Differential and Integral Equations
Volume 17, Numbers 11-12, Pages 1347–1358
November/December 2004

STRONG CONVERGENCE OF THE DIRECTIONAL DERIVATIVE OF THE DECREASING REARRANGEMENT MAPPING AND RELATED QUESTIONS

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Abstract. We make precise a class of functions for which we have the strong convergence of the directional derivative of the rearrangement mapping. We apply this result to give a new approximation of the plasma physics equations in a Stellerator.

Introduction

The first result on the directional derivative of the decreasing rearrangement mapping $u \to u_*$ appears in the well-known paper of Mossino-Temam [13].

Their results imply in particular if (for instance) Ω is a bounded, open set of $\mathbb{R}^N$, $u \in L^1(\Omega)$, $v \in L^p(\Omega)$, and $1 \leqslant p \leqslant +\infty$, then the differential quotient denoted by $\frac{dw_\lambda}{ds} = \frac{(u+\lambda v)_* - u_*}{\lambda}$, $\lambda > 0$ converges weakly if $1 \leqslant p < +\infty$ and weakly-* for $p = +\infty$ to a function $v_{*u} \in L^p\big(0, \text{meas }(\Omega)\big)$ called the relative rearrangement of v with respect to u (see [13], [12], [2], [19], and [20] for instance). As pointed out in [13] and [23], the main motivation of the study of such a directional derivative is the so-called Grad-Shafranov equation (still unsolved) appearing in the frame of plasma physics equations say: Find a function u satisfying

$$-\Delta u(x) + \nu \frac{d^2 u_*}{ds^2}\big(\,|u > u(x)|\,\big) = f(x),\ x \in \Omega,$$

Accepted for publication: April 2004.
AMS Subject Classifications: 46E30, 46E35, 46B70, 26D07, 35J60.

where $|u > t| = \text{meas}\{x \in \Omega : u(x) > t\}$. R. Temam in [23] associated to that problem (when the boundary value is the Dirichlet one) the following optimization problem: Let $K = \{v \in H_0^1(\Omega);\ v_* \in H^1(\Omega_*)\}$, with $\Omega_* = (0, \text{meas }(\Omega) = |\Omega|)$ and

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \nu\int_{\Omega_*} |v_*'(\sigma)|^2\,d\sigma - \int_\Omega f(x)v(x)dx.$$

For $\nu > 0$ and $f \in L^2(\Omega)$, there is at least a function $u \in K : J(u) = \text{Min}\,\{J(v),\ v \in K\}$. Since the regularity of v_* is now well-known (see [22] and [21]), many variants of Temam's functional can be considered. For instance, one can take $K = H_0^1(\Omega)$ and consider the weighted functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \nu\int_\Omega |v_*'(\sigma)|^2 \min\left(\sigma, |\Omega| - \sigma\right)^{2-\frac{2}{N}} d\sigma - \int_\Omega fv\,dx.$$

In all cases, the quantity

$$\frac{d}{ds}\frac{(u+\lambda v)_* - u_*}{\lambda} = \frac{d^2 w_\lambda}{ds^2} \tag{0.1}$$

appears when we look at the directional derivative. And if we want to pass to the limit (in a precise sense), we need to know first if the above quantity remains bounded in some Lebesgue space (for instance).

For this reason, we first ask a weaker question than this boundedness of $\frac{d^2 w_\lambda}{ds^2}$, that is,

Questions: *Does the quotient w_λ' converge strongly to v_{*u}? If not, find the class of functions for which it converges and give, if possible, some examples for which it diverges.*

Some answers: The strong convergence in $L^p(\Omega_*)$, $1 \leqslant p < +\infty$, occurs in the following cases:

a) u is a simple function (that is, there is a finite sequence of pairwise-distinct numbers $(a_j)_{j=1,\ldots,m}$ associated to a sequence of measurable sets $(E_j)_{j=1,\ldots,m}$, pairwise disjoint, such that $\Omega = \bigcup_j E_j$ and $u = \sum_{j=1}^m a_j\chi_{E_j}$, χ_{E_j} denoting the characteristic function of the set E_j), $v \in L^p(\Omega)$.

b) $v \in L^p(\Omega)$ and $u \in W^{1,1}(\Omega)$ with $\text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$.

So our purpose is first to prove the above statements and then derive few consequences of them.

1. Notation and preliminary results

We shall denote by Ω an open set of $\mathbb{R}^N$ (bounded or unbounded).

If $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$, then

$$(x, y) = x \cdot y = \sum_{i=1}^{N} x_i y_i$$

is the Euclidean product and $|x| = (x, x)^{\frac{1}{2}}$ the associated norm. For $1 \leqslant p \leqslant +\infty$, $L^p(\Omega)$ is the usual Lebesgue space endowed with the usual norm, denoted by $|\cdot|_p$. $L^0(\Omega)$ will denote the set of all measurable functions. The usual Sobolev space $W^{1,p}(\Omega)$ is endowed with the following norm: $|u|_{1,p} = |u|_p + |\nabla u|_p$.

For a measurable set E of $\mathbb{R}^N$, we shall denote by $|E|$ its Lebesgue measure, and if $u : \Omega \to \mathbb{R}$ is a measurable function, then

$$\{u > t\} = \{x \in \Omega : u(x) > t\} \text{ and } |u > t| = |\{u > t\}|,$$

$$\{u = t\} = \{x \in \Omega : u(x) = t\}.$$

A plateau of value t is the set $\{u = t\}$ satisfying $|u = t| > 0$. If $u \in L^p(\Omega)$, we set $\mathrm{P(u)} = \bigcup_{\mathrm{t \in D}} \{\mathrm{u = t}\}$ to be the plateau of u (D is at most countable if Ω is bounded or $1 \leqslant p < +\infty$ for unbounded domain Ω). We will say that u has no flat zone if $|P(u)| = 0$.

Definition 1. *Let u be in $L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$ ($u \geqslant 0$ if Ω is unbounded). The distribution function associated to u is the real function*

$$m_u : t \to |\{u > t\}| = |u > t|.$$

Definition 2 (monotone rearrangement)**.** *We define on $[0, |\Omega|]$ the function u_* by setting*

$$u_*(s) = Inf\{t \in \mathbb{R},\ |u > t| \leqslant s\},\ s \in \Omega_*$$

and $u_*(0) = \operatorname*{ess\,sup}_{\Omega} u$, $u_*(|\Omega|) = \operatorname*{ess\,inf}_{\Omega} u$. *The function u_* is called the decreasing rearrangement of u.*

Let $v \in L^p(\Omega)$ and $u \in L^r(\Omega)$, $1 \leqslant p < +\infty$ and $1 \leqslant r < +\infty$. If Ω is unbounded we assume that $u \geqslant 0$, and the restriction of v to $\{u = 0\}$ is nonnegative. Furthermore, if v (respectively u) satisfies the condition $|v > t| < +\infty$ (respectively $|u > t| < +\infty$) $\forall t > 0$, then p (respectively r) can be infinite. For Ω bounded, p or r can be infinite.

Consider the function $w : \overline{\Omega}_* \to \mathbb{R}$, defined by

$$w(s) = \int_{\{u > u_*(s)\}} v(x)dx + \int_0^{s - |u > u_*(s)|} \left(v|_{\{u = u_*(s)\}}\right)_*(\sigma) d\sigma,$$

where $v|_{\{u=u_*(s)\}}$ is the restriction of v to $\{u = u_*(s)\}$. The following result summarizes all those obtained in [13], [12], [6], and [20]; see also [11].

Property 1. *Let Ω be a measurable set in $\mathbb{R}^N$.*

(a) *If Ω is bounded, then one has* $\dfrac{(u+\lambda v)_* - u_*}{\lambda} \underset{\lambda\to 0}{\rightharpoonup} \dfrac{dw}{ds}$ *in $L^p(\Omega_*)$-weak if $1 \leqslant p < +\infty$ and in $L^\infty(\Omega_*)$-weak-star if $p = +\infty$.*

(b) *If Ω is unbounded, then one has*
  i) $w \in W^{1,p}_{loc}([0, +\infty))$,
  ii) $\dfrac{dw}{ds} \in L^p(0, +\infty)$,
  iii) $\dfrac{(u+\lambda v)_{+*} - u_*}{\lambda} \underset{\lambda\to 0}{\rightharpoonup} \dfrac{dw}{ds}$ *in $L^p(0, +\infty)$-weak if $1 < p < +\infty$ (weak-star for $p = +\infty$) and in $L^1(0, M)$-weak, $\forall M$ finite.*

In any case, $\left|\dfrac{dw}{ds}\right|_{L^p(\Omega_*)} \leqslant |v|_{L^p(\Omega)}$.

Definition of relative rearrangement. The function $\frac{dw}{ds}$ is called the relative rearrangement of v with respect to u and is denoted by $v_{*u} = \frac{dw}{ds}$.

Property 2. *The map $v \in L^p(\Omega) \to v_{*u} \in L^p(\Omega_*)$ is a contraction, for $v \in L^1(\Omega)$, $\int_\Omega v\,dx = \int_{\Omega_*} v_{*u} d\sigma$, whenever u has the same regularity as in Property* 1.

Definition 3 (Co-area-regular function)**.** *Let u be in $W^{1,1}_{loc}(\Omega)$. The function u is said to be co-area regular if the distribution derivative of the decreasing function*

$$m_{0,u}(t) = \left|\left\{x \in \Omega : u(x) > t,\ \nabla u(x) = 0\right\}\right|$$

is singular with respect to Lebesgue measure on $\mathbb{R}$.

Here are some useful examples of co-area-regular functions whose proofs can be found in [1], [4], [17], and [8].

Theorem 1.

(1) *If $u \in W^{N,p}_{loc}(\Omega)$ for some $p > 1$, then u is a co-area-regular function.*
(2) *If the set $\{x : \nabla u(x) = 0\}$ is of measure zero, then u is a co-area-regular function.*
(3) *If $u \in W^{1,1}_{loc}(\mathbb{R})$, then u is a co-area-regular function.*

We have shown the following theorem in [15] and [14]:

Theorem 2. *Let $u \in W^{1,p}(\Omega)$, and let u_n be a sequence of elements of $W^{1,p}(\Omega)$, $1 \leqslant p < +\infty$, such that $u_n \underset{n}{\to} u$ in $W^{1,p}(\Omega)$-strong. We assume* *that* Ω *is a bounded, connected, open set with Lipschitz boundary if the boundary trace* $\gamma_0 u_n \not\equiv 0$ *or* $\gamma_0 u \not\equiv 0$*, and* Ω *is an open arbitrary set otherwise (that is, if* $\gamma_0 u = \gamma_0 u_n = 0\ \forall n$*). Assume that* u *is a co-area-regular function and let be* $b \in W_c^{1,1}(\Omega) \cap L_+^\infty(\Omega)$*. Then*

(1) *there exists a subsequence denoted by* $n(j)$ *such that*

$$\liminf_{j\to+\infty} \big[b \left|\nabla u_{n(j)}\right| \big]_{*u_{n(j)}}(s) \geqslant \big[b \left|\nabla u\right| \big]_{*u}(s) \ a.e \ in \ \Omega_*,$$

(2) *the sequence* $[b\,|\nabla u_n|]_{*u_n}$ *converges to* $[b\,|\nabla u|]_{*u}$ *in* $L^p(\Omega_*)$*-strong,*

(3) *the sequence* $[b\,|\nabla u|]_{*u_n}$ *converges to* $[b\,|\nabla u|]_{*u}$ *in* $L^p(\Omega_*)$*-strong,*

where $W_c^{1,1}(\Omega)$ *is the set of functions in* $W^{1,1}(\Omega)$ *with compact support.*

Then, the following corollary can be derived from Theorem 2.

Corollary 2.1. *Under the same conditions as for Theorem* 2*, if* b *is only in* $L^r(\Omega)$*,* $1 \leqslant r < +\infty$*, and* $\mathrm{meas}\,\{\mathrm{x} \in \Omega : \nabla \mathrm{u}(\mathrm{x}) = 0\} = 0$*, then*

$$b_{*u_n} \to b_{*u} \quad in \quad L^r(\Omega_*)\text{-}strong.$$

Remark. For simplicity, if Ω is unbounded we assume that all functions are nonnegative.

2. Main results on strong convergence of the directional derivative

Let us start with the cases where the function u is a simple function:

Theorem 3. *Let* u *be a simple function in* $L^r(\Omega)$*,* $1 \leqslant r < +\infty$*. Then*

$$\frac{(u+\lambda v)_* - u_*}{\lambda} \xrightarrow[\lambda\to 0]{} v_{*u} \ in \ L^p(\Omega_*)\text{-}strong, \ if \ v \in L^p(\Omega).$$

Furthermore, if v *is also a simple function, then there is a number* $\overline{\lambda}(u,v) > 0$ *such that for all* $s \in \Omega_*$ *and* $0 < |\lambda| < \overline{\lambda}(u,v)$*,*

$$\frac{(u+\lambda v)_* - u_*}{\lambda}(s) = v_{*u}(s)$$

(λ *may be negative).*

Proof. Case 1: v is a simple function. Let u be a simple function; then there is a finite sequence of pairwise-distinct numbers $(a_j)_{j=1,\dots,m}$ associated to a sequence of measurable sets $(E_j)_{j=1,\dots,m}$, pairwise disjoint, such that $\Omega = \bigcup_j E_j$ and $u = \sum_{j=1}^m a_j \chi_{E_j}$, χ_{E_j} denoting the characteristic function of the set E_j. Then, the decreasing rearrangement of u is given by the following: for $s \in \Omega_*$,

$$u_*(s) = \sum_{j=1}^{m} a_j^* \chi_{[b_{j-1}, b_j[}(s), \ \textit{with } a_j^* = a_{\sigma(j)},$$

$\sigma : \{1, \dots m\} \to \{1, \dots, m\}$ a decreasing arrangement; that is, $a_{\sigma(1)} > a_{\sigma(2)} > \dots > a_{\sigma(m)}$, $b_0 = 0$, and $b_j = \sum_{k=1}^{j} \left|E_{\sigma(k)}\right|$.

If $v = 0$, the theorem is true.

If $v \neq 0$, then for all $0 < |\lambda| < \frac{\inf_{(j,k), j\neq k} |a_j - a_k|}{2|v|_\infty} = \overline{\lambda}(u, v)$, we have

$$(u + \lambda v)_* = \sum_{j=1}^{m} \delta_j^* \chi_{[b_{j-1}, b_j)}$$

with $\delta_j^* = a_{\sigma(j)} + \lambda v|_{E_{\sigma(j)}}$, $v|_{E_{\sigma(j)}}$ being the constant function restriction of v on $E_{\sigma(j)}$. Indeed, since v is a simple function, then

$$u + \lambda v = \sum_{j=1}^{m} (u + \lambda v|_{E_j}) \chi_{E_j} \doteq \sum_{j=1}^{m} \delta_j \chi_{E_j}.$$

But the choice of λ implies that

$$\delta_{\sigma(j)} = a_{\sigma(j)} + \lambda v|_{E_{\sigma(j)}} < a_{\sigma(j-1)} + \lambda v|_{E_{\sigma(j-1)}} = \delta_{\sigma(j-1)},$$

from which we have statement ii),

$$\frac{(u + \lambda v)_*(s) - u_*(s)}{\lambda} = \sum_{j=1}^{m} v|_{E_{\sigma(j)}} \chi_{[b_{j-1}, b_j[}(s) = v_{*u}(s), \tag{2.1}$$

which gives the result. Notice that in this case p can be infinite.

Case 2: $\mathbf{v \in L^p(\Omega)}$. There exists a sequence v_n of simple functions such that $|v_n - v|_p \xrightarrow[n \to +\infty]{} 0$. By the contraction properties of the mappings $u \to u_*$ and $v \to v_{*u}$, we deduce that $\forall \lambda > 0$

$$\left| \frac{(u + \lambda v)_* - u_*}{\lambda} - v_{*u} \right|_p \leqslant 2 \left|v - v_n\right|_p + \left| \frac{(u + \lambda v_n)_* - u_*}{\lambda} - (v_n)_{*u} \right|_p \tag{2.2}$$

from which we derive the statement i). □

Remark 1. Under the condition of Theorem 3 we recover the expression of v_{*u} when u is a simple function; i.e.,

$$v_{*u}(s) = \left(v|_{E_{\sigma(j)}}\right)_* (s - b_j), \ b_j = \sum_{k=1}^{j} \left|E_{\sigma(k)}\right|.$$

Such an expression can also be derived from Property 1 and was used in [16].

The second theorem concerns the case of a regular function u with meas$\{$x : ∇u(x) $= 0\} = 0$.

Theorem 4. *Let u be in $W^{1,r}(\Omega)$ with* meas$\big\{$x $\in \Omega : \nabla$u(x) $= 0\big\} = 0$, *$1 \leqslant r < +\infty$. We assume that Ω is a bounded, connected, open set with Lipschitz boundary if the boundary trace $\gamma_0 u \not\equiv 0$ and Ω is an open arbitrary set if $\gamma_0 u \equiv 0$. Then, $\forall v \in L^p(\Omega)$, $1 \leqslant p < +\infty$, we have*

$$\left|\frac{(u+\lambda v)_* - u_*}{\lambda} - v_{*u}\right|_p \xrightarrow[\lambda \to 0]{} 0$$

(λ *may be negative*).

The case $r = +\infty$ can be included when Ω is unbounded under the condition given in Property 1; i.e., $|u > t| < +\infty$ $\forall t > 0$, u being nonnegative.

The key lemma for the proof of Theorem 4 is the following:

Lemma 1 (Integral lemma)**.** *Let u be in $L^r(\Omega)$, $1 \leqslant r < +\infty$, and $v \in L^\infty(\Omega)$ with a compact support if Ω is unbounded. Then, for almost every $s \in \Omega_*$, for all $\lambda \in \mathbb{R}$,*

$$(u+\lambda v)_*(s) - u_*(s) = \int_0^\lambda v_{*(u+tv)}(s)dt. \tag{2.3}$$

Proof. For $(t,s) \in \mathbb{R} \times \Omega_*$, we define $G(t,s) = (u+tv)_*(s)$. For all $s \in [0,|\Omega|)$, the map $t \in \mathbb{R} \to G(t,s)$ is a Lipschitzian function. Indeed, by the contraction property, we have for all t and t' in $\mathbb{R}$

$$\left|G(t,s) - G(t',s)\right| \leqslant |v|_\infty \left|t - t'\right|, \ \forall s. \tag{2.4}$$

In particular, we deduce $\forall \lambda \in \mathbb{R}, \ \forall s \in [0,|\Omega|)$,

$$(u+\lambda v)_*(s) - u_*(s) = \int_0^\lambda \frac{\partial G}{\partial t}(t,s)dt. \tag{2.5}$$

To compute $\frac{\partial G}{\partial t}$, we set

$$H(t,s) = \int_0^s (u+tv)_*(\sigma)d\sigma.$$

Then $H \in W^{1,1}_{loc}(\mathbb{R} \times \Omega_*)$.

Indeed, by the contraction property for all $s \in [0,|\Omega|)$, the map

$$t \to \int_0^s (u+tv)_*(\sigma)d\sigma$$

is Lipschitzian and

$$\left|H(t,s) - H(t',s)\right| \leqslant |v|_\infty \left|t - t'\right| s.$$

Clearly, the map

$$s \to \int_0^s (u+tv)_*(\sigma)d\sigma$$

is absolutely continuous and

$$\frac{\partial H}{\partial s} = (u+tv)_* \in L^r(\Omega_*).$$

On the other hand, by the Deny-Lions theorem (see [10]), we deduce in particular that

$$\frac{\partial H}{\partial t}(t,s) = \lim_{\delta t \to 0 \ \delta t > 0} \frac{H(t+\delta t, s) - H(t,s)}{\delta t}$$

by definition

$$= \lim_{\delta t \to 0 \ \delta t > 0} \int_0^s \frac{(u+tv+(\delta t)v)_*(\sigma) - (u+tv)_*(\sigma)}{\delta t} d\sigma.$$

From Property 1, on the directional derivative, this last limit is equal to

$$\int_0^s v_{*(u+tv)}(\sigma)d\sigma \doteq \frac{\partial H}{\partial t}(t,s).$$

We then deduce that in the sense of distribution,

$$\frac{\partial^2 H}{\partial s \partial t} = v_{*(u+tv)}.$$

But

$$\frac{\partial^2 H}{\partial s \partial t} = \frac{\partial^2 H}{\partial t \partial s} = \frac{\partial G}{\partial t}$$

in $\mathcal{D}'(\mathbb{R} \times \Omega_*)$; thus,

$$\frac{\partial G}{\partial t} = v_{*(u+tv)},$$

and with relation (2.5) we get the result. □

Proof of Theorem 4. We begin with the case $v \in C_c^\infty(\Omega)$. Since u satisfies meas $\{\mathrm{x} : \nabla \mathrm{u}(\mathrm{x}) = 0\} = 0$, then from Corollary 2.1 we have

$$v_{*(u+tv)} \xrightarrow[t \to 0]{} v_{*u} \quad in\ L^p(\Omega_*). \tag{2.6}$$

Thus, we deduce from this last relation and Lebesgue's dominated convergence theorem

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \int_0^\lambda \left| v_{*(u+tv)} - v_{*u} \right| dt = 0.$$

From the above integral lemma, we then have

$$\lim_{\lambda \to 0} \left| \frac{(u+\lambda v)_* - u_*}{\lambda} - v_{*u} \right|_p = 0. \tag{2.7}$$

Case where $v \in L^p(\Omega)$**.** Then, we consider a sequence of $v_n \in C_c^\infty(\Omega)$ such that $|v_n - v|_p \xrightarrow[n\to+\infty]{} 0$. From the contraction property of the monotone and relative rearrangement, one has

$$\Big|\frac{(u+\lambda v)_* - u_*}{\lambda} - v_{*u}\Big|_p \leqslant 2\,|v - v_n|_p + \Big|\frac{(u+\lambda v_n)_* - u_*}{\lambda} - (v_n)_{*u}\Big|_p. \quad (2.8)$$

From (2.7) and (2.8), we get

$$\limsup_{\lambda\to 0}\Big|\frac{(u+\lambda v)_* - u_*}{\lambda} - v_{*u}\Big|_p \leqslant 2|v - v_n|_p \xrightarrow[n\to+\infty]{} 0,$$

which gives the result. □

Other convergence results can be found in [15]. Next, in order to give a new approximation of the plasma physics equation studied in [8] (see also [18]), we need the following corollary.

Corollary 4.1 (New approximation)**.** *Under the same conditions as for Theorem* 4*, the function defined by*

$$Q_\lambda(u,v)(x) = \frac{(u+\lambda v)_*(|u > u(x)|) - u(x)}{\lambda}, \quad x \in \Omega,$$

satisfies

i) $|Q_\lambda(u,v)|_p \leqslant |v|_p \qquad \forall \lambda \in \mathbb{R}$,
ii) $Q_\lambda(u,v) \xrightarrow[\lambda\to 0]{} v_{*u}\big(\,|u > u(\cdot)|\,\big)$ *in* $L^p(\Omega)$*-strong,*
iii) *If* $v \in L^\infty(\Omega)$ *and* $\lambda > 0$ *then* $\operatorname{ess\,inf}_{\Omega}\ v \leqslant Q_\lambda(u,v) \leqslant \operatorname{ess\,sup}_{\Omega}\ v$.

Proof. Since $\text{meas}\{x : \nabla u(x) = 0\} = 0$, u has no flat zone. By equimeasurability and the contraction property of the monotone rearrangement, we then have

$$|Q_\lambda(u,v)|_{L^p(\Omega)} = \Big|\frac{(u+\lambda v)_* - u_*}{\lambda}\Big|_p \leqslant |v|_{L^p(\Omega)}\,.$$

Again by equimeasurability and Theorem 4, we have

$$\big|Q_\lambda(u,v) - v_{*u}\big(\,|u > u(x)|\,\big)\big|_p = \Big|\frac{(u+\lambda v)_* - u_*}{\lambda} - v_{*u}\Big|_p \xrightarrow[\lambda\to 0]{} 0.$$

iii) Since the mapping $v \to v_*$ is increasing, for all $s \in \overline{\Omega_*}$

$$u_*(s) + \lambda \operatorname{ess\,inf}_{\Omega}\ v \leqslant (u+\lambda v)_*(s) \leqslant u_*(s) + \lambda \operatorname{ess\,sup}_{\Omega}\ v,$$

from which we get the result. □

3. New approximation of the plasma physics equations in a Stellerator configuration

The following problem is relevant from the viewpoint of a confined plasma in a Stellerator machine, and was introduced in [5] and solved in [8] and [7] (see [3] for its numerical approximation and [9] for the study of the free boundary).

Let Ω be a connected, smooth, open, bounded set of $\mathbb{R}^2$, $\gamma \in (-\infty, 0)$, $\lambda > 0$, $b \in L^\infty_+(\Omega)$, and $a \in L^\infty(\Omega)$.

$(\mathcal{P})$: Find u such that $-\Delta u = F(u)$ and $u-\gamma \in H^1_0(\Omega) \cap W^{2,p}(\Omega)\ \forall p < +\infty$.

Here, F is given by $F(u)(x) = aG(u)(x) + J(u)(x),\ x \in \Omega$, with

$$G(u)(x) = \Big[F_v^2 - 2 \int_{|u>0|}^{|u>u_+(x)|} \Big[p(u_*)\Big]' b_{*u}(\sigma)\ d\sigma\Big]_+^{\frac{1}{2}}$$

$$J(u)(x) = p'\big(u(x)\big)\Big[b(x) - b_{*u}\big(\,|u > u(x)|\,\big)\Big]$$

with $p(t) = \lambda \frac{t_+^2}{2}$, $t_+ = \max(t, 0)$, and F_v a given constant > 0.

Let $\varepsilon > 0$, and consider $Q_\varepsilon(u, b)(x) = \frac{(u+\varepsilon b)_*\big(|u>u(x)|\big) - u(x)}{\varepsilon}$, $x \in \Omega$,

$$J_\varepsilon(u)(x) = p'(u)(x)\big[b(x) - Q_\varepsilon(u, b)(x)\big].$$

Let us define the following new approximation of the problem $(\mathcal{P})$.

$$(\mathcal{P})_\varepsilon \qquad \begin{cases} \text{Find } u^\varepsilon \text{ such that} \\ -\Delta u^\varepsilon = aG(u^\varepsilon)(x) + J_\varepsilon(u^\varepsilon)(x),\ x \in \Omega, \\ u^\varepsilon - \gamma \in H^1_0(\Omega) \cap W^{2,p}(\Omega)\ \forall p < +\infty. \end{cases}$$

The interest resides not only in the fact that $Q_\varepsilon(u, b) \xrightarrow[\varepsilon \to 0]{} b_{*u}\big(\,|u > u(x)|\,\big)$ as stipulated in Corollary 4.1, but also in fact that the approximation is stable in the following sense:

Proposition 1 (stability)**.** *Let u be as in Theorem 4, and $u^\varepsilon \in W^{1,r}(\Omega)$, $1 \leqslant r < +\infty$, such that $u^\varepsilon \xrightarrow[\varepsilon \to 0]{} u$ in $W^{1,r}(\Omega)$-strong. If $b^\varepsilon \xrightarrow[\varepsilon \to 0]{} b$ in $L^p(\Omega)$-strong, $1 \leqslant p < +\infty$, then*

$$Q_\varepsilon(u^\varepsilon, b^\varepsilon) \xrightarrow[\varepsilon \to 0]{} b_{*u}\big(\,|u > u(x)|\,\big) \text{ in } L^p(\Omega)\text{-strong.}$$

Proof. It follows the same scheme as for the corollary so, we sketch it only. First, we notice that

$$|Q_\varepsilon(u^\varepsilon, b^\varepsilon) - Q_\varepsilon(u^\varepsilon, b)|_p \leqslant |b^\varepsilon - b|_p\,. \tag{3.1}$$

Since

$$\left|Q_\varepsilon(u^\varepsilon,b)-b_{*u}\big(|u>u(\cdot)|\big)\right|_p=\left|\frac{(u^\varepsilon+\varepsilon b)_*-u^\varepsilon_*}{\varepsilon}-b_{*u}\right|_{L^p(\Omega_*)},\tag{3.2}$$

so the result follows if we show that

$$\left|\frac{(u^\varepsilon+\varepsilon b)_*-u^\varepsilon_*}{\varepsilon}-b_{*u}\right|_{L^p(\Omega_*)}\xrightarrow[\varepsilon\to0]{}0.\tag{3.3}$$

But if $b\in C_c^\infty(\Omega)$ the result follows from the integral formula lemma since $u^\varepsilon+\varepsilon b\xrightarrow[\varepsilon\to0]{}u$ in $W^{1,r}(\Omega)$-strong, and by density (3.3) follows $\forall b\in L^p(\Omega)$. □

Next, the proof of the existence solution for $(\mathcal{P})$ and $(\mathcal{P}_\varepsilon)$ follows essentially the scheme given in [18], using a Galerkin process, so we announce the results without detailing them.

Theorem 5. *Let* $M_\infty=\dfrac{|a|_\infty F_v|\Omega|}{4\pi}\Big[1+\dfrac{2\lambda_1}{\lambda_1-\lambda\operatorname{osc}_\Omega b}\Big]$, $\dfrac{\lambda|b|\cdot M_\infty^2}{F_v}=\nu$, $d=\operatorname{ess\,inf}_\Omega a^2>0$, *and* $\lambda\operatorname{osc}_\Omega b<\lambda_1$. *Then if* $\nu<1$ *and* $\lambda|b|_\infty<\frac{d(1-\nu)}{4\nu}$, *there exist two functions* u^ε *and* u *in* $H_0^1(\Omega)\cap W^{2,p}(\Omega)$, $\forall p$ *finite, such that* u^ε *solves* $(\mathcal{P}_\varepsilon)$ *and* u *solves* $(\mathcal{P})$. *Furthermore,*

(1) $\operatorname{meas}\{x:\nabla u^\varepsilon(x)=0\}=\operatorname{meas}\{x:\nabla u(x)=0\}=0$,
(2) $u^\varepsilon\xrightarrow[\varepsilon\to0]{}u$ *in* $C^1(\overline{\Omega})$*-strong,*
(3) $|u^\varepsilon-\gamma|_\infty\leqslant M_\infty$, $\forall\varepsilon>0$.

Other approximations might be done; for instance, one may replace in $G(u)$ the term $b_{*u}(s)$ by $\frac{(u+\varepsilon b)_*(s)-u_*(s)}{\varepsilon}$.

Acknowledgments. This research of the second author has been partially supported by "Programa de Recursos Humanos do Plan Galego de Investigaciòne Desenvolvemento Tecnolòxico."

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