Differential and Integral Equations November/December 2004
Volume 17, Numbers 11-12, Pages 1359–1394

OPTIMAL RESULTS FOR THE BREZZI-PITKÄRANTA APPROXIMATION OF VISCOUS FLOW PROBLEMS

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(Submitted by: Herbert Amann)

Abstract. By introducing the term $-\varepsilon^2\Delta p$ into the equation of continuity and an additional Neumann boundary condition for the pressure p, a strongly elliptic system is obtained which is a singular perturbation of the Stokes system. We use parameter-dependent Sobolev norms to derive asymptotically precise estimates for solutions to the perturbed problem as $\varepsilon \searrow 0$. This results in optimal estimates for the difference between solutions to both problems; such estimates are not available by the usually applied energy methods. Under additional regularity assumptions for the data, for the energy estimates, the order of convergence with respect to ε is improved, and convergence in H^{s+1} and H^s norms is obtained for the velocity and pressure with $s \in [0, 3/2)$. We verify the asymptotic precision of the estimates by constructing the boundary layers.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class C^2 at least. For the vector field $u^\varepsilon = (v^\varepsilon, p^\varepsilon) = (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon, p^\varepsilon)$, we investigate a mixed boundary-value problem for the following strongly elliptic system of second-order partial differential equations,

$$\begin{aligned} -\Delta v^\varepsilon + \nabla p^\varepsilon &= f', & -\varepsilon^2\Delta p^\varepsilon + \operatorname{div} v^\varepsilon &= f_4 & &\text{in } \Omega,\\ v^\varepsilon &= g', & \partial_n p^\varepsilon &= g_4 & &\text{on } \partial\Omega, \end{aligned} \tag{S$_\varepsilon$}$$

where we focus our interest on asymptotically precise estimates for the solutions describing their behavior as $\varepsilon \searrow 0$. With $g_4 = 0$, this system has to

Accepted for publication: March 2004.
AMS Subject Classifications: 35Q30, 76D07, 76N20.

be considered as a singular perturbation of the Stokes system

$$\begin{aligned} -\Delta v^0 + \nabla p^0 = f', \qquad & \text{div } v^0 = f_4 \qquad \text{in } \Omega, \\ v^0 = g' \qquad & \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{S$_0$}$$

Formally, (S_0) appears if we set $\varepsilon = 0$ and cancel the Neumann boundary condition for p; we henceforth refer to this condition as the "ghost condition." In particular, the type of ellipticity is changed at $\varepsilon = 0$. Perturbations of this type, usually with $f_4 = 0$ and homogeneous boundary conditions, appear in different contexts, in showing existence of weak solutions for fluid models with shear dependent viscosities [12] as well as in numerical schemes for the Navier-Stokes equations, namely in the so-called *pressure-stabilization methods*, which were introduced by Brezzi and Pitkäranta in [7] (see also [25, 29], e.g.). The energy methods used in these papers lead to the following estimate (see [29], e.g.), $f_4 = 0$, $g = 0$ provided:

$$\|v^\varepsilon - v^0; H^1(\Omega)\| + \|p^\varepsilon - p^0; L^2(\Omega)\| \le C\,\varepsilon \|f'; L^2(\Omega)\|, \tag{E}$$

a result which cannot be amended with energy methods.

Besides treating the full inhomogeneous problem (S_0), we improve this result in two respects: If the data are smooth enough with $g_4 = 0$ in (S_ε), we obtain convergence of the velocity part in $H^{5/2-\delta}$ while the pressure converges in $H^{3/2-\delta}$ and $\delta > 0$ can be arbitrarily small. Moreover, the convergence rate is increased, in the energy norms up to $O(\varepsilon^{3/2})$. The results are sharp, which is proved by the construction of boundary layers. We emphasize that for the pressure p^ε, e.g., from the very beginning this is the optimum we can expect: Any convergence in H^s with $s > 3/2$ would allow for a Neumann condition in the limit p^0, which does not make any sense.

That at least convergence in higher-order Sobolev norms has to be paid for by more regularity assumptions on the data can be already seen by the following arguments. Suppose $f' \in L^2(\Omega)^3$, $f_4 = 0$, and $g = 0$; then there exist solutions $u^\varepsilon \in H^2(\Omega)^4$ and $u^0 \in H^2(\Omega)^3 \times H^1(\Omega)$ to the problems (S_ε) and (S_0), respectively (see also the Propositions 2.2 and 2.1). The difference $u^\varepsilon - u^0$ solves the Stokes system

$$\begin{aligned} -\Delta(v^\varepsilon - v^0) + \nabla(p^\varepsilon - p^0) = 0, \quad & \text{div } (v^\varepsilon - v^0) = -\varepsilon^2 \Delta p^\varepsilon \qquad \text{in } \Omega, \\ v^\varepsilon - v^0 = 0 \quad & \qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{ER}$$

However, to apply the usual $H^2(\Omega)^3 \times H^1(\Omega)$ estimates to the differences now requires $\Delta p^\varepsilon \in H^1(\Omega)$, which holds true only if $f' \in H^1(\Omega)^3$.[1] In order

[1] The inequality (E) shows the same effect; it requires $f' \in L^2(\Omega)^3$, not $f' \in H^{-1}(\Omega)^3$.

to find the optimal convergence result it is now necessary to find optimal estimates for the solutions u^ε.

We briefly describe the organization of the paper, the main achievements, and related results. In Section 2 we recall some basic notation and existence theorems for (S_ε) and (S_0). To derive asymptotically precise estimates, we introduce Sobolev norms $\| \cdot ; H^l_\varkappa(\Omega, \varepsilon)\|$ depending on the small parameter $\varepsilon > 0$ in Section 3. The technique of including the parameter into Sobolev norms is used by many authors for various applications (see, e.g., [4, 18, 20, 10, 8] and papers quoted there). In this context we use stepwise parameter-dependent Sobolev norms proposed in [18] and prove estimates in scales of spaces $H^l_\varkappa(\Omega; \varepsilon)$, where $\varkappa$ indicates the smoothness which is preserved under the limit process $\varepsilon \searrow 0$.

In Section 4 we prove a priori estimates for solutions to (S_0) and (S_ε) in parameter-dependent Sobolev norms. Thereby the main task is to investigate the model problems for u^ε in $\mathbb{R}^3$ (Theorem 4.2) and in the half-space $\mathbb{R}^3_+$ (Theorem 4.3). Collecting the arguments we end up with the first main result, the a priori estimate for a solution to (S_ε) in $H^l_\varkappa(\Omega, \varepsilon)$ spaces (Theorem 4.4). We allow $\varkappa$ to belong to a certain admissible interval I_0 determined by the structures of both the problem (S_ε) and the limit problem (S_0). This is in contrast with the estimates obtained in [4], where $\varkappa = 0$, and in [8], where $\varkappa$ is a fixed integer. Although pure algebraic conditions are found in [10] that provide similar a priori estimates with integer $\varkappa$, we need here to verify directly the parameter-dependent estimates for special solutions of model problems. This is necessary because only the possibility of varying $\varkappa$ within the admissible interval helps to deduce asymptotically precise estimates to the system (S_ε). Moreover, as discovered in [19, 22], those estimates with $\varkappa$ in an open, nonempty interval provide investigations of asymptotic behaviors of solutions to the problem (S_ε) in various classes of unbounded domains, for which we shall obtain estimates of the differences $(v^\varepsilon, p^\varepsilon) - (v^0, p^0)$ in forthcoming papers.

Section 5 is devoted to optimal estimates for the difference $u^\varepsilon - u^0$ in $H^l_\varkappa(\Omega, \varepsilon)$ spaces starting with Theorem 5.1; the conclusions regarding the improvements of (E) are collected in Remark 5.2. To confirm the asymptotic accuracy of our estimates, we construct boundary layers near the surface $\partial\Omega$, while employing the classical asymptotic procedure of Vishik and Lysternik [30] (Section 5.2, Theorem 5.3). This method has been applied in many papers to investigate miscellaneous problems in continuum mechanics, e.g., within the theory of Cosserat's continua (micropolar elasticity) [2, 23, 11], etc., the Reissner-Mindlin theory of plates [21, 5, 6], etc., the theory of shells

[13, 27], and others. Its realization requires additional assumptions on the smoothness properties of the problem data, and moreover, each step of the procedure reduces by one the index related to the differentiability of the remainder. At the same time, the constants in the inequalities derived in Theorem 5.1 and Remark 5.2 depend neither on ε nor on the right-hand side f itself. The latter feature is not maintained in most of the application papers cited above. We also emphasize that our estimate in Theorem 4.4 for the problem (S_ε) is optimal with respect to the smoothness indices and preserves all those independence properties; this is just due to the usage of stepwise parameter-dependent Sobolev norms.

2. Basic notation and elementary existence results

Let Ω be a three-dimensional domain with boundary $\partial\Omega$, which we assume to be of class $C^{\tilde{l}}$ for a suitable $\tilde{l} \in \mathbb{N}$, depending on the context; for $x \in \partial\Omega$ let $n(x)$ denote the exterior normal unit vector. The Sobolev spaces $H^s(\Omega)$, $s \geq 0$ are defined as usual: For $l \in \mathbb{N}$, they consist of all functions $\varphi \in L^2(\Omega)$ with $\partial^\alpha \varphi \in L^2(\Omega)$, $|\alpha| \leq l$. Here we use the common multi-index terminology, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_3^{\alpha_3}$ and $\partial_j = \partial/\partial x_j$. For noninteger $s > 0$, $H^s(\Omega)$ is defined by interpolation (see [17]), or equivalently, $H^s(\Omega) = \{\varphi \in H^{[s]}(\Omega); \|\varphi; H^s(\Omega)\| < \infty\}$ with

$$\|\varphi; H^s(\Omega)\|^2 = \|\varphi; H^{[s]}(\Omega)\|^2 + \sum_{|\alpha|=[s]} \int_\Omega \int_\Omega \frac{|\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)|^2}{|x-y|^{3-(s-[s])/2}} dx\, dy$$

(see, e.g., [1, p. 214]), where $[s]$ stands for the integer part of s and the exponent 3 is related to the space dimension. $(\phi, \psi)_\Omega$ and $(\tilde{\phi}, \tilde{\psi})_{\partial\Omega}$ are the scalar products in $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively, and we use this notation for scalar as well as for vector functions. We indicate the norm in $L^2(\Omega)$ by $\|\cdot\|$, while for other function spaces X we always write $\|\cdot; X\|$ to specify the norm. We also recall the definition of $H^{-s}(\Omega)$ as the (strong) dual space of $\overset{\circ}{H}{}^s(\Omega)$, where $\overset{\circ}{H}{}^s(\Omega)$ represents the closure in $H^s(\Omega)$ of smooth functions with compact supports in Ω.

In the following we need various products of functions spaces for the right-hand sides and the solutions to our problems. We use boldface letters for spaces related to the Stokes problem and gothic letters for spaces related to the perturbed problem (S_ε). In particular we denote the spaces for the solutions by

$$\mathbf{D}^l H(\Omega) = H^{l+1}(\Omega)^3 \times H^l(\Omega), \quad \mathfrak{D}^l H(\Omega) = H^{l+1}(\Omega)^4, \tag{2.1}$$

the spaces for the right-hand sides of the partial differential equations by

$$\mathbf{R}^l H(\Omega) = H^{l-1}(\Omega)^3 \times H^l(\Omega), \quad \mathfrak{R}^l H(\Omega) = H^{l-1}(\Omega)^4, \tag{2.2}$$

and the spaces for the complete data including the boundary conditions by

$$\begin{aligned} \mathbf{R}^l H(\Omega, \partial\Omega) &= \mathbf{R}^l H(\Omega) \times H^{l+1/2}(\partial\Omega)^3, \\ \mathfrak{R}^l H(\Omega, \partial\Omega) &= \mathfrak{R}^l H(\Omega) \times H^{l+1/2}(\partial\Omega)^3 \times H^{l-1/2}(\partial\Omega). \end{aligned} \tag{2.3}$$

With $u = (v, p)$, and $S_0 u = (-\Delta v + \nabla p, \operatorname{div} v)$, the following result for the Stokes problem (S_0) is well known.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class C^{l+2} for some $l \in \mathbb{N}$. Then the system (S_0) defines a Fredholm operator $\mathbf{S}_0$ of index zero from $\mathbf{D}^l H(\Omega)$ into $\mathbf{R}^l H(\Omega, \partial\Omega)$ by $\mathbf{S}_0 u = (S_0 u, v|_{\partial\Omega})$. The kernel of $\mathbf{S}_0$ consists of all $u = (0, 0, 0, c)$ with $c \in \mathbb{R}$, while the range contains all $(f, g') \in \mathbf{R}^l H(\Omega, \partial\Omega)$ which satisfy the compatibility condition*

$$\int_\Omega f_4 \, dx - \int_{\partial\Omega} g' \cdot n \, do = 0. \tag{2.4}$$

Thus, if we introduce $\mathbf{D}^l H(\Omega)_\perp = \{(v, p) \in \mathbf{D}^l H(\Omega) : \int_\Omega p = 0\}$, and $\mathbf{R}^l H(\Omega, \partial\Omega)_\perp = \{(f, g') : (2.4) \text{ holds}\}$, then $\mathbf{S}_0$ defines an isomorphism between $\mathbf{D}^l H(\Omega)_\perp$ and $\mathbf{R}^l H(\Omega, \partial\Omega)_\perp$.

While discussing he system (S_ε), we also shorten the notation. Let S_ε denote the formal differential operator of $(\mathrm{S}_\varepsilon)_1$, and B_ε the boundary operator; then the system $(S_\varepsilon, B_\varepsilon)u^\varepsilon = (f, g)$ is a Petrovsky system (cf. [31, Chapter 10]), and classical results on strongly elliptic systems lead to

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^3$ be defined as in Proposition 2.1; then the system (S_ε) defines a Fredholm operator $\mathbf{S}_\varepsilon$ of index zero between the spaces $\mathfrak{D}^l(\Omega)$ and $\mathfrak{R}^l H(\Omega, \partial\Omega)$. The kernel of $\mathbf{S}_\varepsilon$ coincides with the kernel of $\mathbf{S}_0$, while the right-hand sides of the system (S_ε) must satisfy the compatibility condition*

$$\int_\Omega f_4 \, dx - \int_{\partial\Omega} g' \cdot n \, do + \varepsilon^2 \int_{\partial\Omega} g_4 \, do = 0. \tag{2.5}$$

Thus, with notation similar to that of Proposition 2.1, the operator $\mathbf{S}_\varepsilon$ defines an isomorphism between $\mathfrak{D}^l H(\Omega)_\perp$ and $\mathfrak{R}^l H(\Omega, \partial\Omega)_\perp$.

Proof. To find the kernel and the range of $\mathbf{S}_\varepsilon$, we use Green's formulae. Let $u = (v, p)$ and $U = (V, P)$ be sufficiently smooth; then we obtain

$$\begin{aligned} (S_\varepsilon u, U)_\Omega &= (\nabla v, \nabla V)_\Omega + \varepsilon^2 (\nabla p, \nabla P)_\Omega \\ &\quad + (\nabla p, V)_\Omega + (\operatorname{div} v, P)_\Omega - (\partial_n v, V)_{\partial\Omega} - \varepsilon^2 (\partial_n p, P)_{\partial\Omega} \end{aligned}$$

$$=: \quad a_\varepsilon(u,U) - (\partial_n v, V)_{\partial\Omega} - \varepsilon^2(\partial_n p, P)_{\partial\Omega}. \tag{2.6}$$

Suppose $u \in \ker \mathbf{S}_\varepsilon$; then the classical regularity results of [3] ensure $u \in H^{l+1}(\Omega)^3$ at least; hence (2.6) yields

$$\begin{aligned} 0 = (S_\varepsilon u, u)_\Omega = a_\varepsilon(u,u) &= \|\nabla v\|^2 + \varepsilon\|\nabla p\|^2 - (p, \text{div } v)_\Omega + (\text{div } v, p)_\Omega \\ &= \|\nabla v\|^2 + \varepsilon\|\nabla p\|^2 + 2i\text{Im}\,(p, \text{div } v)_\Omega, \end{aligned}$$

which leads to the equalities $\nabla v = 0$ and $\nabla p = 0$; hence, $v = 0$ and $p = const$ due to the boundary conditions. To describe the range of $\mathbf{S}_\varepsilon$, we have to calculate the kernel of the adjoint problem with respect to the second Green's formula (see, e.g., [17])

$$\begin{aligned} &(S_\varepsilon u, U)_\Omega - (v, \partial_n V + P\,n)_{\partial\Omega} + \varepsilon^2(\partial_n p, P)_{\partial\Omega} = \\ &\quad (u, S_\varepsilon^* U)_\Omega - (\partial_n v - p\,n, V)_{\partial\Omega} + \varepsilon^2(p, \partial_n P)_{\partial\Omega}, \end{aligned}$$

where $S_\varepsilon^* U = (-\Delta V - \nabla P, -\varepsilon^2\Delta P - \text{div } V)$. We have $(f,g) \in \text{ran}\,\mathbf{S}_\varepsilon$ if and only if

$$(f,U)_\Omega - (g', \partial_n V + nP)_{\partial\Omega} + \varepsilon^2(g_4, P)_{\partial\Omega} = 0$$

for all U with $S_\varepsilon^* U = 0$ in Ω, $V = 0$, and $\partial_n P = 0$ on $\partial\Omega$. If $U \in H^2(\Omega)^4$ fulfills this condition, we obtain $0 = (S_\varepsilon^* U, U)_\Omega = a_\varepsilon(U,U) - 2i\text{Im}\,(P, \text{div } V)_\Omega$. Thus, $V = 0$ and $P = const$ with the same arguments as above, and we obtain (2.5). □

3. Parameter-dependent Sobolev norms

In order to obtain asymptotically precise estimates for the solutions to the singularly perturbed problem (S_ε) as ε tends to zero, we pass over to equivalent norms in the H^s spaces. These norms take into account the behavior in ε of the solutions to problems with small parameters at higher-order derivatives.

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain with a smooth boundary, $l \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, $\varkappa \in \mathbb{R}$.*

(i) *For $\varkappa \leq l$, and $u \in H^l(\Omega)$, we set*

$$\|u; H^l_\varkappa(\Omega,\varepsilon)\| = \sum_{k=0}^{l} \varepsilon^{(k-\varkappa)_+}\|u; H^k(\Omega)\| + \|u; H^\varkappa(\Omega)\|, \tag{3.1}$$

where the subscript + denotes the positive part of a number: $t_+ := 2^{-1}(t + |t|)$.

(ii) *For $\varkappa > l$ we choose $t \in \mathbb{N} = \{1,2,\ldots\}$ such that $t + l \geq \varkappa > l$. We introduce the Helmholtz operator $\mathcal{L}_\varepsilon = 1 - \varepsilon^2\Delta$ with domain $\mathfrak{D}(\mathcal{L}_\varepsilon) =$*

$H^2(\mathbb{R}^3)$, *if* $\Omega = \mathbb{R}^3$ *and* $\mathfrak{D}(\mathcal{L}_\varepsilon) = H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$, *if* Ω *is a proper subset of* $\mathbb{R}^3$. *Since the spectrum of* $\mathcal{L}_\varepsilon$ *is contained in the interval* $[1, \infty)$, *there exists* $\mathcal{L}_\varepsilon^{-t/2} u \in H^{l+t}(\Omega)$, *if* $u \in H^l(\Omega)$, *and we set*

$$\|u; H^l_\varkappa(\Omega, \varepsilon)\| = \|\mathcal{L}_\varepsilon^{-t/2} u; H^{l+t}_\varkappa(\Omega, \varepsilon)\|. \tag{3.2}$$

Norms of this type for $\varkappa = 0$ were introduced by Agranovich-Vishik in [4], and for $\varkappa > 0$ in [18]; for $\varkappa = l$ they coincide with the usual H^l norms.

In order to facilitate some proofs and calculations in the following, we introduce an equivalent formulation by means of Fourier transforms. We observe that in the case $\Omega = \mathbb{R}^n$ and $s \in \mathbb{N}_0$ an equivalent formulation of the $H^s_\varkappa$ norm is given by

$$\|u; H^s_\varkappa(\mathbb{R}^n, \varepsilon)\|^2 = \int_{\mathbb{R}^n} (1 + \varepsilon^2 |\xi|^2)^{s-\varkappa} (1 + |\xi|^2)^\varkappa |\hat{u}(\xi)|^2 d\xi, \tag{3.3}$$

where

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} u(x)\, dx$$

denotes the Fourier transform of u, and in this form the definition is easily generalized to the case of a noninteger s.

Now let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^{t+s+1} (with $t = 1$, if $\varkappa \le s$; otherwise, $t > \varkappa - s$). We choose an open covering $\{\mathfrak{O}_j\}_{j=0}^M$ of $\overline{\Omega}$ together with a partition of unity $\{\zeta_j\}_{j=1}^M$ subordinated to this covering. This can be done in such a way that $\overline{\mathfrak{O}}_0 \subset \Omega$ and $\{\mathfrak{O}_j\}_{j=1}^M$ is a covering of $\partial\Omega$, and there exists a system of diffeomorphisms $\Phi_j : \mathfrak{O}_j \to \mathfrak{O}'_j \subset \mathbb{R}^n$ with the property $\Phi_j(\mathfrak{O}_j \cap \partial\Omega) = \mathfrak{O}'_j \cap (\mathbb{R}^{n-1} \times \{0\})$ and $\Phi_j(\mathfrak{O}_j \cap \Omega) = \mathfrak{O}'_j \cap \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the half-space $\{x; x_n > 0\}$. For $u \in H^l(\Omega)$, put $u_0 = \zeta_0 u$ and $u_j = \zeta_j u \circ \Phi_j^{-1}$ for $j = 1, \dots, M$. By means of the canonical extension by zero we get $u_0 \in H^l(\mathbb{R}^n)$ and $u_j \in H^l(\mathbb{R}^n_+)$, respectively. Using the method of "extension by reflection" we have $u_j = \tilde{u}_j|_{\mathbb{R}^n_+}$ (see [17, Theorem 2.2]) with $\|\tilde{u}_j; H^l(\mathbb{R}^n)\| \le C\|u_j; H^l(\mathbb{R}^n_+)\|$. For any $s, \varkappa \in \mathbb{R}$, we now set

$$\|u; H^s_\varkappa(\Omega, \varepsilon)\| = \|u_0; H^s_\varkappa(\mathbb{R}^n, \varepsilon)\| + \sum_{j=1}^M \|\tilde{u}_j; H^s_\varkappa(\mathbb{R}^n, \varepsilon)\|; \tag{3.4}$$

here for the norms on the right-hand side we use definition (3.3). In a rather standard way, one can verify that the norm given by (3.4) is equivalent to (3.1) and (3.2), respectively, while the equivalency constants do not depend on ε. Let us collect some comments and estimates for further use.

Remark 3.2. 1. Note that for $\varkappa \le l$ the definition (3.1) and the restriction $\varepsilon \le 1$ always imply

$$\|u; H^{\varkappa}(\Omega)\| \le \|u; H^l_{\varkappa}(\Omega,\varepsilon)\| \le \|u; H^l(\Omega)\|; \tag{3.5}$$

moreover, for $0 \le \varkappa < l$, the weights $\varepsilon^{(k-\varkappa)_+}$ are significant only for the highest derivatives $\partial^\alpha u$, namely, for $|\alpha| > \varkappa$. Also obvious in this case is the inequality

$$\|u; H^l_{\varkappa}(\Omega,\varepsilon)\| \le \|u; H^{l+\tilde{l}}_{\varkappa+\tilde{\varkappa}}(\Omega,\varepsilon)\|, \qquad \text{if } 0 \le \tilde{\varkappa} \le \tilde{l}. \tag{3.6}$$

2. In the case $\varkappa < 0$ we have

$$\|u; H^l_{\varkappa}(\Omega,\varepsilon)\| = \varepsilon^{-\varkappa}\|u; H^l_0(\Omega,\varepsilon)\| + \|u; H^{\varkappa}(\Omega)\|. \tag{3.7}$$

Let us mention another equivalent formulation of the $H^l_{\varkappa}(\Omega)$ norms with $\varkappa < 0$ for further convenience. Any function $u \in H^l_{\varkappa}(\Omega)$ possesses a (nonunique, of course) representation in the form $u = \sum_{|\alpha|\le -[\varkappa]} \partial^\alpha u^\alpha$ with suitable functions $u^\alpha \in H^{l+|\alpha|}(\Omega)$. We have

$$\|u; H^l_{\varkappa}(\Omega,\varepsilon)\| \sim \inf \sum_{|\alpha|\le -[\varkappa]} \|u^\alpha; H^{l+|\alpha|}_{(\varkappa+|\alpha|)_+}(\Omega,\varepsilon)\|, \tag{3.8}$$

where the equivalency constants do not depend on ε, and the infimum is taken over all representations.

3. For $\varkappa > l$, the inequalities

$$1 \le (1+\varepsilon^2|\xi|^2)^{l-\varkappa}(1+|\xi|^2)^{\varkappa-l} \le \varepsilon^{-2(\varkappa-l)} \le \varepsilon^{-2\varkappa}$$

together with formula (3.3) for the $H^l_{\varkappa}(\Omega,\varepsilon)$ norm lead to (ε-independent) estimates

$$\|u; H^l(\Omega)\| \le c\,\|u; H^l_{\varkappa}(\Omega,\varepsilon)\| \le C\varepsilon^{-(\varkappa-l)}\|u; H^l(\Omega)\| \le C\varepsilon^{-\varkappa}\|u; H^l(\Omega)\|. \tag{3.9}$$

4. Finally, for any $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, with $|\alpha| \le l$, and $\varkappa, \mu \in \mathbb{R}$ we have the estimates

$$\|\varepsilon^{|\mu|}u; H^l_{\varkappa+\mu}(\Omega,\varepsilon)\| \quad\le\quad C\|u; H^l_{\varkappa}(\Omega,\varepsilon)\|, \tag{3.10}$$

$$\|\partial^\alpha u; H^{l-|\alpha|}_{\varkappa-|\alpha|}(\Omega,\varepsilon)\| \quad\le\quad C\|u; H^l_{\varkappa}(\Omega,\varepsilon)\|, \tag{3.11}$$

$$\|\varepsilon^{|\alpha|}\partial^\alpha u; H^{l-|\alpha|}_{\varkappa}(\Omega,\varepsilon) \quad\le\quad C\|u, H^l_{\varkappa}(\Omega,\varepsilon)\|. \tag{3.12}$$

The easiest way to prove (3.10) and (3.11) is to use the formulation (3.4) for the norms and thus, to apply (3.3). Namely the estimate

$$|\varepsilon|^{2\mu}(1+\varepsilon^2|\xi|^2)^{l-(\varkappa+\mu)}(1+|\xi|^2)^{\varkappa+\mu} \le (1+\varepsilon^2|\xi|^2)^{l-(\varkappa)}(1+|\xi|^2)^{\varkappa}$$

leads to (3.10) while

$$|\xi|^{2|\alpha|}(1+\varepsilon^2|\xi|^2)^{l-|\alpha|-(\varkappa-|\alpha|)}(1+|\xi|^2)^{\varkappa-|\alpha|} \le C(1+\varepsilon^2|\xi|^2)^{l-\varkappa}(1+|\xi|^2)^{\varkappa}$$

gives (3.11). Then (3.12) follows from (3.10) and (3.12).

Furthermore, we need information as to how trace and extension operators behave with respect to the parameter ε. The corresponding results are shown in [18]; we recall them for the convenience of the reader. We emphasize that we need the $H^s_\varkappa$ spaces with noninteger s only for the corresponding trace spaces on the boundary. Using the notation from above we see that the family $\{(\mathfrak{O}_j \cap \partial\Omega, \Phi_j|_{\mathfrak{O}_j\cap\partial\Omega})\}_j$ defines a C^{t+s+1} atlas for $\partial\Omega$. We keep the notation Φ_j also for the restrictions on $\mathfrak{O}_j \cap \partial\Omega$. Then for $u \in H^s(\partial\Omega)$, the function $u_j = \zeta_j u \circ \Phi_j^{-1}$ can be extended to a function in $H^s(\mathbb{R}^{n-1})$, and we set

$$\|u; H^s_\varkappa(\partial\Omega,\varepsilon)\|^2 = \sum_{j=1}^{M} \|u_j; H^s_\varkappa(\mathbb{R}^{n-1},\varepsilon)\|^2. \tag{3.13}$$

Clearly, this definition does not depend on the choice of the charts and the partition of unity (up to equivalency); in particular, for $0 \le \varkappa \le s$, we have again

$$\|u; H^\varkappa(\partial\Omega)\| \le \|u; H^s_\varkappa(\partial\Omega,\varepsilon)\| \le \|u; H^s(\partial\Omega)\|. \tag{3.14}$$

Lemma 3.3. [18, Lemma 1.3] *Let $l \in \mathbb{N}$ and $1/2 < \varkappa \le l$; then the trace operator $\gamma : H^l(\Omega) \to H^{l-1/2}(\partial\Omega)$ with $\gamma u = u|_{\partial\Omega}$ for smooth functions obeys the estimate*

$$\|\gamma u; H^{l-1/2}_{\varkappa-1/2}(\partial\Omega,\varepsilon)\| \le C\,\|u; H^l_\varkappa(\Omega,\varepsilon)\|. \tag{3.15}$$

If $\varkappa < 1/2$, then $\gamma u \in H^{l-1/2}_0(\partial\Omega,\varepsilon)$ and

$$\|\gamma u; H^{l-1/2}_0(\partial\Omega,\varepsilon)\| \le C\,\varepsilon^{\varkappa-1/2}\|u; H^l_\varkappa(\Omega,\varepsilon)\|, \tag{3.16}$$

where the constants are independent of u and $\varepsilon \in (0,1]$.

Looking more carefully at the assertions we find that for $\varkappa > 1/2$ the traces of a function $v \in H^l_\varkappa(\Omega,\varepsilon)$ remain stable as $\varepsilon \to 0$, while for $\varkappa < 1/2$ the estimate (3.16) gives nothing as $\varepsilon \to 0$, which is in accordance with the fact that $\|u; H^l_\varkappa(\Omega,\varepsilon)\| \to \|u; H^\varkappa(\Omega)\|$ as $\varepsilon \to 0$. Therefore we change the norms in this case and set

$$|\!|\!| u; \tilde{H}^s_\mu(\partial\Omega,\varepsilon) |\!|\!| = \begin{cases} \|u; H^s_\mu(\partial\Omega,\varepsilon)\| & \text{if } \mu \ge 0; \\ \varepsilon^{-\mu}\|u; H^s_0(\partial\Omega,\varepsilon)\| & \text{if } \mu < 0. \end{cases} \tag{3.17}$$

With this notation the converse result on extension operators is also true.

Lemma 3.4. [18, Lemma 1.4] *For any* $\mu < l + 1/2$, $\mu \neq 0$ *there exists a continuous extension operator* $P_\varepsilon : \tilde{H}^{l-1/2}(\partial\Omega) \to H^l(\Omega)$ *with* $\gamma P_\varepsilon u = u$ *and*

$$\|P_\varepsilon u; H^l_{\mu+1/2}(\Omega, \varepsilon)\| \leq C |\!|\!| u; \tilde{H}^{l-1/2}_\mu(\partial\Omega, \varepsilon) |\!|\!|,$$

where C *depends on* μ *and* l, *but is independent of* u *and* $\varepsilon \in (0, 1]$.

4. A priori estimates in parameter-dependent norms

4.1. **The estimate for solutions to the Stokes system.** Returning back to the problems (S_0) and (S_ε), we now pass to ε-dependent norms in the natural domain and range of the problem operators as defined in (2.1)–(2.3). For fixed $\varepsilon > 0$, we set

$$\begin{aligned} \mathbf{D}^l_\varkappa H(\Omega, \varepsilon) &=: H^{l+1}_{\varkappa+1}(\Omega, \varepsilon)^3 \times H^l_\varkappa(\Omega, \varepsilon), \\ \mathbf{R}^l_\varkappa H(\Omega, \varepsilon) &=: H^{l-1}_{\varkappa-1}(\Omega, \varepsilon)^3 \times H^l_\varkappa(\Omega, \varepsilon). \end{aligned} \tag{4.1}$$

For $\varkappa > -\frac{1}{2}$, the trace estimates of Lemma 3.3 imply $v|_{\partial\Omega} \in H^{l+1/2}_{\varkappa+1/2}(\partial\Omega, \varepsilon)^3$ if $v \in H^{l+1}_{\varkappa+1}(\Omega, \varepsilon)^3$. We include this trace space in the natural range and set

$$\mathbf{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon) =: \mathbf{R}^l_\varkappa H(\Omega, \varepsilon) \times H^{l+1/2}_{\varkappa+1/2}(\Omega, \varepsilon)^3.$$

By $\mathbf{D}^l_\varkappa H(\Omega, \varepsilon)_\perp$ we indicate the subspace of fields (v, p) with $\int_\Omega p = 0$ while $\mathbf{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon)_\perp$ denotes the range of $\mathbf{S}_0$ (see Proposition 2.1).

Lemma 4.1. *For* $\varkappa \in (-1/2, l]$, *the norms of the isomorphism*

$$\mathbf{S}_0 : \mathbf{D}^l_\varkappa H(\Omega, \varepsilon)_\perp \to \mathbf{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon)_\perp$$

as well as as of $\mathbf{S}_0^{-1}$ *are bounded independently of* $\varepsilon \in [0, 1]$.

Proof. The bounds for the norm of $\mathbf{S}_0$ easily follow from the definition of the spaces, inequality (3.11), and the trace estimate (3.15).

To control $\|\mathbf{S}_0^{-1}\|$, we have to show that for all $u \in \mathbf{D}^l H(\Omega)_\perp$, the estimate

$$\|u; \mathbf{D}^l_\varkappa H(\Omega, \varepsilon)\| \leq C \|\mathbf{S}_0 u; \mathbf{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon)\| \tag{4.2}$$

holds true, independent of $\varepsilon \in [0, 1]$. From Proposition 2.1 and standard arguments on increasing the smoothness properties (see [15] and [31, Theorem 10.1.1], e.g.) we have now

$$\|u; \mathbf{D}^\varkappa H(\Omega)\| \leq c \|\mathbf{S}_0 u; \mathbf{R}^\varkappa H(\Omega, \partial\Omega)\| \tag{4.3}$$

for the stable parts of the norms; here the notation of (2.1) and (2.3) carries over to the case of noninteger $\varkappa$ in a natural way. The same inequality with

$\varkappa$ replaced by k, where $k \geq [\varkappa] + 1$, leads to

$$\begin{aligned}&\varepsilon^{k-\varkappa}\Big(\|\nabla^{k+1}v; L^2(\Omega)\| + \|\nabla^k p; L^2(\Omega)\|\Big)\\ &\quad\leq \quad C\varepsilon^{k-\varkappa}\|\mathbf{S}_0 u; \mathbf{R}^k H(\Omega, \partial\Omega)\| \leq C\|\mathbf{S}_0 u; \mathbf{R}^k_\varkappa H(\Omega, \partial\Omega, \varepsilon)\|. \qquad (4.4)\end{aligned}$$

For the last inequality we used $\varepsilon \leq 1$ and the fact that $\varepsilon^{k-\varkappa}$ is the highest-order term (with respect to ε) which appears in the norm of $\mathbf{R}^k_\varkappa H(\Omega, \partial\Omega, \varepsilon)$. Summing up (4.4) for $k = [\varkappa] + 1, \dots, l$, leads to (4.2). □

4.2. **The setup for the problem ($\mathbf{S}_\varepsilon$).** Similar as in (4.1), we introduce the parameter ε in the natural domain and range for the problem (S_ε). We put

$$\begin{aligned}\mathfrak{D}^l_\varkappa H(\Omega, \varepsilon) &=: H^{l+1}_{\varkappa+1}(\Omega, \varepsilon)^3 \times H^{l+1}_\varkappa(\Omega, \varepsilon),\\ \mathfrak{R}^l_\varkappa H(\Omega, \varepsilon) &=: H^{l-1}_{\varkappa-1}(\Omega, \varepsilon)^3 \times H^{l-1}_\varkappa(\Omega, \varepsilon).\end{aligned} \qquad (4.5)$$

We outline that, as in (4.1), the stable parts in the norms are related to the Douglis-Nirenberg indices (see the appendix) of the degenerated problem, i.e., the Stokes problem. Due to (3.11) and (3.12), for any $\varkappa \in \mathbb{R}$, the differential operator $S_\varepsilon : \mathfrak{D}^l_\varkappa H(\Omega, \varepsilon) \to \mathfrak{R}^l_\varkappa H(\Omega, \varepsilon)$ is a continuous linear operator with norm bounded independently of $\varepsilon > 0$. In the following we consider only the case $\varkappa \leq l$; thus, we observe that the norm (3.2) is applied for $H^{l-1}_\varkappa(\Omega, \varepsilon)$ at most.

From the trace inequalities in Lemma 3.3 it follows that the boundary operator

$$\begin{aligned}B_\varepsilon : \mathfrak{D}^l_\varkappa H(\Omega, \varepsilon) \ni u = (v, p) &\to B_\varepsilon u = (v|_{\partial\Omega}, \partial_n p|_{\partial\Omega})\\ &\in \tilde{H}^{l+1/2}_{\varkappa+1/2}(\partial\Omega, \varepsilon)^3 \times \tilde{H}^{l-1/2}_{\varkappa-3/2}(\partial\Omega, \varepsilon)\end{aligned}$$

is bounded, independent of $\varepsilon \in (0, 1]$. We include the trace spaces in the range and set

$$\mathfrak{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon) = \mathfrak{R}^l_\varkappa H(\Omega, \varepsilon) \times \tilde{H}^{l+1/2}_{\varkappa+1/2}(\partial\Omega, \varepsilon)^3 \times \tilde{H}^{l-1/2}_{\varkappa-3/2}(\partial\Omega, \varepsilon). \qquad (4.6)$$

Thus we end up with the following assertion. Let $l \in \mathbb{N}$ and $-1/2 < \varkappa \leq l$; then for any $u \in \mathfrak{D}^l H(\Omega)$ we have

$$\|(S_\varepsilon, B_\varepsilon)u; \mathfrak{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon)\| \leq C\|u; \mathfrak{D}^l_\varkappa H(\Omega, \varepsilon)\|$$

with a constant C independent of $\varepsilon \in (0, 1]$ (and u, of course). What is desirable, and will be one of the main results of our paper, is the reverse inequality, namely,

$$\|u; \mathfrak{D}^l_\varkappa H(\Omega, \varepsilon)\| \leq C\Big(\|(f, g); \mathfrak{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon)\| + \|u; L^2(\Omega)\|\Big), \qquad (4.7)$$

where C is a constant, again independent of $\varepsilon \in (0,1]$. Only three boundary conditions can remain stable under the limiting process $\varepsilon \searrow 0$. Observing the trace estimate (3.15), the Dirichlet conditions $v_j|_{\partial\Omega} = g_j$, $j = 1,2,3$, remain valid at $\varepsilon = 0$ provided $\varkappa > -1/2$. At the same time any estimate which allows us to pass to the limit in the Neumann boundary condition $\partial_n p = g_4$ is impossible, and therefore we end up with the upper bound $3/2$ for the index $\varkappa$. The procedure to derive the estimate (4.7) within the whole admissible interval $(-1/2, 3/2) \ni \varkappa$ is essentially the same as for a general elliptic problem without a parameter (see, e.g., [3, 17]), but unfortunately this procedure loses facilitating features attributed to the case $\varkappa = 0$ and the ellipticity with parameter (cf. [4]).

For our purposes estimates with a large stable part in the norms are of major interest, this means the index $\varkappa$ should be as large as possible. To limit the technical difficulties, we restrict ourselves to the case $\varkappa \in [0, 3/2)$. By using a fine open covering of $\overline{\Omega}$ and the subordinated partition of unity, flattening the boundary and freezing coefficients, estimating solutions of the problem (S_ε) reduces to estimating solutions to the model problems in $\mathbb{R}^3$ and $\mathbb{R}^3_+$. The most important argument in this framework is how the solutions in the half-space depend on the small parameter. An approach, which was based on efficient estimates for solutions of a family of ordinary differential equations related to the model problem in $\mathbb{R}^3_+$, was worked out in [18] for scalar boundary-value problems and outlined in [20] for DN-systems. We emphasize that, for a problem which is elliptic with parameter in the sense of [4], the validity of those auxiliary estimates is evident and results from a simple dilation of coordinates. However, for the general situation and, in particular, for the problem (S_ε), the estimates postulated in [18, 20] still must be verified by direct calculations.

We also point out that in [10, 8, 9] certain simple algebraic conditions have been found to ensure similar estimates but for an integer $\varkappa$ only. As was mentioned in the introduction and will become clear below, we definitely need the estimate (4.7) with noninteger $\varkappa \in (0, 3/2)$ in order to fulfill our further purposes.

4.3. **Estimates for the solutions to $(\mathbf{S}_\varepsilon)$.** The first step to proving the a priori estimate (4.7) is the result for $\Omega = \mathbb{R}^3$. These arguments are standard, and we recall them for the reader's convenience.

Theorem 4.2. *Let $l \in \mathbb{N}$ and $\varkappa \leq l$; then for any $u = (v, p) \in H^{l+1}(\mathbb{R}^3)^4$ the following a priori estimate is valid:*

$$\|u; \mathfrak{D}^l_\varkappa H(\mathbb{R}^3, \varepsilon)\| \leq C \left(\|S_\varepsilon u; \mathfrak{R}^l_\varkappa H(\mathbb{R}^3, \varepsilon)\| + \|u\| \right),$$

with $\mathfrak{D}^l_\varkappa H(\mathbb{R}^3,\varepsilon)$ and $\mathfrak{R}^l_\varkappa H(\mathbb{R}^3,\varepsilon)$ as in (4.1). Here C is a constant which depends on $\varkappa$ and l, but on neither u nor $\varepsilon\in(0,1]$.

Proof. Put $S_\varepsilon u = f$ and apply the Fourier transform in $\mathbb{R}^3$; then

$$S_\varepsilon(\xi)\hat{u}(\xi)=\hat{f}(\xi), \tag{4.8}$$

where

$$S_\varepsilon(\xi)=\begin{pmatrix} |\xi|^2 & 0 & 0 & -i\,\xi_1\\ 0 & |\xi|^2 & 0 & -i\,\xi_2\\ 0 & 0 & |\xi|^2 & -i\,\xi_3\\ -i\,\xi_1 & -i\,\xi_2 & -i\,\xi_3 & \varepsilon^2|\xi|^2\end{pmatrix}$$

and $\det(S_\varepsilon(\xi)) = (\varepsilon^2|\xi|^2+1)|\xi|^6$. Now we choose a cut-off function $\chi\in C^\infty(\mathbb{R}^3)$ with $\chi(\xi)=0$ for $|\xi|<1/2$ and $\chi(\xi)=1$ for $|\xi|\geq 1$. From (4.8) we obtain

$$\hat{u}(\xi)=\chi(\xi)S_\varepsilon^{-1}(\xi)\hat{f}(\xi)+(1-\chi(\xi))\hat{u}(\xi). \tag{4.9}$$

Calculating S_ε^{-1}, we find

$$S_\varepsilon(\xi)^{-1}=\frac{1}{(\varepsilon^2|\xi|^2+1)|\xi|^4}$$
$$\times\begin{pmatrix} (|\xi|^4\varepsilon^2+\xi_3^2+\xi_2^2) & -\xi_1\,\xi_2 & -\xi_3\,\xi_1 & i|\xi|^2\xi_1\\ -\xi_1\xi_2 & (|\xi|^4\varepsilon^2+\xi_3^2+\xi_1^2) & -\xi_3\xi_2 & i|\xi|^2\xi_2\\ -\xi_3\xi_1 & -\xi_3\xi_2 & (|\xi|^4\varepsilon^2+\xi_2^2+\xi_1^2) & i|\xi|^2\xi_3\\ i|\xi|^2\xi_1 & i|\xi|^2\xi_2 & i|\xi|^2\xi_3 & |\xi|^4\end{pmatrix}.$$

From this we derive the following inequalities for $|\xi|>1/2$:

$$|(S_\varepsilon(\xi)^{-1})_{ij}|^2\leq(1+|\xi|^2)^{-2},\qquad i,j=1,2,3,$$
$$|(S_\varepsilon(\xi)^{-1})_{i4}|^2=|(S_\varepsilon(\xi))^{-1}_{4i}|^2\leq C(1+|\xi|^2)^{-2}(1+\varepsilon^2|\xi|^2)^{-2},\quad i=1,2,3,$$
$$|(S_\varepsilon(\xi)^{-1})_{44}|^2\leq C(1+\varepsilon^2|\xi|^2)^{-2}.$$

If we observe that for any μ, $\tilde{\mu}\in\mathbb{R}$ and $\varepsilon\in(0,1]$, the expression

$$(1+\varepsilon^2|\xi|^2)^{\mu}(1+|\xi|^2)^{\bar{\mu}}(1-\chi(\xi))^2\leq\bar{C},$$

then, for $i=1,2,3$, the representation (4.9) delivers

$$\begin{aligned}&(1+\varepsilon^2|\xi|^2)^{l-\varkappa}(1+|\xi|^2)^{\varkappa+1}|\hat{v}_i(\xi)|^2\\ &\leq C\Big(\sum_{j=1}^{3}(1+\varepsilon^2|\xi|^2)^{l-\varkappa}(1+|\xi|^2)^{\varkappa-1}|\hat{f}_j(\xi)|^2\\ &+(1+\varepsilon^2|\xi|^2)^{l-1-\varkappa}(1+|\xi|^2)^{\varkappa}|\hat{f}_4(\xi)|^2\Big)+\bar{C}|\hat{v}_i|^2.\end{aligned}$$

In a similar manner conclude that

$$(1+\varepsilon^2|\xi|^2)^{l+1-\varkappa}(1+|\xi|^2)^{\varkappa}|\hat{p}(\xi)|^2$$
$$\leq C\Big(\sum_{j=1}^{3}(1+\varepsilon^2|\xi|^2)^{l-\varkappa}(1+|\xi|^2)^{\varkappa-1}|\hat{f}_j(\xi)|^2$$
$$+(1+\varepsilon^2|\xi|^2)^{l-1-\varkappa}(1+|\xi|^2)^{\varkappa}|\hat{f}_4(\xi)|^2\Big)+\bar{C}|\hat{p}|^2.$$

Integrating both inequalities over $\mathbb{R}^3$ completes the proof. □

The next step is to verify a priori estimates for the problem in $\mathbb{R}^3_+$. Due to Lemma 3.4, we may restrict ourselves to estimating the solutions of the homogeneous system

$$S_\varepsilon u = 0 \quad \text{in } \mathbb{R}^3_+ \tag{4.10}$$

with inhomogeneous boundary conditions

$$v = g', \quad \partial_n p = \partial_3 p = g_4 \text{ on the plane } x_3 = 0. \tag{4.11}$$

We perform a Fourier transform along the $(x', 0)$ plane; i.e.,

$$\hat{u}(\xi', x_3) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\xi'\cdot x'}u(x', x_3)\,\mathrm{d}x',$$

which changes the problem (4.10), (4.11) into a problem for ordinary differential equations in x_3 with the Fourier parameter $\xi' = (\xi_1, \xi_2)$:

$$\begin{pmatrix} -\partial_3^2+|\xi'|^2 & 0 & 0 & -i\xi_1 \\ 0 & -\partial_3^2+|\xi'|^2 & 0 & -i\xi_2 \\ 0 & 0 & -\partial_3^2+|\xi'|^2 & \partial_3 \\ -i\xi_1 & -i\xi_2 & \partial_3 & \varepsilon^2(|\xi'^2|+\partial_3^2) \end{pmatrix}\hat{u}(\xi', x_3) = 0 \tag{4.12}$$

$$\hat{v}(\xi', 0) = \hat{g}(\xi'), \quad \partial_3\hat{p}(\xi', 0) = \hat{g}_4(\xi'). \tag{4.13}$$

In [18]–[20] a general procedure was outlined as to how the a priori estimate in the half-space can be obtained provided the exponentially decaying solutions to the initial-value problem (4.12), (4.13) fulfill a list of certain inequalities. After some normalization transformations we can get rid of the Fourier parameter on the right-hand side of (4.13), and it turns out to be sufficient if only the unit vectors $\mathbf{e}^q \in \mathbb{R}^4$ are treated as initial conditions. Still the verification of the requirements disposes of the explicit knowledge of four solutions to a system of the form (4.12) and a lot of cumbersome calculations and tricky estimates. We leave this to the appendix, together with the explanation as to how the Agmon-Douglis-Nirenberg indices of both problems enter the application of the general results, and formulate only the result here.

Theorem 4.3. [18, 20] *Let* $\varkappa \in (-1/2, 3/2)$ *and* $l \in \mathbb{N}$ *with* $l \geq \varkappa$. *Then any* $u \in \mathfrak{D}^l_\varkappa(\mathbb{R}^3_+, \varepsilon)$ *with* $S_\varepsilon u = 0$ *obeys the estimate*

$$\|u; \mathfrak{D}^l_\varkappa(\mathbb{R}^3_+, \varepsilon)\| \leq C \left(\|(0, B_\varepsilon u), \mathfrak{R}^l_\varkappa(\mathbb{R}^3_+, \partial\mathbb{R}^3_+, \varepsilon)\| + \|u; L^2(\mathbb{R}^3_+)\| \right), \quad (4.14)$$

where C *is independent of* u *and of* $\varepsilon \in (0, 1]$.

Recalling the notation of (4.5) and Proposition 2.2, we use the notation $\mathfrak{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon)_\perp$ again for all data (f, g) fulfilling the compatibility condition (2.5), while $\mathfrak{D}^l_\varkappa(\Omega; \varepsilon)_\perp$ indicates that the pressure part of the solution is mean-value free.

Theorem 4.4. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with boundary of class* C^{l+2}, $l \in \mathbb{N}$ *and let* $\varkappa \in [0, 3/2)$, $\varkappa \leq l$. *Then, for any* $(f, g) \in \mathfrak{R}^l H(\Omega, \partial\Omega)_\perp$ *there exists a unique solution* $u^\varepsilon = (v^\varepsilon, p^\varepsilon) \in \mathfrak{D}^l H(\Omega)_\perp$ *to the Brezzi-Pitkäranta problem* (S_ε). *This solution fulfills the estimate*

$$\|u^\varepsilon; \mathfrak{D}^l_\varkappa(\Omega; \varepsilon)_\perp\| \leq C \, \|(f, g); \mathfrak{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon)\|, \quad (4.15)$$

where C *is a constant depending on* $\varkappa$, $\partial\Omega$, *and* l, *but on neither* $\varepsilon \in (0, 1]$ *nor the data.*

Proof. For the existence and uniqueness we refer to Proposition 2.2. As already pointed out, Theorems 4.2 and 4.3 together with the usual technique of localization and flattening the boundary leads to the a priori estimate

$$\|u^\varepsilon; \mathfrak{D}^l_\varkappa H(\Omega, \varepsilon)\| \leq C \left(\|(f, g); \mathfrak{R}^l_\varkappa H(\Omega, \varepsilon)\| + \|u^\varepsilon\| \right), \quad (4.16)$$

where C is a constant independent of ε (recall that $\|\cdot\|$ means the $L^2(\Omega)$ norm).

It remains to show that the last term in this inequality can be removed under the condition $\int_\Omega p \, dx = 0$. Using Lemma 3.4, we may assume $g = 0$. The estimates (3.5) and (3.9) imply

$$\|f'; H^{-1}(\Omega)\| \leq C\|f', H^{l-1}_{\varkappa-1}(\Omega, \varepsilon)\|, \qquad \|f_4\| \leq C\|f_2; H^{l-1}_\varkappa(\Omega, \varepsilon)\|.$$

Since $f_4 \in L^2(\Omega)$ and $\int f_4 \, dx = 0$, there exists $w \in \overset{\circ}{H}{}^1(\Omega)$ with div $w = f_4$, and $\|w; H^1(\Omega)\| \leq \|f_4\|$. By subtracting w from v^ε we now may assume $f_4 = 0$. In this case Green's formula leads to

$$\|\nabla v^\varepsilon\|^2 + (\nabla p, v^\varepsilon)_\Omega + \varepsilon^2\|\nabla p\|^2 + (\mathrm{div}\ v^\varepsilon, p)_\Omega = \|\nabla v^\varepsilon\|^2 + \varepsilon^2\|\nabla p\|^2 = (f, v^\varepsilon)_\Omega,$$

which implies already

$$\|v^\varepsilon\| \le C_\Omega \|\nabla v^\varepsilon\| \le C\|f'; H^{-1}(\Omega)\|; \tag{4.17}$$

here we used Poincaré's inequality, too.

It remains to estimate $\|p\|$. We apply an equivalent formulation for the $L^2(\Omega)$ norm in the case $p \in L^2(\Omega)_\perp$:

$$\|p\| \sim \sup\left\{ \frac{|(p, \operatorname{div} \varphi)_\Omega|}{\|\nabla\varphi\|} : \varphi \in \overset{\circ}{H}{}^1(\Omega) \right\}.$$

At the same time, equation $(\mathrm{S}_\varepsilon)_1$ yields

$$\begin{aligned} |(p, \operatorname{div} \varphi)_\Omega| &= |(f', \nabla\varphi)_\Omega + (\nabla v^\varepsilon, \nabla\varphi)_\Omega| \\ &\le \|f'; H^{-1}(\Omega)\|\|\nabla\varphi\| + \|\nabla v^\varepsilon\|\|\nabla\varphi\|; \end{aligned}$$

together with (4.17) this gives $\|u^\varepsilon\| \le C\,\|(f,g); \mathfrak{R}^l_\varkappa H(\Omega, \partial\Omega, \varepsilon)\|$. □

5. Estimates of asymptotic precision for $u^\varepsilon - u^0$

5.1. **The error estimates.** In this section we fix $g_4 = 0$ in problem (S_ε). Let $(f, g') \in \mathbf{R}^l H(\Omega, \partial\Omega)_\perp$ (cf. the notation in (2.1)–(2.3)) be given; i.e., in particular, $f' \in H^{l-1}(\Omega)^3$, $f_4 \in H^l(\Omega)$, and $g' \in H^{l+1/2}(\partial\Omega)$. Then $(f, (g', 0)) \in \mathfrak{R}^l H(\Omega, \partial\Omega)_\perp$, and we obtain unique solutions $u^\varepsilon \in \mathfrak{D}^l H(\Omega)_\perp$ and $u^0 \in \mathbf{D}^l H(\Omega)_\perp$ to problems (S_ε) and (S_0), respectively. The difference $u^\varepsilon - u^0$ is contained in $\mathbf{D}^l H(\Omega)$ and solves the system (ER). However, to apply Lemma 4.1, we need an estimate for $\|\varepsilon^2 \Delta p^\varepsilon; H^l_\varkappa(\Omega, \varepsilon)\|$. Theorem 4.4 makes this available only if $f' \in H^l(\Omega)$, and $g' \in H^{l+3/2}(\Omega)$, which means $(f, (g', 0)) \in \mathfrak{R}^{l+1} H(\Omega, \partial\Omega)_\perp$. If, in addition, $f_4 \in H^{l+1}(\Omega)$, i.e., $(f, g') \in \mathbf{R}^{l+1} H(\Omega, \partial\Omega)_\perp$, then $p^0 \in H^2(\Omega)$ and $\partial_n p^0 \in H^{1/2}(\partial\Omega)$ at least. In this case we may apply Theorem 4.4 directly to $u^\varepsilon - u^0$; as we will see this will lead to a better convergence result.

Theorem 5.1. *Let Ω be defined as in Theorem 4.4, $g_4 = 0$, $\varkappa \in [0, 3/2)$, and*

$$\begin{aligned} &\delta \in [0, \varkappa], \quad l \in \mathbb{N}_0 \text{ with } l \ge \max\{\varkappa - \delta, \varkappa - 1\}, \\ &f \in H^l(\Omega), \quad g' \in H^{l+3/2}(\partial\Omega) \text{ with } \textstyle\int_\Omega f_4\,dx - \int_{\partial\Omega} g' \cdot n\,do = 0. \end{aligned} \tag{5.1}$$

Let $u^\varepsilon \in H^{l+2}(\Omega)^4$ and $u^0 \in H^{l+1}(\Omega)^3 \times H^l(\Omega)$ be the unique solutions to (S_0) and (S_ε) with $\int_\Omega p^\varepsilon = \int_\Omega p^0 = 0$. Then the difference satisfies the following inequality with a constant independent of $\varepsilon \in (0, 1]$ and the data:

$$\|u^\varepsilon - u^0; \mathbf{D}^l_{\varkappa-\delta}(\Omega; \varepsilon)\| \le C\varepsilon^\delta \left(\|f; H^l(\Omega)^4\| + \|g'; H^{l+3/2}\| \right), \tag{5.2}$$

where $\mathbf{D}^l_{\varkappa-\delta}(\Omega;\varepsilon)$ *is defined as in* (4.1)*. If, in addition to* (5.1)*, the requirements*

$$l \geq \varkappa, \quad f_4 \in H^{l+1}(\Omega) \tag{5.3}$$

are met, then using notation in (4.5) *the inequality* (5.2) *can be strengthened to*

$$\begin{aligned} &\|u^\varepsilon - u^0; \mathfrak{D}^l_\varkappa(\Omega;\varepsilon)\| \\ &\leq C\varepsilon^{3/2-\varkappa} \left(\|f'; H^l(\Omega)^3\| + \|f_4; H^{l+1}(\Omega)\| + \|g'; H^{l+3/2}\| \right). \end{aligned} \tag{5.4}$$

Proof. As we know already from the introduction, the difference $u^\varepsilon - u^0 =: (r,q)$ satisfies

$$-\Delta r + \nabla q = 0, \quad \operatorname{div} r = \varepsilon^2 \Delta p^\varepsilon \quad \text{in } \Omega, \qquad r = 0 \quad \text{on } \partial\Omega. \tag{5.5}$$

The a priori estimate for solutions of (S_0), Lemma 4.1, applied with $\tilde{\varkappa} = \varkappa - \delta$ (therefore we require $l \geq \varkappa - \delta$), implies

$$\|(r,q); \mathbf{D}^l_{\varkappa-\delta}(\Omega,\varepsilon)\| \leq C\|\varepsilon^2\Delta p^\varepsilon; H^l_{\varkappa-\delta}(\Omega,\varepsilon)\|.$$

With inequalities (3.10) and (3.11), we obtain

$$\|\varepsilon^2\Delta p^\varepsilon; H^l_{\varkappa-\delta}(\Omega,\varepsilon)\| = \varepsilon^\delta\|\varepsilon^{2-\delta}\Delta p^\varepsilon; H^l_{\varkappa-\delta}(\Omega,\varepsilon)\| \leq C\varepsilon^\delta\|p^\varepsilon; H^{l+2}_\varkappa(\Omega,\varepsilon)\|.$$

Now we use Theorem 4.4 with l replaced by $l+1$ (here we need $l \geq \varkappa - 1$), and arrive at

$$\begin{aligned} \|p^\varepsilon; H^{l+2}_\varkappa(\Omega,\varepsilon)\| &\leq C\|(f,(g',0)); \mathfrak{R}^{l+1}_\varkappa H(\Omega,\partial\Omega,\varepsilon)\| \\ &\leq C\left(\|f; H^l(\Omega)\| + \|g'; H^{l+3/2}(\partial\Omega)\| \right); \end{aligned}$$

for the second inequality we used $\varepsilon \leq 1$ again. Collecting the last inequalities gives (5.2).

To obtain (5.4), we now use

$$S_\varepsilon(u^\varepsilon - u^0) = (0, \varepsilon^2\Delta p^0), \quad B_\varepsilon(u^\varepsilon - u^0) = (0, \partial_n p^0). \tag{5.6}$$

Note that due to $l \geq 1$, $p^0 \in H^2(\Omega)$ at least, and thus $\partial_n p^0 \in H^{1/2}(\partial\Omega)$. To apply Theorem 4.4, we have to estimate

$$\|\varepsilon^2\Delta p^0; H^{l-1}_\varkappa(\Omega,\varepsilon)\| \quad \text{and} \quad \|\partial_n p^0; \tilde{H}^{l-1/2}_{\varkappa-3/2}(\partial\Omega,\varepsilon)\|.$$

Due to our assumptions and Cattabriga's inequality (see [3], e.g.), we have

$$\|p^0; H^{l+1}(\Omega)\| \leq C\left(\|f'; H^l(\Omega)^3\| + \|f_4; H^{l+1}(\Omega)\| + \|g'; H^{l+3/2}\| \right). \tag{5.7}$$

For $\varkappa > l-1$, we use (3.9) and obtain

$$\|\varepsilon^2\Delta p^0; H^{l-1}_\varkappa(\Omega,\varepsilon)\| \;\leq\; \varepsilon^2\varepsilon^{-\varkappa}\|\Delta p^0; H^{l-1}(\Omega)\| \leq \varepsilon^{2-\varkappa}\|p^0; H^{l+1}(\Omega)\|.$$

If $\varkappa \le l-1$, then by (3.5),

$$\begin{aligned}\|\varepsilon^2\Delta p^0; H^{l-1}_\varkappa(\Omega,\varepsilon)\| &\le \varepsilon^2\|\Delta p^0; H^{l-1}_\varkappa(\Omega,\varepsilon)\| \le \varepsilon^2\|\Delta p^0; H^{l-1}(\Omega)\| \\ &\le \varepsilon^2\|p^0; H^{l+1}(\Omega)\|.\end{aligned}$$

Finally, for $\partial_n p^0$, we use $(3.17)_2$ together with (3.14) and obtain

$$\begin{aligned}\|\partial_n p^0; \tilde H^{l-1/2}_{\varkappa-3/2}(\partial\Omega,\varepsilon)\| &= \varepsilon^{3/2-\varkappa}\|\partial_n p^0; H^{l-1/2}_0(\partial\Omega,\varepsilon)\| \\ &\le \varepsilon^{3/2-\varkappa}\|\partial_n p^0; H^{l-1/2}(\partial\Omega)\| \le \varepsilon^{3/2-\varkappa}\|p^0; H^{l+1}(\Omega)\|.\end{aligned}$$

Combining the last estimates with (5.7) finishes the proof. □

Remark 5.2. For $f_4 = 0$ and $g' = 0$, let us look at some particular choices of $\varkappa$, δ, and l. If we take $l = 0$ and $\varkappa = \delta = 1$, then we obtain the inequality (E) again, which can also be derived by simple energy methods and is the same as used in [25, 26, 7].

If in addition, $f' \in H^1(\Omega)^3$, then we may apply (5.4) with $\varkappa = 0$ (and $l = 1$)

$$\|v^\varepsilon - v^0; H^1(\Omega)\| + \|p^\varepsilon - p^0; L^2(\Omega)_\perp\| \le C\,\varepsilon^{3/2}\,\|f'; H^1(\Omega)\|. \tag{5.8}$$

The same argument, applied to $\varkappa = l = 1$, gives

$$\|v^\varepsilon - v^0; H^2(\Omega)\| + \|p^\varepsilon - p^0; H^1(\Omega)_\perp\| \le C\,\varepsilon^{1/2}\,\|f'; H^1(\Omega)\|. \tag{5.9}$$

Taking $l = 2$ it is possible to fix $\varkappa = 3/2 - \delta$ and $\delta > 0$ arbitrarily small, and we obtain

$$\|v^\varepsilon - v^0; H^{5/2-\delta}(\Omega)\| + \|p^\varepsilon - p^0; H^{3/2-\delta}(\Omega)_\perp\| \le C\,\varepsilon^\delta\,\|f'; H^2(\Omega)\|.$$

With some additional technical efforts, namely generalizing the theory to noninteger l, it is possible to diminish the required regularity property to $f' \in H^\varkappa(\Omega)^3$. Finally we emphasize that these inequalities provide the optimal convergence result, as we will see in the next section.

5.2. **Construction of the boundary layers.** To see that the estimates of the previous section are asymptotically precise, even if the data are arbitrarily smooth, we use the construction of boundary layers (cf. [30]). In the following let $f \in C^\infty(\overline{\Omega})^4$ and $g \in C^\infty(\partial\Omega)^4$ be fixed smooth data with $g_4 = 0$. Let u^ε, $\varepsilon > 0$ and u^0 be solutions to the problem $\mathbf{S}_\varepsilon u^\varepsilon = (f, g)$ and to $\mathbf{S}_0 u^0 = (f, g')$, respectively, according to Propositions 2.1 and 2.2. We construct the main terms of the asymptotic expansion of u^ε as $\varepsilon \to 0$, where we choose u^0 as the main asymptotic term. From (5.6) it is clear that the main discrepancy in the right-hand side of (S_ε) appears in the Neumann condition, i.e., in that boundary condition which is a ghost condition for the degenerated problem at $\varepsilon = 0$. We will see that this term can be removed with the help of a boundary layer, which in addition leads to a well-posed boundary-value problem of the type (S_0) for the next asymptotic term.

To catch the main idea, we consider again the half-space $\mathbb{R}^3_+$, and assume that f and g have compact supports. Let u^ε and u^0 be defined as above. We look for $u^{(1)}$ as the main part (as $\varepsilon \to 0$) of the solution U to the homogeneous system $S_\varepsilon U = 0$; moreover, we claim that, for compensating for the discrepancy in the ghost condition, the pressure component $p^{(1)}$ satisfies

$$\frac{\partial}{\partial x_3} p^{(1)}(x', 0) = -\frac{\partial}{\partial x_3} p^0(x', 0), \qquad \text{with } x' = (x_1, x_2), \tag{5.10}$$

together with the decay condition $|u^{(1)}(x)| \to 0$ as $x_3 \to \infty$. To calculate this $u^{(1)}$, we perform a rescaling by the transformations

$$(x', x_3) = (y', \varepsilon y_3),\ U(x) = (\varepsilon^2 \tilde{V}, \varepsilon \tilde{P})(y),\ (\tilde{V}, \tilde{P})(y) = (\varepsilon^{-2} V, \varepsilon^{-1} P)(y', \varepsilon y_3).$$

Since we suppose $S_\varepsilon U = 0$ in $\mathbb{R}^3_+$, we find for $(\tilde{V}, \tilde{P})$ the following system, with $\Delta'_y = \partial^2_{y_1} + \partial^2_{y_2}$:

$$\begin{aligned} -\varepsilon^2 \Delta'_y \tilde{V}_i(y) - \partial_3^2 \tilde{V}_i(y) + \varepsilon \partial_i \tilde{P}(y) &= 0, \quad i = 1, 2, \\ -\varepsilon^2 \Delta'_y \tilde{V}_3(y) - \partial_3^2 \tilde{V}_3(y) + \partial_3 \tilde{P}(y) &= 0, \\ -\varepsilon^3 \Delta'_y \tilde{P}(y) - \varepsilon \partial_3^2 \tilde{P}(y) + \varepsilon^2 (\partial_1 \tilde{V}_1(y) + \partial_2 \tilde{V}_1(y)) + \varepsilon \partial_3 \tilde{V}_3(y) &= 0. \end{aligned}$$

We divide the last equation by ε and take into account only the leading terms with respect to ε. This gives a system of ordinary differential equations with respect to $y_3 > 0$ for the (rescaled) main part $\tilde{u}^{(1)}$, while y' appears as a parameter:

$$\begin{aligned} -\partial_3^2 \tilde{v}_i^{(1)}(y', y_3) &= 0, \qquad i = 1, 2, \\ -\partial_3^2 \tilde{v}_3^{(1)}(y', y_3) + \partial_3 \tilde{p}^{(1)}(y', y_3) &= 0, \\ -\partial_3^2 \tilde{p}^{(1)}(y', y_3) + \partial_3 \tilde{v}_3^{(1)}(y', y_3) &= 0. \end{aligned} \tag{5.11}$$

With condition (5.10) we obtain

$$\tilde{v}_{1,2}^{(1)}(y) = 0, \qquad \tilde{v}_3^{(1)}(y) = -\partial_3 p^0(y', 0)\, e^{-y_3}, \qquad \tilde{p}^{(1)}(y) = \partial_3 p^0(y', 0)\, e^{-y_3}; \tag{5.12}$$

hence,

$$u^{(1)}(x) = \Big(\ 0, 0,\ -\varepsilon^2 \partial_3 p^0(x', 0)\, e^{(-\varepsilon^{-1} x_3)},\ \varepsilon\, \partial_3 p^0(x', 0)\, e^{(-\varepsilon^{-1} x_3)}\ \Big). \tag{5.13}$$

Note that here ∂_3 is the normal derivative with respect to the interior normal vector.

We now return to the assumption that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, and we denote by $\mathbf{n}(\bar{x})$ the interior normal vector in $\bar{x} \in \partial\Omega$. For $\delta > 0$, put $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$. For any $x \in \Omega$, we can choose a point $\bar{x} \in \partial\Omega$ with $\mathrm{dist}(x, \partial\Omega) = |x - \bar{x}|$. In the sequel we fix a $\delta > 0$ such that $x \in \Omega_\delta$ implies that $\bar{x}$ is uniquely determined by x (and depends smoothly on it). For $p^0 \in H^{l+2}(\Omega)$, and $x \in \Omega_\delta$, we define boundary-layer terms analogous to (5.12):

$$\begin{aligned} \mathcal{V}(\varepsilon, x) &= -\mathbf{n}(\bar{x})\partial_{\mathbf{n}} p^0(\bar{x}) \exp(-\varepsilon^{-1}|x - \bar{x}|) \\ \mathcal{P}(\varepsilon, x) &= \partial_{\mathbf{n}} p^0(\bar{x}) \exp(-\varepsilon^{-1}|x - \bar{x}|). \end{aligned} \tag{5.14}$$

Furthermore, we fix a cut-off function $\chi \in C^\infty(\overline{\Omega})$ with $\chi = 1$ on $\Omega_{\delta/2}$, and $\chi = 0$ on $\Omega \setminus \Omega_\delta$.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let $\varkappa \in [0, 3/2)$ and $l \in \mathbb{N}$ with $l \geq \varkappa$ be fixed. For given data $(f, g') \in \mathbf{R}^{l+3}H(\Omega, \partial\Omega)_\perp$, let $u^\varepsilon \in \mathfrak{D}^{l+2}H(\Omega)_\perp$ be the solution to the problem* (S_ε) *(with $g_4 = 0$) and $u^0 \in \mathbf{D}^{l+3}H(\Omega)_\perp$ the solution to the corresponding Stokes problem* (S_0). *Then it holds that*

$$v^\varepsilon(x) = v^0(x) - \varepsilon^2\, \mathcal{V}(\varepsilon, x)\chi(x) + \varepsilon^2 v^{(2)}(x) + \mathfrak{V}(x), \tag{5.15}$$

$$p^\varepsilon(x) = p^0(x) - \varepsilon\, \mathcal{P}(\varepsilon, x)\chi(x) + \varepsilon^2 p^{(2)}(x) + \mathfrak{P}(x), \tag{5.16}$$

where $\mathcal{V}$ and $\mathcal{P}$ are defined as in (5.14), *and $(v^{(2)}, p^{(2)}) \in \mathbf{D}^{l+1}H(\Omega)_\perp$ is the solution to the Stokes problem*

$$-\Delta v^{(2)} + \nabla p^{(2)} = 0, \quad \mathrm{div}\, v^{(2)} = \Delta p^0 \quad \text{in } \Omega, \quad v^{(2)} = \mathbf{n}\partial_{\mathbf{n}} p^0 \ \text{on } \partial\Omega. \tag{5.17}$$

The remainder $\mathfrak{U} = (\mathfrak{V}, \mathfrak{P})$ can be estimated by

$$\|\mathfrak{U}; \mathfrak{D}^l_\varkappa H(\Omega)\| \leq C\, \varepsilon^{5/2-\varkappa} \|(f, g'); \mathbf{R}^{l+3}H(\Omega, \partial\Omega)\|, \tag{5.18}$$

where $\varkappa \in [0, 3/2)$ and C is independent of $\varepsilon \in (0, 1]$.

Remark. Note that the second term on the right-hand side of (5.15) and (5.16) is well defined due to the choice of the cut-off function χ. Furthermore, we observe that the compatibility condition in Proposition 2.1 is fulfilled for the right-hand sides of (5.17), if we keep in mind that now $\mathbf{n}(\bar{x})$ is the interior normal vector. Hence we obtain a unique solution $u^{(2)} \in \mathfrak{D}^l H(\Omega)$. This solution removes instantaneously the discrepancy in f_4 produced by p^0 and in the boundary values produced by the boundary layer.

For the proof of Theorem 5.3 we need $H^s(\Omega_\delta)$ estimates for functions of boundary-layer type. We recall that $\bar{x}$ denotes the projection of $x \in \Omega_\delta$ onto $\partial\Omega$. By choosing a suitable basis $\{\mathbf{t}_1(\bar{x}), \mathbf{t}_2(\bar{x})\}$ of the tangent space to $\partial\Omega$ in $\bar{x} \in \partial\Omega$, on Ω_δ each differential operator ∂^α can be expressed in terms $\partial_{\mathbf{t}}$ of tangential derivatives and the normal derivative $\partial_{\mathbf{n}}$ with smooth coefficients $c^\alpha = c^\alpha(\bar{x})$. Moreover, we may assume that $(\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2)$ is a smooth orthonormal vector field.

Lemma 5.4. *Let Ω_δ be specified as before Theorem* 5.3*. On Ω_δ, let ψ be defined by*

$$\psi(\varepsilon, x) = \phi(\bar{x}) \exp(-\varepsilon^{-1}|x - \bar{x}|) \tag{5.19}$$

with $\phi \in H^l(\partial\Omega)$; then, for $s \leq l$, it holds that

$$\|\psi; H^s(\Omega_\delta)\| \leq C\varepsilon^{1/2-s}\|\phi; H^l(\partial\Omega)\|. \tag{5.20}$$

Step 1: $s = 0, \dots, l$. Clearly, we have the following equivalency of norms for functions of the boundary-layer type (5.19):

$$\|\psi; L^2(\Omega_\delta)\|^2 \sim \Big(\int_0^\delta e^{-2\varepsilon^{-1}z} dz \int_{\partial\Omega} |\phi(\bar{x})|^2 do(\bar{x}) \Big) \leq c\,\varepsilon\|\phi; L^2(\partial\Omega)\|^2. \tag{5.21}$$

For $s = 1$, we observe that

$$\|\nabla\psi; L^2(\Omega_\delta)\|^2 \leq C \ \Big(\|\partial_{\mathbf{n}}\psi; L^2(\Omega_\delta)\|^2 + \|\partial_{\mathbf{t}}\psi; L^2(\Omega_\delta)\|^2 \Big),$$

and from the special form of ψ we have $\partial_{\mathbf{n}}\psi = -\varepsilon^{-1}\psi$, while $\partial_{\mathbf{t}}\psi(\varepsilon, x) = \partial_{\mathbf{t}}\phi(\bar{x}) \exp(-\varepsilon^{-1}|x - \bar{x}|)$, which leads to

$$\begin{aligned} \|\psi; H^1(\Omega_\delta)\| &\leq C \ \Big(\|\psi; L^2(\Omega_\delta)\| + \|\nabla\psi; L^2(\Omega_\delta)\| \Big) \\ &\leq c \Big(\varepsilon^{1/2}\|\phi; L^2(\partial\Omega)\| + \|\nabla_{\mathbf{t}}\phi; L^2(\partial\Omega)\| + \varepsilon^{1/2-1}\|\phi; L^2(\partial\Omega)\| \Big) \\ &\leq C\,\varepsilon^{1/2-1}\|\phi; H^1(\partial\Omega)\|. \end{aligned} \tag{5.22}$$

We obtain (5.20) by induction now.

Step 2: $s \geq 0$ and noninteger. Here we recall the interpolation inequality (see [17, Chapters I.9.1 and I.2.5])

$$\|w; H^s(\Omega)\| \leq C\,\|w; H^m(\Omega)\|^{1-\theta}\|w\|^\theta, \quad (1 - \theta)m = s. \tag{5.23}$$

Let $s \in (0, 1)$; then by (5.23) for $m = 1$

$$\begin{aligned} \|\psi; H^s(\Omega_\delta)\| &\leq C\|\psi; H^1(\Omega_\delta)\|^s\|\psi; L^2(\Omega_\delta)\|^{1-s} \\ &\leq C\varepsilon^{(1/2-1)s}\varepsilon^{1/2(1-s)}\|\psi; H^1(\partial\Omega)\| = C\varepsilon^{1/2-s}\|\psi; H^1(\partial\Omega)\|. \end{aligned} \tag{5.24}$$

For $s \in (k - 1, k)$, we use the estimate (5.20) for $s = k$, the formula (5.21), and the interpolation inequality (5.23) with $m = k$.

Step 3: $s<0$. For $s<0$, we use the relationship (compare (3.8))

$$\|\psi;H^s(\Omega_\delta)\|\sim\inf\Big\{\sum_{|\alpha|\le -[s]}\|\psi^\alpha;H^{s+|\alpha|}(\Omega_\delta)\|:\psi=\sum_{|\alpha|\le -[s]}\partial^\alpha\psi^\alpha\Big\}.$$

Here we choose a special representation

$$\psi(\varepsilon,x)=\partial_{\mathbf{n}}^{-[s]}\Psi(\varepsilon,x),\quad \Psi(\varepsilon,x)=(-\varepsilon)^{-[s]}\partial_{\mathbf{n}}^{-[s]}\psi(\varepsilon,x).$$

Owing to Step 1 and Step 2, we conclude the formula

$$\begin{aligned}\|\psi;H^s(\Omega_\delta)\|&\le C\|\Psi;H^{s-[s]}(\Omega_\delta)\|=C\varepsilon^{-[s]}\|\psi;H^{s-[s]}(\Omega_\delta)\|\\ &\le C\varepsilon^{-[s]}\varepsilon^{1/2-s+[s]}\|\phi;H^1(\partial\Omega)\|=C\varepsilon^{1/2-s}\|\phi;H^1(\partial\Omega)\|,\end{aligned}$$

which proves the assertion. □

Proof of Theorem 5.3. By definition of u^0 and $u^{(2)}$, and $\mathcal{V}$ and $\mathcal{P}$, the remainder $\mathfrak{U}=(\mathfrak{V},\mathfrak{P})$ satisfies the problem $\mathbf{S}_\varepsilon\mathfrak{U}=(\mathfrak{F},\mathfrak{G})$ with

$$\begin{array}{rcl}\mathfrak{F}'&=&\varepsilon^2\Delta(\chi\mathcal{V})-\varepsilon\nabla(\chi\mathcal{P}),\\ \mathfrak{F}_4&=&-\varepsilon^4\Delta p^{(2)}+\varepsilon^3\Delta(\chi\mathcal{P})-\varepsilon^2\mathrm{div}\,(\chi\mathcal{V})\\ \mathfrak{G}'&=&0,\\ \mathfrak{G}_4&=&\varepsilon^2\partial_{\mathbf{n}}p^{(2)}.\end{array}\tag{5.25}$$

In order to apply Theorem 4.4 we have to estimate $\|(\mathfrak{F},\mathfrak{G});\mathfrak{R}^l_\varkappa(\Omega,\varepsilon)\|$ by

$$C\varepsilon^{\frac52-\varkappa}\Big(\|f';H^{l+2}(\Omega)^3\|+\|f_4;H^{l+3}(\Omega)\|+\|g;H^{l+7/2}(\partial\Omega)\|\Big)=:C\varepsilon^{\frac52-\varkappa}\mathfrak{M}.$$

All the terms on the right-hand side of (5.25) are calculated by means of p^0; thus, our starting point is the well-known estimate for p^0 following from the regularity results on elliptic systems [3],

$$\|p^0;H^{l+3}(\Omega)\|\le C\,\mathfrak{M}.\tag{5.26}$$

Let us start with the terms related to $p^{(2)}$, which means to estimate (compare the proof of (5.4))

$$\|\varepsilon^4\Delta p^{(2)};H^{l-1}_{\varkappa-1}(\Omega,\varepsilon)\|,\quad \|\varepsilon^2\partial_{\mathbf{n}}p^{(2)};\tilde H^{l-1/2}_{\varkappa-3/2}(\partial\Omega,\varepsilon)\|.$$

By virtue of (5.17), the estimates of [3] for $p^{(2)}$ and (5.26) imply here

$$\|p^{(2)};H^{l+1}(\Omega)\|\le C\Big(\|\partial_{\mathbf{n}}p^0;H^{l+3/2}(\partial\Omega)\|+\|\Delta p^0;H^l(\Omega)\|\Big)\le C\,\mathfrak{M}.\tag{5.27}$$

From (3.12) with $|\alpha|=1$, (3.5), and finally (5.27), we obtain

$$\begin{aligned}\|\varepsilon^4\Delta p^{(2)};H^{l-1}_\varkappa(\Omega,\varepsilon)\|&\le C\varepsilon^3\|\nabla p^{(2)};H^l_\varkappa(\Omega,\varepsilon)\|\\ &\le C\varepsilon^3\|p^{(2)};H^{l+1}(\Omega)\|\le C\varepsilon^3\mathfrak{M}.\end{aligned}$$

For the discrepancy term generated by $p^{(2)}$ in the boundary values of $\mathfrak{U}$ we have to use $(3.17)_2$ for the trace norm, since $\varkappa < 3/2$, then we apply (3.14) and end up with

$$\|\varepsilon^2\partial_{\mathbf{n}}p^{(2)}; \tilde{H}^{l-1/2}_{\varkappa-3/2}(\partial\Omega,\varepsilon)\| = \varepsilon^2\varepsilon^{-\varkappa+3/2}\|\partial_{\mathbf{n}}p^{(2)}; H^{l-1/2}_0(\partial\Omega,\varepsilon)\| \tag{5.28}$$
$$\leq C\varepsilon^{7/2-\varkappa}\|\partial_{\mathbf{n}}p^{(2)}; H^{l-1/2}(\partial\Omega)\| \leq C\varepsilon^{7/2-\varkappa}\|p^{(2)}; H^{l+1}(\Omega)\| \leq C\varepsilon^{7/2-\varkappa}\mathfrak{M}.$$

Now the remaining task is to control the boundary-layer terms $\mathfrak{F}_{bl} := \mathfrak{F} - (0, -\varepsilon^4\Delta p^{(2)})$ on the right-hand side of (5.25). Since χ vanishes outside of Ω_δ, we have

$$\|\mathfrak{F}_{bl}; \mathfrak{R}^l_\varkappa(\Omega,\varepsilon)\| \leq \|\mathfrak{F}_{bl}; \mathfrak{R}^l_\varkappa(\Omega_{\delta/2},\varepsilon)\| + \|\mathfrak{F}_{bl}; \mathfrak{R}^l_\varkappa(\Omega_\delta\setminus\Omega_{\delta/4},\varepsilon)\|.$$

Thus, the remaining estimates for the boundary-layer terms can be split into estimates in the interior part $\Omega_\delta\setminus\Omega_{\delta/4}$ of Ω, and estimates near the boundary. The interior estimates are the easier part. On $\Omega_\delta\setminus\Omega_{\delta/4}$, we have $\varepsilon^N\exp(-\varepsilon^{-1}|x-\bar{x}|) \leq C(N)\exp(-(8\varepsilon)^{-1}\delta)$ for any $N\in\mathbb{Z}$, and the special form of $(\mathcal{V},\mathcal{P})$ together with (5.26) ensures that

$$\begin{aligned}\|\mathfrak{F}; \mathfrak{R}^l_\varkappa H(\Omega_\delta\setminus\Omega_{\delta/4},\varepsilon)\| &\leq C\,\exp(-(8\varepsilon)^{-1}\delta)\|\partial_{\mathbf{n}}p^0; H^{l-1}(\partial\Omega)\|\\ &\leq C\,\exp(-(8\varepsilon)^{-1}\delta)\,\mathfrak{M}.\end{aligned} \tag{5.29}$$

The estimates of the boundary-layer terms on $\Omega_{\delta/2}$ require more subtle arguments. Let $(\mathbf{n},\mathbf{t})$ be defined as in Lemma 5.4. Apparently we have $\mathcal{V} = \mathbf{n}\mathcal{V}_{\mathbf{n}}$ on Ω_δ. If we express the Laplacian and the gradient in terms of $\partial_{\mathbf{n}}$ and $\partial_{\mathbf{t}}$, then $\nabla = \mathbf{n}\partial_{\mathbf{n}} + \mathbf{t}_1\partial_{\mathbf{t}_1} + \mathbf{t}_2\partial_{\mathbf{t}_2}$, while the Laplacian is decomposed in the form $\Delta = \partial^2_{\mathbf{n}} + \mathcal{L}(\bar{x},\mathbf{n},\nabla_{\mathbf{t}},\partial_{\mathbf{n}})$. Here $\mathcal{L}$ is still a second-order differential operator; at each point $\mathcal{L}$ is a linear combination of $\partial_{\mathbf{n}}$, $\nabla_{\mathbf{t}}$, and $\nabla^2_{\mathbf{t}}$ (without mixed derivatives of the form $\partial_{\mathbf{n}}\partial_{\mathbf{t}}$). From the definition of $\mathcal{V}$ and $\mathcal{P}$, we obtain the equalities

$$\varepsilon^2\partial^2_{\mathbf{n}}\mathcal{V}_{\mathbf{n}} - \varepsilon\partial_{\mathbf{n}}\mathcal{P} = 0, \qquad \varepsilon^3\partial^3_{\mathbf{n}}\mathcal{P} + \varepsilon^2\partial_{\mathbf{n}}\mathcal{V} = 0, \tag{5.30}$$

which cancel certain terms on the right-hand side of $(5.25)_{1,2}$. As a result, it remains to estimate the expressions

$$\varepsilon^2\partial^h_{\mathbf{t}}\mathcal{V},\quad h=0,1,2,\quad \varepsilon\partial_{\mathbf{t}}\mathcal{P}, \varepsilon^2\partial_{\mathbf{n}}\mathcal{V}_{\mathbf{n}} \quad \text{in } H^{l-1}_{\varkappa-1}(\Omega_{\delta/2},\varepsilon), \tag{5.31}$$
$$\varepsilon^3\partial^h_{\mathbf{t}}\mathcal{P},\quad h=0,1,2 \quad \varepsilon^2\partial_{\mathbf{t}}\mathcal{V} \quad \text{in } H^{l-1}_{\varkappa}(\Omega_{\delta/2},\varepsilon). \tag{5.32}$$

Since $\mathcal{V} = \mathbf{n}\mathcal{V}_{\mathbf{n}}$, $-\mathcal{V}_{\mathbf{n}} = \mathcal{P}$, and $\varepsilon^2\partial_{\mathbf{n}}\mathcal{V}_{\mathbf{n}} = -\varepsilon\mathcal{P}$, we need only to look at $\partial^h_{\mathbf{t}}\mathcal{P}$, $h = 0,1,2$ in $H^{l-1}_{\varkappa-1}(\Omega_{\delta/2})$ for the estimates related to the terms in (5.31). If $l = 1$, due to the assumption $l \geq \varkappa$ we have $\varkappa - 1 \leq 0$. From (3.7), (5.20), and (5.26) it follows that

$$\|\partial^h_{\mathbf{t}}\mathcal{P}; H^0_{\varkappa-1}(\Omega_{\delta/2},\varepsilon)\| \leq \varepsilon^{-(\varkappa-1)}\|\partial^h_{\mathbf{t}}\mathcal{P}; L^2(\Omega_{\delta/2})\| + \|\partial^h_{\mathbf{t}}\mathcal{P}; H^{\varkappa-1}(\Omega_{\delta/2})\|$$

$$\leq \quad C\varepsilon^{1/2-(\varkappa-1)}\|\partial_{\mathbf{n}}p^0; H^2(\partial\Omega)\| \leq C\varepsilon^{3/2-\varkappa}\|p^0; H^4(\partial\Omega)\| \leq C\varepsilon^{3/2-\varkappa}\mathfrak{M}.$$

Now let $l \geq 2$. Then

$$\begin{aligned}
&\|\partial_{\mathbf{t}}^h\mathcal{P}; H_{\varkappa-1}^{l-1}(\Omega_{\delta/2}, \varepsilon)\| \\
&= \sum_{k=0}^{l-1} \varepsilon^{(k-(\varkappa-1))_+}\|\partial_{\mathbf{t}}^h\mathcal{P}; H^k(\Omega_{\delta/2})\| + \|\partial_{\mathbf{t}}^h\mathcal{P}; H^{\varkappa-1}(\Omega_{\delta/2})\| \\
&\leq C\sum_{k=0}^{l-1} \varepsilon^{(k-(\varkappa-1))_+}\varepsilon^{1/2-k}\|\partial_{\mathbf{t}}^h(\partial_{\mathbf{n}}p^0); H^k(\partial\Omega)\| \\
&\qquad + \varepsilon^{1/2-(\varkappa-1)}\|\partial_{\mathbf{t}}^h(\partial_{\mathbf{n}}p^0); H^1(\partial\Omega)\|;
\end{aligned}$$

here we used (5.20) again. Clearly, for $h \leq 2$ and $k \leq l-1$

$$\|\partial_{\mathbf{t}}^h(\partial_{\mathbf{n}}p^0); H^k(\partial\Omega)\| \leq C\|p^0; H^{l+3}(\Omega)\| \leq C\mathfrak{M};$$

furthermore, it follows for $\varkappa \in [0,1]$ that $(k-(\varkappa-1))_+ = k+1-\varkappa$, while for $\varkappa \in (1, 3/2)$

$$\varepsilon^{(k-(\varkappa-1))_+} = \left\{ \begin{array}{ll} \varepsilon^{k-(\varkappa-1)} & \text{for } k \geq 1 \\ 1 & \text{for } k = 0 \end{array} \right\} \leq \varepsilon^{k+1-\varkappa}; \tag{5.33}$$

thus, we finally arrive at

$$\|\partial_{\mathbf{t}}^h\mathcal{P}; H_{\varkappa-1}^{l-1}(\Omega_{\delta/2}, \varepsilon)\| \leq C\varepsilon^{3/2-\varkappa}\,\mathfrak{M}. \tag{5.34}$$

Since each of the terms in (5.31) is multiplied at least by ε, we obtain the assertion. A completely analogous argument gives

$$\|\partial_{\mathbf{t}}^h\mathcal{P}; H_{\varkappa}^{l-1}(\Omega_{\delta/2}, \varepsilon)\| \leq C\varepsilon^{1/2-\varkappa}\,\mathfrak{M}, \tag{5.35}$$

but in (5.32) all terms are multiplied at least by ε^2, which finally proves

$$\|\mathfrak{F}; \mathfrak{R}_{\varkappa}^l H(\Omega_{\delta/2}, \varepsilon)\| \leq C\,\varepsilon^{5/2-\varkappa}\mathfrak{M}.$$

5.3. **Comments and conclusions.** To demonstrate the asymptotic precision of our estimates, let u^ε and u^0 be defined as in Remark 5.2. Looking carefully at the arguments used in the proof of Lemma 5.4, and especially at formula (5.21), we find for functions of the type $\psi(\varepsilon, x) = \phi(\bar{x})\exp(-\varepsilon^{-1}|x - \bar{x}|)$, with a smooth ϕ, the estimate from below:

$$\|\psi; H^\mu(\Omega_\delta)\| \geq C\sum_{k\leq\mu}\|\partial_n^k\psi; L^2(\Omega_\delta)\| \geq C\varepsilon^{1/2-\mu}\|\phi; L^2(\partial\Omega)\|, \tag{5.36}$$

which is valid for any $\mu \in \mathbb{N}_0$, with a constant $C > 0$ independent of $\varepsilon \in (0,1]$. Therefore the representation (5.14) leads to

$$\begin{aligned} \|(\varepsilon^2\chi\mathcal{V}, \varepsilon\chi\mathcal{P}); \mathfrak{D}^l_\tau(\Omega,\varepsilon)\| &\geq \varepsilon^2\|\chi\mathcal{V}; H^{\tau+1}(\Omega)\| + \varepsilon\|\chi\mathcal{P}; H^\tau(\Omega)\| \\ &\geq C\,\varepsilon^{3/2-\tau}\|\partial_n p^0; L^2(\partial\Omega)\|; \end{aligned} \tag{5.37}$$

this is valid for any $\tau, l \in \mathbb{N}_0$ with $0 \leq \tau \leq l$. Obviously, it holds that

$$\|(\varepsilon^2 v^{(2)}, \varepsilon^2 p^{(2)}); H^{\tau+1}(\Omega)^3 \times H^\tau(\Omega)\| \leq \|(\varepsilon^2 v^{(2)}, \varepsilon^2 p^{(2)}); \mathfrak{D}^l_\tau H(\Omega,\varepsilon)\| \leq C\varepsilon^2 \tag{5.38}$$

for any $0 \leq \tau \leq l$. If we now calculate the difference $u^\varepsilon - u^0$ according to (5.15) and (5.16), we find with $\tau = l = 0$ in (5.37) that

$$\|u^\varepsilon - u^0; H^1(\Omega)^3 \times L^2(\Omega)_\perp\| \geq C\,\varepsilon^{3/2}\|\partial_{\mathbf{n}} p^0; L^2(\partial\Omega)\| - c\varepsilon^2; \tag{5.39}$$

here $c > 0$ is a bound for the corresponding norms of $(v^{(2)}, p^{(2)})$ and $\varepsilon^{-2}\mathfrak{U}$ according to (5.38) and (5.18). As for the estimate (5.9), we put $\tau = l = 1$ in (5.37) and obtain, by repeating the above arguments,

$$\|u^\varepsilon - u^0; H^2(\Omega)^3 \times H^1(\Omega)_\perp\| \geq C\,\varepsilon^{1/2}\|\partial_{\mathbf{n}} p^0; L^2(\partial\Omega)\| - c\varepsilon^2. \tag{5.40}$$

In both cases the exponents of the main asymptotic term can be achieved by sending $\varkappa \nearrow 3/2$ in (5.8) and (5.9), respectively. In order to annul the main asymptotic term on the right-hand sides of (5.39) and (5.40), the expression $\partial_n p^0$ must vanish for almost every $x \in \partial\Omega$ that needs a noncountable family of compatibility conditions for the data.

The estimate (5.18) in Theorem 5.3 is asymptotically sharp, too. Namely, the next asymptotic terms in (5.15), (5.16) constructed along the Vishik-Lyusternik method take the form $\varepsilon^3\chi\mathcal{V}^1$ and $\varepsilon^2\chi\mathcal{P}^1$, where $\mathcal{U}^1 = (\mathcal{V}^1, \mathcal{P}^1)$ have exponential behavior near the boundary $\partial\Omega$ similar to $(\mathcal{V}, \mathcal{P})$ in (5.14). By (5.37), it follows that

$$\begin{aligned} \|(\varepsilon^3\mathcal{V}^1, \varepsilon^2\mathcal{P}^1); \mathfrak{D}^l_\tau H(\Omega,\varepsilon)\| &\geq C\varepsilon^3\|\chi\mathcal{V}^1; H^{\tau+1}(\Omega)^3\| + \varepsilon^2\|\chi\mathcal{P}^1; H^\tau(\Omega)\| \\ &\geq C\,\varepsilon^{5/2-\tau}, \end{aligned}$$

as long as $0 \leq \tau \leq l$.

The restriction $\varkappa < 3/2$ appears only in Theorem 4.3 and Lemma A2 related to the model problem in $\mathbb{R}^3_+$, while Theorem 4.2 on the model problem in $\mathbb{R}^3$ remains valid with any $\varkappa \leq l$. This observation provides that a better estimate *inside* the domain Ω is predictable and weighted Sobolev norms are fit to subdue boundary layers near $\partial\Omega$ which are responsible for the bound 3/2. Delineating the main ideas only, we introduce the following weighted parameter-dependent norm in the space $H^l(\Omega)$:

$$\Big(\|v;H^{\varkappa}(\|\Omega)\|^2+\|d_\varepsilon^{\tau-\varkappa}v;H^\tau(\Omega)+\sum_{k=0}^{l}\varepsilon^{2(k-\tau)_+}\|d_\varepsilon^{(k-\varkappa)_+-(k-\tau)_+}\nabla^k v;L^2(\Omega)\|^2\Big)^{\frac{1}{2}}. \tag{5.41}$$

Here $d_\varepsilon(x)=\varepsilon+d_0(x)$, and d_0 is a positive, smooth function in Ω such that $d_0(x)=\operatorname{dist}(x,\partial\Omega)$ in Ω_δ. The space $H^l(\Omega)$ with the norm (5.41) is denoted by $V^l_{\varkappa,\tau}(\Omega,\varepsilon)$. We emphasize that $\|\cdot;V^l_{\varkappa,\tau}(\Omega_\delta,\varepsilon)\|\sim\|\cdot;H^l_\tau(\Omega_\delta,\varepsilon)\|$ for any fixed $\delta>0$ and $\|\cdot;V^l_{\varkappa,\tau}(\Omega_\varepsilon,\varepsilon)\|\sim\|\cdot;H^l_\varkappa(\Omega_\varepsilon,\varepsilon)\|$. The latter keeps the results of Lemma 3.3 (τ has to be ignored in the trace space). Moreover, Lemma 3.4 remains valid if $H^l_{\mu+1/2}(\Omega,\varepsilon)$ is replaced by $V^l_{\mu+1/2,\tau}(\Omega,\varepsilon)$ with any $\tau>\mu+1/2$. Repetition of the methods applied in Section 3.2 and 3.3 would lead to the a priori estimate

$$\begin{aligned}&\|(v^\varepsilon,p^\varepsilon);V^{l+1}_{\varkappa+1,\tau+1}(\Omega,\varepsilon)^3\times V^{l+1}_{\varkappa}(\Omega,\varepsilon)\|\\&\le C\ \Big(\ \|f;V^{l-1}_{\varkappa-1,\tau-1}(\Omega,\varepsilon)^3\times V^{l-1}_{\varkappa,\tau}(\Omega,\varepsilon)\|\\&+\|g;V^{l+1/2}_{\varkappa+1/2}(\partial\Omega,\varepsilon)^3\times\tilde H^{l-1/2}_{\varkappa-3/2}(\partial\Omega,\varepsilon)\|+\|(v^\varepsilon,p^\varepsilon);L^2(\Omega)^4\|\ \Big)\,.\end{aligned} \tag{5.42}$$

Continuing with arguments used in the proof of Theorem 4.4, the last term on the right-hand side can be removed. If we compare the solutions to the Brezzi-Pitkäranta problem (S_ε) and the solutions of the Stokes problem (S_0) again, we find by means of the error equation (5.5) that

$$\begin{aligned}&\|(v^\varepsilon-v^0,p^\varepsilon-p^0);V^{l+1}_{\varkappa+1,\tau+1}(\Omega,\varepsilon)^3\times V^{l+1}_{\varkappa,\tau}(\Omega,\varepsilon)\|\\&\le C\ \Big(\ \varepsilon^2\|\Delta p^0;V^{l-1}_{\varkappa,\tau}(\Omega,\varepsilon)\|+\|\partial_n p^0;\tilde H^{l-1/2}_{\varkappa-3/2}(\partial\Omega,\varepsilon)\|\ \Big)\\&\le C\ \Big(\ \varepsilon^2\|\Delta p^0;H^{l-1}(\Omega)\|+\varepsilon^{3/2-\varkappa}\|\partial_n p^0;\tilde H^{l-1/2}(\partial\Omega)\|\ \Big)\,.\end{aligned} \tag{5.43}$$

As already mentioned, with some additional work, the a priori estimates (4.15) as well as (5.42) can be extended to the case $\varkappa\in(-1/2,0]$, too. Thus, the estimate (5.43) would finally yield

$$\|(v^\varepsilon-v^0,p^\varepsilon-p^0);V^{l+1}_{\varkappa+1,\tau+1}(\Omega,\varepsilon)^3\times V^{l+1}_{\varkappa,\tau}(\Omega,\varepsilon)\|\le\ C\,\varepsilon^{2-\gamma}\,\|f;H^l(\Omega)^3\|$$

with $\gamma>0$, but arbitrarily small. The definition of these spaces now induces that the difference between u^ε and u^0 is asymptotically of order ε^2 at a distance of $\partial\Omega$, while it enlarges in vicinity of $\partial\Omega$.

Acknowledgment. This work was supported by grants of the DFG and the IST, Lisbon.

Appendix: The Agmon-Douglis-Nirenberg indices and the proof of Theorem 4.3. We first recall the notation of the Agmon-Douglis-Nirenberg indices related to the problems (S_0) and (S_ε). If $(L_{ij}(\partial_x))_{i,j=1}^k$ is a matrix of differential operators, then integers s_i and t_j must be chosen such that $s_i + t_j \geq \mathrm{ord} L_{ij}$. Let us denote the indices associated to the differential operator S_ε by $\{s_i, t_j\}$ for $\varepsilon > 0$ and by $\{\sigma_i, \tau_j\}$ for S_0 and write tables for the orders of S_ε and S_0:

$j\backslash i$		1	2	3	4
	$s\backslash t$	1	1	1	1
1	1	2	2	2	2
2	1	2	2	2	2
3	1	2	2	2	2
4	1	2	2	2	2

$j\backslash i$		1	2	3	4
	$\sigma\backslash\tau$	1	1	1	0
1	1	2	2	2	1
2	1	2	2	2	1
3	1	2	2	2	1
4	0	1	1	1	0

(A1)

To the boundary operators, we also assign similar tables of ADN-indices. If we rewrite the boundary conditions as $B_\varepsilon u = \sum_j (B_\varepsilon(x, \nabla)_{ij} u_j)_{i=1}^4$, then we have to choose r_j such that $\mathrm{ord}(B_\varepsilon)_{ij} \leq t_1 + r_j$, where t_i is taken from (A1). With the boundary operator at $\varepsilon = 0$, we also connect four indices ρ_j and include them in tables, too:

$j\backslash i$		1	2	3	4		$j\backslash i$		1	2	3	4
	$r\backslash t$	1	1	1	1			$\rho\backslash\tau$	1	1	1	0
1	-1	0	0	0	0	$\leftrightarrow v_1 \leftrightarrow$	1	-1	0	0	0	-1
2	-1	0	0	0	0	$\leftrightarrow v_2 \leftrightarrow$	2	-1	0	0	0	-1
3	-1	0	0	0	0	$\leftrightarrow v_3 \leftrightarrow$	3	-1	0	0	0	-1
4	0	0	0	0	1	$\leftrightarrow \partial_n p$	4	1	0	0	0	1

(A2)

A negative order in the table means that the corresponding component B_{ij} of the boundary operator vanishes. The number of necessary boundary conditions providing an elliptic boundary-value problem has to be calculated as (cf. [3]) $m = 2^{-1}\sum_i (s_i + t_i) = 4$ for the operator S_ε, $\varepsilon > 0$ and as $\mu = 2^{-1}\sum_i(\sigma_i + \tau_i) = 3$ for the operator S_0. Thus, the fourth boundary condition related to S_0 in the table (A2) is void; we refer to this condition as a *ghost condition* (cf. [20]).

The desired a priori estimates in $\mathbb{R}^3_+$ can be calculated with the help of the exponentially decaying solutions to the initial-value problem (4.12), (4.13). We reduce (4.12) to the case $|\xi'| = 1$ by performing the following transformations: We change the coordinates from ξ' to $\theta = \xi'/|\xi'|$, x_3 to $y = |\xi'|\, x_3$, and the parameter ε to $\zeta = \varepsilon\,|\xi'|$; simultaneously we pass from $\hat{u}$ and $\hat{g}$ in (4.12), (4.13) to $U = (|\xi'|\,\hat{u}', \hat{u}_4)$ and $G = (|\xi'|\,\hat{g}', |\xi'|^{-1}\hat{g}_4)$; this turns (4.12), (4.13) into

$$\begin{aligned} L(\zeta, \theta, \partial_y)U = 0, \quad U = (U_1, \ldots, U_4), \\ B(\zeta, \theta, \partial_y)U = (U'(0), \partial_y U_4(0)) = G, \end{aligned} \tag{A3}$$

with

$$L = \begin{pmatrix} -\partial_y^2+1 & 0 & 0 & -i\,\theta_1 \\ 0 & -\partial_y^2+1 & 0 & -i\,\theta_2 \\ 0 & 0 & -\partial_y^2+1 & \partial_y \\ -i\,\theta_1 & -i\,\theta_2 & \partial_y & \zeta^2(1+\partial_y^2) \end{pmatrix}.$$

Let $\mathbf{w}^q$, $q=1,\dots,4$ denote the exponentially decaying solution of (A3) with $G=e^q$, where e^q is the q-th unit vector in $\mathbb{R}^4$. Furthermore, we introduce the following notation: For $\varphi \in L^2(\mathbb{R}_+)$ let φ_{ev} denote the even extension on the whole line $\mathbb{R}$. As known (cf. [14]), the norm in $H^\delta(\mathbb{R}_+)$ can be chosen as follows—here the hat indicates the one-dimensional Fourier transform:

$$\|\varphi; H^\delta(\mathbb{R}_+)\|^2 = \int_{\mathbb{R}} (1+|\eta|)^{2\delta} |\widehat{\varphi_{ev}}|^2 \,\mathrm{d}\eta, \quad 0<\delta<1.$$

In previous papers the first author has proved the following result.

Theorem A1. [18, 20] *Let $\mathbf{w}^1,\dots,\mathbf{w}^4$ denote the exponentially decaying solutions of the system* (A3) *with $G=e^q$. Let*

$$W_j^q = (1+\zeta^2)^{(t_j+r_q-\tau_j-\rho_q)/2}\mathbf{w}_j^q, \quad j,q=1,\dots,4. \tag{A4}$$

Suppose $W^1,\dots,W^4$ satisfy the following estimates for $j=1,\dots,4$:

$$\|\partial_y^p W_j^q; L^2(\mathbb{R}_+)\|^2 \le C_p \left(1+(1+\zeta^{-2})^{p-\varkappa-\tau_j}\right), \quad q=1,2,3, \tag{A5}$$

$$\|\partial_y^p W_j^4; L^2(\mathbb{R}_+)\|^2 \le C_p\,(1+\zeta^{-2})^{p-\tau_j-1-1/2}\left(1+(1+\zeta^{-2})^{\varkappa+\tau_j-p}\right), \tag{A6}$$

where $p\in\mathbb{N}_0$ and

$$\varkappa \in (\rho_\mu+\frac{1}{2}, \rho_{\mu+1}+\frac{1}{2}) = (-\frac{1}{2},\frac{3}{2}).$$

Additionally, if $\varkappa$ is not an integer, we assume that W_j^q also fulfills

$$\|\partial_y^{[\varkappa]+\tau_j} W_j^q; H^{\varkappa-[\varkappa]}\|^2 \le C, \quad q=1,2,3, \quad j=1,\dots,4, \tag{A7}$$

$$\|\partial_y^{[\varkappa]+\tau_j} W_j^4; H^{\varkappa-[\varkappa]}\|^2 \le C(1+\zeta^{-2})^{\varkappa-3/2}, \quad j=1,\dots,4, \tag{A8}$$

where $[\varkappa]$ again denotes the integer part of $\varkappa$. Then any $u\in\mathfrak{D}^l_\varkappa(\mathbb{R}^3_+,\varepsilon)$ with $S_\varepsilon u=0$ obeys the estimate

$$\|u;\mathfrak{D}^l_\varkappa(\mathbb{R}^3_+,\varepsilon)\| \;\le\; C\;\left(\|(0,Bu),\mathfrak{R}^l_\varkappa(\mathbb{R}^3_+,\partial\mathbb{R}^3_+,\varepsilon)\| + \|u;L^2(\mathbb{R}^3_+)\|\right), \tag{A9}$$

where C is independent of u and of $\varepsilon\in(0,1]$.

The rest of this appendix contains the verification of the estimates (A5)–(A8). Elementary but lengthy calculations, for example by means of the computer-algebra system MAPLE, provide the following representations of the solutions $\mathbf{w}^q$:

$$\begin{aligned}\mathbf{w}_1^1(\theta,\zeta,y) = Q_1(\zeta)\,\Big(\,(-\theta_1^2\,y+1)e^{(-y)} + 2\,\zeta^2\,\Big(\,e^{(-y)} - \frac{\zeta}{\sqrt{1+\zeta^2}}\,e^{(-\sqrt{1+\zeta^{-2}}\,y)}\,\Big)\\ -\,2\,\frac{\zeta^3}{\sqrt{1+\zeta^2}}\theta_2^2\,\Big(\,e^{(-y)} -\; e^{(-\sqrt{1+\zeta^{-2}}\,y)}\,\Big)\Big),\end{aligned}$$

$$
\begin{aligned}
Q_1(\zeta) &= \frac{1+\zeta^2}{2\,\zeta^4 - 2\,\zeta^3\,\sqrt{1+\zeta^2} + 3\,\zeta^2 + 1},\\
\mathbf{w}_2^1(\theta,\zeta,y) &= \theta_1\,\theta_2\,Q_1(\zeta)\left((-\,y)\,e^{(-y)} + 2\,\frac{\zeta^3}{\sqrt{1+\zeta^2}}\left(e^{(-y)} -\, e^{(-\sqrt{1+\zeta^{-2}}\,y)}\right)\right),\\
\mathbf{w}_3^1(\theta,\zeta,y) &= i\theta_1\,Q_2(\zeta)\,\Big(\,y\,e^{(-y)} - 2\zeta^2\,\Big(\,e^{(-y)} -\, e^{(-\sqrt{1+\zeta^{-2}}\,y)}\,\Big)\Big),\\
Q_2 &= \frac{2\,\zeta^4 + 2\,\zeta^3\,\sqrt{1+\zeta^2} + 3\,\zeta^2 + 1}{8\,\zeta^4 + 5\,\zeta^2 + 1},\\
\mathbf{w}_4^1(\theta,\zeta,y) &= 2\,i\,\theta_1\,Q_2(\zeta)\,\Big(\,e^{(-y)} - \frac{\zeta}{\sqrt{1+\zeta^2}}\,e^{(-\sqrt{1+\zeta^{-2}}\,y)}\,\Big),\\
\mathbf{w}_1^2(\theta,\zeta,y) &= \mathbf{w}_2^1,\\
\mathbf{w}_2^2(\theta,\zeta,y) &= Q_1(\zeta)\,\Big(\,(-\theta_2^2\,y+1)e^{(-y)} + 2\,\zeta^2\,\Big(\,e^{(-y)} - \frac{\zeta}{\sqrt{1+\zeta^2}}\,e^{(-\sqrt{1+\zeta^{-2}}\,y)}\,\Big)\\
&\quad - 2\,\frac{\zeta^3}{\sqrt{1+\zeta^2}}\theta_1^2\,\Big(\,e^{(-y)} -\, e^{(-\frac{\sqrt{1+\zeta^2}}{\zeta}y)}\,\Big)\\
&= \mathbf{w}_1^1(\theta_2,\theta_1,\zeta,y),\\
\mathbf{w}_3^2(\theta,\zeta,y) &= \frac{\theta_2}{\theta_1}\mathbf{w}_3^1(\theta,\zeta,y), \quad \mathbf{w}_4^2(\theta,\zeta,y) = \frac{\theta_2}{\theta_1}\mathbf{w}_4^1(\theta,\zeta,y),\\
\mathbf{w}_1^3(\theta,\zeta,y) &= i\theta_1\,Q_2(\zeta)\,\Big(\,y\,e^{(-y)} - 2\,\frac{\zeta^3}{\sqrt{1+\zeta^2}}(e^{(-y)} - e^{(-\frac{\sqrt{1+\zeta^2}}{\zeta})\,y})\,\Big),\\
\mathbf{w}_2^3(\theta,\zeta,y) &= \frac{\theta_2}{\theta_1}\mathbf{w}_1^3(\theta,\zeta,y),\\
\mathbf{w}_3^3(\theta,\zeta,y) &= Q_1(\zeta)\Big((y+1)e^{(-y)} - 2\frac{\zeta^3}{\sqrt{1+\zeta^2}}\Big(e^{(-y)} - \frac{\sqrt{1+\zeta^2}}{\zeta}\,e^{(-\sqrt{1+\zeta^{-2}}\,y)}\Big)\Big),\\
\mathbf{w}_4^3(\theta,\zeta,y) &= 2\,Q_1(\zeta)\,\Big(\,e^{(-y)} - \frac{\zeta}{\sqrt{1+\zeta^2}}\,e^{(-\sqrt{1+\zeta^{-2}}\,y)}\,\Big),\\
\mathbf{w}_1^4(\theta,\zeta,y) &= i\,\theta_1\,\zeta^2\,\frac{3\,\zeta^2 - \zeta\,\sqrt{1+\zeta^2}+1}{8\,\zeta^4 + 5\,\zeta^2 + 1}\,\Big(\,-ye^{(-y)} -\\
&\qquad (\zeta^2 + \zeta\,\sqrt{1+\zeta^2})(e^{(-y)} -\, e^{(-\sqrt{1+\zeta^{-2}}\,y)})\,\Big),\\
\mathbf{w}_2^4(\theta,\zeta,y) &= \frac{\theta_2}{\theta_1}\mathbf{w}_1^4(\theta,\zeta,y),\\
\mathbf{w}_3^4(\theta,\zeta,y) &= Q_1(\zeta)\,\zeta^2\,\Big(\,(\frac{\zeta}{\sqrt{1+\zeta^2}} - 1)\,ye^{(-y)} - (e^{(-y)} -\, e^{(-\sqrt{1+\zeta^{-2}}\,y)})\,\Big),\\
\mathbf{w}_4^4(\theta,\zeta,y) &= Q_1(\zeta)\,\zeta\,\Big(\,2\,\zeta(\frac{\zeta}{\sqrt{1+\zeta^2}} - 1)e^{(-y)} - \frac{1}{\sqrt{1+\zeta^2}}\,e^{(-\sqrt{1+\zeta^{-2}}\,y)}\,\Big)\,.
\end{aligned}
$$

We observe that there exist constants c_1 and c_2 such that

$$0 < c_1 \le |Q_1(\zeta)|,\ |Q_2(\zeta)| \le c_2 \text{ for all } \zeta \in \mathbb{R}_+. \tag{A10}$$

Lemma A2. *Let* $\mathbf{w}^1, \dots \mathbf{w}^4$ *be defined as above, and* $W^1, \dots, W^4$ *by* (A4)*; moreover,* $\varkappa \in (-1/2, 3/2)$*. Then the estimates* (A5) *and* (A6) *are valid.*

Proof. Due to the distribution of the ADN-indices we can divide the estimates into blocks of the same type.
Step 1: $q, j = 1, 2, 3$. In this case we have $t_j + r_q - \tau_j - \rho_q = 0$; hence, $W_j^q = \mathbf{w}_j^q$. Since $1 + \zeta^{-2} \ge 1$ for all $\zeta \in \mathbb{R}$, we conclude $(1+\zeta^{-2})^{p-3/2-1} \le (1+\zeta^{-2})^{p-\varkappa-1}$ for all $\varkappa < 3/2$ and $p \in \mathbb{N}_0$; therefore, it is sufficient to prove

$$\|\partial_y^p W_j^q; L^2(\mathbb{R}_+)\| \le C_p(1 + (1+\zeta^{-2})^{p-5/2}) \tag{A11}$$

in order to obtain (A6). Inspecting the formulas for $\mathbf{w}_j^q$, $q, j \le 3$, we find the following representation for $\mathbf{w}_j^q$:

$$\begin{aligned}\mathbf{w}_j^q(y,\zeta) &= G_1(\zeta) y e^{(-y)} + G_2(\zeta) e^{(-y)} \\ &+ \ \zeta^2 \Big(G_3(\zeta)(e^{(-y)} - e^{(-\sqrt{1+\zeta^{-2}}\, y)}) \\ &\quad + G_4(\zeta)(e^{(-y)} - (1+\zeta^{-2})^{-1/2}\, e^{(-\sqrt{1+\zeta^{-2}}\, y)}) \\ &\quad + G_5(\zeta)(e^{(-y)} - (1+\zeta^{-2})^{1/2}\, e^{(-\sqrt{1+\zeta^{-2}}\, y)}) \Big),\end{aligned} \tag{A12}$$

where $|G_i(\zeta)| \le c$ for all $\zeta \ge 0$ and θ due to (A10). Observe that $G_5(\zeta)$ appears only in $\mathbf{w}_3^3$. We set $f_p(\zeta) = 1 + (1+\zeta^{-2})^{p-5/2}$, $\zeta \in \mathbb{R}_+$ and distinguish now the cases $\zeta > 1$ and $\zeta \le 1$.

For $\zeta > 1$, we have $1 < f_p(\zeta) < \max\{2, 1 + 2^{p-5/2}\}$; therefore, we are ready if we prove

$$\|\partial_y^p W_j^q(\zeta); L^2(\mathbb{R}_+)\| \le C_p \text{ independent of } \zeta > 1. \tag{A13}$$

From (A12) we obtain

$$\begin{aligned}\|\partial_y^p W_j^q; L^2(\mathbb{R}_+)\|^2 \le \ c \ \Big(&\int_{\mathbb{R}_+} |\partial_y^p (e^{(-y)}(y+1))|^2 dy \\ + \sum_{k=0}^{2} &\int_0^\infty \Big| \zeta^2 \partial_y^p [e^{(-y)} - (1+\zeta^{-2})^{(k-1)/2}\, e^{(-\sqrt{1+\zeta^{-2}}\, y)}] \Big|^2 dy \Big) .\end{aligned}$$

The first integral in this formula is clearly bounded independently of ζ. In order to estimate the other three integrals we investigate the term $\partial_y^p[\ldots]$. We find

$$\begin{aligned}&|\partial_y^p [e^{(-y)} - (1+\zeta^{-2})^{(k-1)/2}\, e^{(-\sqrt{1+\zeta^{-2}}\, y)}]| \\ &= |e^{(-y)} - (1+\zeta^{-2})^{p/2}(1+\zeta^{-2})^{(k-1)/2}\, e^{(-\sqrt{1+\zeta^{-2}}\, y)}| =: |\varphi(0) - \varphi(x)|,\end{aligned} \tag{A14}$$

where φ is defined by $\varphi(x) = (1+x)^\alpha e^{-\sqrt{1+x}\,y}$, $\alpha = (p+k-1)/2$, $x = \zeta^{-2}$. We use the well-known inequality

$$|\varphi(0) - \varphi(x)| \le x \max_{\xi\in[0,1]} |\varphi'(\xi)| \qquad \text{for } x \in [0,1]$$

to estimate the right-hand side of (A14). We have

$$\varphi'(x) = e^{-\sqrt{1+x}y}(1+x)^{\alpha-1}\,(\alpha - \frac{y}{2}\sqrt{1+x}),$$

which gives $\max_{\xi\in[0,1]} |\varphi(\xi)| \le C\, e^{(-y)}(y+1)$. Therefore, the right-hand side of (A14) satisfies the estimate

$$|\partial_y^p[\ldots]| \le C_p\, \zeta^{-2}\, e^{(-y)}(y+1),$$

which leads to (A13). For $\zeta \le 1$, we rewrite the representation of $\mathbf{w}_j^q$ in the form

$$\mathbf{w}_j^q(y,\zeta) = \tilde{G}_1(\zeta) e^{(-y)} + \tilde{G}_2(\zeta)\, y e^{(-y)} + \zeta^2 \tilde{G}_3(\zeta)\; e^{(-\sqrt{1+\zeta^{-2}}\,y)}, \tag{A15}$$

where the functions $\tilde{G}_i(\zeta) = \tilde{G}_i(\zeta,\theta)$ remain bounded independently of ζ and θ for $\zeta \le 1$. Since

$$\|\partial_y^p\, e^{(-\sqrt{1+\zeta^{-2}}\,y)}; L^2(\mathbb{R}_+)\|^2 = (1+\zeta^{-2})^p \int_0^\infty e^{(-\sqrt{1+\zeta^{-2}}\,2y)} dy = \frac{1}{2}(1+\zeta^{-2})^{p-1/2}, \tag{A16}$$

while all the other integrals keep the boundedness as $\zeta \to 0$, we gain

$$\|\partial_y^p W_j^q; L^2(\mathbb{R}_+)\|^2 \le C_p(1+\zeta^{5-2p}) \text{ for } \zeta \le 1.$$

This implies (A11) and therefore (A5) for $q, j = 1,2,3$.

Step 2: $q = 1,2,3$, $j = 4$. Then $t_j + r_q - \tau_j - \rho_q = 1$; hence, $W_4^q = (1+\zeta^2)^{1/2}\mathbf{w}_4^q$, and, to obtain (A5), we have to prove

$$\|\partial_y^p W_4^q; L^2(\mathbb{R}_+)\|^2 \le c\,(1 + (1+\zeta^{-2})^{p-3/2}). \tag{A17}$$

For $\zeta > 1$, this means again that $\|\partial_y^p W_4^q; L^2(\mathbb{R}_+)\|$ remains bounded. Since $W_4^q(y,\zeta) = G_q(\zeta)\big[e^{(-y)} - (1+\zeta^{-2})^{-1/2}\, e^{(-\sqrt{1+\zeta^{-2}}\,y)}\big]$, where $|G_q(\zeta)| \le c$ for $\zeta > 1$, we obtain the assertion by the same arguments as for the estimation of (A14) in Step 1. For $\zeta \le 1$, we use $W_4^q(y,\zeta) = \tilde{G}_1(\zeta) e^{(-y)} + \tilde{G}_2(\zeta)\zeta\, e^{(-\sqrt{1+\zeta^{-2}}\,y)}$; then (A10) ensures the boundedness of $\tilde{G}_1$ and $\tilde{G}_2$ as $\zeta \searrow 0$, and finally (A17) follows from (A16).

Step 3: $q = 4$. In this case we have

$$t_j + r_q - \tau_j - \rho_q = \begin{cases} -1 & \text{for} \quad j = 1,2,3, \\ \;\;0 & \text{for} \quad j = 4. \end{cases}$$

The inequalities (A6) turn into

$$\|\partial_y^p W_j^4; L^2(\mathbb{R}_+)\|^2 \le c \begin{cases} \tilde{f}_{p,1}(\zeta) & \text{for} \quad j = 1,2,3, \\ \tilde{f}_{p,2}(\zeta) & \text{for} \quad j = 4 \end{cases} \tag{A18}$$

with

$$\tilde{f}_{p,1}(\zeta) = (1+\zeta^{-2})^{p-5/2}(1+(1+\zeta^{-2})^{\varkappa+1-p}),$$
$$\tilde{f}_{p,2}(\zeta) = (1+\zeta^{-2})^{p-3/2}(1+(1+\zeta^{-2})^{\varkappa-p}).$$

Again we consider the asymptotic behavior of the right-hand side of (A18) at infinity as well as at zero. For $\zeta > 1$, it is sufficient once more to prove

$$\|\partial_y^p W_j^q; L^2(\mathbb{R}_+)\|^2 \le C_p, \quad j = 1, \ldots, 4,$$

independent of $\zeta > 1$. This follows from the representation of $\mathbf{w}_j^4$ with $|1-(1+\zeta^{-2})^{-1/2}| < \zeta^{-2}$. For $\zeta \le 1$, we use the Taylor expansion of $(1+x)^\alpha$ with $x = \zeta^2$, $\alpha = p - 5/2, \varkappa - 3/2$ to calculate

$$\begin{aligned} f_{p,1}(\zeta) &= \zeta^{3-2\varkappa}(\zeta^{2(\varkappa+1-p)}(1+\zeta^2)^{p-5/2} + (1+\zeta^2)^{\varkappa-3/2}) \\ &= \zeta^{5-2p} + \zeta^{3-2\varkappa} + O(\zeta^\beta) \end{aligned} \tag{A19}$$

with $\beta > \max\{5-2p, 3-2\varkappa\}$. From (A19) we conclude that the leading term for $p = 0$ is $\zeta^{3-2\varkappa} \ge \zeta^4$ for $\varkappa > -1/2$, while for $p \ge 1$ the leading term is ζ^{5-2p}. Since for $j = 1, 2, 3$

$$W_j^4(y,\zeta) = \zeta^2 \left(g_1(\zeta)\, y,\, e^{(-y)} + g_2(\zeta)\, e^{(-y)} + g_3(\zeta)\; e^{(-\sqrt{1+\zeta^{-2}}\, y)} \right) \tag{A20}$$

with $|g_i(\zeta)| \le c$ independent of $\zeta < 1$, we see immediately that $\|W_j^4; L^2(\mathbb{R}_+)\| \le c\,\zeta^4$. For $p \ge 1$, we use (A20) and (A16) to obtain $\|\partial_y^p W_j^4; L^2(\mathbb{R}_+)\| \le c\,\zeta^{5-2p}$. For $j = 4$, similar calculations deliver the formula $f_{p,2}(\zeta) = \zeta^{3-2\varkappa} + \zeta^{3-2p} + O(\zeta^\beta)$ with $\beta \ge \max\{3-2\varkappa, 3-2p\}$. In this case the leading term is always ζ^{3-2p}, and with $W_4^4(\zeta, y) = \zeta \left(\tilde{g}_1(\zeta) e^{(-y)} + \tilde{g}_2(\zeta)\, e^{(-\sqrt{1+\zeta^{-2}}\, y)} \right)$ the estimate

$$\|\partial_y^p W_4^4; L^2(\mathbb{R}_+)\| \le c\,\zeta^{3-2p}$$

follows from (A10) and (A16). This completes the proof of (A6). □

In the next lemma we want to prove the inequalities (A7) and (A8) with noninteger $\varkappa$. For fixed ζ and θ, the even extension of $\partial_y^p W_j^q$ is a linear combination of $(e^{(-y)})_{ev}$, $(ye^{(-y)})_{ev}$, and $(\, e^{(-\sqrt{1+\zeta^{-2}}\, y)})_{ev}$. We calculate the corresponding Fourier transforms

$$\begin{aligned} \mathfrak{F}_1(\eta) &= \mathfrak{F}_{y\to\eta}(e^{(-y)})_{ev} = \frac{1}{\sqrt{2\pi}}\left(\int_0^\infty e^{-iy\eta} e^{(-y)} dy + \int_{-\infty}^0 e^{-iy\eta} e^{y} dy\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+i\eta} + \frac{1}{1-i\eta}\right) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\eta^2}, \end{aligned} \tag{A21}$$

$$\begin{aligned} \mathfrak{F}_2(\eta) &= \mathfrak{F}_{y\to\eta}(ye^{(-y)})_{ev} = \frac{1}{\sqrt{2\pi}}\left(\int_0^\infty e^{-iy\eta} y e^{(-y)} dy - \int_{-\infty}^0 e^{-iy\eta} y e^{(-y)} dy\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{(1+i\eta)^2} + \frac{1}{(1-i\eta)^2}\right) = \sqrt{\frac{2}{\pi}}\frac{1-\eta^2}{(1+\eta)^2}, \end{aligned} \tag{A22}$$

$$\begin{aligned}\mathfrak{F}_3(\eta,\zeta) &= \mathfrak{F}_{y\to\eta}(\, e^{(-\sqrt{1+\zeta^{-2}}\,y)})_{ev} = \frac{1}{2\pi}\Big(\frac{1}{\sqrt{1+\zeta^{-2}}+i\,\eta}+\frac{1}{\sqrt{1+\zeta^{-2}}-i\,\eta}\Big)\\ &= \sqrt{\frac{2}{\pi}}\frac{\zeta\sqrt{1+\zeta^2}}{1+\zeta^2+\zeta^2\,\eta^2}. \end{aligned} \tag{A23}$$

Note that $\|\mathfrak{F}_i; L^2_\delta\| < \infty$ for all $\delta \in [0,1)$ and $\zeta \in \mathbb{R}_+$. With the help of these formulas we prove

Lemma A3. *Let W^q_j, $q, j = 1,\dots,4$ be defined as in Theorem* 4.3*. Then estimates* (A7) *and* (A8) *hold true.*

We make a similar fall differentiation as in the previous lemma.
Step 1. $q = 1,2,3$, $j = 1,2,3$. In this case we have to show (A7) for $p = 0,1,2$. We put $\mathfrak{F}_{y\to\eta}(\partial^p_y W^q_j)_{ev} = \mathfrak{W}^{p,q}_j$. From (A12) and (A21)–(A23) we obtain for $\delta\in[0,1)$

$$\begin{aligned}\|\mathfrak{W}^{p,q}_j; L^2_\delta\|^2 &= \int_{\mathbb{R}} |\eta|^{2\delta}|\mathfrak{W}^{p,q}_j(\eta,\zeta)|^2 d\eta \\ &\le c\left(\ \|\mathfrak{F}_1; L^2_\delta\|^2 + \|\mathfrak{F}_2; L^2_\delta\|^2 + \zeta^4\sum_{k=0}^{3}\|\mathfrak{F}_1 - (1+\zeta^{-2})^{(p+k-1)/2}\mathfrak{F}_3; L^2_\delta\|^2\ \right).\end{aligned} \tag{A24}$$

Again the first two terms are bounded uniformly with respect to ζ. To estimate the remaining term, we put $\tilde p = p + k - 1$ and start with the case $\zeta \ge 1$. We have

$$\begin{aligned}&|\mathfrak{F}_1(\zeta) - (1+\zeta^{-2})^{\tilde p/2}\mathfrak{F}_3(\eta,\zeta)| = \sqrt{\frac{2}{\pi}}\Big|\frac{1}{1+\eta^2} - (1+\zeta^{-2})^{\tilde p/2}\frac{\zeta\sqrt{1+\zeta^2}}{1+\zeta^2(1+\eta^2)}\Big| \\ &\le c\Big\{\frac{1}{1+\eta^2}\Big|1-(1+\zeta^{-2})^{\tilde p/2}\frac{\zeta\sqrt{1+\zeta^2}}{(1+\zeta^2)}\Big| \\ &\quad + (1+\zeta^{-2})^{\tilde p/2}\zeta\sqrt{1+\zeta^2}\,\Big|\frac{1}{(1+\eta^2)(1+\zeta^2)} - \frac{1}{1+\zeta^2(1+\eta^2)}\Big|\Big\} \\ &\le \Big\{\frac{1}{1+\eta^2}\big|1-(1+\zeta^{-2})^{(\tilde p-1)/2}\big| + (1+\zeta^{-2})^{(\tilde p-1)/2}\frac{1}{\zeta^2(1+\eta^2)}\Big\} \le c\zeta^{-2}(1+\eta^2)^{-2}.\end{aligned} \tag{A25}$$

Here we used the inequalities

$$\eta^2(1+\eta^2)^{-1} < 1 \quad \text{for all } \eta,$$

$$\left.\begin{array}{l}(1+\zeta^2(1+\eta^2))^{-1} \le \zeta^2(1+\eta^2)^{-1},\\ (1+\zeta^{-2})^{(\tilde p-1)/2} \le\ c,\\ |1-(1+\zeta^{-2})^{(\tilde p-1)/2}| \le c\zeta^{-2},\end{array}\right\} \text{ for } \zeta 1.$$

Inequality (A8) with $\zeta \ge 1$ follows now from (A24) and (A25).

For $\zeta \le 1$, we use the representation (A15) and observe that $\zeta^2(1+\zeta^{-2})^{\tilde p/2} \le c$ for $\zeta \le 1$ as long as $\tilde p \le 2$.
Step 2: $q = 1,2,3$, $j = 4$. In this case we have only to confirm (A7) at $p = 0$ and $p = 1$, and we may use the same arguments as in Step 1. Recall that $W^q_4 = (1+\zeta^2)^{1/2}\mathbf{w}^q_4$, for $\zeta \to \infty$, the factor $(1+\zeta^2)^{1/2}$ is cancelled by the difference

$e^{(-y)} - (1+\zeta^{-2})^{-1/2}\, e^{(-\sqrt{1+\zeta^{-2}}\, y)}$. For $\zeta \to 0$, we use again the representation $W_4^q(\zeta, y) = \tilde{G}_1(\zeta) e^{(-y)} + \tilde{G}_2(\zeta)\zeta\, e^{(-\sqrt{1+\zeta^{-2}}\, y)}$ and formula (A16).

Step 3: $q = 4$. Our last task to verify the assumptions of Theorem 4.3 is to prove the estimate (A8). For $j = 1, 2, 3$ we have $\tau_j = 1$. Since $(1+\zeta^{-2})^{\bar{\varkappa}-3/2} \geq (1+\zeta^{-2})^{\varkappa-3/2}$ as long as $\bar{\varkappa} \geq \varkappa$ in this cases inequality (A8) holds true if we show

$$\|W_j^4; H^{\varkappa+1}\|^2 \leq c\,(1+\zeta^{-2})^{-2}, \quad \varkappa \in (-1/2, 0), \tag{A26}$$

$$\|\partial_y^1 W_j^4; H^{\varkappa}\|^2 \leq c\,(1+\zeta^{-2})^{-3/2}, \quad \varkappa \in (0, 1), \tag{A27}$$

$$\|\partial_y^2 W_j^4; H^{\varkappa-1}\|^2 \leq c\,(1+\zeta^{-2})^{-1/2}, \quad \varkappa \in (1, 3/2). \tag{A28}$$

For $\zeta > 1$, this means again that $\|\mathfrak{W}_j^{p,4}, L_\delta^2\|$ remains bounded for $p = 0, 1, 2$ and $\delta \in [0, 1)$, which follows with the same arguments as in Step 1. For $\zeta \leq 1$, we observe that the representation of $\mathbf{w}_j^4$ induces the formula

$$\mathfrak{W}_j^{p,4}(\eta, \zeta) = \zeta^2 \Big(G_1(\zeta)\mathfrak{F}_1(\zeta) + G_2(\zeta)\mathfrak{F}_2(\zeta) + G_3(\zeta)\,(1+\zeta^{-2})^{p/2}\,\mathfrak{F}_3(\eta, \zeta)\Big), \tag{A29}$$

where $G_i(\zeta)$ remains bounded as $\zeta \to 0$. Therefore the leading term on the right-hand side is $\zeta^2\, G_3(\zeta)\,(1+\zeta^{-2})^{p/2}\,\mathfrak{F}_3(\eta, \zeta)$ and (A26)–(A28) are proved if we show that, for any $\delta \in (0, 1)$,

$$\|\mathfrak{F}_3(\zeta); L_\delta^2\|^2 \leq c\zeta \quad \text{uniformly in } \zeta \leq 1. \tag{A30}$$

For this purpose, we use the inequalities

$$\begin{aligned} &\|\mathfrak{F}_3(\zeta); L_\delta^2\|^2 = \int_{\mathbb{R}} |\eta|^{2\delta} |\mathfrak{F}_3(\eta, \zeta)|^2 d\eta \qquad \text{(A31)}\\ &= \int_{|\eta|<1} |\eta|^{2\delta} |\mathfrak{F}_3(\eta, \zeta)|^2 d\eta + \int_{|\eta|\geq 1} |\eta|^{2\delta} |\mathfrak{F}_3(\eta, \zeta)|^2 d\eta \\ &\leq c\,\Big(\int_{|\eta|<1} d\eta + \int_{|\eta|\geq 1} \eta^2 |\mathfrak{F}_3(\eta, \zeta)|^2 d\eta \Big) \leq c\,\Big(\zeta + 2\int_0^\infty \frac{\eta^2}{(1+\zeta^{-2}+\eta^2)^2} d\eta \Big) \\ &= c\,\Big(\zeta + \frac{\pi}{\sqrt{1+\zeta^{-2}}} \Big) \leq c\zeta, \end{aligned}$$

which is nothing but (A30). For $j = 4$, we have $\tau_j = 0$; thus, to obtain (A8) it is sufficient to have

$$\|W_4^4; H^{\varkappa}\|^2 \leq c\,(1+\zeta^{-2})^{-3/2}, \quad \varkappa \in (0, 1), \tag{A32}$$

$$\|\partial_y^1 W_j^4; H^{\varkappa-1}\|^2 \leq c\,(1+\zeta^{-2})^{-1/2}, \quad \varkappa \in (1, 3/2). \tag{A33}$$

This follows for $\zeta \geq 1$ by the arguments of Step 1 and for $\zeta < 1$ from the representation

$$\mathfrak{W}_4^{p,4}(\eta, \zeta) = \zeta\, \Big(\tilde{G}_1(\zeta)\mathfrak{F}_1(\zeta) + (1+\zeta^{-2})^{p/2}\tilde{G}_2(\zeta)\,\mathfrak{F}_3(\eta, \zeta) \Big)$$

by means of (A31), which finishes the proof of the lemma. □

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