $1 \le i,j \le m$, $0 \le a \le 3$, and $1 \le k,l \le 3$.

We also assume that there exists a positive constant M such that

$$M^{-1}|\xi|^2 \le \sum_{i,j=1}^m \sum_{k,l=1}^3 S_{k,l}^{i,j}(t,x)\xi_{i,k}\xi_{j,l} \le M|\xi|^2$$

holds for any $(t,x) \in [0,\infty)\times\mathbb{R}^3$ and any $\xi = \big(\xi_{i,k}\big)_{\substack{i=1,\dots,m\\ k=1,2,3}} \in \mathbb{R}^{3m}$. Then we have

$$\begin{aligned}\|\partial v(t,\cdot)\|_{L^2} \le & C(\|f\|_{H^1} + \|g\|_{L^2}) \\ & + C\int_0^t (\|\partial S(\tau,\cdot)\|_{L^\infty}\|\partial v(\tau,\cdot)\|_{L^2} + \|\Phi(\tau,\cdot)\|_{L^2})\,d\tau,\end{aligned} \tag{2.1}$$

where $S = \Big(S_{k,a}^{i,j}\Big)_{\substack{i,j=1,\dots,m\\ k=1,2,3\\ a=0,1,2,3}}$, $\Phi = (\Phi_i)_{i=1,\dots,m}$.

The above energy inequality can be easily obtained by the standard argument.

The following lemma is due to Strauss [25].

Lemma 2.2. *Let $\Phi \in C\big([0,T);\mathfrak{S}(\mathbb{R}^3)\big)$. Suppose $f,\ g \in \mathfrak{S}(\mathbb{R}^3)$. Then, for $i \in \{1,\dots,m\}$, we have*

$$\big\|U_i[\Phi](t,\cdot)\big\|_{L^2} \le C\int_0^t \|\Phi(\tau,\cdot)\|_{L^{6/5}}d\tau \text{ for } t\in[0,T), \tag{2.2}$$

and

$$\big\|U_i^*[f,g](t,\cdot)\big\|_{L^2} \le C\big(\|f\|_{L^2}+\|g\|_{L^{6/5}}\big) \text{ for } t\in[0,T). \tag{2.3}$$

Proof. Since (2.2) is an immediate consequence of (2.3) by Duhamel's principle, it suffices to prove (2.3). For simplicity, we assume $c_i = 1$. By the explicit representation of the solution to the wave equation in the Fourier space, and then by Plancherel's formula, we have

$$\begin{aligned}\|U_i^*[f,g](t,\cdot)\|_{L^2(\mathbb{R}^3)} \le & \Big\|\cos(|\cdot|t)\hat f(\cdot)\Big\|_{L^2(\mathbb{R}^3)} + \left\|\frac{\sin(|\cdot|t)}{|\cdot|}\hat g(\cdot)\right\|_{L^2(\mathbb{R}^3)} \\ \le & \|\hat f\|_{L^2(\mathbb{R}^3)} + \left\|\frac{\hat g(\cdot)}{|\cdot|}\right\|_{L^2(\mathbb{R}^3)} \le \|f\|_{L^2(\mathbb{R}^3)} + \|g\|_{\dot H^{-1}(\mathbb{R}^3)},\end{aligned} \tag{2.4}$$

and this completes the proof, because $L^{6/5}(\mathbb{R}^3)$ is continuously embedded in $\dot H^{-1}(\mathbb{R}^3)$ by Sobolev's embedding theorem. □

Observing

$$U_i[\partial_a\Phi] = \partial_a U_i[\Phi] - \delta_{0,a}U_i^*[0,\Phi(0,\cdot)], \quad a=0,1,2,3 \tag{2.5}$$

with Kronecker's delta $\delta_{b,c}$, from the energy inequality and (2.3) we get the following:

Lemma 2.3. *Let $a \in \{0,1,2,3\}$. Then for $t>0$, we have*

$$\|U_i[\partial_a\Phi](t,\cdot)\|_{L^2} \le C\left(\int_0^t \|\Phi(\tau,\cdot)\|_{L^2}d\tau + \delta_{0,a}\|\Phi(0,\cdot)\|_{L^{6/5}}\right). \tag{2.6}$$

Now we turn our attention to decay estimates of solutions. We start with the homogeneous case.

Lemma 2.4. *Let $f,\ g\in\mathfrak{S}(\mathbb{R}^3)$. Then for $(t,x)\in[0,\infty)\times\mathbb{R}^3$, we have*

$$\begin{aligned}&(1+t+|x|)\big(1+\big|c_it-|x|\big|\big)|U_i^*[f,g](t,x)| \\ &\quad\le C\Bigg(\sum_{|\alpha|\le 1}\Big\|(1+|\cdot|)^3\,\partial_x^\alpha f(\cdot)\Big\|_{L^\infty(\mathbb{R}^3)} + \Big\|(1+|\cdot|)^3\,g(\cdot)\Big\|_{L^\infty(\mathbb{R}^3)}\Bigg).\end{aligned} \tag{2.7}$$

For the proof, see Proposition 3.3 in Kubota-Yokoyama [22] (see also Asakura [4]).

Next, we consider the inhomogeneous case. In the following lemma, we summarize the L^∞–L^∞ estimates due to Kubota-Yokoyama [22], which will be used in the proof of our theorem. To state the lemma, we introduce some notation: For a sufficiently smooth function $v(t,x)$, a nonnegative integer s, and a constant $\rho > 0$, we define

$$\langle v(t,x)\rangle_s^{(i)} = (1+t+|x|)\left(1+\log\frac{1+c_it+|x|}{1+\big|c_it-|x|\,\big|}\right)^{-1}|v(t,x)|_s, \tag{2.8}$$

$$[v(t,x)]_{\rho,s}^{(i)} = (1+|x|)\left(1+\big|c_it-|x|\,\big|\right)^{\rho}|v(t,x)|_s. \tag{2.9}$$

We define a family of sets Λ_i ($i \in \{0,1,\dots,m\}$) of $[0,\infty)\times[0,\infty)$ associated with the given speeds $c_1,\dots,c_m$ by

$$\Lambda_i = \{(\tau,\lambda)\in[1,\infty)\times[1,\infty); |\lambda - c_i\tau| \le c_{m+1}\tau\} \text{ for } i=1,\dots,m, \tag{2.10a}$$

$$\Lambda_0 = \big\{[0,\infty)\times[0,\infty)\big\}\setminus\bigcup_{j=1}^m \Lambda_j, \tag{2.10b}$$

where we have set

$$c_0 = 0, \tag{2.11a}$$

$$c_{m+1} = \frac{1}{3}\min_{j\in I}(c_j - c_{j-1}) \text{ with } I = \big\{j\in\{1,\dots,m\}; c_j \ne c_{j-1}\big\}. \tag{2.11b}$$

Lemma 2.5 (Kubota-Yokoyama [22])**.** *Let $\Phi(t,x)$ be a smooth function decaying sufficiently fast at spatial infinity.*
(i) *Suppose $\mu > 0$ and $\nu \ge 0$. Let s be a nonnegative integer. Then we have*

$$\begin{aligned} w_+(t,|x|)^{-\nu}\langle U_i[\Phi](t,x)\rangle_s^{(i)} \le{}& C\sup_{\substack{\tau\in[0,t)\\ y\in\mathbb{R}^3}} |y| z_\mu(\tau,|y|)\, w_+(\tau,|y|)^{-\nu}|\Phi(\tau,y)|_s \\ &+ C\sum_{|\alpha|\le s-1}\sup_{y\in\mathbb{R}^3}|\Phi_\alpha(y)|(1+|y|)^3, \end{aligned} \tag{2.12}$$

$$\begin{aligned} w_+(t,|x|)^{-\nu}\big[\partial U_i[\Phi](t,x)\big]_{1,s}^{(i)} \le{}& C\sup_{\substack{\tau\in[0,t)\\ y\in\mathbb{R}^3}} |y| z_\mu(\tau,|y|)\, w_+(\tau,|y|)^{-\nu}|\Phi(\tau,y)|_{s+1} \\ &+ C\sum_{|\alpha|\le s}\sup_{y\in\mathbb{R}^3}|\Phi_\alpha(y)|(1+|y|)^3, \end{aligned} \tag{2.13}$$

where $w_+(\tau,\lambda)$ and $z_\mu(\tau,\lambda)$ are functions on $[0,\infty)\times[0,\infty)$ which are given by

$$w_+(\tau,\lambda) = 1+\tau+\lambda \textit{ for } (\tau,\lambda)\in[0,\infty)\times[0,\infty), \tag{2.14}$$

and

$$z_\mu(\tau,\lambda) = (1+\tau+\lambda)^{1+\mu}\,(1+|c_j\tau-\lambda|)^{1-\mu} \tag{2.15}$$

for $(\tau,\lambda)\in\Lambda_j$ *with* $0\le j\le m$*, respectively, while* Φ_α *is given by* $\Phi_\alpha(x) = \Gamma^\alpha\Phi(t,x)\big|_{t=0}$ *for any multi-index* α.

(ii) *For* $0<\rho\le 1$ *and* $\mu>0$*, we have*

$$\begin{aligned}\left[\partial U_i[\Phi](t,x)\right]^{(i)}_{\rho,s} \le & C \sup_{\substack{\tau\in[0,t)\\ y\in\mathbb{R}^3}} |y| z^i_{\mu,\rho}(\tau,|y|)\,|\Phi(\tau,y)|_{s+1} \\ & + C\sum_{|\alpha|\le s}\sup_{y\in\mathbb{R}^3}|\Phi_\alpha(y)|(1+|y|)^3,\end{aligned} \tag{2.16}$$

where

$$z^i_{\mu,\rho}(\tau,\lambda) = \begin{cases}(1+\tau+\lambda)^{\rho+\mu}(1+|c_j\tau-\lambda|)^{1-\mu} & \text{for } (\tau,\lambda)\in\Lambda_j \text{ if } j\notin I(i),\\ (1+\tau+\lambda)^{1+\mu}(1+|c_i\tau-\lambda|)^{\rho-\mu} & \text{for } (\tau,\lambda)\in\Lambda_i,\end{cases} \tag{2.17}$$

while Φ_α *is defined as before.*

(iii) *Let* $k\in\{1,\dots,m\}$ *and* $\mu>0$*. Assume that* $k\notin I(i)$*. Then we have*

$$\begin{aligned}\langle U_i[\Phi](t,x)\rangle^{(i)}_s \le & C \sup_{\substack{\tau\in[0,t)\\ y\in\mathbb{R}^3}} |y|\widetilde{z}^k_\mu(\tau,|y|)\,|\Phi(\tau,y)|_s \\ & + C\sum_{|\alpha|\le s-1}\sup_{y\in\mathbb{R}^3}|\Phi_\alpha(y)|(1+|y|)^3,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\left[\partial U_i[\Phi](t,x)\right]^{(i)}_{1,s} \le & C \sup_{\substack{\tau\in[0,t)\\ y\in\mathbb{R}^3}} |y|\widetilde{z}^k_\mu(\tau,|y|)\,|\Phi(\tau,y)|_{s+1} \\ & + C\sum_{|\alpha|\le s}\sup_{y\in\mathbb{R}^3}|\Phi_\alpha(y)|(1+|y|)^3,\end{aligned} \tag{2.19}$$

where

$$\widetilde{z}^k_\mu(\tau,\lambda) = \begin{cases}(1+\tau+\lambda)^{1+\mu}(1+|c_j\tau-\lambda|)^{1-\mu} & \text{for } (\tau,\lambda)\in\Lambda_j \text{ if } j\notin I(k),\\ (1+\tau+\lambda)(1+|c_k\tau-\lambda|)^{1+\mu} & \text{for } (\tau,\lambda)\in\Lambda_k.\end{cases} \tag{2.20}$$

Lemma 2.5 (in more general form) is proved in Kubota-Yokoyama [22]. (i) is the special case of Corollary 3.6 in [22]. (ii) and (iii) are essentially $(3.31)_2$ in Theorem 3.4 and the special case of Theorem 3.7 (especially $(3.34)_1$ and $(3.35)_1$) in [22], respectively.

To prove the almost-global existence result, we use the following variation of Lemma 2.5.

Lemma 2.6. *Let $\Phi(t,x)$ be a smooth function decaying sufficiently fast at spatial infinity, and s be a nonnegative integer. Then we have*

$$\langle U_i[\Phi](t,x)\rangle_s^{(i)} \le C\log(2+t) \sup_{\substack{\tau\in[0,t)\\ y\in\mathbb{R}^3}} |y| z_0\left(\tau,|y|\right)|\Phi(\tau,y)|_s + C\sum_{|\alpha|\le s-1} \sup_{y\in\mathbb{R}^3} |\Phi_\alpha(y)|(1+|y|)^3, \tag{2.21}$$

$$\left[\partial U_i[\Phi](t,x)\right]_{1,s}^{(i)} \le C\log(2+t) \sup_{\substack{\tau\in[0,t)\\ y\in\mathbb{R}^3}} |y| z_0\left(\tau,|y|\right)|\Phi(\tau,y)|_{s+1} + C\sum_{|\alpha|\le s} \sup_{y\in\mathbb{R}^3} |\Phi_\alpha(y)|(1+|y|)^3, \tag{2.22}$$

where w_+ and Φ_α are defined as in Lemma 2.5, *and*

$$z_0(\tau,\lambda) = (1+\tau+\lambda)\left(1+|c_j\tau-\lambda|\right) \ \text{for } (\tau,\lambda)\in\Lambda_j \ (j=0,1,\dots,m). \tag{2.23}$$

Lemma 2.6 follows from a slight modification of the proof of Lemma 2.5. To illustrate the modification, here we prove only (2.21) with $s=0$ after giving a sketch of the proof for (2.12) with $s=0$. Other estimates can be shown by similar modification of proof for corresponding estimates in Lemma 2.5, and we omit the details.

Outline of proof for (2.12) **and** (2.21) **with** $s=0$**.** Fix some $i\in\{1,\dots,m\}$. By an apparent change of variables, the general case can be reduced to the special case where $c_i=1$ and $x=(0,0,r)$ with $r\ge 0$. Then, recalling the explicit representation formula of the solution which is due to John [10], we have

$$U_i[\Phi](t,x) = \frac{1}{4\pi r}\int_0^t d\tau \int_{|r-(t-\tau)|}^{r+t-\tau} \lambda\, d\lambda \int_0^{2\pi} \Phi\Big(\tau, \lambda\Theta\big(\theta(\tau,\lambda;t,r),\varphi\big)\Big) d\varphi, \tag{2.24}$$

where $x=(0,0,r)$ and

$$\Theta(\theta,\varphi) = \big(\Theta_1(\theta,\varphi),\Theta_2(\theta,\varphi),\Theta_3(\theta,\varphi)\big) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

while $\theta(\tau,\lambda;t,r)$ is a function which satisfies $0\le\theta(\tau,\lambda;t,r)\le\pi$ and is defined implicitly by

$$\cos\theta(\tau,\lambda;t,r) = \frac{\lambda^2+r^2-(t-\tau)^2}{2r\lambda}. \tag{2.25}$$

For $\mu \geq 0$ and $\nu \geq 0$, we set

$$I_{\mu,\nu}(t,r) = \frac{1}{r} \iint_{(\tau,\lambda)\in D(t,r)} \frac{(1+\tau+\lambda)^\nu}{z_\mu(\tau,\lambda)} d\tau\, d\lambda \tag{2.26}$$
$$= \sum_{j=0}^{m} \frac{1}{r} \iint_{(\tau,\lambda)\in D(t,r)\cap\Lambda_j} \frac{(1+\tau+\lambda)^\nu}{(1+\tau+\lambda)^{1+\mu}(1+|c_j\tau-\lambda|)^{1-\mu}} d\tau\, d\lambda,$$

where $D(t,r) = \{(\tau,\lambda) \in \mathbb{R}\times\mathbb{R};\, 0 \leq \tau \leq t \text{ and } |r-(t-\tau)| \leq \lambda \leq r+(t-\tau)\}$. Now it is easy to see that (2.12) with $s = 0$ is an immediate consequence of (2.24) if we have the following inequality:

$$I_{\mu,\nu}(t,r) \leq C(1+t+r)^{\nu-1} \log\left(\frac{1+t+r}{1+|t-r|}\right) \tag{2.27}$$

for $\mu > 0$ and $\nu \geq 0$. Similarly, (2.21) with $s = 0$ follows from

$$I_{0,0}(t,r) \leq C \log(2+t)(1+t+r)^{-1} \log\left(\frac{1+t+r}{1+|t-r|}\right). \tag{2.28}$$

First we prove (2.27). Since we have $1+\tau+\lambda \leq 1+t+r$ for $(\tau,\lambda) \in D(t,r)$, it suffices to show the case $\nu = 0$. Fix some $\mu > 0$ and set

$$I_j(t,r) = \frac{1}{r} \iint_{(\tau,\lambda)\in D(t,r)} \frac{1}{(1+\tau+\lambda)^{1+\mu}(1+|c_j\tau-\lambda|)^{1-\mu}} d\tau\, d\lambda$$

for $j \in \{0, 1, \ldots, m\}$. By introducing new variables $p = \tau+\lambda$ and $q = \lambda - c_j\tau$, we obtain

$$I_j(t,r) = \frac{1}{(c_j+1)r} \int_{|t-r|}^{t+r} (1+p)^{-1-\mu} dp \int_{p_j}^{p} (1+|q|)^{-1+\mu} dq, \tag{2.29}$$

where $2p_j = (1-c_j)p + (1+c_j)(r-t)$. Observing that we have $p_j \geq -c_j p$ for $p \geq |r-t|$, we get

$$I_j(t,r) \leq \frac{C}{r} \int_{|t-r|}^{t+r} (1+p)^{-1-\mu} dp \int_{-c_j p}^{p} (1+|q|)^{-1+\mu} dq \leq \frac{C}{r} \int_{|t-r|}^{t+r} (1+p)^{-1} dp$$
$$= \frac{C}{r} \log \frac{1+t+r}{1+|t-r|}. \tag{2.30}$$

Since $\log(1+\alpha) \leq \alpha$ for any $\alpha \geq 0$, we obtain

$$\frac{1}{r} \log \frac{1+t+r}{1+|t-r|} \leq \frac{1}{r} \log\left(1 + \frac{2r}{1+|t-r|}\right) \leq \frac{2}{1+|t-r|},$$

which, together with (2.30), leads to

$$I_j(t,r) \leq C(1+|t-r|)^{-1}. \tag{2.31}$$

Since we have $1+t+r \le C(1+t-r)$ for $2r \le t$, (2.31) implies

$$I_j(t,r) \le C(1+t+r)^{-1} \log\left(\frac{1+t+r}{1+|t-r|}\right) \tag{2.32}$$

for $2r \le t$. Similarly, since $1+t+r \le 1+2r+|t-r| \le 2(1+|t-r|)$ holds for $2r \le 1$, (2.32) follows from (2.31) also for this case. On the other hand, since we have $1+t+r \le 5r$ for $2r \ge \max\{t,1\}$, (2.32) is an immediate consequence of (2.30) for this case. Therefore (2.32) holds for any t and r, and this completes the proof of (2.27) with $\nu = 0$.

Now let us turn our attention to (2.28). (2.29) remains true for $\mu = 0$, but (2.30) is replaced by

$$\begin{aligned} I_j(t,r) &\le \frac{C}{r}\int_{|t-r|}^{t+r}(1+p)^{-1}dp \int_{-c_j p}^{p}(1+|q|)^{-1}dq \\ &\le \frac{C}{r}\int_{|t-r|}^{t+r}(1+p)^{-1}\log(1+p)dp \le \frac{C}{r}\log(1+t+r)\log\frac{1+t+r}{1+|t-r|}. \end{aligned}$$

Now it is easy to show

$$I_j(t,r) \le C(1+t+r)^{-1}\log(1+t+r)\log\left(\frac{1+t+r}{1+|t-r|}\right) \tag{2.33}$$

instead of (2.32), and this implies (2.28) if $r \le (2+c_m+c_{m+1})t$, say, because we have $\log(1+t+r) \le C\log(2+t)$ for such t and r (see (2.11b) for the definition of c_{m+1}).

On the other hand, if $r \ge (2+c_m+c_{m+1})t$, then we have $D(t,r)\cap\Lambda_j = \emptyset$ for $j \in \{1,\dots,m\}$, and (2.26) implies that $I_{0,0}(t,r)$ is dominated by $I_0(t,r)$ with $\mu = 0$. Since $2p_0 = p + (r-t) \ge p$, we find $\int_{p_0}^{p}(1+q)^{-1}dq \le \log 2$, and going back to (2.29), we get

$$I_0(t,r) \le \frac{C}{r}\int_{r-t}^{t+r}(1+p)^{-1}dp \le \frac{C}{r}\log\left(\frac{1+t+r}{1+|t-r|}\right),$$

which is just the same conclusion as (2.30), and we obtain

$$I_{0,0}(t,r) \le I_0(t,r) \le C(1+t+r)^{-1}\log\left(\frac{1+t+r}{1+|t-r|}\right)$$

for $r \ge (2+c_m+c_{m+1})t$. This completes the proof of (2.28). □

3. Basic estimates for nonlinearity

In this section, we give some pointwise basic estimates for nonlinear terms. For simplicity of exposition, we introduce

$$w_+(t,r) = 1+t+r, \tag{3.1}$$

$$w_j(t,r) = 1 + |c_j t - r| \quad (0 \le j \le m) \tag{3.2}$$

for $(t,r) \in [0,\infty) \times [0,\infty)$, where, as in Section 2, $c_1, \dots, c_m$ are given constants and $c_0 = 0$.

Lemma 3.1. *Let s be a nonnegative integer, and $i \in \{1, \dots, m\}$.*
(i) *For any $\delta \in (0,1)$, there exists a positive constant C such that*

$$|v(t,x)|_s \le C w_+(t,|x|)^{-1+\delta} w_j(t,|x|)^{-\delta} \langle v(t,x) \rangle_s^{(i)} \qquad \text{if } (t,|x|) \in \Lambda_j \tag{3.3}$$

for each $j \in \{0, 1, \dots, m\}$, provided that v is sufficiently smooth.
(ii) *For any $\rho > 0$, we have*

$$|v(t,x)|_s \le \begin{cases} C w_+(t,|x|)^{-1} w_i(t,|x|)^{-\rho} [v(t,x)]_{\rho,s}^{(i)}, & \text{if } (t,|x|) \in \Lambda_i, \\ C w_+(t,|x|)^{-\rho} w_0(t,|x|)^{-1} [v(t,x)]_{\rho,s}^{(i)}, & \text{if } (t,|x|) \in \Lambda_0, \\ C w_+(t,|x|)^{-(1+\rho)} [v(t,x)]_{\rho,s}^{(i)}, & \text{otherwise.} \end{cases} \tag{3.4}$$

Especially, we have

$$|v(t,x)|_s \le C w_+(t,|x|)^{-1} w_j(t,|x|)^{-1} [v(t,x)]_{1,s}^{(i)} \qquad \text{if } (t,|x|) \in \Lambda_j \tag{3.5}$$

for any $j \in \{0, 1, \dots, m\}$, provided that v is sufficiently smooth.

Proof. First we claim

$$C^{-1}(1 + c_i\tau + \lambda) \le 1 + \left| c_i\tau - \lambda \right| \le C(1 + c_i\tau + \lambda) \tag{3.6}$$

for $(\tau, \lambda) \notin \Lambda_i$. The second part of (3.6) is obvious, so we prove the first part. Suppose that τ and λ satisfy $|c_i\tau - \lambda| \le c_{m+1}\tau$, and that either $0 \le \tau < 1$ or $0 \le \lambda < 1$ is fulfilled. Then (τ, λ) obviously belongs to some bounded set, and the first half of (3.6) is trivial. Next, suppose that $c_i\tau - \lambda > c_{m+1}\tau$ is satisfied. Then we have $c_i\tau + \lambda < (2c_i - c_{m+1})\tau$, and since (2.11b) implies $2c_i - c_{m+1} > 0$ for any $i \in \{1, \dots, m\}$, we obtain

$$1 + c_i\tau - \lambda > 1 + c_{m+1}\tau > 1 + \frac{c_{m+1}}{2c_i - c_{m+1}}(c_i\tau + \lambda) \ge C(1 + c_i\tau + \lambda).$$

Finally, suppose that τ and λ satisfy $c_i\tau - \lambda < -c_{m+1}\tau$. Then, since we have $\lambda > (c_i + c_{m+1})\tau$ or equivalently $\lambda + c_i\tau < \big(1 + c_i/(c_i + c_{m+1})\big)\lambda$, we obtain

$$1 + \lambda - c_i\tau > 1 + \frac{c_{m+1}}{c_i + c_{m+1}}\lambda > C(1 + c_i\tau + \lambda).$$

This completes the proof of (3.6).

Now we are going to prove (3.3). If we have $(t,|x|) \in \Lambda_i$, (3.3) is an immediate consequence of the definition (2.8), because for any $\delta > 0$, there exists a positive constant C_δ such that $1 + \log X \le C_\delta X^\delta$ holds for any $X \ge 1$. If $(t,|x|) \notin \Lambda_i$, from (3.6) and (2.8) we have

$$|v(t,x)| \le C(1 + t + |x|)^{-1} \langle v(t,x) \rangle_s^{(i)} = w_+(t,x)^{-1} \langle v(t,x) \rangle_s^{(i)},$$

which is better than (3.3) for $(t,|x|) \in \Lambda_j$ with $j \notin I(i)$. The proof of (3.3) is done.

Observing that $(1+\tau+\lambda) \le C(1+\lambda)$ for $(\tau,\lambda) \notin \Lambda_0$, we obtain (3.4) immediately from the definition (2.9) and (3.6). (3.5) is a consequence of (3.4) with $\rho = 1$, because $w_j(t,|x|) \le Cw_+(t,|x|)$. □

Now we want to investigate estimates for the nonlinear terms. For simplicity of exposition, we sometimes abbreviate $F\big(u(t,x), \partial u(t,x), \partial_x \partial u(t,x)\big)$ as $F(u,\partial u, \partial_x\partial u)(t,x)$, or more simply as $F(t,x)$ and so on.

For a multi-index α, we define

$$F_{i,\alpha}(t,x) = \Gamma^\alpha F_i(u,\partial u,\partial_x\partial u)(t,x) - \sum_{j,k,a} c^{ij}_{ka}(t,x)\partial_k\partial_a(\Gamma^\alpha u_j)(t,x) \tag{3.7}$$

and

$$\widetilde{F}_{i,\alpha}(t,x) = \Gamma^\alpha F_i(u,\partial u,\partial_x\partial u)(t,x) - \sum_{j,k,a} \partial_k\big(c^{ij}_{ka}(t,x)(\Gamma^\alpha \partial_a u_j)(t,x)\big), \tag{3.8}$$

where

$$c^{ij}_{ka}(t,x) = \frac{\partial F_i}{\partial w_{j,ka}}(u,\partial u,\partial_x\partial u)(t,x) \tag{3.9}$$

for $i,j \in \{1,\dots,m\}$, $1 \le k \le 3$ and $0 \le a \le 3$.

Lemma 3.2. *Assume that F satisfies the condition* (1.9). *Let s be a positive integer, and α be a multi-index satisfying $|\alpha| = s$. Suppose $|u(t,x)|_{[\frac{s}{2}]} + |\partial u(t,x)|_{[\frac{s}{2}]+1}$ to be sufficiently small. Then we have*

$$\begin{aligned} |\Gamma^\alpha F_i(t,x)| \le& C|\partial u(t,x)|_{[\frac{s}{2}]+1}|\partial u(t,x)|_{s+1} \\ &+ C\left(|u(t,x)|_{[\frac{s}{2}]} + |\partial u(t,x)|_{[\frac{s}{2}]+1}\right)^2 \big(|u(t,x)|_s + |\partial u(t,x)|_{s+1}\big), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |F_{i,\alpha}(t,x)|,\ |\widetilde{F}_{i,\alpha}(t,x)| \le& C|\partial u(t,x)|_{[\frac{s}{2}]+1}|\partial u(t,x)|_s \\ &+ C\left(|u(t,x)|_{[\frac{s}{2}]} + |\partial u(t,x)|_{[\frac{s}{2}]+1}\right)^2 \big(|u(t,x)|_s + |\partial u(t,x)|_s\big). \end{aligned} \tag{3.11}$$

Proof. (3.10) is an immediate consequence of the Leibniz formula and (1.9). Observing that we need $|\partial u|_{s+1}$ only for the control of $c^{ij}_{ka}\Gamma^\alpha(\partial_k\partial_a u_j)$ in (3.10), we find that $\big|\Gamma^\alpha F_i - \sum_{j,k,a} c^{ij}_{ka}\Gamma^\alpha(\partial_k\partial_a u_j)\big|$ is dominated by the right-hand side of (3.11). This completes the proof, because (1.13) and (1.14) imply that

$$|c^{ij}_{ka}\Gamma^\alpha(\partial_k\partial_a)u_j - c^{ij}_{ka}\partial_k\partial_a\Gamma^\alpha u_j| \text{ and } |c^{ij}_{ka}\Gamma^\alpha(\partial_k\partial_a)u_j - \partial_k(c^{ij}_{ka}\Gamma^\alpha\partial_a u_j)|$$

are dominated by the right-hand side of (3.11). □

Let K be an integer, and ν_1 and ν_2 be positive constants, while ρ_1, $\rho_2 \in (0,1)$. For given K, ν_1, ν_2, ρ_1, and ρ_2, we define

$$e_1[u](t,x) = \sum_{i=1}^{m} \left(\langle u_i(t,x) \rangle_{K+2}^{(i)} + [\partial u_i(t,x)]_{1,K+1}^{(i)} \right), \tag{3.12a}$$

$$e_2[u](t,x) = \sum_{i=1}^{m} [\partial u_i(t,x)]_{\rho_1,K+3}^{(i)}, \tag{3.12b}$$

$$e_3[u](t,x) = \sum_{i=1}^{m} w_+(t,|x|)^{-\rho_2} \left(\langle u_i(t,x) \rangle_{2K-3}^{(i)} + [\partial u_i(t,x)]_{1,2K-4}^{(i)} \right), \tag{3.12c}$$

and also

$$E_j(T) = E_j[u](T) = \sup_{0 \le t < T} \| e_j[u](t,\cdot) \|_{L^\infty(\mathbb{R}^3)} \quad \text{for } j = 1,2,3, \tag{3.12d}$$

$$E_4(T) = E_4[u](T) = \sup_{0 \le t < T} (1+t)^{-\nu_1} \| u(t,\cdot) \|_{2K,2}, \tag{3.12e}$$

$$E_5(T) = E_5[u](T) = \sup_{0 \le t < T} (1+t)^{-\nu_2} \| \partial u(t,\cdot) \|_{2K,2}. \tag{3.12f}$$

In the rest of this section, $\sum_{k=1}^5 E_k[u](T)$ is always assumed to be sufficiently small. We abbreviate $E_j[u](T)$ to $E_j(T)$ in what follows, as far as it does not cause any confusion.

Lemma 3.3. *Let $\delta \in (0,1/2)$. Suppose $\nu_1 \ge \nu_2$. Assume that F satisfies the condition* (1.9)*. Then we have*

$$\sum_{|\alpha|=2K} \| F_{i,\alpha}(t,\cdot) \|_{L^2(\mathbb{R}^3)} + \| F(t,\cdot) \|_{2K-1,2} \tag{3.13}$$
$$\le C(1+t)^{\nu_2-1} E_1(T) E_5(T) + C(1+t)^{\nu_1+2\delta-2} E_1(T)^2 \big(E_4(T) + E_5(T) \big)$$

for $0 \le t < T$, where C is a constant independent of T. Moreover, we have

$$\sum_{|\alpha| \le 1} \sum_{j,k,a} \| \partial^\alpha c_{ka}^{ij}(t,\cdot) \|_{L^\infty(\mathbb{R}^3)} \le C(1+t)^{-1} E_1(T), \tag{3.14}$$

$$\sum_{|\alpha|=2K} \sum_{j,k,a} \| c_{ka}^{ij}(t,\cdot) \Gamma^\alpha \partial_a u_i(t,\cdot) \|_{L^2(\mathbb{R}^3)} \le C(1+t)^{\nu_2-1} E_1(T) E_5(T). \tag{3.15}$$

Proof. By Lemma 3.1, we have

$$\| u(t,\cdot) \|_{K,\infty} \le (1+t)^{-1+\delta} E_1(T) \text{ and } \| \partial u(t,\cdot) \|_{K+1,\infty} \le (1+t)^{-1} E_1(T).$$

Now (3.13) is obtained immediately from Lemma 3.2 and Hölder's inequality.

Observing that $c^{ij}_{ka} = O\big(|\partial u| + |\partial_x \partial u| + |u|^2\big)$, we get

$$|\partial^\alpha c^{ij}_{ka}(t,x)| \le C(1+t)^{-1}E_1(T) + C(1+t)^{-2+2\delta}E_1(T)^2$$

for $|\alpha| \le 1$. Since $-2+2\delta \le -1$ and $E_1(T)^2 \le E_1(T)$, we obtain (3.14). (3.15) is an immediate consequence of Hölder's inequality and (3.14). □

The following proposition plays an important role in our L^2 estimates.

Proposition 3.4. *Let $c \ge 0$. Suppose $0 \le \delta < 1/6$. Then*

$$\int_{x\in\mathbb{R}^3} (1+t+|x|)^{-6+6\delta}\left(1+\big|ct-|x|\,\big|\right)^{-6\delta} dx \le C(1+t)^{-3} \tag{3.16}$$

holds for any $t > 0$. Here C is a constant depending only on δ and c.

Proof. Though the proof is elementary and simple, we give it here because of its importance in L^2 estimates of the solution.

By changing to polar coordinates, we have

$$\begin{aligned}
&\int_{x\in\mathbb{R}^3} (1+t+|x|)^{-6+6\delta}\left(1+\big|ct-|x|\,\big|\right)^{-6\delta} dx \qquad (3.17)\\
&\qquad = 4\pi \int_0^\infty (1+t+r)^{-6+6\delta}\,(1+|ct-r|)^{-6\delta}\, r^2 dr\\
&\qquad \le 4\pi \int_0^\infty (1+t+r)^{-4+6\delta}\,(1+|ct-r|)^{-6\delta}\, dr.
\end{aligned}$$

We split the integral on the right-hand side of (3.17) into three parts: $\int_0^\infty = \int_0^{ct} + \int_{ct}^{(c+1)t} + \int_{(c+1)t}^\infty \equiv I_1 + I_2 + I_3$. Note that $1-6\delta > 0$ and $-3+6\delta < 0$ by the assumption. Noting that $1+t \le 1+t+r$, we have

$$\begin{aligned}
I_1 &\le (1+t)^{-4+6\delta} \int_0^{ct} (1+ct-r)^{-6\delta} dr\\
&\le C(1+t)^{-4+6\delta}(1+ct)^{1-6\delta} \le C(1+t)^{-3}.
\end{aligned} \tag{3.18}$$

Similarly, I_2 can be estimated as

$$\begin{aligned}
I_2 &\le (1+t)^{-4+6\delta} \int_{ct}^{(c+1)t} (1+r-ct)^{-6\delta} dr\\
&\le C(1+t)^{-4+6\delta}(1+t)^{1-6\delta} \le C(1+t)^{-3}.
\end{aligned} \tag{3.19}$$

Finally, since $1+r-ct \ge 1+t$ for $r \ge (c+1)t$, we obtain

$$\begin{aligned}
I_3 &\le (1+t)^{-6\delta} \int_{(c+1)t}^\infty (1+t+r)^{-4+6\delta} dr\\
&\le C(1+t)^{-6\delta}\,(1+(c+2)t)^{-3+6\delta} \le C(1+t)^{-3}.
\end{aligned} \tag{3.20}$$

This completes the proof. □

Lemma 3.5. *Assume that F satisfies the condition* (1.9), *and let $|\alpha| = 2K$. Suppose $\nu_1 \geq \nu_2$. Then we have*

$$\big\|\widetilde{F}_{i,\alpha}(t)\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|F_i(t)\|_{2K-1,\frac{6}{5}} \leq C(1+t)^{-\frac{1}{3}+\frac{5}{3}\nu_2} E_1(T)^{\frac{1}{3}} E_5(T)^{\frac{5}{3}} + C(1+t)^{\nu_1-1} E_1(T)^2 \big(E_4(T)+E_5(T)\big) \tag{3.21}$$

for $0 \leq t < T$, where C is a constant independent of T.

Proof. By Lemma 3.2 and Hölder's inequality, we have

$$\big\|\widetilde{F}_{i,\alpha}(t)\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|F_i(t)\|_{2K-1,\frac{6}{5}} \leq C\|\partial u(t)\|_{K+1,3}\|\partial u(t)\|_{2K,2} + C\,\big\|\big(|u(t)|_K^2 + |\partial u(t)|_{K+1}^2\big)\big\|_{L^3}\big(\|u(t)\|_{2K,2} + \|\partial u(t)\|_{2K,2}\big). \tag{3.22}$$

Since we have

$$\|\partial u(t)\|_{K+1,3}^3 \leq \|\partial u(t)\|_{K+1,\infty}\|\partial u(t)\|_{K+1,2}^2 \leq (1+t)^{-1+2\nu_2} E_1(T) E_5(T)^2$$

and $\|\partial u(t)\|_{2K,2} \leq (1+t)^{\nu_2} E_5(T)$, the first term on the right-hand side of (3.22) is dominated by $C(1+t)^{-\frac{1}{3}+\frac{5}{3}\nu_2} E_1(T)^{\frac{1}{3}} E_5(T)^{\frac{5}{3}}$.

Now we are going to estimate the second term of the right-hand side of (3.22). Choose some $\delta \in (0, 1/6)$. For $0 \leq t < T$, since Lemma 3.1 implies

$$|u(t,x)|_K + |\partial u(t,x)|_{K+1} \leq C w_+(t,|x|)^{-1+\delta} w_j(t,|x|)^{-\delta} E_1(T)$$

when $(t,|x|) \in \Lambda_j$ $(j = 0, 1, \ldots, m)$, we obtain

$$\big\|\big(|u(t,\cdot)|_K^2 + |\partial u(t,\cdot)|_{K+1}^2\big)\big\|_{L^3}^3 \leq C E_1(T)^6 \sum_{j=0}^{m} \int_{\mathbb{R}^3} w_+(t,|x|)^{-6+6\delta} w_j(t,|x|)^{-6\delta} dx. \tag{3.23}$$

Now Proposition 3.4 leads to

$$\big\|\big(|u(t,\cdot)|_K^2 + |\partial u(t,\cdot)|_{K+1}^2\big)\big\|_{L^3} \leq C(1+t)^{-1} E_1(T)^2. \tag{3.24}$$

Since $\nu_1 \geq \nu_2$, we have

$$\|u(t)\|_{2K,2} + \|\partial u(t)\|_{2K,2} \leq (1+t)^{\nu_1}\big(E_4(T) + E_5(T)\big),$$

and we see that the second term of the right-hand side of (3.22) is dominated by $C(1+t)^{\nu_1-1} E_1(T)^2 \big(E_4(T) + E_5(T)\big)$. This completes the proof. □

In the remainder of this section, we will derive some L^∞ estimates for nonlinear terms. We define

$$\Lambda_{j,T} = \big\{(t,x) \in [0,T) \times \mathbb{R}^3;\ (t,|x|) \in \Lambda_j\big\} \tag{3.25}$$

for $j = 0, 1, \ldots, m$, where Λ_j is given by (2.10a) and (2.10b).

Lemma 3.6. *Let $j \in \{0, 1, \dots, m\}$, and $\delta \in (0, 1/6)$. Suppose $K \geq 9$. If F satisfies the condition* (1.9)*, then we have*

$$\begin{aligned} r|F_i(t,x)|_{2K-3} \leq & C w_+(t,r)^{-1+\nu_2} w_j(t,r)^{-1} E_1(T) E_5(T) \\ & + C w_+(t,r)^{-\frac{3}{2}+\nu_2+2\delta} w_j(t,r)^{-2\delta} E_1(T)^2 E_5(T) \end{aligned} \tag{3.26}$$

for any $(t,x) \in \Lambda_{j,T}$, where $r = |x|$.

Proof. Suppose $(t,r) \in \Lambda_j$. By (3.3), (3.5), and Lemma 3.2, we have

$$\begin{aligned} r|F_i(t,x)|_{2K-3} \leq & C w_+(t,r)^{-1} w_j(t,r)^{-1} E_1(T) r |\partial u(t,x)|_{2K-2} \\ & + C w_+(t,r)^{-2+2\delta} w_j(t,r)^{-2\delta} E_1(T)^2 \\ & \times r\big(|u(t,x)|_{2K-3} + |\partial u(t,x)|_{2K-2}\big). \end{aligned} \tag{3.27}$$

Making use of the Sobolev-type inequalities

$$\sqrt{r}|u(t,x)|_s \leq C\|\partial u(t,x)\|_{s+2,2}, \tag{3.28}$$

$$r|\partial u(t,x)|_s \leq C\|\partial u(t,x)\|_{s+2,2}, \tag{3.29}$$

we obtain

$$r|\partial u(t,x)|_{2K-2} \leq C\|\partial u(t,\cdot)\|_{2K,2} \leq C(1+t)^{\nu_2} E_5(T), \tag{3.30}$$

$$r|u(t,x)|_{2K-3} \leq C\sqrt{r}\|\partial u(t,\cdot)\|_{2K-1,2} \leq C(1+t+r)^{\frac{1}{2}+\nu_2} E_5(T) \tag{3.31}$$

(see Lemma 5.2 and Corollary 5.3 in Kubota-Yokoyama [22] for the proof of (3.28) and (3.29); see also Klainerman [18] and Klainerman-Sideris [19]). Now (3.26) follows from (3.27), (3.30), and (3.31). □

The structure of nonlinearity which we have used in the proof so far is only (1.9). Now we are going to take advantage of the null condition. First we introduce null forms Q_0 and Q_{ab}:

$$Q_0(\phi, \psi; c) = (\partial_t \phi)(\partial_t \psi) - c^2 \sum_{j=1}^{3} (\partial_j \phi)(\partial_j \psi), \tag{3.32}$$

$$Q_{ab}(\phi, \psi) = (\partial_a \phi)(\partial_b \psi) - (\partial_b \phi)(\partial_a \psi) \quad \text{for } 0 \leq a < b \leq 3. \tag{3.33}$$

If the nonlinear term F satisfies the condition (1.9) and the null condition, it is easy to see that for each $i \in \{1, \dots, m\}$, F_i has the following form (see Yokoyama [26] for instance):

$$F_i(u, \partial u, \partial_x \partial u) = N_i(\partial u, \partial_x \partial u) + R_i(\partial u, \partial_x \partial u) + H_i(u, \partial u, \partial_x \partial u), \tag{3.34}$$

where

$$N_i(\partial u, \partial_x \partial u) = \sum_{(j,k) \in I(i) \times I(i)}{}' \Big(\sum_{0 \leq |\alpha|, |\beta| \leq 1}{}' Q_0(\partial^\alpha u_j, \partial^\beta u_k; c_i) \Big)$$

$$+\sum_{(j,k)\in I(i)\times I(i)}{}' \Big(\sum_{\substack{0\le|\alpha|,|\beta|\le 1\\ 0\le a<b\le 3}}{}' Q_{ab}(\partial^\alpha u_j, \partial^\beta u_k)\Big), \tag{3.35}$$

$$R_i(\partial u, \partial_x\partial u) = \sum_{(j,k)\notin I(i)\times I(i)}{}' \Big(\sum_{1\le|\alpha|,|\beta|\le 2}{}' (\partial^\alpha u_j)(\partial^\beta u_k)\Big), \tag{3.36}$$

and H_i is a smooth function satisfying

$$H_i(u, \partial u, \partial_x\partial u) = O(|u|^3 + |\partial u|^3 + |\partial_x\partial u|^3). \tag{3.37}$$

We split R_i further into two parts:

$$R_i(\partial u, \partial_x\partial u) = \sum_{q\notin I(i)} R_{i,q}(\partial u, \partial_x\partial u) + \widetilde{R}_i(\partial u, \partial_x\partial u), \tag{3.38}$$

where

$$R_{i,q}(\partial u, \partial_x\partial u) = \sum_{j,k\in I(q)}{}' \Big(\sum_{1\le|\alpha|,|\beta|\le 2}{}' (\partial^\alpha u_j)(\partial^\beta u_k)\Big), \tag{3.39}$$

$$\widetilde{R}_i(\partial u, \partial_x\partial u) = \sum_{j,\,k \text{ with } I(j)\neq I(k)}{}' \Big(\sum_{1\le|\alpha|,|\beta|\le 2}{}' (\partial^\alpha u_j)(\partial^\beta u_k)\Big). \tag{3.40}$$

To start with, we estimate the higher nonlinearity.

Lemma 3.7. *Let $\delta \in (0, 1/6)$ and $j \in \{0, 1, \dots, m\}$. Suppose $K \ge 9$. Then we have*

$$r|H_i(t,x)|_{K+4} \le C w_+(t,r)^{-2+\rho_2+3\delta} w_j(t,r)^{-3\delta} E_1(T)^2 E_3(T), \tag{3.41}$$

$$\begin{aligned} r|H_i(t,x)|_{K+2} \le & C w_+(t,r)^{-2+3\delta} w_j(t,r)^{-3\delta} E_1(T)^3 \\ & + C w_+(t,r)^{-2+\rho_2+2\delta} w_j(t,r)^{-1-2\delta} E_1(T)^2 E_3(T) \end{aligned} \tag{3.42}$$

for any $(t,x) \in \Lambda_{j,T}$, where $r = |x|$.

Proof. Let $s \le K+4$. Since H_i is a function of cubic degree, similarly to Lemma 3.2 we obtain

$$\begin{aligned} |H_i|_s \le & C \left(|u|_{[\frac{s}{2}]} + |\partial u|_{[\frac{s}{2}]+1}\right)^2 \left(|u|_s + |\partial u|_{s+1}\right) \\ \le & C \left(|u|_{K+2} + |\partial u|_{K+1}\right)^2 \left(|u|_s + |\partial u|_{s+1}\right). \end{aligned} \tag{3.43}$$

Here we have used $\left[\frac{s}{2}\right] \le K$, which is true for $s \le K+4$ and $K \ge 3$.

Suppose $(t,r) \in \Lambda_j$. From (3.3) and (3.5) in Lemma 3.1, we get

$$|u(t,x)|_{K+2} + |\partial u(t,x)|_{K+1} \le C w_+(t,r)^{-1+\delta} w_j(t,r)^{-\delta} E_1(T), \tag{3.44a}$$

$$|u(t,x)|_{2K-4} \le C w_+(t,r)^{-1+\rho_2+\delta} w_j(t,r)^{-\delta} E_3(T), \quad (3.44b)$$

$$|\partial u(t,x)|_{2K-4} \le C w_+(t,r)^{-1+\rho_2} w_j(t,r)^{-1} E_3(T). \quad (3.44c)$$

Since $K+5 \le 2K-4$ for $K \ge 9$, we have

$$|u(t,x)|_s + |\partial u(t,x)|_{s+1} \le \begin{cases} |u(t,x)|_{2K-4} + |\partial u(t,x)|_{2K-4} & \text{for } s = K+4, \\ |u(t,x)|_{K+2} + |\partial u(t,x)|_{2K-4} & \text{for } s = K+2. \end{cases} \quad (3.45)$$

Now, observing that $r \le w_+(t,r)$, we obtain (3.41) and (3.42) immediately from (3.43)–(3.45). □

Next, we consider the quadratic terms.

Lemma 3.8. *Suppose $K \ge 9$, and $1 \le |\alpha|, |\beta| \le 2$.*
(i) *Let k, $l \in \{1, \dots, m\}$. Then we have*

$$r\big|(\partial^\alpha u_k)(\partial^\beta u_l)\big|_{K+4} \le C w_+(t,x)^{-2+\rho_2} w_0(t,x)^{-1} E_1(T) E_3(T) \quad (3.46)$$

for any $(t,x) \in \Lambda_{0,T}$, where $r = |x|$. We also have

$$r\big|(\partial^\alpha u_k)(\partial^\beta u_l)\big|_{K+4} \le C w_+(t,x)^{-1+\rho_2} w_j(t,x)^{-2} E_1(T) E_3(T) \quad (3.47)$$

for any $(t,x) \in \Lambda_{j,T}$ with $j \in \{1, \dots, m\}$.
(ii) *Let $k \in I(q)$ for some $q \in \{1, \dots, m\}$, and $l \in \{1, \dots, m\}$. Suppose $j \in \{1, \dots, m\}$. If $j \notin I(q)$, then we have*

$$r\big|(\partial^\alpha u_k)(\partial^\beta u_l)\big|_{K+4} \le C w_+(t,x)^{-2+\rho_2} w_j(t,x)^{-1} E_1(T) E_3(T) \quad (3.48)$$

for any $(t,x) \in \Lambda_{j,T}$, where $r = |x|$.

Proof. Since $K \ge 9$, we have

$$\begin{aligned}\big|(\partial^\alpha u_k)(\partial^\beta u_l)\big|_{K+4} \le & C|\partial u_k|_{[\frac{K+4}{2}]+1} |\partial u_l|_{K+5} + C|\partial u_l|_{[\frac{K+4}{2}]+1} |\partial u_k|_{K+5} \\ \le & C|\partial u_k|_{K+1} |\partial u_l|_{2K-4} + C|\partial u_l|_{K+1} |\partial u_k|_{2K-4}. \quad (3.49)\end{aligned}$$

For $(t,x) \in \Lambda_{0,T}$, the second line of (3.4) with $\rho = 1$ immediately implies (3.46), because $r \le w_0(t,r)$. (3.47) can be obtained from (3.5) because $r \le w_+(t,r)$.

Similarly, observing that $\Lambda_j \cap \Lambda_k = \emptyset$ holds for $k \in I(q)$ and $j \notin I(q)$, we can apply the third line of (3.4) with $\rho = 1$ to u_k and (3.5) to u_l, respectively, and (3.48) is shown. □

Lemma 3.9. *Let $j \in \{0, 1, \dots, m\}$. Suppose $K \ge 9$. Then we have*

$$r\big|\widetilde{R}_i(t,x)\big|_{K+4} \le C w_+(t,r)^{-2+\rho_2} w_j(t,r)^{-1} E_1(T) E_3(T) \quad (3.50)$$

for any $(t,x) \in \Lambda_{j,T}$, where $r = |x|$.

Proof. It suffices to prove (3.50) for $\widetilde{R}_i = (\partial^\alpha u_k)(\partial^\beta u_l)$, where $I(k) \neq I(l)$, and $1 \leq |\alpha|, |\beta| \leq 2$. If $j = 0$, the result is nothing else than (3.46).

Since $I(k) \neq I(l)$, either $I(k)$ or $I(l)$ differs from $I(j)$. Therefore (3.50) for $j \neq 0$ is an immediate consequence of (3.48). □

Lemma 3.10. *Let $j \in \{0, 1, \ldots, m\}$ and $q \in \{1, \ldots, m\}$. Suppose $K \geq 9$. Then we have*

$$r|R_{i,q}(t,x)|_{K+4} \leq C w_+(t,r)^{-1+\rho_2} w_q(t,r)^{-2} E_1(T) E_3(T), \tag{3.51}$$

$$r|R_{i,q}(t,x)|_{K+2} \leq C w_+(t,r)^{-1} w_q(t,r)^{-1-\rho_1} E_1(T) E_2(T) \tag{3.52}$$

for $(t,x) \in \Lambda_{q,T}$, and also

$$r|R_{i,q}(t,x)|_{K+4} \leq C w_+(t,r)^{-2+\rho_2} w_j(t,r)^{-1} E_1(T) E_3(T) \tag{3.53}$$

for $(t,x) \in \Lambda_{j,T}$ with $j \notin I(q)$, where $r = |x|$.

Proof. It suffices to prove the result for $R_{i,q} = \partial^\alpha u_k \partial^\beta u_l$ with $k, l \in I(q)$, and $1 \leq |\alpha|, |\beta| \leq 2$. Similarly to (3.49), we obtain

$$\big|(\partial^\alpha u_k)(\partial^\beta u_l)\big|_{K+2} \leq C|\partial u_k|_{K+1} |\partial u_l|_{K+3} + C|\partial u_l|_{K+1} |\partial u_k|_{K+3}. \tag{3.54}$$

Since $k, l \in I(q)$, we have $c_k = c_l = c_q$, and hence $\Lambda_k = \Lambda_l = \Lambda_q$. Therefore the first line of (3.4) in Lemma 3.1 leads to

$$|\partial u_k(t,x)|_{K+3} + |\partial u_l(t,x)|_{K+3} \leq C w_+(t,r)^{-1} w_q(t,r)^{-\rho_1} E_2(T) \tag{3.55}$$

in $\Lambda_{q,T}$. Remembering

$$|\partial u(t,x)|_{K+2} \leq C w_+(t,r)^{-1} w_q(t,r)^{-1} E_1(T)$$

in $\Lambda_{q,T}$ by (3.5), we obtain (3.52). (3.51) and (3.53) are obtained immediately from (3.47) and (3.48) in Lemma 3.8, respectively. □

Finally, we give an estimate for null terms, which is due to Yokoyama [26] (see also Kubota-Yokayama [22]).

Lemma 3.11. *Let $i \in \{1, \ldots, m\}$, and s be a positive integer. Then we have*

$$\begin{aligned} |Q_0(\phi_1, \phi_2; c_i)(t,x)|_s \leq & C w_+(t,|x|)^{-1} w_i(t,|x|) |\partial\phi|_{[\frac{s}{2}]} |\partial\phi|_s \\ & + w_+(t,|x|)^{-1} \big(|\partial\phi|_{[\frac{s}{2}]} |\phi|_{s+1} + |\phi|_{[\frac{s}{2}]+1} |\partial\phi|_s\big), \end{aligned} \tag{3.56}$$

$$|Q_{ab}(\phi_1, \phi_2)(t,x)|_s \leq w_+(t,|x|)^{-1} \big(|\partial\phi|_{[\frac{s}{2}]} |\phi|_{s+1} + |\phi|_{[\frac{s}{2}]+1} |\partial\phi|_s\big) \tag{3.57}$$

for any (t,x) with $(t,|x|) \in \Lambda_i$, and for any smooth function

$$\phi(t,x) = \big(\phi_1(t,x), \phi_2(t,x)\big).$$

Proof. Since we can prove

$$\Gamma^\alpha Q_0(\phi_1,\phi_2;c_i)={\sum_{|\beta|+|\gamma|\le|\alpha|}}' Q_0(\Gamma^\beta\phi_1,\Gamma^\gamma\phi_2;c_i), \tag{3.58a}$$

$$\Gamma^\alpha Q_{ab}(\phi_1,\phi_2)={\sum_{\substack{|\beta|+|\gamma|\le|\alpha|\\ 0\le c<d\le 3}}}' Q_{cd}(\Gamma^\beta\phi_1,\Gamma^\gamma\phi_2), \tag{3.58b}$$

it suffices to prove (3.56) and (3.57) for $s=0$. By direct computation we have

$$\begin{aligned} Q_{0j}(\phi_1,\phi_2)=&\frac{x_j}{tr}[\Gamma_0,\partial_r](\phi_1,\phi_2)+\sum_{k\neq j}\frac{x_k}{tr^2}[\Gamma_0,\Omega_{kj}](\phi_1,\phi_2)\\ &+\sum_{k\neq j}\frac{x_k}{tr}[\partial_r,\Omega_{kj}](\phi_1,\phi_2),\\ Q_{jk}(\phi_1,\phi_2)=&\sum_{l\neq k}\frac{x_jx_l}{r^3}[\partial_r,\Omega_{lk}](\phi_1,\phi_2)+\sum_{l\neq j}\frac{x_kx_l}{r^3}[\Omega_{lj},\partial_r](\phi_1,\phi_2)\\ &+\sum_{\substack{l_1\neq j\\ l_2\neq k}}\frac{x_{l_1}x_{l_2}}{r^4}[\Omega_{l_1j},\Omega_{l_2k}](\phi_1,\phi_2) \end{aligned}$$

for $j,k\in\{1,2,3\}$, where $\partial_r=\sum_{j=1}^3(x_j/|x|)\partial_j$, and $[A,B](\phi_1,\phi_2)$ is given by

$$[A,B](\phi_1,\phi_2)=(A\phi_1)(B\phi_2)-(B\phi_1)(A\phi_2)$$

for any vector fields A and B. Observing that $t,r\ge C(1+t+r)$ for any $(t,r)\in\Lambda_i$, and that $|\Gamma^\alpha v|/(1+t+r)\le C|\partial v|$ holds for any function v and any multi-index α with $|\alpha|=1$, we obtain (3.57) with $s=0$.

Similarly, from the expression

$$\begin{aligned} Q_0(\phi_1,\phi_2;c_i)=&\frac{1}{t^2}\big(\Gamma_0\phi_1+(c_it-r)\partial_r\phi_1\big)\big(\Gamma_0\phi_2-(c_it+r)\partial_r\phi_2\big)\\ &+c_i\sum_{j=1}^3\frac{x_j}{r}Q_{0j}(\phi_1,\phi_2)+c_i^2\sum_{j=1}^3\sum_{k\neq j}\frac{x_k}{r^2}(\partial_j\phi_1)(\Omega_{jk}\phi_2), \end{aligned}$$

we obtain (3.56). □

From Lemma 3.11, we immediately obtain the following.

Corollary 3.12. *Let $i\in\{1,\dots,m\}$, and $j\in\{0,\dots,m\}$. Suppose $\delta\in(0,1/6)$ and $K\ge 9$. Then we have*

$$r|N_i(t,x)|_{K+4}\le Cw_+(t,r)^{-2+\rho_2+\delta}w_i(t,r)^{-1-\delta}E_1(T)E_3(T) \tag{3.59}$$

for $(t,x)\in\Lambda_{i,T}$, where $r=|x|$.

If $j \notin I(i)$, we also have

$$r|N_i(t,x)|_{K+4} \le Cw_+(t,r)^{-2+\rho_2} w_j(t,r)^{-1} E_1(T)E_3(T) \tag{3.60}$$

for $(t,x) \in \Lambda_{j,T}$ with $j \notin I(i)$.

Proof. First, suppose $(t,x) \in \Lambda_{i,T}$. By (3.35) and Lemma 3.11, we obtain

$$\begin{aligned}|N_i|_{K+4} \le & Cw_+^{-1} w_i |\partial u|_{[\frac{K+4}{2}]+1} |\partial u|_{K+5} + Cw_+^{-1} |\partial u|_{[\frac{K+4}{2}]+1} (|u|_{K+5} + |\partial u|_{K+5}) \\ & + Cw_+^{-1} \big(|u|_{[\frac{K+4}{2}]+1} + |\partial u|_{[\frac{K+4}{2}]+1}\big) |\partial u|_{K+5} \\ \le & Cw_+^{-1} w_i |\partial u|_{K+1} |\partial u|_{2K-4} + w_+^{-1} |\partial u|_{K+1} \big(|u|_{2K-4} + |\partial u|_{2K-4}\big) \\ & + Cw_+^{-1} \big(|u|_{K+1} + |\partial u|_{K+1}\big) |\partial u|_{2K-4} \\ \le & Cw_+^{-3+\rho_2} w_i^{-1} E_1(T)E_3(T) + Cw_+^{-3+\rho_2+\delta} w_i^{-1-\delta} E_1(T)E_3(T).\end{aligned}$$

Observing $w_+^{-\delta} \le Cw_i^{-\delta}$ and $r \le w_+(t,r)$, we get (3.59) immediately from the above inequality.

(3.60) is an immediate consequence of (3.48) in Lemma 3.8. □

4. Proof of Theorem 1.1

For nonnegative integers s and q, we define the weighted Sobolev space $H^{s,q}$ by $H^{s,q}(\mathbb{R}^3) = \{\phi \in L^1_{loc}(\mathbb{R}^3); \|\phi\|_{H^{s,q}(\mathbb{R}^3)} < \infty\}$, where

$$\|\phi\|^2_{H^{s,q}(\mathbb{R}^3)} = \sum_{|\alpha| \le s} \|\langle | \cdot | \rangle^q \partial_x^\alpha \phi(\cdot)\|^2_{L^2(\mathbb{R}^3)},$$

and $\langle X \rangle = \sqrt{1+X^2}$ for $X \in \mathbb{R}$. We write $H^s(\mathbb{R}^3)$ for $H^{s,0}(\mathbb{R}^3)$.

The following local existence theorem is classical.

Proposition 4.1. *Consider the Cauchy problem (1.1) in $(0,\infty) \times \mathbb{R}^3$ with initial data*

$$u(0,x) = \phi(x), \ (\partial_t u)(0,x) = \psi(x) \text{ for } x \in \mathbb{R}^3. \tag{4.1}$$

Assume ϕ, $\psi \in \mathfrak{S}(\mathbb{R}^3)$. Suppose that (1.3) with $p = 2$, (1.4), and (1.6) are fulfilled. We also suppose that c^{ij}_{ka} defined by (1.5) satisfies

$$\frac{1}{C}|\xi|^2 \le \sum_{\substack{1 \le i,j \le m \\ 1 \le k,l \le 3}} \Big(\delta_{i,j}\delta_{k,l}c_i - c^{ij}_{kl}(u,v,w)\Big) \xi_{i,k}\xi_{j,l} \le C|\xi|^2 \tag{4.2}$$

for any (u,v,w) and any $\xi \in \mathbb{R}^{3m}$, with some positive constant C and the Kronecker delta $\delta_{i,j}$. Then we have the following:

(i) *There exists* $T > 0$ *such that* (1.1) *with* (4.1) *admits a unique local solution* $u \in C^\infty([0,T] \times \mathbb{R}^3)$. *More precisely, we have*

$$u \in \bigcap_{s,q=0}^{\infty} C^\infty\big([0,T]; H^{s,q}(\mathbb{R}^3)\big). \tag{4.3}$$

(ii) *Let* T^* *be the supremum of all* T *such that the above Cauchy problem* (1.1) *with* (4.1) *admits a solution* $u \in C^\infty\left([0,T] \times \mathbb{R}^3\right)$. *If* $T^* < \infty$, *then we have*

$$\sup_{(t,x)\in[0,T^*)\times\mathbb{R}^3} \sum_{|\alpha|\le 3} |\partial^\alpha u(t,x)| = \infty.$$

Remark. From Sobolev's lemma and (4.3), we can see $\sup_{x\in\mathbb{R}^3} |x|^q |\partial^\alpha u(t,x)|$ is continuous with respect to $t \in [0,T]$ for any nonnegative integer q and any multi-index α.

Outline of proof. By introducing a new variable $\widetilde{u} = (\widetilde{u}_j)_{1\le j\le 4m} = (u, \partial_x u)$, we can reduce the system (1.1) to a quasi-linear system at the cost of one regularity. Therefore it suffices to consider the quasi-linear case

$$F_i(u, \partial u, \partial_x \partial u) = \sum_{j,k,a} c_{ka}^{ij}(u, \partial u) \partial_k \partial_a u_j + \widetilde{F}_i(u, \partial u).$$

Namely we consider

$$\Box_i u_i - \sum_{j,k,a} c_{ka}^{ij}(u, \partial u)\, \partial_k \partial_a u_j = \widetilde{F}_i(u, \partial u) \quad (1 \le i \le m), \tag{4.4}$$

$$(u, \partial_t u) = (\phi, \psi) \text{ at } t = 0. \tag{4.5}$$

First we note that, by considering the energy inequality in some backward light cone, we have the uniqueness of C^2 solutions for the above quasi-linear system (and thus we get the uniqueness of C^3 solutions for our original system).

Let $s \ge 3$. From the proof of Theorem 6.4.11 in [7] with an apparent modification, we see that (4.4)–(4.5) admits a (unique) local solution $u \in X^s(T_s)$ for some $T_s > 0$, where $X^s(T)$ is defined by

$$X^s(T) = L^\infty\left([0,T]; H^{s+1}(\mathbb{R}^3)\right) \cap C^{0,1}\left([0,T]; H^s(\mathbb{R}^3)\right) \cap C^1\left([0,T]; H^{s-1}(\mathbb{R}^3)\right)$$

for $T > 0$. Here $C^{0,1}$ denotes the space of the Lipschitz-continuous functions. We also see that if $T_s^* < \infty$, then we have

$$\sup_{(t,x)\in[0,T_s^*)\times\mathbb{R}^3} \sum_{|\alpha|\le 2} |\partial^\alpha u(t,x)| = \infty,$$

where $T_s^* = \sup\{T_s; (1.1) \text{ with } (4.1) \text{ admits a solution } u \in X^s(T_s)\}$ (refer to the argument in the last part of the proof for Theorem 6.4.11 in [7]).

Let u be a solution in $X^3(T)$ with some $T > 0$. We are going to prove that this u has the regularity

$$u \in \bigcap_{s=0}^{\infty} C^{\infty}\left([0,T]; H^s(\mathbb{R}^3)\right). \tag{4.6}$$

The Sobolev's embedding theorem implies

$$\begin{aligned} &\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \sum_{|\alpha|\le 2} |\partial^\alpha u(t,x)| \qquad (4.7)\\ &\quad \le C \sup_{t\in[0,T]} \left(\|u(t,\cdot)\|_{H^4} + \|\partial_t u(t,\cdot)\|_{H^3} + \|\partial_t^2 u(t,\cdot)\|_{H^2}\right). \end{aligned}$$

Since we have $\partial_t^2 u \in L^\infty\left([0,T]; H^2\right)$ by (4.4) for $u \in X^3(T)$, we find

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \sum_{|\alpha|\le 2} |\partial^\alpha u(t,x)| \le M, \tag{4.8}$$

with some positive constant M. We already know from the above existence result that $u \in X^s(T_s)$ with some $T_s > 0$ for $s \ge 4$, and (4.8) implies $T_s^* > T$. Hence we find that u in fact belongs to $X^s(T)$ for any $s \ge 4$. Therefore we find $u \in \bigcap_{s=0}^{\infty} X^s(T)$. Using (4.4), we can also see the regularity for $\partial_t^l u$ with $l \ge 2$, and we obtain (4.6). Note that, by Sobolev's lemma, (4.6) implies $u \in C^\infty\left([0,T]\times\mathbb{R}^3\right)$. Therefore we have $T^* = T_3^*$. This completes the proof for the first half of (i), and (ii).

Finally, we are going to show that (4.3) holds for the solution $u \in X^3(T)$. Since $\widetilde{F}_i(0,0) = 0$, we can find smooth functions g_{ij} and g_{ija} such that

$$\widetilde{F}_i(u, \partial u) = \sum_j g_{ij}(u,\partial u)\, u_j + \sum_{j,a} g_{ija}(u,\partial u)\, \partial_a u_j.$$

Let $u \in X^3(T)$ be the solution to (4.4)–(4.5). For $1 \le i \le m$, we define

$$G_i(t,x,V,\partial V) = \sum_j g_{ij}(u,\partial u)(t,x) V_j + \sum_{j,a} g_{ija}(u,\partial u)(t,x)\partial_a V_j.$$

We also define

$$\Phi_{i,q}(t,x) = [\Box_i, \langle |x|\rangle^q]\, u_i(t,x) + \sum_{j,a} g_{ija}(u,\partial u)(t,x)\, [\langle |x|\rangle^q, \partial_a]\, u_j(t,x),$$

where $[A,B] = AB - BA$.

Let $q \geq 1$. Assume that we know

$$u \in \bigcap_{s=0}^{\infty} C^{\infty}\left([0,T]; H^{s,q-1}(\mathbb{R}^3)\right) \tag{4.9}$$

(note that this is true for $q = 1$). Set $V(t,x) = \langle |x| \rangle^q u(t,x)$. From (4.4), we can verify that V satisfies the following linear system:

$$\Box_i V_i - \sum_{j,k,a} c_{ka}^{ij}\left(u, \partial u\right)(t,x) \partial_k \partial_a V_j = G_i(t,x,V,\partial V) + \Phi_{i,q}(t,x), \tag{4.10}$$

$$V(0,x) = \langle |x| \rangle^q \phi(x),\ \partial_t V(0,x) = \langle |x| \rangle^q \psi(x). \tag{4.11}$$

From the assumption (4.9), we find that the coefficients g_{ij} and g_{ija} for V_j and $\partial_a V_j$ in G_i are bounded, smooth functions on $[0,T] \times \mathbb{R}^3$, and

$$\Phi_{i,q} \in \bigcap_{s=0}^{\infty} C^{\infty}\left([0,T]; H^{s}(\mathbb{R}^3)\right).$$

Now the theory of linear hyperbolic equations implies that

$$\langle |\cdot| \rangle^q u \in \bigcap_{s=0}^{\infty} C^{\infty}\left([0,T]; H^{s}(\mathbb{R}^3)\right). \tag{4.12}$$

Inductively we conclude that (4.12) is valid for all $q \geq 0$, and this implies (4.3) immediately. $\Box$

Let $u(t,x)$ be a solution to the Cauchy problem (1.1)–(1.2) for $(t,x) \in [0,T) \times \mathbb{R}^3$ with some $T > 0$. Using $E_j[u](T)$ $(1 \leq j \leq 5)$ given by (3.12d)–(3.12f), we introduce

$$E(T) = E[u](T) = \sum_{j=1}^{5} E_j[u](T). \tag{4.13}$$

We choose the constants K, ν_1, ν_2, ρ_1, and ρ_2 in (3.12a)–(3.12f) so that the following conditions are satisfied:

$$0 < \nu_2 \ll 1 \big(\nu_2 = \frac{1}{100}, \text{ say}\big),\ \nu_1 = \frac{2}{3} + \frac{5}{3}\nu_2, \tag{4.14}$$

$$K \geq 9, \tag{4.15}$$

$$\frac{1}{2} + \nu_2 < \rho_2 < 1, \quad 0 < \rho_1 < 1 - \rho_2. \tag{4.16}$$

We remark here that, by (4.14), we have

$$\nu_1 + 2\delta - 2 \leq \nu_2 - 1 \tag{4.17}$$

for sufficiently small $\delta > 0$.

For the proof of Theorem 1.1, it suffices to prove the following:

Proposition 4.2. *Let* $u \in C^\infty\big([0,T)\times\mathbb{R}^3;\mathbb{R}^m\big)$ *be a solution to the Cauchy problem* (1.1)–(1.2). *Suppose that the assumptions in Theorem* 1.1 *are fulfilled. Then there exist three positive constants* B, ε_1, *and* C_0, *which are independent of* T, *such that* $E[u](T) \le B$ *implies*

$$E[u](T) \le C_0\big(\varepsilon + E[u](T)^2\big), \tag{4.18}$$

provided $\varepsilon \le \varepsilon_1$. *Here the constants* B, ε_1, *and* C_0 *are determined by some* (*weighted*) *norms of the data, but do not depend explicitly on the size of support of the data.*

For a while, assume that Proposition 4.2 is established.

Proof of Theorem 1.1. In view of Proposition 4.1, our task is to show that $\sup_{|\alpha|\le 3}\|\partial^\alpha u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}$ stays small as far as the solution exists. Note that we can modify the definition of $F(u,v,w)$ for large (u,v,w) so that (4.2) holds, and this modification does not affect the solution at all because we are just considering small solutions.

Since we have (4.3) for the local solution by Proposition 4.1, we can easily verify that $E[u](t)$ is a continuous function with respect to t.

Choose a sufficiently large constant M which satisfies

$$M \ge 4C_0, \text{ and } E[u](0) \le \frac{M\varepsilon}{2} \text{ for any } \varepsilon \in (0,\varepsilon_1].$$

Set $\varepsilon_0 = \min\left\{\frac{1}{4C_0M}, \frac{B}{M}, \varepsilon_1\right\}$, and assume that $\varepsilon \in (0,\varepsilon_0]$. We define T_ε by $T_\varepsilon = \sup\{t \in (0,T); E[u](t) \le M\varepsilon\}$. Note that the set $\{t \in (0,T); E[u](t) \le M\varepsilon\}$ is not empty and we have $T_\varepsilon > 0$ for any fixed $\varepsilon \in (0,\varepsilon_0]$, because $E[u](0) \le M\varepsilon/2$ and $E[u](t)$ is continuous with respect to t. Since $E[u](T_\varepsilon) \le M\varepsilon \le B$, by Proposition 4.2 we conclude that

$$E[u](T_\varepsilon) \le C_0\big(\varepsilon + E[u](T_\varepsilon)^2\big) \le C_0\varepsilon + (C_0M\varepsilon)M\varepsilon \le \frac{M}{2}\varepsilon \tag{4.19}$$

holds for each $\varepsilon \in (0,\varepsilon_0]$. We see $T_\varepsilon = T$ from (4.19). In fact, if we assume $T_\varepsilon < T$, then by (4.19) and the continuity of $E[u](t)$ with respect to t, we see that $E[u](T_*) \le M\varepsilon$ for some $T_* \in (T_\varepsilon, T)$. This contradicts the definition of T_ε. Therefore we conclude that $T_\varepsilon = T$. Now, since we have proved that $\sum_{|\alpha|\le 3}\|\partial^\alpha u(t,\cdot)\|_{L^\infty} \le E[u](t) \le M\varepsilon$ holds as far as the local solution exists, Proposition 4.1 implies the existence of a global solution, provided $\varepsilon \in (0,\varepsilon_0]$. This completes the proof of Theorem 1.1. □

The remainder of this section is devoted to the proof of Proposition 4.2. We assume $E(T) \le B \ll 1$, so that we can make use of all the lemmas in Section 3. We also assume $\varepsilon \ll 1$.

Let α be a multi-index satisfying $|\alpha| \le 2K$. From (1.15), we have

$$\Box_i(\Gamma^\alpha u_i) = \Gamma^\alpha F_i + {\sum_{|\beta|\le 2K-1}}' \Gamma^\beta F_i. \tag{4.20}$$

Hence we get

$$\Box_i(\Gamma^\alpha u_i) - \sum_{j,k,a} c_{ka}^{ij}\partial_k\partial_a(\Gamma^\alpha u_j) = F_{i,\alpha} + {\sum_{|\beta|\le 2K-1}}' \Gamma^\beta F_i, \tag{4.21}$$

where $F_{i,\alpha}$ and c_{ka}^{ij} are given by (3.7) and (3.9), respectively. Thanks to (3.14) in Lemma 3.3 and the assumptions (1.4) and (1.6), we can apply Lemma 2.1 to (4.21) and, by using Lemma 3.3 again, we get

$$\begin{aligned} \|\partial(\Gamma^\alpha u)(t,\cdot)\|_{L^2} \le C\varepsilon + C\int_0^t \sum_{i,j,k,a} \|\partial c_{ka}^{ij}(\tau,\cdot)\|_{L^\infty}\|\partial(\Gamma^\alpha u)(\tau)\|_{L^2}d\tau \\ + C\sum_{i=1}^m \int_0^t \big(\|F_{i,\alpha}(\tau,\cdot)\|_{L^2} + \|F_i(\tau,\cdot)\|_{2K-1,2}\big)d\tau \\ \le C\varepsilon + C\big(E(T)^2 + E(T)^3\big)\int_0^t (1+\tau)^{\nu_2-1}d\tau \le C(1+t)^{\nu_2}\big(\varepsilon + E(T)^2\big) \end{aligned} \tag{4.22}$$

for $0 \le t < T$. Here we have also used (4.17). Therefore we obtain

$$E_5(T) \le C\big(\varepsilon + E(T)^2\big). \tag{4.23}$$

To estimate $E_4(T)$, using $\widetilde{F}_{i\alpha}$ defined by (3.8), we rewrite (4.20) as

$$\Box_i(\Gamma^\alpha u_i) = \widetilde{F}_{i,\alpha} + {\sum_{|\beta|\le 2K-1}}' \Gamma^\beta F_i + \sum_{j,k,a} \partial_k\big(c_{ka}^{ij}\Gamma^\alpha\partial_a u_j\big). \tag{4.24}$$

From Lemmas 2.2 and 3.5, remembering $\nu_1 = 2/3 + 5\nu_2/3$, we find

$$\begin{aligned} \Big\|U_i\big[\widetilde{F}_{i,\alpha}\big](t,\cdot)\Big\|_{L^2} + {\sum_{|\beta|\le 2K-1}}' \Big\|U_i\big[\Gamma^\beta F_i\big](t,\cdot)\Big\|_{L^2} \\ \le C\int_0^t \Big(\big\|\widetilde{F}_{i,\alpha}(\tau,\cdot)\big\|_{L^{\frac{6}{5}}} + \|F_i(\tau,\cdot)\|_{2K-1,\frac{6}{5}}\Big)\,d\tau \\ \le CE(T)^2\int_0^t (1+\tau)^{-\frac{1}{3}+\frac{5}{3}\nu_2}d\tau + E(T)^3\int_0^t (1+\tau)^{\nu_1-1}d\tau \le C(1+t)^{\nu_1}E(T)^2. \end{aligned} \tag{4.25}$$

On the other hand, Lemma 2.3 and (3.15) in Lemma 3.3 lead to

$$\Big\|U_i\left[\partial_k\big(c_{ka}^{ij}\Gamma^\alpha\partial_a u_j\big)\right](t)\Big\|_{L^2} \le C\varepsilon^2 + C\int_0^t \Big\|\big(c_{ka}^{ij}\Gamma^\alpha\partial_a u_j\big)(\tau)\Big\|_{L^2}\,d\tau$$

$$\leq C\varepsilon^2 + CE(T)^2 \int_0^t (1+\tau)^{\nu_2-1} d\tau \tag{4.26}$$
$$\leq C\varepsilon^2 + CE(T)^2(1+t)^{\nu_2} \leq C(1+t)^{\nu_1}\big(\varepsilon + E(T)^2\big),$$

because $\nu_2 \leq \nu_1$. By Lemma 2.2, it is easy to see

$$\big\|U_i^*[\Gamma^\alpha u_i(0), \partial_t \Gamma^\alpha u_i(0)](t,\cdot)\big\|_{L^2} \leq C\varepsilon. \tag{4.27}$$

Finally, (4.24), (4.25), (4.26), and (4.27) imply

$$E_4(T) \leq C\big(\varepsilon + E(T)^2\big). \tag{4.28}$$

Next we are going to get control of $E_3(T)$. Making use of Lemma 2.4 and Lemma 2.5 (i) with $\nu = \rho_2$, we have

$$E_3(T) \leq C\varepsilon + \sup_{\substack{\tau\in[0,T) \\ y\in\mathbb{R}^3}} z_\mu(\tau,|y|) w_+(\tau,|y|)^{-\rho_2} |y| |F_i(\tau,y)|_{2K-3} \tag{4.29}$$

for $\mu > 0$. We claim that

$$\sup_{\substack{\tau\in[0,T) \\ y\in\mathbb{R}^3}} z_\mu(\tau,|y|) w_+(\tau,|y|)^{-\rho_2} |y| |F_i(\tau,y)|_{2K-3} \leq CE(T)^2, \tag{4.30}$$

provided that μ is sufficiently small. For that purpose, by virtue of Lemma 3.6, it suffices to show that the inequalities

$$W_1(\tau,\lambda) = z_\mu(\tau,\lambda) w_+(\tau,\lambda)^{-\rho_2-1+\nu_2} w_j(\tau,\lambda)^{-1} \leq C, \tag{4.31a}$$
$$W_2(\tau,\lambda) = z_\mu(\tau,\lambda) w_+(\tau,\lambda)^{-\rho_2-\frac{3}{2}+\nu_2+2\delta} w_j(\tau,\lambda)^{-2\delta} \leq C \tag{4.31b}$$

hold for $(\tau,\lambda) \in \Lambda_j$ $(j = 0, 1, \ldots, m)$, provided that μ and δ are small. Since $Cw_+ \geq w_j$ and we may assume $1 - \mu - 2\delta > 0$, the definition of z_μ leads to

$$W_1 = w_+^{\nu_2+\mu-\rho_2} w_j^{-\mu} \text{ and } W_2 = w_+^{\nu_2+2\delta+\mu-\frac{1}{2}-\rho_2} w_j^{1-\mu-2\delta} \leq w_+^{\frac{1}{2}+\nu_2-\rho_2} \text{ in } \Lambda_j.$$

Observing that $\frac{1}{2} + \nu_2 - \rho_2 < 0$ by (4.16), and $\nu_2 + \mu - \rho_2 < -\frac{1}{2} + \mu < 0$ for sufficiently small $\mu (> 0)$, we obtain (4.31a) and (4.31b), and consequently (4.30). Now (4.29) and (4.30) lead to

$$E_3(T) \leq C\big(\varepsilon + E(T)^2\big). \tag{4.32}$$

Next, we are going to prove

$$\big\langle U_i[N_i + \widetilde{R}_i](t,x)\big\rangle_{K+4}^{(i)} + \big[\partial U_i[N_i + \widetilde{R}_i](t,x)\big]_{1,K+3}^{(i)} \leq C\big(\varepsilon + E(T)^2\big) \tag{4.33}$$

and

$$\big\langle U_i^*[\varepsilon f, \varepsilon g](t,x)\big\rangle_{K+4}^{(i)} + \big[\partial U_i^*[\varepsilon f, \varepsilon g](t,x)\big]_{1,K+3}^{(i)} \leq C\varepsilon \tag{4.34}$$

for $(t,x) \in [0,T) \times \mathbb{R}^3$.

By Lemma 2.4, it is easy to see (4.34) holds. In order to prove (4.33), by virtue of Lemma 2.5 (i) with $\nu = 0$, Lemma 3.9, and Corollary 3.12, it suffices to prove

$$W_3(\tau,\lambda) = z_\mu(\tau,\lambda) w_+(\tau,\lambda)^{-2+\rho_2+\delta} w_j(\tau,\lambda)^{-1-\delta} \le C \tag{4.35}$$

for $(\tau,\lambda) \in \Lambda_j$ $(j = 0, 1, \ldots, m)$. By the definition of z_μ, we have

$$W_3 = w_+^{\rho_2+\delta+\mu-1} w_j^{-\mu-\delta} \le w_+^{\rho_2+\delta+\mu-1}$$

in Λ_j. Since $\rho_2 < 1$, we can find δ and μ which are small enough to satisfy $\rho_2 + \delta + \mu - 1 < 0$. This completes the proof of (4.35).

Now we claim that

$$E_2(T) \le C\big(\varepsilon + E(T)^2\big). \tag{4.36}$$

Because of (4.33) and (4.34), we have only to prove

$$\sum_{q \notin I(i)} \big[\partial U_i[R_{i,q}](t,x)\big]^{(i)}_{\rho_1,K+3} + \big[\partial U_i[H_i](t,x)\big]^{(i)}_{\rho_1,K+3} \le C\big(\varepsilon + E(T)^2\big). \tag{4.37}$$

For the proof of (4.37), by applying Lemma 2.5 (ii) with $\rho = \rho_1$, we see that it suffices to prove

$$\sup_{\substack{0 \le \tau < T \\ y \in \mathbb{R}^3}} |y| z^i_{\mu,\rho_1}(\tau,|y|) \Big(\sum_{q \notin I(i)} |R_{i,q}(\tau,y)|_{K+4} + |H_i(\tau,y)|_{K+4} \Big) \le C E(T)^2. \tag{4.38}$$

For $q \notin I(i)$, Lemma 3.10 implies

$$\begin{aligned} &|y||R_{i,q}(\tau,y)|_{K+4} \\ &\qquad \le \begin{cases} C w_+(\tau,|y|)^{-2+\rho_2} w_i(\tau,|y|)^{-1} E(T)^2, & (\tau,|y|) \in \Lambda_i, \\ C w_+(\tau,|y|)^{-1+\rho_2} w_j(\tau,|y|)^{-2} E(T)^2, & (\tau,|y|) \in \Lambda_j \text{ with } j \notin I(i). \end{cases} \end{aligned} \tag{4.39}$$

(4.38) follows immediately from (4.39) and (3.41) in Lemma 3.7, if once we establish

$$W_4 = z^i_{\mu,\rho_1}\big(w_+^{-2+\rho_2} w_i^{-1} + w_+^{-2+\rho_2+3\delta} w_i^{-3\delta}\big) \le C \text{ in } \Lambda_i, \tag{4.40a}$$

$$W_5 = z^i_{\mu,\rho_1}\big(w_+^{-1+\rho_2} w_j^{-2} + w_+^{-2+\rho_2+3\delta} w_j^{-3\delta}\big) \le C \text{ in } \Lambda_j,\ j \notin I(i) \tag{4.40b}$$

for small μ and δ. From the definition of z^i_{μ,ρ_1}, we get

$$W_4 = w_+^{\mu+\rho_2-1} w_i^{\rho_1-1-\mu} + w_+^{\rho_2+3\delta+\mu-1} w_i^{\rho_1-3\delta-\mu} \le w_+^{\mu+\rho_2-1} + w_+^{\rho_1+\rho_2-1}$$

in Λ_i, because we may assume $\rho_1 - 3\delta - \mu > 0$ and we have $\rho_1 - 1 - \mu < -\mu < 0$. Now we see W_4 is bounded, since $\rho_1 + \rho_2 - 1 < 0$ by (4.16), and $\mu + \rho_2 - 1 < 0$ for small $\mu > 0$.

Similarly, since we may assume $1-\mu-3\delta>0$, we have

$$W_5 = w_+^{\rho_1+\rho_2+\mu-1} w_j^{-\mu-1} + w_+^{\rho_1+\rho_2+\mu+3\delta-2} w_j^{1-\mu-3\delta} \le w_+^{\rho_1+\rho_2+\mu-1} + w_+^{\rho_1+\rho_2-1}$$

in Λ_j $(j \notin I(i))$. W_5 is bounded because we have $\rho_1+\rho_2+\mu-1<0$ for small $\mu>0$ by (4.16). This completes the proof of (4.36).

Finally, we will prove

$$E_1(T) \le C\big(\varepsilon + E(T)^2\big). \tag{4.41}$$

Because of (4.33) and (4.34) again, it suffices to prove

$$\sum_{q \notin I(i)} \Big(\big\langle U_i[R_{i,q}](t,x)\big\rangle_{K+2}^{(i)} + \big[\partial U_i[R_{i,q}](t,x)\big]_{1,K+1}^{(i)} \Big) \le C\big(\varepsilon + E(T)^2\big), \tag{4.42}$$

$$\big\langle U_i[H_i](t,x)\big\rangle_{K+2}^{(i)} + \big[\partial U_i[H_i](t,x)\big]_{1,K+1}^{(i)} \le C\big(\varepsilon + E(T)^2\big). \tag{4.43}$$

First we prove (4.43). As before, by virtue of Lemma 2.5 (i) with $\nu=0$ and (3.42) in Lemma 3.7, it suffices to prove

$$W_6 = z_\mu \big(w_+^{-2+3\delta} w_j^{-3\delta} + w_+^{-2+2\delta+\rho_2} w_j^{-1-2\delta} \big) \le C \tag{4.44}$$

in Λ_j $(j \in \{0,1,\dots,m\})$ for sufficiently small δ and μ. Since we may assume $1-\mu-3\delta>0$, we get

$$W_6 = w_+^{\mu+3\delta-1} w_j^{1-\mu-3\delta} + w_+^{\rho_2+2\delta+\mu-1} w_j^{-2\delta-\mu} \le C + w_+^{\rho_2+2\delta+\mu-1}. \tag{4.45}$$

We have $\rho_2+2\delta+\mu-1<0$ for sufficiently small μ and δ, because $\rho_2<1$. Hence the boundedness of W_6 is ensured by (4.45). This completes the proof of (4.43).

Next, we prove (4.42). For each $U_i[R_{i,q}]$, we apply Lemma 2.5 (iii) with $k=q$. Then (3.52) and (3.53) in Lemma 3.10 imply (4.42), because we have

$$\widetilde{z}_\mu^{q}(\tau,\lambda) w_+(\tau,\lambda)^{-1} w_q(\tau,\lambda)^{-1-\rho_1} = w_q(\tau,\lambda)^{\mu-\rho_1} \le C \text{ in } \Lambda_q$$

for $0<\mu\le\rho_1$, and

$$\widetilde{z}_\mu^{q}(\tau,\lambda) w_+(\tau,\lambda)^{-2+\rho_2} w_j(\tau,\lambda)^{-1} = w_+(\tau,\lambda)^{-1+\rho_2+\mu} w_j^{-\mu} \le C$$

in Λ_j $(j \notin I(q))$ for $0<\mu\le 1-\rho_2$.

Now, (4.18) follows immediately from (4.23), (4.28), (4.32), (4.36), and (4.41). This completes the proof of Proposition 4.2. □

5. Proof of Theorem 1.2

In this section, instead of $E(T)$ in Section 4, we will get an *a priori* bound for a rather simple quantity $E^*(T)$, which is defined by

$$E^*(T) = E^*[u](T) = E_1[u](T) + E_4[u](T) + E_5[u](T), \tag{5.1}$$

where $E_j[u](T)$ are given by (3.12d)–(3.12f), but this time we choose $K \geq 5$, $\nu_1 = 2/3$ in (3.12e), and $\nu_2 = 0$ in (3.12f).

Similarly to the proof of Theorem 1.1, what we need for Theorem 1.2 is the following proposition.

Proposition 5.1. *Suppose that the assumptions in Theorem* 1.2 *are fulfilled. Let $u \in C^\infty\big([0,T) \times \mathbb{R}^3; \mathbb{R}^m\big)$ be a solution to the Cauchy problem* (1.1)–(1.2)*. Then there exist three positive constants B, ε_1, and C_0, which are independent of T, such that $E^*[u](T) \leq B$ implies*

$$E^*[u](T) \leq C_0\big(\varepsilon + \log(2+T)E^*[u](T)^2\big), \tag{5.2}$$

provided $\varepsilon \leq \varepsilon_1$. Here B, ε_1, and C_0 are determined by f and g, but do not depend explicitly on the size of support of f and g.

Proof. First we estimate $E_4(T)$ and $E_5(T)$. Remember the estimates of $E_4(T)$ and $E_5(T)$ in the proof of Proposition 4.2 (see (4.20)–(4.28)). Observing that we have used only the assumptions (1.4), (1.6), and (1.9) in the derivation of (4.20)–(4.28), and that only the quantities $E_1(T)$, $E_4(T)$, and $E_5(T)$ have been used there (see Lemmas 3.3 and 3.5), we conclude that (4.20)–(4.22) and (4.24)–(4.27) are still valid if we replace $E(T)$ by $E^*(T)$, except for the integral of $(1+\tau)^{\nu_2 - 1}$ from 0 to t which appears in (4.22) and (4.26). Since $\nu_2 = 0$ in our present choice, the integral is not $(1+t)^{\nu_2}$ but $\log(1+t)$, and the last line of (4.22) must be replaced by "$\leq C\big(\varepsilon + \log(1+t)E^*(T)^2\big)$," while the conclusion of (4.26) does not have to be changed, because we have $\log(1+t) \leq (1+t)^{\nu_1}$. In this way, we get

$$E_5(T) \leq C\big(\varepsilon + \log(1+T)E^*(T)^2\big), \tag{5.3}$$

and

$$E_4(T) \leq C\big(\varepsilon + E^*(T)^2\big). \tag{5.4}$$

It remains for us to estimate $E_1(T)$. By virtue of Lemma 2.6, it turns out that if we have

$$\sup_{\substack{\tau \in [0,T) \\ y \in \mathbb{R}^3}} |y| z_0(\tau, |y|) |F_i(u, \partial u, \partial_x \partial u)(\tau, y)|_{K+2} \leq C E^*(T)^2, \tag{5.5}$$

then we get

$$E_1(T) \leq C\big(\varepsilon + \log(2+T)E^*(T)^2\big). \tag{5.6}$$

If $K \geq 5$, Lemma 3.2 and the Sobolev-type inequality (3.29) imply

$$\begin{aligned}|y|\,|F_i(\tau,y)|_{K+2} \leq{}& C|y|\,|\partial u(\tau,y)|_{K+1}|\partial u(\tau,y)|_{K+3}\\ &+ C\left(|u(\tau,y)|_{K+2} + |\partial u(\tau,y)|_{K+1}\right)^2 \left(|y|\,|u(\tau,y)|_{K+2} + |y|\,|\partial u(\tau,y)|_{K+3}\right)\\ \leq{}& C|\partial u(\tau,y)|_{K+1}\|\partial u(\tau,\cdot)\|_{2K,2} + C\left(|u(\tau,y)|_{K+2} + |\partial u(\tau,y)|_{K+1}\right)^2\\ &\times \left(|y|\,|u(\tau,y)|_{K+2} + \|\partial u(\tau,\cdot)\|_{2K,2}\right).\end{aligned} \tag{5.7}$$

Choose small $\delta > 0$, and let $(\tau,|y|) \in \Lambda_j$ for some $j \in \{0,1,\dots,m\}$. Using (3.3) and (3.5) in Lemma 3.1, we can proceed to

$$\begin{aligned}|y|\,|F_i(\tau,y)|_{K+2} \leq{}& Cw_+(\tau,|y|)^{-1}w_j(\tau,|y|)^{-1}E^*(T)^2\\ &+ Cw_+(\tau,|y|)^{-2+3\delta}w_j(\tau,|y|)^{-3\delta}E^*(T)^3.\end{aligned} \tag{5.8}$$

Since $z_0 = w_+w_j$ in Λ_j and $w_j \leq Cw_+$, we have $z_0w_+^{-1}w_j^{-1} = 1$ and $z_0w_+^{-2+3\delta}w_j^{-3\delta} = w_+^{-1+3\delta}w_j^{1-3\delta} \leq C$. Now (5.5) is an immediate consequence of (5.8). This completes the proof of Proposition 5.1. □

Proof of Theorem 1.2. Choose a sufficiently large constant M which satisfies $M \geq 4C_0$, and $E^*[u](0) \leq \frac{M\varepsilon}{2}$. Let $\varepsilon \leq \varepsilon_0 = \min\{\frac{B}{M}, \varepsilon_1\}$. If $E^*[u](T) \leq M\varepsilon$, Proposition 5.1 implies

$$E^*[u](T) \leq C_0\varepsilon + \{C_0M\varepsilon\log(2+T)\}M\varepsilon \leq \frac{M\varepsilon}{2}, \tag{5.9}$$

provided that

$$C_0M\varepsilon\log(2+T) \leq 1/4, \text{ or equivalently, } 2+T \leq \exp\frac{1}{4C_0M}\varepsilon^{-1}. \tag{5.10}$$

Therefore, an argument similar to the proof of Theorem 1.1 implies existence of a solution to our problem in the time interval $[0,T)$, provided that (5.10) is satisfied. This implies Theorem 1.2 immediately. □

Remark. Define $\langle v(t,x)\rangle_{1,s}^{(i)} = w_+(t,|x|)w_i(t,|x|)|v(t,x)|_s$. Similarly to Lemma 2.6, we can show

$$\begin{aligned}\langle U_i[\Phi](t,x)\rangle_{1,s}^{(i)} \leq{}& C\log(2+t)\sup_{\tau\in[0,t]}|y|w_+(\tau,|y|)z_0(\tau,|y|)|\Phi(\tau,y)|_s\\ &+ C\sum_{|\alpha|\leq s-1}\sup_{y\in\mathbb{R}^3}|\Phi_\alpha(y)|(1+|y|^3),\end{aligned} \tag{5.11}$$

where $s \geq 0$, and Φ_α is defined as in Lemma 2.6.

Using this estimate we can show the almost-global existence for the system of the following type:

$$\Box_i u_i = \sum_{j,k=1,\dots,m} c_{jk}^i u_j u_k + H_i(u) \text{ in } (0,\infty)\times\mathbb{R}^3 \tag{5.12}$$

for $i = 1, \ldots, m$ with $H_i(u) = O(|u|^3)$ near $u = 0$, provided that

$$c_j = c_k \text{ implies } c^i_{jk} = 0 \text{ for any } i \in \{1, \ldots, m\} \,. \tag{5.13}$$

In fact, set

$$M_K(T) = \sup_{t\in[0,T)} \sum_{i=1}^{m} \left\| \langle u_i(t,\cdot)\rangle^{(i)}_{1,K} \right\|_{L^\infty(\mathbb{R}^3)}$$

with some large integer K. By (5.13), we see that

$$\sum_{j,k} \left|c^i_{jk} u_j u_k(t,x)\right|_K \le C w_+(t,|x|)^{-3} w_l(t,|x|)^{-1} M_K(T)^2$$

for all (t,x) satisfying $(t,|x|) \in \Lambda_l$ $(0 \le l \le m)$. The higher nonlinear terms $H_i(u)$ also enjoy the same estimate. Now (5.11) leads to

$$M_K(T) \le C\left(\varepsilon + \log(2+T) M_K(T)^2\right),$$

which implies an *a priori* bound for M_K as before, and the almost-global existence of u follows from this *a priori* bound.

This example suggests that the almost-global existence result should hold for much more general nonlinearities than those treated in Theorem 1.2.

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