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STABILIZATION OF STAR-SHAPED NETWORKS OF STRINGS

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Abstract. We study the energy decay of a network of vibrating elastic strings where the strings are coupled at a common end in a star-shaped configuration. We prove that the solutions are not exponentially stable in the energy space. Moreover, we give explicit polynomial decay estimates valid for regular initial data. These estimates depend on the diophantine approximations properties of the lengths of the strings of the network.

1. Introduction

We study the pointwise feedback stabilization of network of N strings, where $2 \leq N \in \mathbb{N}$. More precisely we consider the following initial- and boundary-value problems:

$$\frac{\partial^2 u_i}{\partial t^2}(x,t) - \frac{\partial^2 u_i}{\partial x^2}(x,t) = 0, \quad 0 < x < l_i,\ t > 0, \tag{1.1}$$

$$u_i(l_i,t) = 0, \quad t > 0, \tag{1.2}$$

$$u_i(0,t) = u_j(0,t), \quad t > 0, \tag{1.3}$$

$$\sum_{i=1}^{N} \frac{\partial u_i}{\partial x}(0,t) = \frac{\partial u_1}{\partial t}(0,t), \quad t > 0, \tag{1.4}$$

$$u_i(x,0) = u_i^0(x), \frac{\partial u_i}{\partial t}(x,0) = u_i^1(x), \quad 0 < x < l_i, \tag{1.5}$$

for $i,j = 1,\dots,N$ and where $u_i : [0,l_i] \times (0,+\infty) \to \mathbb{R}$, $i = 1,\dots,N$, are the displacement of the string of length l_i. Denote $L = \sum_{i=1}^{N} l_i$.

In the present paper we prove that the solutions of a strings network are not exponentially stable in the energy space. Moreover, we give explicit decay estimates for regular initial data. These estimates depend on the

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diophantine approximations properties of l_i/L, $i = 1, \dots, N$. Our approach (see [2]), avoiding the frequency domain method, is based on a sharp trace regularity result combined with an observability inequality valid for solutions of an appropriate conservative problem, where the observability inequality does not use Ingham inequalities but uses an adaptation of an idea in [7] such as the d'Alembert representation formula.

The plan of the paper is as follows. In Section 2 we give precise statements of the main results. Section 3 contains some trace regularity results needed in the following sections. In Section 4 we prove an exact pointwise observability result for the associated undamped problem. The proof of the main results are given in Section 5.

2. Statement of the main results

The skew-adjoint operator corresponding to (3.8)–(3.12) can be diagonalized over the orthonormal basis of eigenvectors $\Phi_k = (\Phi_{k,1}, \Phi_{k,2}, \dots, \Phi_{k,N})$, $k \in \mathbb{N}^*$, with associated eigenvalues (see [5] and [11]) $\lambda_k = i w_k \sim i\dfrac{k\pi}{L}$, $k \to \infty$.

We define the energy of a u_i, $i = 1, \dots, N$ of (1.1)–(1.5) at the instant t by

$$E(t) = \sum_{i=1}^{N} \frac{1}{2} \int_0^{l_i} \left(\left| \frac{\partial u_i}{\partial t}(x,t) \right|^2 + \left| \frac{\partial u_i}{\partial x}(x,t) \right|^2 \right) dx. \tag{2.1}$$

We show that a sufficiently smooth solution of (1.1)–(1.5) satisfies the energy estimate

$$E(0) - E(t) = \int_0^t \left| \frac{\partial u_1}{\partial t}(0,s) \right|^2 ds. \tag{2.2}$$

The wellposedness space for (1.1)–(1.5) is $X = V \times \prod_{i=1}^N L^2(0, l_i)$, where

$$V = \left\{ \phi = (\phi_i)_{i=1,\dots,N},\ \phi_i \in H^1(0, l_i) : \phi(l_i) = 0,\ \phi_i(0) = \phi_j(0) \right\}.$$

Denote

$$\begin{aligned} \mathcal{D}(A) = \Big\{ & ((u_i)_{i=1,\dots,N}, (v_i)_{i=1,\dots,N}) \in \Big[V \cap \prod_{i=1}^N H^2(0, l_i) \Big] \times V, \\ & \sum_{i=1}^N \frac{du_i}{dx}(0) = v_1(0) \Big\}. \end{aligned} \tag{2.3}$$

The corresponding operator A is defined by

$$A \begin{pmatrix} u_1 \\ \dots \\ u_N \\ v_1 \\ \dots \\ v_N \end{pmatrix} = \begin{pmatrix} v_1 \\ \dots \\ v_N \\ \frac{d^2 u_1}{dx^2} \\ \dots \\ \frac{d^2 u_N}{dx^2} \end{pmatrix}, \qquad \forall\, (u, v) \in \mathcal{D}(A).$$

The wellposedness and strong stability properties are summarized in the result below.

Proposition 2.1. *The following assertions hold true:*

(1) *If* $(u^0, u^1) \in X$, *then the problem* (1.1)–(1.5) *admits a unique solution*

$$u \in C(0, +\infty; V) \cap C^1(0, +\infty; \prod_{i=1}^{N} L^2(0, l_i))$$

such that, for all $T > 0$, $u_1(0, \cdot) \in H^1(0, T)$ *and*

$$\|u_1(0, \cdot)\|^2_{H^1(0,T)} \leq C \sum_{i=1}^{N} \Big(\|u_i^0\|^2_{H^1(0,l_i)} + \|u_i^1\|^2_{L^2(0,l_i)} \Big), \tag{2.4}$$

where the constant $C > 0$ *depends only on* T. *Moreover* $(u_i)_{i=1,..,N}$ *satisfies the energy estimate* (2.2).

(2) *The estimate* $\lim_{t \to \infty} E(t) = 0$ *holds true for any finite energy solution of* (1.1)–(1.5) *if and only if, for all* $i = 1, \dots, N$,

$$\frac{l_i}{l_j} \notin \mathbb{Q}, \ \forall\, i, j = 1, \dots, N. \tag{2.5}$$

Denote by $\mathbf{Q}$ the set of all rational numbers. Let us also denote by $\mathcal{S}$ the set of all numbers ρ such that $\rho \notin \mathbf{Q}$ and if $[0, a_1, \dots, a_n, \dots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. Let us notice that $\mathcal{S}$ is obviously uncountable and, by classical results on diophantine approximation (cf. [6, p. 120]), its Lebesgue measure is equal to zero. Roughly speaking the set $\mathcal{S}$ contains the irrationals which are "badly" approximable by rational numbers. In particular, by the Euler-Lagrange theorem (cf. [9, p. 57]) $\mathcal{S}$ contains all l_i/L, $i = 1, \dots, N$ such that l_i/L is an irrational quadratic number (i.e., satisfying a second-degree equation with rational coefficients). According to a classical result if $l_i/L \in \mathcal{S}$, $i = 1, \dots, N$, then there exists a constant $C_i > 0$ such that

$$|\sin(n\pi l_i/L)| \geq \frac{C_i}{n}, \qquad \forall\, n \geq 1. \tag{2.6}$$

Our main results can now be stated as follows.

Theorem 2.2.

(1) *The system described by* (1.1)–(1.5) *is not exponentially stable in the energy space.*

(2) *Suppose that* $l_i/l_j \notin \mathbb{Q}$, $\forall\, i, j = 1, \ldots, N$. *Then, for all* $l_i/L \in \mathcal{S}$, $i = 1, \ldots, N$ *and for all* $t \geq 0$ *we have*

$$E(t) \leq \frac{C}{(t+1)^{\frac{1}{N}}} \|(u^0, u^1)\|^2_{\mathcal{D}(A)}, \qquad \forall\, (u^0, u^1) \in \mathcal{D}(A), \tag{2.7}$$

where $C > 0$ *is a constant depending only on* l_i.

(3) *Suppose that* $l_i/l_j \notin \mathbb{Q}$, $\forall\, i, j = 1, \ldots, N$. *Then, for all* $\epsilon > 0$ *we have for almost all* l_i/L, $i = 1, \ldots, N$ *and for all* $t \geq 0$

$$E(t) \leq \frac{C_\epsilon}{(t+1)^{\frac{1}{N(1+\epsilon)}}} \|(u^0, u^1)\|^2_{\mathcal{D}(A)}, \qquad \forall\, (u^0, u^1) \in \mathcal{D}(A), \tag{2.8}$$

where $C_\epsilon > 0$ *is a constant depending only on* l_i *and* ϵ.

3. Some regularity results

Consider the initial- and boundary-value problem

$$\frac{\partial^2 v_i}{\partial t^2}(x,t) - \frac{\partial^2 v_i}{\partial x^2}(x,t) = 0, \quad 0 < x < l_i,\ t > 0, \tag{3.1}$$

$$v_i(l_i, t) = 0, \quad t > 0, \tag{3.2}$$

$$v_i(0,t) = v_j(0,t), \quad \forall t > 0, \tag{3.3}$$

$$\sum_{i=1}^{N} \frac{\partial v_i}{\partial x}(0,t) = k(t), \quad \forall t > 0, \tag{3.4}$$

$$v_i(x,0) = 0,\ \frac{\partial v_i}{\partial t}(x,0) = 0, \quad 0 < x < l_i, \tag{3.5}$$

for $i, j = 1, \ldots, N$. The equations above are models for the vibrations of an undamped network of strings, in the presence of a pointwise force. The following proposition gives regularity properties of the solutions of (3.1)–(3.5).

Proposition 3.1. *Let $T < 0$ be fixed, and suppose that $k \in L^2(0,T)$. Then the problem* (3.1)–(3.5) *admits a unique solution having the regularity*

$$v \in C(0,T;V) \cap C^1(0,T;\prod_{i=1}^{N} L^2(0,l_i)). \tag{3.6}$$

Moreover, $v_1(0,\cdot) \in H^1(0,T)$ and there exists a constant $C > 0$, depending only on T, such that

$$\|v_1(0,\cdot)\|_{H^1(0,T)} \leq C\|k\|_{L^2(0,T)}, \qquad \forall\, k \in L^2(0,T). \tag{3.7}$$

In order to prove Proposition 3.1 we first study the case of free vibrations of an undamped network of strings; i.e., we consider the initial- and boundary-value problem

$$\frac{\partial^2 \phi_i}{\partial t^2}(x,t) - \frac{\partial^2 \phi_i}{\partial x^2}(x,t) = 0, \quad 0 < x < l_i,\ t > 0, \tag{3.8}$$

$$\phi_i(l_i,t) = 0, \quad t > 0, \tag{3.9}$$

$$\phi_i(0,t) = \phi_j(0,t), \quad t > 0, \tag{3.10}$$

$$\sum_{i=1}^{N} \frac{\partial \phi_i}{\partial x}(0,t) = 0, \quad t > 0, \tag{3.11}$$

$$\phi_i(x,0) = u_i^0(x), \frac{\partial \phi_i}{\partial t}(x,0) = u_i^1(x), \quad 0 < x < l_i. \tag{3.12}$$

The following result, besides showing that problem above is well-posed in the natural energy space, gives a sharp inequality on the trace of ϕ_1 at the point 0. It is easy to see by the semigroup method [12], that the problem (3.8)–(3.12) is well-posed in the energy space. The inequality (3.14) is a consequence of (3.7); see [2] for details.

Lemma 3.2. *Suppose that $(u^0, u^1) \in X$. Then the initial- and boundary-value problem* (3.8)–(3.12) *admits a unique solution*

$$\phi \in C(0,T+\infty;V) \cap C^1(0,T+\infty;\prod_{i=1}^{N} L^2(0,l_i)), \tag{3.13}$$

satisfying, for all $T > 0$, $\phi_1(0,\cdot) \in H^1(0,T)$. Moreover, there exists a constant $C > 0$, depending only on T, such that

$$\|\phi_1(0,\cdot)\|^2_{H^1(0,T)} \leq C \sum_{i=1}^{N} \Big(\|u_i^0\|^2_{H^1(0,l_i)} + \|u_i^1\|^2_{L^2(0,l_i)} \Big). \tag{3.14}$$

In order to prove Proposition 3.1 we need the following technical result:

Lemma 3.3. *Let $\beta > 0$ be a fixed real number and $C_\beta = \{\lambda \in \mathbb{C} : Re(\lambda) = \beta\}$. Then, the function*

$$f(\lambda) = \frac{1}{\sum_{i=1}^N \coth(\lambda l_i)} \tag{3.15}$$

is bounded on C_β.

Proof. We have, for $\lambda = \beta + i\alpha \in C_\beta$,

$$\Big|\sum_{i=1}^N \coth(\lambda l_i)\Big|^2$$
$$= \Big|\sum_{i=1}^N \frac{e^{4\beta l_i} - 1}{e^{4\beta l_i} - 2\, e^{2\beta l_i}\, \cos(\alpha l_i) + 1}\Big|^2 + \Big|\sum_{i=1}^N \frac{2\, e^{2\beta l_i} \sin(\alpha l_i)}{e^{4\beta l_i} - 2\, e^{2\beta l_i}\, \cos(\alpha l_i) + 1}\Big|^2,$$

which implies

$$\Big|\sum_{i=1}^N \frac{e^{4\beta l_i} - 1}{e^{4\beta l_i} - 2\, e^{2\beta l_i}\, \cos(\alpha l_i) + 1}\Big| \geq \sum_{i=1}^N \coth(\beta l_i);$$

thus,

$$|f(\lambda| \leq \frac{1}{\displaystyle\sum_{i=1}^N \coth(\beta l_i)}, \quad \forall\, \lambda \in C_\beta. \qquad \square$$

Proof of Proposition 3.1. For proving (3.6), it suffices to use the method of transposition (see [4] and [10]).

We still have to prove the trace regularity property (3.7).

As equation (3.1) is time-reversible, after extending k by zero for $t \in \mathbb{R} \setminus [0, T]$ we can solve (3.1)–(3.5), for $t \in \mathbb{R}$. In this way we obtain a function, denoted also by v_i, $i = 1, \ldots, N$, such that

$$\begin{cases} v \in C_B(\mathbb{R}; V) \cap C_B^1(\mathbb{R}; \prod_{i=1}^N L^2(0, l_i)), \\ v(x,t) = 0, \qquad \forall\, t \leq 0, \end{cases} \tag{3.16}$$

and v_i, $i = 1, \ldots, N$ satisfies (3.1)–(3.5) for all $(x,t) \in [0, l_i] \times \mathbb{R}$.

Let $\widehat{v_i}(x, \lambda)$, where $\lambda = \gamma + i\eta,\ \gamma > 0$ and $\eta \in \mathbb{R}$, be the Laplace (with respect to time) transform of v_i, $i = 1, \ldots, N$ (above we denote by $C_B^k(\mathbb{R}, X)$ the space of C^k functions from $\mathbb{R}$ into X which are bounded on $\mathbb{R}$ together with their derivatives up to the order k). Since v_1 satisfies (3.16), estimate

(3.7) is equivalent to the fact that the function $t \to e^{-\gamma t} v_1(0,t)$ belongs to $H^1(\mathbb{R})$ and that there exists a constant $M_1 > 0$ such that

$$\|e^{-\gamma \cdot} v_1(0,\cdot)\|^2_{H^1(-\infty,\infty)} \leq M_1 \|k\|^2_{L^2(-\infty,\infty)}.$$

Equivalently, by the Parseval identity (see for instance [8, p. 212]), it suffices to prove that the function $\eta \to (\gamma + i\eta)\widehat{v}_1(0, \gamma + i\eta)$ belongs to $L^2(\mathbb{R}_\eta)$, for some $\gamma > 0$, and that there exists a constant $M_2 > 0$ such that

$$\|(\gamma + i\eta)\widehat{v}_1(0, \gamma + i\eta)\|^2_{L^2(\mathbb{R}_\eta)} \leq M_2 \int_{-\infty}^{\infty} |k(\gamma + i\eta)|^2 d\eta. \tag{3.17}$$

It can be easily checked that $\widehat{v}$ satisfies

$$\lambda^2 \widehat{v_i}(x,\lambda) - \frac{d^2 \widehat{v}_i}{dx^2}(x,\lambda) = 0,\ x \in (0, l_i),\ Re\lambda > 0, \tag{3.18}$$

$$\widehat{v_i}(l_i, \lambda) = 0,\ Re\lambda > 0, \tag{3.19}$$

$$\widehat{v_i}(0,\lambda) = \widehat{v_j}(0,\lambda),\ Re\lambda > 0, \tag{3.20}$$

$$\sum_{i=1}^{N} \frac{d\widehat{v}_i}{dx}(0) = \widehat{k}(\lambda),\ Re\lambda > 0, \tag{3.21}$$

for $i, j = 1, \ldots, N$. We deduce that, for every $\lambda \in \mathbb{C},\ Re\lambda > 0$, we can find $H(\lambda) \in \mathbb{C}$, such that

$$\lambda\, \widehat{v_1}(0,\lambda) = H(\lambda)\widehat{k}(\lambda),\ \forall Re\,\lambda > 0. \tag{3.22}$$

In order to compute $H(\lambda)$ we notice that the solutions of (3.18)–(3.19) have the form

$$\widehat{v_i}(x,\lambda) = A_i sh[\lambda(x - l_i)],\ x \in (0, l_i),$$

where A_i is a constant. Consequently, the solutions of (3.18)–(3.21) have the following form:

$$\widehat{v_1}(x,\lambda) = \frac{\widehat{k}}{\lambda sh(\lambda l_1) \sum_{i=1}^{N} \coth(\lambda l_i)}\, sh[\lambda(x - l_1)],\ x \in (0, l_1). \tag{3.23}$$

Then, using (3.21) and (3.22) we obtain

$$H(\lambda) = -f(\lambda), \tag{3.24}$$

where f is defined by (3.15). We can now apply Lemma 3.3 to obtain the existence of a constant $M_2 > 0$ such that (3.17) holds true. This ends the proof of the fact that (3.7) holds for all solutions of (3.1)–(3.5).

4. Observability inequality

We have for ϕ a solution of (3.8)–(3.12), for all $t \geq 0$, the following inequality:

$$E_\phi(0) = E_\phi(t) \leq C \sum_{i=1}^{N} \int_{t-l_i}^{t+l_i} \Big(\Big|\frac{\partial \phi_1}{\partial t}(0,\tau)\Big|^2 + \Big|\frac{\partial \phi_i}{\partial x}(0,\tau)\Big|^2 \Big)\, d\tau,$$

where

$$E_\phi(t) = \frac{1}{2} \sum_{i=1}^{N} \int_0^{l_i} \Big(\Big|\frac{\partial \phi_i}{\partial t}(x,t)\Big|^2 + \Big|\frac{\partial \phi_i}{\partial x}(x,t)\Big|^2 \Big) dx.$$

Denote $R = \prod_{i=1}^N l_i^-$ and $R_j = \prod_{i \neq j}^N l_i^-$, where $l_i^\pm f(t) = \frac{f(t+l_i) \pm f(t-l_i)}{2}$.

Since $R\phi$ is solution of (3.8)–(3.12), we have

$$E_{R\phi}(0) = E_{R\phi}(t) \leq C \sum_{i=1}^{N} \int_{t-l_i}^{t+l_i} \Big(\Big| R\Big(\frac{\partial \phi_1}{\partial t}\Big)(0,\tau)\Big|^2 + \Big| R_i l_i^- \Big(\frac{\partial \phi_i}{\partial x}\Big)(0,\tau)\Big|^2 \Big) d\tau.$$

But the D'Alembert formula implies that

$$l_i^+ \Big(\frac{\partial \phi_i}{\partial t}\Big)(0,t) - l_i^- \Big(\frac{\partial \phi_i}{\partial x}\Big)(0,t) = 0, \qquad \forall\ 1 \leq i \leq N;$$

thus,

$$\begin{aligned} E_{R\phi}(t) &\leq C \sum_{i=1}^{N} \int_{t-l_i}^{t+l_i} \Big(\Big| R\Big(\frac{\partial \phi_1}{\partial t}\Big)(0,\tau)\Big|^2 + \Big| R_i l_i^+ \Big(\frac{\partial \phi_i}{\partial t}\Big)(0,\tau)\Big|^2 \Big) d\tau \\ &\leq C \sum_{i=1}^{N} \int_{t-l_i-L}^{t+l_i-L} \Big|\frac{\partial \phi_1}{\partial t}(0,\tau)\Big|^2 d\tau, \end{aligned}$$

which implies

$$E_{R\phi}(t) \leq C \int_{t-2L}^{t+2L} \Big|\frac{\partial \phi_1}{\partial t}(0,\tau)\Big|^2 d\tau;$$

let else

$$E_{R\phi}(0) = E_{R\phi}(2L) \leq C \int_0^{4L} \Big|\frac{\partial \phi_1}{\partial t}(0,\tau)\Big|^2 d\tau.$$

Now, if we denote $B = \sum_{i=1}^N l_i^+ R_i$, then $B(\frac{\partial \phi_1}{\partial t})(0,t) = 0$ (see [7, Proposition 6], with $c_L = \dfrac{N}{2^N} \neq 0$), which implies that

$$E_{R\phi}(0) \leq C \int_0^{2L} \Big|\frac{\partial \phi_1}{\partial t}(0,\tau)\Big|^2 d\tau. \tag{4.1}$$

If we put

$$u^0(x) = \sum_{k\geq 1} a_k \Phi_k(x),\ u^1(x) = \sum_{k\geq 1} b_k \Phi_k(x),$$

with $(w_k a_k)_k$, $(b_k)_k \in l^2$, then

$$\phi(x,t) = \sum_{k\geq 1} \left(a_k \cos(w_k t) + \frac{b_k}{w_k} \sin(w_k t) \right) \Phi_k(x); \tag{4.2}$$

thus,

$$R\phi(x,t) = \sum_{k\geq 1} \gamma_k \Big[(-1)^N a_k \sin(w_k t) + \frac{b_k}{w_k} \cos(w_k t) \Big] \Phi_k(x),$$

where $\gamma_k = \prod_{i=1}^N \sin(w_k l_i)$. The inequality (4.1) can be written as

$$\sum_{k\geq 1} \gamma_k^2 \left(|w_k a_k|^2 + |b_k|^2 \right) \leq C \int_0^{2L} \left| \frac{\partial \phi_1}{\partial t}(0,\tau) \right|^2 d\tau. \qquad \square \tag{4.3}$$

The observability inequality concerning the solutions of (3.8)–(3.12) can be stated as follows.

Proposition 4.1. *Let $T > 0$ be fixed and $\mathcal{S}$ be the set introduced in Section 2. Then*

(1) *For all $l_i, i = 1, \ldots, N$, there does not exist a constant $C > 0$ such that the solutions ϕ of (3.8)–(3.12) satisfy*

$$\int_0^T \left| \frac{\partial \phi_1}{\partial t}(0,t) \right|^2 dt \geq C \|(u^0, u^1)\|_X^2, \qquad \forall (u^0, u^1) \in X. \tag{4.4}$$

(2) *Suppose that $l_i/l_j \notin \mathbb{Q}$, $\forall\, i, j = 1, \ldots, N$. Then, for all $l_i/L \in \mathcal{S}$, $i = 1, \ldots, N$, the solution $(\phi_i)_{i=1,\ldots,N}$ of (3.8)–(3.12) satisfies*

$$\int_0^T \left| \frac{\partial \phi_1}{\partial t}(0,t) \right|^2 dt \geq C \sum_{i=1}^N \left(\|u_i^0\|^2_{H^{-N+1}(0,l_i)} + \|u_i^1\|^2_{H^{-N}(0,l_i)} \right), \tag{4.5}$$

$\forall (u^0, u^1) \in X$, where $C > 0$ is a constant depending only on l_i.

(3) *Suppose that $l_i/l_j \notin \mathbb{Q}$, $\forall\, i, j = 1, \ldots, N$. Then for all $\epsilon > 0$ we have for almost all l_i/l_j, $i, j = 1, \ldots, N$, that the solution $(\phi_i)_{i=1,\ldots,N}$ of (3.8)–(3.12) satisfies, $\forall (u^0, u^1) \in X$,*

$$\int_0^T \left| \frac{\partial \phi_1}{\partial t}(0,t) \right|^2 dt \geq C_\epsilon \sum_{i=1}^N \left(\|u_i^0\|^2_{H^{-N(1+\epsilon)+1}(0,l_i)} + \|u_i^1\|^2_{H^{-N(1+\epsilon)}(0,l_i)} \right), \tag{4.6}$$

where $C_\epsilon > 0$ is a constant depending only on l_i and ϵ.

Proof. Since, for all $\frac{l_i}{L}$, there does not exist a constant $C > 0$ such that

$$\left|\sin(k\pi\frac{l_i}{L})\right| \geq C, \ \forall\, k \in \mathbb{Z}^*,$$

we get the existence of a sequence $(p_m) \subset \mathbb{N}$, $\lim_{m\to\infty} p_m = \infty$ such that

$$\lim_{m\to\infty} \sin\left[p_m\pi\frac{l_1}{L}\right] = 0. \tag{4.7}$$

If we denote by $\phi_{i,m}$ the solution of (3.8)–(3.12) with initial data

$$\phi_{i,m}(x,0) = \frac{\sin(w_{p_m}(x-l_i))}{sin(w_{p_m}l_i)}, \ \ \frac{\partial\phi_{i,m}}{\partial t}(x,0) = 0, \quad \forall x \in (0,l_i),$$

a simple calculation using (4.7) implies that

$$\lim_{m\to\infty} \frac{\int_0^T \left|\frac{\partial\phi_{1,m}}{\partial t}(0,t)\right|^2 dt}{\left\|\left(\phi_m(x,0), \frac{\partial\phi_m}{\partial t}(x,0)\right)\right\|_X^2} = \lim_{m\to\infty} \frac{\int_0^T \left|\frac{\partial\phi_{1,m}}{\partial t}(0,t)\right|^2 dt}{\left\|\left(\phi_m(x,0), \frac{\partial\phi_m}{\partial t}(x,0)\right)\right\|_X^2}$$
$$= \lim_{m\to\infty} \sin^2(w_{p_m}l_1) = \lim_{m\to\infty} \sin^2(\frac{p_m\pi}{L}l_1) = 0,$$

so (4.4) is false for any l_i, $i = 1,\ldots,N$.

According to (4.3), for all $T > 2\,L$, there exists a constant $C_T > 0$ such that

$$\int_0^T \left|\frac{\partial\phi_1}{\partial t}(0,t)\right|^2 dt \geq C_T \sum_{n=1}^{\infty} \gamma_n^2 \left[w_n^2|a_n|^2 + |b_n|^2\right]. \tag{4.8}$$

Suppose that $l_i/l_j \notin \mathbb{Q}$, $\forall\, i,j = 1,\ldots,N$, and $\frac{l_i}{L}$, $i = 1,\ldots,N$, belongs to the set $\mathcal{S}$ defined in Section 2. Then relations (4.8) and (2.6) imply the existence of a constant $K_{T,N} > 0$ such that

$$\int_0^T \left|\frac{\partial\phi_1}{\partial t}(0,t)\right|^2 dt \geq K_{T,N} \sum_{n\geq 1} n^{-2N} \left[w_n^2|a_n|^2 + |b_n|^2\right],$$

$\forall l_i/L \in \mathcal{S}$, $i = 1,\ldots,N$, which is exactly (4.5).

In order to prove (4.6) we use a result in [6, p. 120] to get that, for all $\epsilon > 0$, there exists a set $B_\epsilon \subset (0,1) \subset \mathbb{Q}$ having Lebesgue measure equal to 1 and a constant $C > 0$, such that for any $\rho \in B_\epsilon$

$$|\sin(n\pi\rho)| \geq \frac{C}{n^{1+\epsilon}}, \qquad \forall\, n \geq 1. \tag{4.9}$$

Let us notice that by Roth's theorem B_ϵ contains all numbers having the property that l_i/L, $i = 1, \dots, N$, is an algebraic irrational (see for instance [6, p. 104]). Inequalities (4.8) and (4.9) obviously imply (4.6).

5. Proof of the main results

Proof of Proposition 2.1. The existence and uniqueness of finite energy solutions of (1.1)–(1.5) can be obtained by standard semigroup methods. In order to prove estimate (2.2) and the trace regularity property (2.4) it suffices to remark that, by a simple integration by parts, they hold true for regular solutions (i.e., $(u, \frac{\partial u}{\partial t}) \in C(0,T;\mathcal{D}(A))$). We can then use the density of $\mathcal{D}(A)$ in X.

As the imbedding of $\mathcal{D}(A)$ in X is obviously compact, by LaSalle's invariance principle the strong stability estimate at the end of Proposition 2.1 holds if the relations

$$\begin{cases} \dfrac{\partial^2 v_i}{\partial t^2} - \dfrac{\partial^2 v_i}{\partial x^2} = 0 \quad \text{in } (0,l_i)\times(0,T), \\ v_i(l_i,t) = 0,\ \dfrac{\partial v_i}{\partial t}(0,t) = 0,\ t \in (0,T), \\ v_i(0,t) = v_j(0,t),\ t\in(0,T), \\ \displaystyle\sum_{i=1}^N \dfrac{\partial v_i}{\partial x}(0,t) = 0,\ t\in(0,T), \\ v_i(x,0) = v_i^0(x),\ \dfrac{\partial v_i}{\partial t}(x,0) = v_i^1(x),\ x\in(0,l_i), \end{cases}$$

for $i, j = 1, \dots, N$, imply that $v_i = 0$, $\forall\, i = 1, \dots, N$.

In order to prove the first unique continuation result we put

$$v^0(x) = \sum_{n\geq 1} c_n \Phi_n(x),\ v^1(x) = \sum_{n\geq 1} d_n \Phi_n(x),$$

with $(w_n\, c_n), (d_n) \subset l^2$. Thus,

$$v(x,t) = \sum_{k\geq 1}\Big(c_k \cos(w_k t) + \frac{d_k}{w_k}\sin(w_k t)\Big)\Phi_k(x).$$

Then, according to (4.8) and (2.5) we have $c_n = d_n = 0$, $\forall n \in \mathbb{N}^*$, so $v^0 \equiv v^1 \equiv 0$, which implies the strong stability of the solutions of (1.1)–(1.5).

If we suppose that there exist $i, j = 1, \dots, N$ such that $l_i/l_j = \frac{p}{q}$, $p, q \in \mathbb{N}^*$, one can easily check that the solution u of (1.1)–(1.5), such that $u_i(x,t) = \sin(\frac{p\pi}{l_i}x)\ \cos(\frac{p\pi}{l_i}t)$, $u_j(x,t) = -\ \sin(\frac{q\pi}{l_j}x)\ \cos(\frac{q\pi}{l_j}t)$, $u_k(x,t) = 0$, $\forall\, k \neq i$, $k \neq$

$j,\ k = 1, \dots, N$, satisfies $E(t) = E(0),\ \forall\, t \geq 0$. Then (2.5) is a necessary condition for the strong stability of the solutions of (1.1)–(1.5). □

Let $u \in C(0,T;V) \cap C^1(0,T;\prod_{i=1}^N L^2(0,l_i))$ be the solution of (1.1)–(1.5). Then u can be written as

$$u_i = \phi_i + \psi_i,\ i = 1, \dots, N, \tag{5.1}$$

where $\phi_i,\ i = 1, \dots, N$, is the solution of (3.8)–(3.12) and $\psi_i,\ i = 1, \dots, N$, satisfies

$$\frac{\partial^2 \psi_i}{\partial t^2} - \frac{\partial^2 \psi_i}{\partial x^2} = 0 \quad \text{in } (0,l_i) \times (0,T), \tag{5.2}$$

$$\psi_i(l_i,t) = 0,\ t \in (0,T), \tag{5.3}$$

$$\psi_i(0,t) = \psi_j(0,t),\ t \in (0,T), \tag{5.4}$$

$$\sum_{i=1}^N \frac{\partial \psi_i}{\partial x}(0,t) = \frac{\partial u_1}{\partial t}(0,t),\ t \in (0,T), \tag{5.5}$$

$$\psi_i(x,0) = 0,\ \frac{\partial \psi_i}{\partial t}(x,0) = 0,\ x \in (0,l_i), \tag{5.6}$$

for $i,j = 1, \dots, N$.

The main ingredient of the proof of Theorem 2.2 is the following result.

Lemma 5.1. *Let $T > 0$ be fixed, and suppose that $(u^0,u^1) \in X$. Then the solutions $u_i,\ i = 1, \dots, N$, of (1.1)–(1.5) and the solution $\phi_i,\ i = 1, \dots, N$, of (3.8)–(3.12) satisfy*

$$C_1 \int_0^T \left|\frac{\partial \phi_1}{\partial t}(0,t)\right|^2 dt \leq \int_0^T \left|\frac{\partial u_1}{\partial t}(0,t)\right|^2 dt \leq 4 \int_0^T \left|\frac{\partial \phi_1}{\partial t}(0,t)\right|^2 dt, \tag{5.7}$$

where $C_1 > 0$ is a constant independent of (u^0,u^1).

Proof. We prove (5.7) for $u_i,\ i = 1, \dots, N$, satisfying (1.1)–(1.5) and ϕ_i, $i = 1, \dots, N$, a solution of (3.8)–(3.12).

Relation (5.1) implies that

$$\int_0^T \left|\frac{\partial \phi_1}{\partial t}(0,t)\right|^2 dt \leq 2\Big\{ \int_0^T \left|\frac{\partial u_1}{\partial t}(0,t)\right|^2 dt + \int_0^T \left|\frac{\partial \psi_1}{\partial t}(0,t)\right|^2 dt \Big\}.$$

The estimate above combined with inequality (3.7) in Proposition 3.1 implies the existence of a constant $C_1 > 0$, independent of $(u_i^0, u_i^1),\ i = 1, \dots, N$, such that

$$C_1 \int_0^T \left|\frac{\partial \phi_1}{\partial t}(0,t)\right|^2 dt \leq \int_0^T \left|\frac{\partial u_1}{\partial t}(0,t)\right|^2 dt. \tag{5.8}$$

On the other hand, according to relation (5.1) we have that $\frac{\partial \phi_1}{\partial t}(0,\cdot) \in L^2(0,T)$. This means that (5.2)–(5.6) can be rewritten as

$$\begin{gathered}\frac{\partial^2 \psi_i}{\partial t^2}(x,t) - \frac{\partial^2 \psi_i}{\partial x^2}(x,t) = 0 \quad \text{in } (0,l_i)\times(0,T),\\ \psi_i(l_i,t) = 0,\ t\in(0,T),\\ \psi_i(0,t) = \psi_j(0,t),\ t\in(0,T),\\ \sum_{i=1}^{N} \frac{\partial \psi_i}{\partial x}(0,t) - \frac{\partial \psi_1}{\partial t}(0,t) = \frac{\partial \phi_1}{\partial t}(0,t),\ t\in(0,T),\\ \psi_i(x,0) = 0,\ \frac{\partial \psi_i}{\partial t}(x,0) = 0,\ x\in(0,l_i).\end{gathered} \tag{5.9}$$

If we formally multiply (5.9) by $\frac{\partial \bar{\psi}_i}{\partial t}$, $i = 1,\ldots,N$, (this can be done rigorously by considering a regularizing sequence; see [2] for details) we obtain

$$\int_0^T \Big|\frac{\partial \psi_1}{\partial t}(0,t)\Big|^2 dt \le \Big|\int_0^T \frac{\partial \phi_1}{\partial t}(0,t)\frac{\partial \bar{\psi}_1}{\partial t}(0,t)dt\Big|,$$

which obviously yields

$$\Big\|\frac{\partial \psi_1}{\partial t}(0,t)\Big\|^2_{L^2(0,T)} \le \Big\|\frac{\partial \phi_1}{\partial t}(0,t)\Big\|^2_{L^2(0,T)}.$$

Relation (5.1) and the inequality above imply that

$$\Big\|\frac{\partial u_1}{\partial t}(0,t)\Big\|^2_{L^2(0,T)} \le 4\Big\|\frac{\partial \phi_1}{\partial t}(0,t)\Big\|^2_{L^2(0,T)}. \tag{5.10}$$

Inequalities (5.8) and (5.10) obviously yield the conclusion (5.7). □

Before giving the proof of the main results we need one more technical lemma. This lemma was proved in [3] (see also [2]).

Lemma 5.2. *Let $(\mathcal{E}_k)$ be a sequence of positive real numbers satisfying*

$$\mathcal{E}_{k+1} \le \mathcal{E}_k - C\mathcal{E}_{k+1}^{2+\alpha},\ \forall k \ge 0, \tag{5.11}$$

where $C > 0$ and $\alpha > -1$ are constants. Then there exists a positive constant M (depending on α and C) such that

$$\mathcal{E}_k \le \frac{M}{(k+1)^{\frac{1}{1+\alpha}}},\ \forall k \ge 0. \tag{5.12}$$

We can now prove the main results.

Proof of Theorem 2.2. Suppose that the solutions of (1.1)–(1.5) satisfy the estimate

$$E(t) \leq M e^{-\omega t} E(0), \qquad \forall\, t \geq 0, \tag{5.13}$$

where $M, \omega > 0$ are constants depending only on l_i. Relation (5.13) is equivalent to the existence of a time $T > 0$ and of a constant $C > 0$ (depending on T) such that

$$E(0) - E(T) \geq C E(0), \qquad \forall\, (u^0, u^1) \in X.$$

The relation above combined with (2.2) yields

$$\int_0^T \left| \frac{\partial u_1}{\partial t}(0,s) \right|^2 ds \geq C E(0), \qquad \forall\, (u^0, u^1) \in X;$$

by Lemma 5.1, the inequality above is equivalent to the fact that the solution $(\phi_i)_{i=1,\ldots,N}$ of (3.8)–(3.12) satisfies

$$\int_0^T \left| \frac{\partial \phi_1}{\partial t}(0,s) \right|^2 ds \geq \frac{C}{4} E(0), \qquad \forall\, (u^0, u^1) \in X.$$

The inequality above clearly contradicts assertion 1 in Proposition 4.1. We complete in this way the proof of the first assertion of Theorem 2.2.

We pass now to the proof of the second assertion of this theorem. Suppose that $l_i/l_j \notin \mathbb{Q}$, $\forall\, i,j = 1,\ldots,N$, and let $l_i/L \in \mathcal{S}$, $i = 1,\ldots,N$. By Proposition 4.1 and Lemma 5.1, the solution $(u_i)_{i=1,\ldots,N}$ of (1.1)–(1.5) satisfies the inequality, for all $T > 2L$,

$$\int_0^T \left| \frac{\partial u_1}{\partial t}(0,t) \right|^2 dt \geq K_1 \|(u^0, u^1)\|^2_{X_{-N}}, \ \forall (u^0, u^1) \in X,$$

where $K_1 > 0$ is a constant and

$$\|(u^0, u^1)\|^2_{X_{-N}} = \sum_{i=1}^N \left(\|u_i^0\|^2_{H^{-N+1}(0,l_i)} + \|u_i^1\|^2_{H^{-N}(0,l_i)} \right).$$

The relation above and (2.2) imply that

$$\|(u(T), u'(T))\|^2_X \leq \|(u^0, u^1)\|^2_X - K_1 \|(u^0, u^1)\|^2_{X_{-N}}, \quad \forall (u^0, u^1) \in X, \tag{5.14}$$

where

$$\|(u(t), u'(t))\|^2_X = \sum_{i=1}^N \|(u_i(t), u_i'(t))\|^2_{H^1(0,l_i) \times L^2(0,l_i)}.$$

By using a simple interpolation inequality (cf. [10, p. 49]), the fact that the function $t \to E(t)$ is nonincreasing, and the relation (5.14), we obtain the existence of a constant $K_2 > 0$ such that

$$\|(u(T), u'(T))\|_X^2 \leq \|(u^0, u^1)\|_X^2 - K_2 \frac{\|(u(T), u'(T))\|_X^{2N+2}}{\|(u^0, u^1)\|_{\mathcal{D}(A)}^{2N}}. \tag{5.15}$$

We follow now the method used in [3]. Estimate (5.15) remains valid in successive intervals $[kT, (k+1)T]$. So, for all $k \geq 0$, we have

$$\begin{aligned} &\|(u((k+1)T), u'((k+1)T))\|_X^2 \\ &\leq \|(u(kT), u'(kT))\|_X^2 - K_2 \frac{\|(u((k+1)T), u'((k+1)T))\|_X^{2N+2}}{\|(u(kT), u'(kT)\|_{\mathcal{D}(A)}^{2N}}. \end{aligned} \tag{5.16}$$

If we adopt now the notation

$$\mathcal{E}_k = \frac{\|(u(kT), u'(kT))\|_X^2}{\|(u^0, u^1)\|_{\mathcal{D}(A)}^2}, \tag{5.17}$$

relation (5.16) gives

$$\mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^{N+1}, \ \forall k \geq 0. \tag{5.18}$$

By applying Lemma 5.2 for $\alpha = N - 1$ and using relation (5.18) we obtain the existence of a constant $M > 0$ such that

$$\|(u(kT), u'(kT))\|_X^2 \leq \frac{M\|(u^0, u^1)\|_{\mathcal{D}(A)}^2}{(k+1)^{\frac{1}{N}}}, \ \forall k \geq 0.$$

The conclusion (2.7) follows now by simply using the fact that the function $t \to E(t)$ is nonincreasing.

Suppose that $l_i/l_j \notin \mathbb{Q}$, $\forall\, i, j = 1, \ldots, N$. Let us now suppose that $\epsilon > 0$ and that l_i/L, $i = 1, \ldots, N$ belongs to the set B_ϵ, introduced in Section 4. From (2.2), (4.6), and Lemma 5.1 it follows that

$$\|(u(T), u'(T))\|_X^2 \leq \|(u^0, u^1)\|_X^2 - C\|(u^0, u^1)\|_{X_{-N-\epsilon}}^2,$$

where

$$\|(u^0, u^1)\|_{X_{-N-\epsilon}}^2 = \sum_{i=1}^{N} \|(u_i^0, u_i^1)\|_{H^{-N(1+\epsilon)+1}(0,l_i) \times H^{-N(1+\epsilon)}(0,l_i)}^2.$$

Using now the same method as above and the interpolation theorem from [10, p. 81] we obtain that the sequence $\mathcal{E}_k$, defined by (5.17), satisfies

$$\mathcal{E}_{k+1} \leq \mathcal{E}_k - K\mathcal{E}_{k+1}^{N(1+\epsilon)+1}, \ \forall k \geq 1.$$

The relation above and Lemma 5.2 (with $\alpha = N(1+\epsilon) - 1$) give

$$\mathcal{E}_k \leq \frac{M}{(k+1)^{\frac{1}{N(1+\varepsilon)}}}, \ \forall k \geq 1,$$

which obviously implies (2.8).

6. RELATED QUESTION

A related question is to extend this result to general trees [1].

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