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# EXISTENCE AND STABILITY FOR SOME PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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**Abstract.** In this work we study the existence and stability for some neutral partial functional differential equations. We suppose that the linear part is not necessarily densely defined and satisfies the Hille-Yosida condition on a Banach space $X$. The nonlinear term takes its values in a space larger than $X$, namely the extrapolated Favard class corresponding to the extrapolated semigroup corresponding to the linear part. Our approach is based on the theory of the extrapolation spaces.

## 1. Introduction

Consider the partial functional differential equation of neutral type

$$\begin{cases} \dfrac{d}{dt}D\left(x_t\right) = AD\left(x_t\right) + F(x_t), \text{ for } t \geq 0 \\ x_0 = \varphi \in C = C\left([-r,0]\,;X\right), \end{cases} \tag{1.1}$$

where $A$ is a nondensely defined linear operator on a Banach space $X$ and satisfies the Hille-Yosida condition; that is, there exists $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda-\omega_0)^n\left|(\lambda-A)^{-n}\right| : n \in \mathbb{N},\ \ \lambda > \omega_0\right\} < \infty,$$

where $\rho(A)$ is the resolvent set of $A$. $C$ is the space of continuous functions from $[-r,0]$ into $X$ endowed with the uniform norm topology. $F$ is a continuous function from $C$ with values in a larger space than $X$, namely the extrapolated Favard class corresponding to $A$. $D$ is a bounded linear

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operator from $C$ into $X$ defined by $D(\varphi) = \varphi(0) - D_0(\varphi)$, for $\varphi \in C$, where $D_0$ is given by

$$D_0(\varphi) = \int_{-r}^{0} d\eta(\theta)\varphi(\theta), \text{ for } \varphi \in C,$$

where $\eta : [-r, 0] \to L(X)$ is of bounded variation and nonatomic at zero:

$$\underset{[-\varepsilon,0]}{var}\, \eta \to 0 \text{ as } \varepsilon \to 0.$$

For every $t \geq 0$, the history function $x_t \in C$ is defined by

$$x_t(\theta) = x(t + \theta), \text{ for } \theta \in [-r, 0].$$

We will discuss the existence, the regularity of solutions, and the stability of an equilibrium. When $F$ takes its values in $X$, equation (1.1) has been studied by several authors; we cite [1], [2], [4], [9], and [15] . In [15] Wu and Xia have proved the existence of an oscillatory solution for the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\widetilde{D}(x_t) = \dfrac{\partial^2}{\partial x^2}\widetilde{D}(x_t) + f(x_t), \text{ for } t \geq 0 \\ x_0 = \varphi \in C = C([-r, 0]; E), \end{cases} \tag{1.2}$$

where $d > 0$, $E = H^1(S^1)$, and $\widetilde{D}$ is a bounded linear operator from $C\left([-r, 0]; H^1(S^1)\right)$ into $H^1(S^1)$ given by

$$\widetilde{D}(\varphi)(\xi) = \varphi(0, \xi) - \int_{-r}^{0} d\alpha(\theta)\varphi(\theta, \xi), \xi \in S^1,$$

where $\alpha$ is of bounded variation on $[-r, 0]$ and nonatomic at zero. In [9] Hale studied the properties of the solution map, the existence of the global attractor, and the behavior near an equilibrium. Recently, in [1] and [2], the existence and the regularity of solutions of equation (1.1), when $F$ takes its values in $X$ and the operator $A$ is nondensely defined and satisfies the Hille-Yosida condition, were established. The authors proved the existence of the integral solution and its regularity, and they gave a variation-of-constants formula. This work is motivated by [8], where the authors have studied the equation

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + G(t, x_t), \text{ for } t \geq 0 \\ x_0 = \varphi \in C([-r, 0]; X), \end{cases}$$

where $A$ generates a $C_0$ semigroup on $X$ and $G$ is a continuous function on $\mathbb{R}^+ \times C([-r, 0]; X)$ with values in the extrapolated Favard class of $A$.

In Section 2, we recall some preliminary results about the extrapolation spaces and Favard class which will be used throughout the paper. In Section 3, we will establish the existence and the regularity of solutions. In Section 4, we will prove the linearized stability, and the last section will be dedicated to an application for some neutral partial functional differential equations with delay.

## 2. Extrapolation spaces and Favard class

Assume that
($\mathbf{H_1}$) $A$ is a Hille-Yosida operator.

**Lemma 1.** ([7], Corollary II.3.21) *The part $(A_0, D(A_0))$ of $A$ in $X_0 = \overline{D(A)}$ is given by*

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ A_0 x = Ax, \ \textit{for } x \in D(A_0) \end{cases}$$

*generates a $C_0$ semigroup $(T_0(t))_{t\geq 0}$ on $X_0$ and $|T_0(t)| \leq M_0 e^{\omega t}$, for $t \geq 0$. Moreover, $\rho(A) \subset \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)/_{X_0}$, for $\lambda \in \rho(A)$.*

For a fixed $\lambda_0 \in \rho(A)$, define on $X_0$ a new norm by

$$\|x\|_{-1} = |R(\lambda_0, A_0)x|, \text{ for } x \in D(A_0).$$

The completion $X_{-1}$ of $(X_0, \|\cdot\|_{-1})$ is called the extrapolation space of $X$ associated with $A$. Note that for $\lambda \in \rho(A)$ the norms on $X_0$ given by $|R(\lambda, A_0)x|$ and $\|\cdot\|_{-1}$ are equivalent. The operator $T_0(t)$ has a unique bounded linear extension $T_{-1}(t)$ to the Banach space $X_{-1}$, and $(T_{-1}(t))_{t\geq 0}$ is a $C_0$ semigroup on $X_{-1}$. $(T_{-1}(t))_{t\geq 0}$ is called the extrapolated semigroup of $(T_0(t))_{t\geq 0}$; we denote by $(A_{-1}, D(A_{-1}))$ its generator.

**Lemma 2.** [8] *The following properties hold:*

i) $|T_{-1}(t)|_{L(X_{-1})} = |T_0(t)|_{L(X_0)}$,
ii) $D(A_{-1}) = X_0$,
iii) $A_{-1} : X_0 \to X_{-1}$ *is the unique continuous extension of* $A_0 : D(A_0) \subset (X_0, |\cdot|) \to (X_{-1}, \|\cdot\|_{-1})$ *and* $(\lambda_0 - A_{-1})$ *is an isometry from* $(X_0, |\cdot|)$ *to* $(X_{-1}, \|\cdot\|_{-1})$,
iv) *If* $\lambda \in \rho(A_0)$, *then* $(\lambda - A_{-1})$ *is invertible and* $(\lambda - A_{-1})^{-1} \in L(X_{-1})$. *In particular,* $\lambda \in \rho(A_{-1})$ *and* $R(\lambda, A_{-1})/_{X_0} = R(\lambda, A_0)$.
v) *The space* $X_0 = \overline{D(A)}$ *is dense in* $(X_{-1}, \|\cdot\|_{-1})$. *Hence the extrapolation space* $X_{-1}$ *is also the completion of* $(X, \|\cdot\|_{-1})$ *and* $X \hookrightarrow X_{-1}$.

vi) *The operator $A_{-1}$ is an extension of $A$. In particular, if $\lambda \in \rho(A)$, then $R(\lambda, A_{-1})/_X = R(\lambda, A)$ and $R(\lambda, A_{-1})X = D(A)$.*

**Definition 1.** [8] *Let $(S(t))_{t\geq 0}$ be a $C_0$ semigroup with generator $(B, D(B))$ on a Banach space $Y$ such that $|S(t)| \leq Ne^{\nu t}$, for $t \geq 0$. The Favard class of $(S(t))_{t\geq 0}$ is the Banach space*

$$\mathbb{F} = \left\{ x \in Y : \sup_{t>0} \frac{1}{t} \left| e^{-\nu t} S(t)x - x \right| < \infty \right\}$$

*equipped with the norm*

$$|x|_{\mathbb{F}} = |x| + \sup_{t>0} \frac{1}{t} \left| e^{-\nu t} S(t)x - x \right|.$$

We can see that $\mathbb{F}$ is invariant under$(S(t))_{t\geq 0}$ and $D(B) \subset \mathbb{F}$. If the space $Y$ is reflexive, then $\mathbb{F} = D(B)$. Furthermore, if we denote by $|\cdot|_B$ the graph norm of $B$, then $|\cdot|_B$ and $|\cdot|_{\mathbb{F}}$ are equivalent norms on $D(B)$. In the sequel, we denote by $F_1 \subset X_0$ the Favard class of the $C_0$ semigroup $(T_0(t))_{t\geq 0}$ and $F_0 \subset X_{-1}$ the Favard class of $(T_{-1}(t))_{t\geq 0}$.

**Lemma 3.** [8] *For the Favard classes $F_0$ and $F_1$ the following hold.*

i) $(\lambda_0 - A_{-1})F_1 = F_0$,
ii) $T_{-1}(t)F_0 \subset F_0$, $t \geq 0$.
iii) $D(A_0) \hookrightarrow D(A) \hookrightarrow F_1 \hookrightarrow X_0 \hookrightarrow X \hookrightarrow F_0 \hookrightarrow X_{-1}$, *where $D(A)$ is equipped with the graph norm.*

**Proposition 1.** [8] *For $f \in L^1_{loc}(\mathbb{R}^+, F_0)$, let*

$$(T_{-1} * f)(t) = \int_0^t T_{-1}(t-s) f(s) ds, \ t \geq 0.$$

*Then*

i) $(T_{-1} * f)(t) \in X_0$,
ii) $|(T_{-1} * f)(t)| \leq M \int_0^t e^{w(t-s)} |f(s)|_{F_0} ds$, *for some $M$ independent of $f$ and $t$,*
iii) $\lim_{t\to 0} |(T_{-1} * f)(t)| = 0$.

## 3. Existence and regularity of solutions

Assume that $F$ takes its values in $F_0$ and satisfies the Lipschitz condition:

$(\mathbf{H}_2)$ $\qquad |F(\varphi_1) - F(\varphi_2)|_{F_0} \leq K |\varphi_1 - \varphi_2|, \quad$ for $\varphi_1, \varphi_2 \in C$.

Let us recall some notions of solutions associated with equation (1.1).

**Definition 2.** *A continuous function $x : [-r, \infty) \to X$ is called a mild solution of equation* (1.1) *if*

$$\begin{cases} D(x_t) = T_0(t)D(\varphi) + \int_0^t T_{-1}(t-s)F(x_s)ds, \ t \geq 0 \\ x_0 = \varphi. \end{cases}$$

Note that when $F$ takes its values in $X$, the mild solution coincides with the integral solution given in [1] and [2]. For the existence of that solution, we have

**Theorem 1.** *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Then for* $\varphi \in C$ *such that* $D(\varphi) \in \overline{D(A)}$*, equation* (1.1) *has a unique mild solution which is defined for all* $t \geq 0$.

**Proof.** Let $T > 0$ and $\varphi \in C$ such that $D(\varphi) \in \overline{D(A_0)}$, and consider the set

$$\Lambda = \{x \in C([-r, T]; X) : x(\theta) = \varphi(\theta), \text{ for } \theta \in [-r, 0]\}.$$

$\Lambda$ is a closed subset of $C([-r, T]; X)$ endowed with the uniform norm topology. Let $\mathcal{J}$ be the operator defined on $C([-r, T]; X)$ by

$$\mathcal{J}(x)(t) = \begin{cases} \int_{-r}^0 d\eta(\theta)x(t+\theta) + T_0(t)D(\varphi) + \int_0^t T_{-1}(t-s)F(x_s)ds, \\ \hfill \text{if } t \in [0, T] \\ \varphi(t), \quad \text{if } t \in [-r, 0]. \end{cases}$$

Then, by Proposition 1 we have $\mathcal{J}(\Lambda) \subset \Lambda$. Furthermore, for $x, y \in \Lambda$ and $t \in [0, T]$ one has

$$\begin{aligned} (\mathcal{J}(x) - \mathcal{J}(y))(t) = & \int_{-r}^0 d\eta(\theta)\left(x(t+\theta) - y(t+\theta)\right) \\ & + \int_0^t T_{-1}(t-s)(F(x_s) - F(y_s))ds. \end{aligned}$$

Now take $T = \varepsilon \in (0, r]$; then for $t \in [0, T]$ one has

$$\begin{aligned} (\mathcal{J}(x) - \mathcal{J}(y))(t) = & \int_{-\varepsilon}^0 d\eta(\theta)\left(x(t+\theta) - y(t+\theta)\right) \\ & + \int_0^t T_{-1}(t-s)(F(x_s) - F(y_s))ds. \end{aligned}$$

$\varepsilon$ is chosen such that $\underset{[-\varepsilon,0]}{var}\,\eta < 1$. By using Proposition 1, we have

$$\left| \int_0^t T_{-1}(t-s)(F(x_s) - F(y_s))ds \right| \leq M \int_0^t e^{w(t-s)} \left|F(x_s) - F(y_s)\right|_{F_0} ds.$$

Thus, by assumption $(\mathbf{H}_2)$

$$\Big|\int_0^t T_{-1}(t-s)(F(x_s)-F(y_s))ds\Big| \leq MK\int_0^t e^{w(t-s)}\left|x_s-y_s\right|ds.$$

Moreover,

$$\Big|\int_{-\varepsilon}^0 d\eta(\theta)\left(x(t+\theta)-y(t+\theta)\right)\Big| \leq \big(\underset{[-\varepsilon,0]}{var}\,\eta\big)|x-y|.$$

Without loss of generality we suppose that $\omega>0$; then

$$|(\mathcal{J}(x)-\mathcal{J}(y))\,(t)| \quad \leq \Big(\underset{[-\varepsilon,0]}{var}\,\eta + Me^{\omega t}tK\Big)\left|x-y\right|.$$

If we choose $\varepsilon$ such that $\underset{[-\varepsilon,0]}{var}\,\eta + Me^{\omega\varepsilon}\varepsilon K<1$, then $\mathcal{J}$ is a strict contraction in $\Lambda$ and it has a unique fixed point in $\Lambda$ which is the unique mild solution of the equation on $[0,T]$. Proceeding inductively, we extend the solution uniquely and continuously to $[-r,+\infty)$. □

For the regularity of the mild solution, we have

**Theorem 2.** *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Furthermore, assume that* $F:C\to F_0$ *is continuously differentiable and* $F'$ *is locally Lipschitz. Let* $\varphi\in C$ *such that*

$$\varphi'\in C,\;\; D(\varphi)\in F_1,\;\; D(\varphi')\in\overline{D(A)},\;\; \textit{and } D\left(\varphi'\right)=A_{-1}D(\varphi)+F(\varphi).$$

*Then, the mild solution* $x$ *of equation* (1.1) *is continuously differentiable on* $\mathbb{R}^+$*and* $D\left(x.\right)$ *belongs to* $C^1\left(\mathbb{R}^+,X\right)\cap C\left(\mathbb{R}^+,F_1\right)$ *and satisfies*

$$\begin{cases} \dfrac{d}{dt}D\left(x_t\right)=A_{-1}D\left(x_t\right)+F(x_t),\;\; t\geq 0 \\ x_0=\varphi. \end{cases} \tag{3.1}$$

The proof of the theorem 2 is based on the following lemma.

**Lemma 4.** [12] *For* $u_0\in F_1$ *and* $f\in W^{1,1}\left(\mathbb{R}^+,F_0\right)$ *such that* $A_{-1}u_0+f(0)\in\overline{D(A)}$, *the Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}u(t)=A_{-1}u(t)+f(t),\;\; t\geq 0 \\ u(0)=u_0 \end{cases}$$

*has a unique solution* $u\in C^1\left(\mathbb{R}^+,X\right)\cap C\left(\mathbb{R}^+,F_1\right)$.

**Proof of Theorem 2.** Let $T > 0$ and $x$ be the unique mild solution of equation (1.1) on $[0,T]$. Consider the equation

$$D(y_t) = T_0(t)\,D(\varphi') + \int_0^t T_{-1}(t-s)\,F'(x_s)\,y_s\,ds, \quad y_0 = \varphi'. \tag{3.2}$$

Then, using the same reasoning as in the proof of Theorem 1, one can see that equation (3.2) has a unique continuous solution $y$ on $[-r,T]$. Let $z \in C([-r,T];X)$ be defined by

$$z(t) = \begin{cases} \varphi(t), & \text{if } t \in [-r,0], \\ \varphi(0) + \int_0^t y(s)\,ds, & \text{if } t \in [0,T]. \end{cases}$$

Then,

$$z_t = \varphi + \int_0^t y_s\,ds, \quad \text{for } t \in [0,T]. \tag{3.3}$$

We will show that $x = z$ on $[0,T]$. From equation (3.2), we get

$$\int_0^t D(y_s)\,ds = \int_0^t T_0(s)\,D(\varphi')ds + \int_0^t \int_0^s T_{-1}(s-\sigma)\,F'(x_\sigma)\,y_\sigma\,d\sigma\,ds. \tag{3.4}$$

On the other hand, for $0 \le t \le T$ in $X_{-1}$ norm, we have

$$\frac{d}{dt}\int_0^t T_{-1}(t-s)\,F(z_s)\,ds = T_{-1}(t)\,F(\varphi) + \int_0^t T_{-1}(t-s)\,F'(z_s)y_s\,ds.$$

Hence,

$$\begin{aligned} &\int_0^t T_{-1}(s)\,F(\varphi)ds \\ &= \int_0^t T_{-1}(t-s)\,F(z_s)\,ds - \int_0^t \int_0^s T_{-1}(s-\sigma)\,F'(z_\sigma)y_\sigma\,d\sigma\,ds. \end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned} D(z_t) &= D(\varphi) + \int_0^t T_0(s)\,(A_{-1}D(\varphi) + F(\varphi))\,ds \\ &\quad + \int_0^t \int_0^s T_{-1}(s-\sigma)\,F'(x_\sigma)\,y_\sigma\,d\sigma\,ds \\ &= D(\varphi) + \int_0^t T_{-1}(s)\,(A_{-1}D(\varphi) + F(\varphi))\,ds \\ &\quad + \int_0^t \int_0^s T_{-1}(s-\sigma)\,F'(x_\sigma)\,y_\sigma\,d\sigma\,ds \end{aligned}$$

$$= T_{-1}(t)D(\varphi) + \int_0^t T_{-1}(s)\, F(\varphi) ds + \int_0^t \int_0^s T_{-1}(s-\sigma)\, F'(x_\sigma)\, y_\sigma \, d\sigma \, ds.$$

Using (3.5), we get

$$D(z_t) = T_{-1}(t)D(\varphi) + \int_0^t T_{-1}(t-s)\, F(z_s)\, ds$$
$$+ \int_0^t \int_0^s T_{-1}(s-\sigma) \left(F'(x_\sigma) - F'(z_\sigma)\right) y_\sigma \, d\sigma \, ds.$$

On the other hand, $D(\varphi) \in F_1$. Thus, $T_{-1}(t)D(\varphi) = T_0(t)D(\varphi)$, for all $t \geq 0$, and

$$D(x_t - z_t) = \int_0^t T_{-1}(t-s)\left(F(x_s) - F(z_s)\right) ds$$
$$+ \int_0^t \int_0^s T_{-1}(s-\sigma)\left(F'(x_\sigma) - F'(z_\sigma)\right) y_\sigma \, d\sigma \, ds.$$

To complete the proof, the following lemma is needed.

**Lemma 5.** ([2], Lemma 5) *There exist positive constants $a$, $b$, and $c$ such that if $w \in C([0,+\infty); X)$ is a solution of*

$$\begin{cases} D(w_t) = \upsilon(t), & t \geq 0, \\ w_0 = \varphi \in C, \end{cases}$$

*where $\upsilon$ is a continuous function from $[0,+\infty)$ to $X$, then*

$$|w_t| \leq \Big(a\,|\varphi| + b \sup_{0\leq s\leq t} |\upsilon(s)|\Big) e^{ct}, \ t \geq 0. \tag{3.6}$$

By Lemma 5 and the local Lipschitz condition of $F'$, there exists a positive constant $k_0$ such that $|x_t - z_t| \leq k_0 \int_0^t |x_s - z_s|\, ds$, for $t \in [0,T]$. By Gronwall's lemma we deduce that $x_t = z_t$ for $t \in [0,T]$ and $x$ is continuously differentiable on $[0,T]$, for every $T > 0$. Hence, $F(x\cdot) \in C^1(\mathbb{R}^+, F_0)$, and by Lemma 4, we conclude that $D(x.)$ belongs to $C^1(\mathbb{R}^+, X) \cap C(\mathbb{R}^+, F_1)$ and satisfies equation (3.1).

**Remark 1.** In the nondistributed delay, $F(\varphi) = G_1(\varphi(-r))$, where $G_1$ is a Lipschitz-continuous function from $X$ to $F_0$. The local Lipschitz condition of $F'$ is not needed, since in this case

$$F'(x_\sigma) - F'(z_\sigma) = G_1'(x(\sigma - r)) - G_1'(z(\sigma - r)).$$

If we proceed by steps and choose $\varepsilon \in [0,r]$, then $F'(x_\sigma) - F'(z_\sigma) = 0$ and the regularity of the mild solution is obtained without assuming the local Lipschitz condition of $F'$.

## 4. The solution semigroup and the linearized stability

Let $\mathcal{H}$ be the phase space of equation (1.1) given by

$$\mathcal{H} = \{\varphi \in C : D(\varphi) \in \overline{D(A)}\}.$$

Define the operator $U(t)$ on $\mathcal{H}$ for $t \geq 0$ by $U(t)(\varphi) = x_t(\cdot, \varphi)$, where $x(\cdot, \varphi)$ is the mild solution of equation (1.1). Then we have

**Proposition 2.** *The family $(U(t))_{t\geq 0}$ is a nonlinear $C_0$ semigroup on $\mathcal{H}$; that is,*

i) $U(0) = I$,
ii) $U(t+s) = U(t)U(s)$, *for* $t, s \geq 0$,
iii) *for all* $\varphi \in \mathcal{H}$, $U(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* $\mathcal{H}$,
iv) *for all* $t \geq 0$, $U(t)$ *is Lipschitzian from* $\mathcal{H}$ *to* $\mathcal{H}$,
v) $(U(t))_{t\geq 0}$ *satisfies the translation property: for* $t \geq 0$ *and* $\theta \in [-\tau, 0]$

$$(U(t)(\varphi))(\theta) = \begin{cases} (U(t+\theta)(\varphi))(0), & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta), & \text{if } t+\theta \leq 0. \end{cases}$$

**Proof.** *i*) and *ii*) are just consequences of the definition and the uniqueness of the solution. *iii*) comes from the fact that the solution is continuous for every $t \geq 0$. Let $\varphi_1$ and $\varphi_2$ be in $\mathcal{H}$; then

$$\begin{aligned} D(U(t)\varphi_1 - U(t)\varphi_2) &= T_0(t)D(\varphi_1 - \varphi_2) \\ &\quad + \int_0^t T_{-1}(t-s)\left(F(U(s)\varphi_1) - F(U(s)\varphi_2)\right) ds,\ t \geq 0. \end{aligned}$$

Applying Lemma 5, we obtain that for every $T > 0$, there exist some positive constants $a_1$ and $b_1$ such that for $t \in [0, T]$

$$|U(t)\varphi_1 - U(t)\varphi_2| \leq a_1 |\varphi_1 - \varphi_2| + b_1 \int_0^t |U(s)\varphi_1 - U(s)\varphi_2|\, ds.$$

By Gronwall's lemma we obtain that for every $t \geq 0$, $U(t)$ is Lipschitzian, and this prove *iv*). *v*) as a consequence of the definition of $U$. $\square$

By an equilibrium of equation (1.1), we mean a constant mild solution $x^*$ of equation (1.1) . Without loss of generality we suppose that $x^* = 0$ and $F(0) = 0$.

($\mathbf{H}_3$) $F$ is differentiable at zero.

Then, the linearized equation at zero is given by

$$\begin{cases} \dfrac{d}{dt}D\left(y_t\right) = AD\left(y_t\right) + L(y_t), & \text{for } t \geq 0, \\ y_0 = \varphi \in C, \end{cases} \tag{4.1}$$

where $L = F'(0)$. Let $(T(t))_{t\geq 0}$ be the solution $C_0$ semigroup associated with equation (4.1). Then,

**Theorem 3.** *Assume that conditions* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, *and* $(\mathbf{H}_3)$ *hold. Then for* $t \geq 0$ *the derivative at zero of* $U(t)$ *is* $T(t)$.

**Proof.** We have

$$\begin{aligned} D\left[U(t)\varphi - T(t)\varphi\right] = &\int_0^t T_{-1}(t-s)\left(F(U(s)(\varphi)) - F(T(s)\varphi)\right)ds, \\ &+ \int_0^t T_{-1}(t-s)\big(F(T(s)(\varphi)) - F'(0)(T(s)\varphi)\big)\,ds. \end{aligned}$$

Let $\varepsilon > 0$; then there exists $\delta > 0$ such that

$$\left|F(T(s)\varphi) - F'(0)(T(s)\varphi)\right|_{F_0} \leq \varepsilon\left|\varphi\right|, \ \text{ for } \ |\varphi| \leq \delta \text{ and } s \in [0,t]\,.$$

Applying Lemma 5, we have

$$|U(t)\varphi - T(t)\varphi| \leq \quad bMe^{\omega t}e^{ct}\left(t\varepsilon\left|\varphi\right| + K\textstyle\int_0^t e^{-\omega s}\left|U(s)\varphi - T(s)\varphi\right|ds\right).$$

By Gronwall's lemma, we obtain

$$|U(t)\varphi - T(t)\varphi| \leq bM\varepsilon t\left|\varphi\right|\exp\left[\left(bMKe^{ct} + \omega + c\right)t\right]$$

and conclude that $U(t)$ is differentiable at 0 and $D_\varphi U(t)(0) = T(t)$, for each $t \geq 0$. $\square$

**Theorem 4.** *Assume that conditions* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, *and* $(\mathbf{H}_3)$ *hold. If the zero equilibrium of* $(T(t))_{t\geq 0}$ *is exponentially stable, then the zero equilibrium of* $(U(t))_{t\geq 0}$ *is locally exponentially stable in the sense that there exist* $\delta > 0$, $\mu > 0$, *and* $k \geq 1$ *such that*

$$|U(t)(\varphi)| \leq ke^{-\mu t}\left|x\right|, \ \textit{for } \varphi \in \mathcal{H} \textit{ with } |\varphi| \leq \delta \textit{ and } t \geq 0.$$

*Moreover, if* $\mathcal{H}$ *can be decomposed as* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, *where* $\mathcal{H}_i$ *are* $T$*-invariant subspaces of* $\mathcal{H}$ *and* $\mathcal{H}_1$ *is finite-dimensional and with* $\omega = \lim\limits_{h\to\infty}\frac{1}{h}\log|T(h)/\mathcal{H}_2|$, *we have*

$$\inf\left\{|\lambda| : \lambda \in \sigma\left(T(t)/\mathcal{H}_1\right)\right\} > e^{\omega t}.$$

*Then, zero is not stable in the sense that there exist* $\varepsilon > 0$ *and sequences* $(\varphi_n)_n$ *converging to* 0 *and* $(t_n)_n$ *of positive reals such that* $|U(t_n)\varphi_n| > \varepsilon$.

The proof of the above theorem is based on Theorem 3 and the following result.

**Theorem 5.** [5] *Let $(V(t))_{t\geq 0}$ be a nonlinear $C_0$ semigroup on a subset $\Omega$ of a Banach space $Z$, and assume that $x_0 \in \Omega$ is an equilibrium of $(V(t))_{t\geq 0}$ such that $V(t)$ is differentiable at $x_0$ for each $t \geq 0$, with $W(t)$ the derivative at $x_0$ of $V(t)$, $t \geq 0$. Then, $(W(t))_{t\geq 0}$ is a $C_0$ semigroup of bounded linear operators on $Z$. If the zero equilibrium of $(W(t))_{t\geq 0}$ is exponentially stable, then $x_0$ is locally exponentially stable of $(V(t))_{t\geq 0}$. Moreover, if $Z$ can be decomposed as $Z = Z_1 \oplus Z_2$, where $Z_i$ are $W$-invariant subspaces of $Z$ and $Z_1$ is finite-dimensional and with $\omega = \lim_{h\to\infty} \frac{1}{h} \log |W(h)/Z_2|$, we have*

$$\inf \{|\lambda| : \lambda \in \sigma (W(t)/Z_1)\} > e^{\omega t}.$$

*Then, the equilibrium $x_0$ is not stable in the sense that there exist $\varepsilon > 0$ and sequences $(x_n)_n$ converging to $x_0$ and $(t_n)_n$ of positive reals such that $|V(t_n)x_n - x_0| > \varepsilon$.*

## 5. Application

Consider the partial functional differential equations of neutral type,

$$\begin{cases} \dfrac{\partial}{\partial t}\left[v(t,x) - \displaystyle\int_{-1}^{0} \beta(\theta)v(t+\theta,x)d\theta\right] \\ = -\dfrac{\partial}{\partial x}\left[v(t,x) - \displaystyle\int_{-1}^{0} \beta(\theta)v(t+\theta,x)d\theta\right] \\ \qquad +m(x)h(v(t-1,x)),\ \text{for } x \in [0,1]\,,\ t \geq 0, \\ v(t,0) = \displaystyle\int_{-1}^{0} \beta(\theta)v(t+\theta,0)d\theta,\ t \geq 0, \\ v(\theta,x) = v_0(\theta,x),\ \text{for } (\theta,x) \in [-1,0] \times [0,1]\,, \end{cases} \tag{5.1}$$

where $v_0 \in C([-1,0] \times [0,1]\,;\mathbb{R})$, $\beta : [-1,0] \to \mathbb{R}$ is a continuous function, and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function. Furthermore, suppose that $m \in L^\infty(0,1)$ and let $A$ be the operator defined on $X = C([0,1]\,;\mathbb{R})$ by

$$\begin{cases} D(A) = \left\{g \in C^1([0,1]\,;\mathbb{R}) : g(0) = 0\right\} \\ Ag = -g'. \end{cases}$$

Then, $\overline{D(A)} = C_0([0,1]\,;\mathbb{R}) = \{g \in C([0,1]\,;\mathbb{R}) : g(0) = 0\}\,.$

**Lemma 6.** [12] *The operator $A$ is a Hille-Yosida operator on $X$. The part $A_0$ of $A$ in $C_0([0,1]\,;\mathbb{R})$ generates a $C_0$ semigroup $(T_0(t))_{t\geq 0}$ on $C_0([0,1]\,;\mathbb{R})$,* *which is given for* $g \in C_0([0,1];\mathbb{R})$ *by*

$$(T_0(t)g)(x) = \begin{cases} g(x-t), & \text{if } t \le x \\ 0, & \text{if } t > x. \end{cases}$$

Let $Lip_0[0,1]$ be the space of Lipschitz-continuous functions on $[0,1]$ vanishing at zero. Then

**Lemma 7.** [12] *The Favard class $F_0$ of the extrapolated semigroup $(T_{-1}(t))_{t\ge 0}$ is given by $F_0 = L^\infty(0,1)$. The Favard class $F_1$ of the extrapolated semigroup $(T_0(t))_{t\ge 0}$ is given by $F_1 = Lip_0[0,1]$, with the norm*

$$|g|_{Lip} = \sup_{0\le x_1 < x_2 \le 1} \frac{|g(x_1) - g(x_2)|}{x_1 - x_2}.$$

*The extrapolated operator $A_{-1}$ coincides on $F_1$ with the almost-everywhere derivative.*

Let $F : C([-1,0];C([0,1];\mathbb{R})) \to L^\infty(0,1)$ be defined by

$$(F(\varphi_1))(x) = m(x)h((\varphi_1)(-1)(x)), \text{ for } x \in [0,1]$$

and $D$ be the bounded linear operator from $C([-1,0];C([0,1];\mathbb{R})$ to $C([0,1];\mathbb{R})$ defined by

$$D(\varphi_1)(x) = \varphi_1(0)(x) - \int_{-1}^{0} \beta(\theta)\varphi_1(\theta)(x)d\theta, \text{ for } x \in [0,1].$$

Then, equation (5.1) takes the abstract form

$$\begin{cases} \dfrac{d}{dt}D(x_t) = AD(x_t) + F(x_t), \text{ for } t \ge 0 \\ x_0 = \varphi \in C([-1,0];X), \end{cases} \tag{5.2}$$

where the function $\varphi$ is given by $\varphi(\theta)(x) = v_0(\theta, x)$, for $\theta \in [-1,0]$ and $x \in [0,1]$. For $\varphi_1, \varphi_2 \in C([-1,0];C([0,1];\mathbb{R})$, we have

$$\begin{aligned} |F(\varphi_1) - F(\varphi_2)|_{L^\infty(0,1)} &= \underset{x\in[0,1]}{esssup} |m(x)| \, |h((\varphi_1)(-1)(x)) - h((\varphi_2)(-1)(x))| \\ &\le |m|_{L^\infty(0,1)} \, \alpha \, |\varphi_1 - \varphi_2|_{C([-1,0],C([0,1],\mathbb{R}))}, \end{aligned}$$

where $\alpha$ is the Lipschitz constant of $h$. It follows that assumption $(\mathbf{H}_2)$ is satisfied. Moreover, if we assume that $v_0(0,0) = \int_{-1}^{0} \beta(\theta)v_0(\theta,0)d\theta$. Then, $D(\varphi) \in \overline{D(A)}$. As a consequence, equation (5.2) has a unique mild solution on $\mathbb{R}^+$. For the regularity of the mild solution, suppose that $h$

is continuously differentiable; then $F$ is continuously differentiable from $C([-1,0]\,; C\,([0,1]\,;\mathbb{R})$ into $L^{\infty}(0,1)$ and the derivative $F'$ is given by

$$\left(F'(\varphi_1)\,(\varphi_2)\right)(x) = m(x)h'(\varphi_1(-1)(x))\varphi_2(-1)(x), \text{ for a.e. } x \in [0,1]\,.$$

Let $v_0 \in C([-1,0] \times [0,1]\,, \mathbb{R})$ such that $\frac{\partial v_0}{\partial \theta}$ exists, is continuous on $[-1,0] \times [0,1]$, and satisfies

a) $v_0(0,\cdot) - \int_{-1}^{0} \beta(\theta) v_0(\theta,\cdot) d\theta \in Lip_0\,[0,1]\,,$

b) $\frac{\partial v_0(0,x)}{\partial \theta} - \int_{-1}^{0} \beta(\theta) \frac{\partial v_0(\theta,x)}{\partial \theta} d\theta = -\frac{\partial}{\partial x}\left[v_0(0,x) - \int_{-1}^{0} \beta(\theta) v_0(\theta,x) d\theta\right] + m(x)h\,(v_0(-1,x))\,,$ for a.e. $x \in [0,1]\,,$

c) $\frac{\partial v_0(0,0)}{\partial \theta} = \int_{-1}^{0} \beta(\theta) \frac{\partial v_0(\theta,0)}{\partial \theta} d\theta.$

From assumptions $a)$, $b)$, and $c)$ we deduce that

$$\varphi' \in C,\ D(\varphi) \in F_1,\ D(\varphi') \in \overline{D(A)}, \text{ and } D\left(\varphi'\right) = A_{-1}D(\varphi) + F(\varphi).$$

Since all the assumptions of Theorem 2 are satisfied, we have

**Proposition 3.** *Let $v$ be the mild solution of equation* (5.2). *Then, the function*

$$t \to \left[v(t,\cdot) - \int_{-1}^{0} \beta(\theta) v(t+\theta,\cdot) d\theta\right]$$

*belongs to* $C^1(\mathbb{R}^+, C_0([0,1]\,;\mathbb{R})) \cap C(\mathbb{R}^+, Lip_0\,[0,1]\,)$ *and satisfies*

$$\begin{cases} \dfrac{\partial}{\partial t}\left[v(t,x) - \displaystyle\int_{-1}^{0} \beta(\theta) v(t+\theta,x) d\theta\right] \\ = -\dfrac{\partial}{\partial x}\left[v(t,x) - \displaystyle\int_{-1}^{0} \beta(\theta) v(t+\theta,x) d\theta\right] \\ \qquad + m(x)h(v(t-1,x)),\ \text{for a.e. } x \in [0,1]\,,\ t \geq 0\ , \\ v(t,0) = \displaystyle\int_{-1}^{0} \beta(\theta) v(t+\theta,0) d\theta,\ t \geq 0, \\ v(\theta,x) = v_0(\theta,x),\ \text{for } \theta \in [-1,0]\ \text{and } x \in [0,1]\,. \end{cases}$$

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