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STABLE SOLUTIONS ON R^n AND THE PRIMARY BRANCH OF SOME NON-SELF-ADJOINT CONVEX PROBLEMS

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Abstract. We prove that if $n = 2$ or 3, the problem $-\Delta u = e^u$ on R^n has no stable negative solution. We then use this to remove self-adjointness conditions in a paper of Crandall and Rabinowitz on the primary branch of positive solutions of a nonlinear boundary-value problem.

In this paper, we achieve two things. Firstly we prove the nonexistence of weakly stable negative solutions of $-\Delta u = e^u$ on R^n for $n = 2$ or 3. We use some ideas from [6] though these ideas need considerable modification because our solutions can not be bounded. We also show that our ideas are useful for some higher-dimensional problems.

We then use this result, results in [1], and blow-up techniques to generalize classical results of Crandall-Rabinowitz [5] on the primary branch of

$$\begin{gathered} -\Delta u + a(x) \cdot \nabla u = \lambda g(x) f(u) \text{ on } \Omega \\ u = 0 \end{gathered} \tag{1}$$

if $\Omega \subseteq R^2$ or R^3, f is convex, and f behaves like a power or an exponential at infinity. In particular, we prove that the primary branch must bend back. The original Crandall-Rabinowitz proof used strongly the self-adjointness of the linear part (but applied in all dimensions and also to more general f). We actually show that the results hold in more general situations. Note that the cases of 2 or 3 dimensions are the ones of primary interest in applications.

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1. No stable solution on $\mathbf{R^n}$

In this section we prove that there is no nonpositive weakly stable solution of $-\Delta u = e^u$ on R^n for $n = 2$ or 3. More formally, we say that u is weakly

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stable if $\int_{R^n} |\nabla h|^2 - e^u h^2 \geq 0$ for all $h \in C_o^\infty(R^n)$. To be more precise this should possibly be called linearized stability. Note that by scalings, we could assume that u is only bounded from above. It is convenient to use the transformation $v = -u$ and consider instead nonnegative weakly stable solutions of

$$\Delta v = e^{-v} \tag{2}$$

on R^n. We use ideas from [2], [6], and [11].

Theorem 1. *If $n = 2$ or 3, there is no nonnegative C^2 weakly stable solution of (2) on R^n.*

We need the following lemma. Here v is a nonnegative weakly stable solution of (2). Note that this result applies if e^{-v} is replaced by $g(v)$, where g is C^1.

Lemma 1. *Assume that there exists a sequence $R_j \longrightarrow \infty$ as $j \longrightarrow \infty$ and $c \geq 0$ such that $\int_{B_{R_j}} |\nabla v|^2 \leq c\, R_j^2$ for all large j. Then after a rotation of coordinates, $v = v(x_1)$ and v is monotone in x_1.*

Proof. This is a variant of Theorem 1 in [6]. We merely sketch it. Since v is stable on compact sets, we can easily use the Harnack inequality to prove that there exists $\psi > 0$ on R^n and $\overline{\lambda} \geq 0$ such that $-\Delta\psi = -e^{-v}\psi + \overline{\lambda}\psi$ on R^n. As in [6], we then see by an easy calculation that $\sigma_i = \frac{\partial v}{\partial x_i}/\psi$ solves a linear equation (as on p. 730 of [2] or in [6]).

We can then repeat the proof of Proposition 2.1 in [2] to deduce that σ_i is constant. (Here it is easy to examine their proof to see that we only need $\int_{B_R} |\nabla v|^2 \leq cR^2$ for a sequence R_i tending to infinity rather than all large R. This has been observed by several people. Moreover, we do not need the assumption in [6] that v is bounded.) Hence we see that $\frac{\partial v}{\partial x_i} = C_i\psi$, where C_i is constant. As in the proof of Theorem 1 in [6], we can easily make a rotation of axes so that all the C_i's except C_1 are zero. Hence $\frac{\partial v}{\partial x_i} = C_i\psi \equiv 0$ for $i > 1$ and thus $v = v(x_1)$. Since $\frac{\partial v}{\partial x_1} = C_1\psi$, $\frac{\partial v}{\partial x_1}$ has fixed sign (or vanishes identically), and hence our claim follows. This completes the proof of Lemma 1.

We return to the proof of Theorem 1. If $v = v(x_1)$, $v'' > 0$ and hence v' is strictly increasing. Thus, by Lemma 1, $v' > 0$ for all x_1 or $v' < 0$ for all x_1. We consider the former case. The other is similar. Since v' is strictly increasing, $\lim_{x_1 \longrightarrow -\infty} v'(x_1) = \mu$, where μ is finite and $\mu \geq 0$.

If $\mu > 0, v$ is negative for x_1 large negative, which contradicts our assumptions. Hence $v'(x_1) \longrightarrow 0$ as $x_1 \longrightarrow -\infty$. Since v is nonnegative and increasing, $\lim_{x_1 \longrightarrow -\infty} v(x_1) \equiv v(-\infty)$ exists and is finite. Then $v''(x_1) =$

$e^{-v(x_1)} \longrightarrow e^{-v(-\infty)} > 0$ as $x_1 \longrightarrow -\infty$, and hence $v'(x_1) \longrightarrow -\infty$ as $x_1 \longrightarrow -\infty$, which is a contradiction. Thus no one-dimensional monotone solution exists.

By this argument and Lemma 1, we see that the proof of Theorem 1 reduces to proving that there exists $R_j \longrightarrow \infty$ such that $\int_{R_j} |\nabla v|^2 \leq c\, R_j^2$ for all j. Note that the argument in [6] for bounded solutions uses very much that v is bounded on R^3 and hence ∇v is bounded on R^3. Thus it is not usable here. We consider only the case $n = 3$. The case $n = 2$ is similar but much easier and is in fact also covered in [8].

If we multiply (2) by v and integrate over the ball B_R we see that

$$\int_{B_R} |\nabla v|^2 \;=\; -\int_{B_R} ve^{-v} \;+\; \int_{\partial B_R} v\frac{\partial v}{\partial r},$$

where $r = |x|$. Since v is nonnegative, the first term on the right-hand side is nonpositive, and hence it suffices to estimate $R^2 \int_S v(R,\Theta)\frac{\partial v(R,\Theta)}{\partial r}|_{r=R}$, where Θ is the angle variable on the unit sphere S of R^3. Equivalently, we need to estimate $\frac{1}{2}R^2\frac{d}{\partial R}W(R)$, where $W(R) = \int_S v(R,\Theta)^2 d\Theta$. If we can prove that there exists a sequence $\widetilde{R}_j \longrightarrow \infty$ such that $W(\widetilde{R}_j) \leq c\widetilde{R}_j$ for all j, it follows easily from the mean-value theorem that there exists a sequence $R_j \longrightarrow \infty$ such that $W'(R_j)$ is bounded above, and our claim follows.

Hence our result reduces to proving the estimate for $W(\widetilde{R}_j)$. We will prove that W has at most logarithmic growth.

Our proof of this uses essentially the weak stability. We choose radial functions $\phi_{\alpha,\mu}(r) = r^{-\beta} \sin\,(\alpha \ln r + \mu)$ between two successive positive zeros, t_1 and t_2. Here $\beta = \frac{1}{2}(n-2)$. Note that $\Delta\phi_{\alpha,\mu} = \widetilde{C}r^{-2}\phi_{\alpha,\mu}$ for $r > 0$ by a simple computation. Here $\widetilde{C} = \widetilde{C}(\alpha)$. We use the test function, which is $\phi_{\alpha,\mu}(r)$ if $t_1 < r < t_2$ and is zero otherwise. Then the inequality $\int_{R^3} |\nabla\phi|^2 - e^{-v}\phi^2 \geq 0$ becomes that

$$\int_{t_1}^{t_2} \overline{(e^{-v})}(\phi_{\alpha,\mu}(r))^2 r^2 dr \leq \widetilde{C}\int_{t_1}^{t_2} (\phi_{\alpha,\mu}(r)^2 dr\;.$$

Here $\overline{e^{-v}}$ is the average of $e^{-v(r,\Theta)}$ on the sphere of radius r and we used that $n = 3$.

We evaluate the integral on the right-hand side by the substitution $z = \ell n\, r$. Noting that $t_1 = \exp\,(\frac{n\pi-\mu}{\alpha})$ and $t_2 = \exp\,(\frac{(n+1)\pi-\mu}{\alpha})$, we find that

$$\int_{\widetilde{t}(n)}^{\widetilde{t}(n+1)} \overline{e^{-v}}r(\sin(\alpha\ell nr + \mu))^2 dr \;\leq\; K\widetilde{C}(\alpha),$$

where $\widetilde{t}(n) = \exp\ (\frac{n\pi-\mu}{\alpha})$. By fixing α and using that $r \leq \widetilde{t}(n+1)$ on $[\widetilde{t}(n), \widetilde{t}(n+1)]$, we find that

$$\int_{\widetilde{t}(n)}^{\widetilde{t}(n+1)} \overline{e^{-v}}\ r^2 (\ \sin\ (\alpha \ell n r + \mu))^2 dr\ \leq\ K_1(\alpha)\ \exp\ (\frac{n\pi}{\alpha})\ .$$

Adding and summing the geometric progression we find that

$$\int_{\widetilde{t}(1)}^{\widetilde{t}(n)} \overline{e^{-v}}\ r^2\ (\sin\ (\alpha \ell n r + \mu))^2 dr\ \leq\ K_1(\alpha)\ \exp\ (\frac{n\pi}{\alpha}),$$

and thus for large x

$$\int_1^x \overline{e^{-v}}\ r^2\ \sin^2(\alpha \ell n r + \mu)\ dr\ \leq\ K_1(\alpha) x\ .$$

Remember that for large n, $\widetilde{t}(n+1) \leq K\widetilde{t}(n)$.

Since we could do the same thing for cos rather than sin and adding, we find that

$$\int_1^x \overline{e^{-v}} r^2\ dr\ \leq\ K_2(\alpha) x \tag{3}$$

for large x.

Now $r^{-2}(r^2\overline{v}')' = \overline{e^{-v}}$ by averaging (2) over spheres.

By using (3) and integrating, we see that $r^2\overline{v}'(r) \leq Kr$ for large r. Hence $\overline{v}'(r) \leq Kr^{-1}$ for large r. Integrating, $\overline{v}(r) \leq K\ell n r$ for large r.

By applying Theorem 3.19 in Hayman and Kennedy [13] with $\rho = \frac{1}{2}r$ we see that $\max_\Theta v(r,\Theta) \leq K\ell n(2r)$ for large r. Hence $\max_\Theta v(r,\Theta) \leq K_1 \ell n r$ for large r.

Hence $\int_S v^2(r,\Theta) d\Theta \leq K_2(\ell n\ r)^2$ for r large, and we have completed the proof.

Remark. If we examine the proof, we see that max $v(r,\Theta) \leq K_1 \ell n r$ even if we assume only that the solution v has finite Morse index (which implies that v is stable on $R^n \backslash B_q$, where q is large). We suspect that there is no positive finite Morse-index solution of (2) if $n = 3$.

Our ideas have some uses for the equation

$$-\Delta u = f(u)$$

in R^4. We sketch this. We look for weakly stable bounded positive solutions u assuming that $f(0) = 0$ and $f'(y) > 0$ for $0 < y < \|u\|_\infty$ and $f'(y) \sim c\ y^{p-1}$ as $y \longrightarrow 0$, where $p > 1$ and $c > 0$. Hence we see that there exist $C_1, C_2 > 0$

such that $C_2 y^{p-1} \geq f'(y) \geq C_1 y^{p-1}$ on $(0, \|u\|_\infty]$. Arguing much as in the proof above, we find that

$$\int_0^x \overline{u^{p-1}}\, r^3\, dr \;\leq\; K_2 x^2, \tag{4}$$

and hence $\int_{B_R} uf(u) \;\leq\; K_2 R^2$ (since u is bounded).

Now it follows easily from (4) and Jensen's inequality that there is a sequence $\widetilde{R}_j \longrightarrow \infty$ such that $\overline{u}(\widetilde{R}_j) \longrightarrow 0$ as $\widetilde{R}_j \longrightarrow \infty$, and hence much as before (but more easily since u is bounded) we have a sequence $R_j \longrightarrow \infty$ such that $W'(R_j) \leq 0$ for all j. Hence we see that

$$\int_{B_{R_j}} |\nabla u|^2 \leq \int_{B_{Rj}} uf(u) \leq CR_j^2\ .$$

Thus, as before, we find that u is a monotone function of x_1, and we use the first integral of the ordinary differential equation to obtain a contradiction unless u vanishes identically. If $n > 5$, we can obtain some results for the case of a pure power $f(y) = y^p$ by using the estimate from [3] and [8] that for weakly stable solutions $\int_{B_R} uf(u) \leq CR^{n-2(p+1)/(p-1)}$, and hence we see that much as before there are no weakly stable solutions if $n-2(p+1)/(p-1) \leq 2$, i.e., if $p \leq n/(n-4)$. With care, both of these two results also apply to sign-changing solutions if f is odd (and the oddness can be weakened considerably).

2. Applications to non-self-adjoint problems on bounded domains

We consider the problem

$$\begin{aligned} -\Delta u + a(x)\cdot\nabla u \;&=\; \lambda g(x) f(u) \;\text{ in }\; \Omega \\ u \;&=\; 0 \;\text{ on }\; \partial\Omega, \end{aligned} \tag{5}$$

where f is C^1 and $y^{1-p} f'(y) \longrightarrow pa > 0$ as $y \longrightarrow \infty$ or $y^{1-q} f'(y) e^{-y} \longrightarrow a > 0$ as $y \longrightarrow \infty$, where $n = 2$ or 3, $p > 1$, g and a are continuous on $\overline{\Omega}$, and g is positive on $\overline{\Omega}$. Here Ω is a smooth bounded domain in R^n. We are interested in a sequence of linearized stable positive solutions (u_i, λ_i) such that $\|u_i\|_\infty \longrightarrow \infty$ and $\lambda_i \longrightarrow \overline{\lambda} > 0$ (possibly plus infinity) as $i \longrightarrow \infty$. In particular, we wish to prove this cannot occur. Here the solution (u_i, λ_i) is said to be linearized stable if the principal eigenvalue α of

$$\begin{aligned} -\Delta h + a(x)\cdot\nabla h - \lambda_i g(x) f'(u_i) h \;&=\; \alpha h \;\text{ in }\; \Omega \\ h \;&=\; 0 \;\text{ on }\; \partial\Omega \end{aligned}$$

is nonnegative.

Theorem 2. *No such linearized stable solutions* u_i *of* (5) *exist if* $n = 3$ *and* $p > 1$ *or if* $n = 2, 3$ *and* $q \in Z$.

Proof. We first do the case of power growth and then return to the more difficult exponential case afterwards. We write $u_i = \|u_i\|_\infty v_i$. Then

$$-\Delta v_i + a(x) \cdot \nabla v_i = \lambda_i (\|u_i\|_\infty)^{-1} g(x) f \|u_i\|_\infty v_i) \ .$$

We blow up near a maximum x_i of v_i. We use rescaled variables $X_i = \alpha_i(x - x_i)$, where $\alpha_i^2 = a\lambda_i \|u_i\|_\infty^{p-1} g(x_i)$. Here $y^{-p} f(y) \longrightarrow a > 0$ as $y \longrightarrow \infty$. In the new variables, $\|v_i\|_\infty = 1$, $v_i(0) = 1$,

$$-\Delta v_i + \alpha_i^{-1} a(x) \cdot \nabla v_i = g(x) g(x_i)^{-1} (v_i^p + (\|u_i\|_\infty^{1-p} r(\|u_i\|_\infty v_i)))$$

in Ω, where $f(y) = a(y^p + r(y))$. First assume that in the new variables $d(0, \widetilde{\Omega}_i) \longrightarrow \infty$ as $i \longrightarrow \infty$, where $\widetilde{\Omega}_i$ is Ω rescaled (possibly after taking a subsequence). Standard estimates show that the v_i are bounded in $W^{2,s}$ on compact sets. Thus a standard limiting argument shows that a subsequence of v_i converges uniformly on compact sets to a positive solution $\overline{v}$ of

$$-\Delta \overline{v} = \overline{v}^p \ \text{ on } \ R^n, \|\overline{v}\|_\infty = 1 = \overline{v}(0) \ .$$

There is one point to be noted here. $\|u_i\|_\infty^{-p} r(\|u_i\|_\infty a) \longrightarrow 0$ as $i \longrightarrow \infty$ uniformly in a for $a \in [0, 1]$ (since $y^{-p} r(y) \longrightarrow 0$ as $y \longrightarrow \infty$). Moreover, $\overline{v}$ is stable. To see this, we first note that the principal eigenvalue of

$$-\Delta h + c(x) \cdot \nabla h + q(x) h = \alpha h \tag{6}$$

on $\widetilde{\Omega}$ with $h = 0$ on $\partial\widetilde{\Omega}$ is a monotone decreasing function of $\widetilde{\Omega}$. This is well known, but for completeness we sketch a proof in Lemma 2 below. Since v_i is stable on $\widetilde{\Omega}_i$, Lemma 2 implies that v_i is stable on B_R for i large (so that $B_R \subseteq \widetilde{\Omega}_i$). Since, as is well known, the principal eigenvalue of (6) depends continuously on c and q if c and q are given the L^q norm for $q > n$, it follows that $\overline{v}$ is stable on B_R. Since R was arbitrary, it follows easily that $\overline{v}$ is stable. By Theorem 1 in [6], $\overline{v} = \overline{v}(x_1)$ after a rotation of coordinates and $\overline{v}$ is monotone. By an easy argument similar to that in the proof of Theorem 1, $\overline{v} \equiv 0$, which gives a contradiction.

The second possibility is that $\{d(0, \partial\widetilde{\Omega}_i)\}_{i=1}^\infty$ has positive upper and lower bounds, possibly after taking a subsequence. In this the blowing-up argument gives a solution $\overline{v}$ of the same equation on a half space T with $v = 0$ on ∂T, $\|v\|_\infty = 1$, and v achieves its maximum. (Similar arguments appear in [9].) This contradicts Theorem 1 in [10].

The third possibility is that $d(0,\partial\overline{\Omega}_i) \longrightarrow 0$ as $i \longrightarrow \infty$ (again possibly after taking subsequences). Once again, v_i is bounded locally in $W^{2,s}$ and thus in C^1 (if $s > n$). This is impossible since $v_i(0) = 1$, and $v_i(\partial\Omega) = 0$. Thus we have a contradiction, and hence the case where f is asymptotically y^p does not occur.

We now consider the case where $f(y)y^{-q}e^{-y} \longrightarrow a$ as $y \longrightarrow \infty$. This is partially by a similar type blow-up argument. We write $v_i(x) = v_{i,m}+w_i(x)$, where $v_{i,m}$ is the maximum value of v_i on Ω (which occurs at x_i) and $w_i \le 0$. Again we use a rescaling $X_i = \alpha_i(x - x_i)$, where $\alpha_i^2 = a\lambda_i v_{i,m}^q e^{v_{i,m}} g(x_i)$. In the new variables $-v_{i,m} \le w_i \le 0$, $w_i(0) = 0$, and

$$0 = L_i w_i = \Delta w_i + \alpha_i^{-1} a(x) \cdot \nabla w_i =$$
$$(v_{i,m})^{-q} g(x)(g(x_i))^{-1}(1 + v_{i,m} - w_i(x))^q e^{w_i(x)}(1 + r(v_{i,m} + w_i(x)),$$

where $r(y) = a^{-1}(1+y)^{-q}$ $(\exp{-y})f(y) - 1$. (Thus $r(y) \longrightarrow 0$ as $y \longrightarrow \infty$.) Since $0 \le -w_i(x) \le v_{i,m}$, we easily see that the right-hand side $\ell_i(x)$ of our equation is uniformly bounded on $\widetilde{\Omega}_i$. Consider a fixed ball B_R with center zero. By solving the Dirichlet problem $L_i w = \ell_i(x)$ in B_R, $w = 0$ on ∂B_R, we obtain a solution $\widetilde{w}_i$ uniformly bounded in C^1. By adding a constant C independent of i to the solution, we obtain a solution $\widetilde{w}_i \ge 0$ on B_R of the same equation except that the boundary condition is $\widetilde{w}_i = C$ on ∂B_R. Moreover the $\widetilde{w}_i$ are uniformly bounded. Then $w_i - \widetilde{w}_i$ is a nonpositive solution of $L_i w = 0$ and $|(w_i - \widetilde{w}_i)(0)|$ is uniformly bounded. Thus by the Harnack inequality (cp. [12]), $w_i - \widetilde{w}_i$ is uniformly bounded on $B_{\frac{1}{2}R}$. Hence we see that w_i is uniformly bounded on $B_{\frac{1}{2}R}$ and thus by the equation w_i is uniformly bounded in C^1 on $B_{\frac{1}{2}R}$. Since R was arbitrary, $\{w_i\}$ is uniformly bounded in C^1 on compact sets in R^n. Thus we can find a subsequence of w_i converging weakly in $W^{2,s}$ and strongly in $C(R^n)$ on compact subsets of R^n to w, where $w \le 0$, $w(0) = 0$, and $-\Delta w = \exp w$. Here we have used that w_i converges uniformly on compact sets, $r(y) \longrightarrow 0$ as $y \longrightarrow \infty$, and that in the blow-up variables $g(x)g(x_i)^{-1}$ is uniformly close to 1 on compact sets. We can argue much as in the previous case to prove that w is stable. This is impossible by Theorem 1. Hence we see that $\{d(0,\widetilde{\Omega}_i)\}_{i=1}^{\infty}$ is uniformly bounded.

We need to use a different argument in this case because our change of variable does not respect the boundary condition. In fact, we find w_i is a large negative constant $-C_i$ on $\partial\widetilde{\Omega}_i$ by our rescaling. If $d(0,\widetilde{\Omega}) \le K$ for all i, we can obtain a nicer neighbourhood N_i of part of $\partial\widetilde{\Omega}_i$ and containing 0 by using a tubular neighbourhood of the boundary $\partial\Omega$ in Ω (by following normals to $\partial\Omega$ into Ω) to obtain N_i which consists of a "good" neighbourhood Z_i of the point of $\partial\widetilde{\Omega}_i$ closest to zero, and then we obtain N_i by moving a distance up to $2K$ along the normal to $\partial\widetilde{\Omega}_i$ at each $z_i \in Z_i$. This neighbourhood is "almost" a rectangle (since $\partial\widetilde{\Omega}_i$ is rather flat). Then $C_i \geq -w_i \geq 0$ in N_i, $w_i(0) = 0$, $w_i = -C_i$ on $\partial N_i \cap \partial\widetilde{\Omega}_i$. Moreover,

$$-\Delta w_i + o(1) \cdot \nabla w_i = e^{w_i} + o(1)$$

in N_i (as earlier). Thus $\widetilde{w}_i = -w_i/C_i$ satisfies that $\widetilde{w}_i = 1$ on $\partial N_i \cap \partial\widetilde{\Omega}_i$, $0 \leq \widetilde{w}_i \leq 1$ in N_i, $\widetilde{w}_i(0) = 0$, and $\Delta\widetilde{w}_i + o(1) \cdot \nabla\widetilde{w}_i = o(1)$ in N_i. We can pass to the limit through subsequences at the expense of shrinking Z_i slightly, and we obtain a solution Y of $\Delta w = 0$ in $\widetilde{Z} \times [-K, K]$ such that $Y \geq 0$ on $\widetilde{Z} \times [-K, K]$, $Y = 0$ at an interior point (corresponding to zero), and $Y = 1$ on $\widetilde{Z} \times \{-K\}$. This is impossible by the maximum principle, and hence we see that $\{d(0, \partial\widetilde{\Omega}_i\}$ can not be bounded.

This completes the proof.

Remark. 1) We did not use the stability in the case where the maximum occurs very near the boundary.

2) The results hold for rather more general equations. We could allow a term $g(x, u) \cdot \nabla u$, where g is uniformly bounded and continuous, we could allow the top-order terms to depend on x, we could allow terms $r(x, \nabla u)$ where r is uniformly bounded, and we could allow more general dependence on x and u than simply $g(x)f(u)$.

3) Our techniques can also be used to prove the analogue of Theorem 2 for Neumann or Robin boundary conditions. In fact, these are a little easier because for maxima near the boundary, the blow-up is a half-space Neumann problem which gives a full-space solution (after a reflection). We can argue as in the proof of Theorem 2 in [6] to see that the whole-space solution is weakly stable, and then we can proceed as before. By changing the rescaling slightly, we could prove our results for nonlinearities asymptotically like $(\log y)^s y^p$ or $(\log y)^s y^q \exp y$. Note that the blow-up equations are unchanged. Note also that our results for the cases of subcritical power growth (but $p > 1$) are uninteresting because the blow-up argument shows that all solutions are bounded.

Lemma 2. *Suppose that $\Omega_1 \subset \Omega_2$, where Ω_1 and Ω_2 are smooth domains in R^n and $\lambda_1(\Omega_i)$ denote the principal eigenvalue of* (6) *on Ω_i for Dirichlet boundary condition on $\partial\Omega_i$. Then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$.*

Proof. If h_1 is the positive eigenfunction of (6) on Ω_1 corresponding to the eigenvalue $\lambda_1(\Omega_1)$, then $\frac{\partial h_1}{\partial \nu} \leq 0$ on $\partial\Omega_1$, where ν is the outward normal. If we extend h_1 to be zero on $\Omega_2 \backslash \Omega_1$, then by Berestycki and Lions [4] the extended h_1 is a nontrivial nonnegative subsolution of (6) on Ω_2 (for $\alpha = \lambda_1(\Omega_1)$). Thus, by the maximum principle, $\lambda_1(\Omega_1) L^{-1} \widetilde{h}_1 \geq \widetilde{h}_1$, where $\widetilde{h}_1$ is our extended function, L is defined by the left-hand side of (6), and L^{-1} is the inverse on Ω_2. Hence $\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1)$ (by the Krein-Rutman theorem).

Remark. In fact, strict monotonicity holds.

The main interest of the above result is the problem

$$\begin{aligned} -\Delta u + a(x) \cdot \nabla u &= \lambda g(x) f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{7}$$

where in addition to our earlier assumptions we assume f is strictly convex positive and $f(0) > 0$. Then it is easy to prove (cp. [1]) that there exists $\lambda_* > 0$ such that (7) has a smooth solution $u(\lambda)$ for $0 < \lambda < \lambda_*$, no positive smooth solution for $\lambda > \lambda_*$, and either $\|u_\lambda\|_\infty \longrightarrow \infty$ as $\lambda \longrightarrow \lambda_*$ or there is a smooth positive solution for $\lambda = \lambda_*$ (and the branch "bends" at λ_*). We show that under the above assumptions the first possibility does not occur and the branch bends back. (Thus there is a turning point at the end of the curve of minimal solutions.) That the first case can not occur follows immediately from Theorem 2 if we note that the minimal solution is weakly stable. As before, we could allow rather more general operators.

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