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ON THE DIRAC-KLEIN-GORDON EQUATIONS IN ONE SPACE DIMENSION

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Abstract. We establish local and global existence results for Dirac-Klein-Gordon equations in one space dimension, employing a null-form estimate and a fixed-point argument.

0. Introduction and main results

In the present work, we would like to study the Cauchy problem for the Dirac-Klein-Gordon equations. The unknown quantities are a spinor field $\psi : \mathbb{R} \times \mathbb{R}^1 \mapsto \mathbb{C}^4$ and a scalar field $\phi : \mathbb{R} \times \mathbb{R}^1 \mapsto \mathbb{R}$. The evolution equations for the fields are given below:

$$\mathcal{D}\psi = \phi\psi; \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^1 \tag{0.1a}$$

$$\Box\phi = \overline{\psi}\psi; \tag{0.1b}$$

$$\psi(0,x) := \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x), \tag{0.1c}$$

where $\mathcal{D}$ is the Dirac operator, $\mathcal{D} := -i\gamma^\mu\partial_\mu$, $\mu = 0, 1$, and γ^μ are the Dirac matrices, the wave operator $\Box = -\partial_{tt} + \partial_{xx}$, and $\overline{\psi} = \psi^\dagger\gamma^0$, and $\dagger$ is the complex conjugate transpose.

The purpose of this work is to demonstrate the usefulness of a null-form estimate, by employing the solution representations in Fourier transform of the DKG equations. We will take full advantage of the null-form structure depicted in the nonlinear term $\overline{\psi}\psi$, which has been observed as possessing such a structure; see [11] and [3].

For the DKG system, there are many conserved quantities which are not positive definite, such as the energy. Therefore they are not applicable to deriving a priori estimates. However, the known positive conserved quantity

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is the law of conservation of charge,

$$\int |\psi(t)|^2 \, dx \; = \text{constant}, \tag{0.2}$$

which leads to the global existence result, once the local existence result is established; see [3] and [7].

In 1973, Chadam showed that the Cauchy problem for the DKG equations has a global unique solution for $\psi_0 \in H^1$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$; see [4]. In 1993, Zheng proved that there exists a global weak solution to the Cauchy problem of modified DKG equations, based on the technique of compensated compactness, with $\psi_0 \in L^2$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$; see [14]. In 2000, Bournaveas derived a new proof of a global existence for the DKG equations, based on a null-form estimate, if $\psi_0 \in L^2$, $\phi_0 \in H^1$, and $\phi_1 \in L^2$; see [1]. In 2002, Fang gave a direct proof for (0.1), based on a variant null-form estimate, which is more straightforward, and the result is parallel to Bournaveas'; see [7].

The outline of this paper is as follows. First we derive some solution representations in Fourier transform. Next we prove some a priori estimates of solutions for the Dirac equation and for the wave equation. Then we show a local result for (0.1), employing the null-form estimate together with other estimates derived previously, and a fixed-point argument. Finally we show the key estimate, namely the null-form estimate.

The main result in this work is as follows.

Theorem 0.1 (Local Existence). *Let $0 < \epsilon \le \frac{1}{4}$ and $0 < \delta \le 2\epsilon$. If the initial data of* (0.1) *is $\psi_0 \in H^{-\frac{1}{4}+\epsilon}$, $\phi_0 \in H^{\frac{1}{2}+\delta}$, and $\phi_1 \in H^{-\frac{1}{2}+\delta}$, then there is a unique local solution for* (0.1).

Theorem 0.2 (Global Existence). *Let $\delta > 0$. If the data of* (0.1) *is $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}+\delta}$, and $\phi_1 \in H^{-\frac{1}{2}+\delta}$, then there is a unique global solution for* (0.1).

Remarks. 1) The DKG equations follow from the Lagrangian

$$\int_{\mathbb{R}^{1+1}} \Big\{ |\nabla\phi|^2 - |\phi_t|^2 - \overline{\psi}\mathcal{D}\psi - \phi\overline{\psi}\psi \Big\} dx\, dt. \tag{0.3}$$

2) The Dirac-Klein-Gordon system must be

$$\begin{cases} \mathcal{D}\psi = \phi\psi; \\ \Box\phi + m^2\phi = \overline{\psi}\psi, \end{cases} \tag{0.4}$$

and the proof works for this system too.

3) $\widehat{\mathcal{D}}^2 = \widehat{\Box} I$, where I is the 4×4 identity matrix.

4) $\overline{\psi}\psi = \psi^\dagger\gamma^0\psi = |\psi_1|^2+|\psi_2|^2-|\psi_3|^2-|\psi_4|^2$, where ψ_j are the component functions of the vector function ψ, which take values in $\mathbb{C}$.

The case $\delta = 0$ is critical in the following sense. Assuming that the initial data (ϕ_0, ϕ_1) are in $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$ does not imply that $\phi(t, \cdot)$ is bounded. In fact, it is a BMO function. One of the motivations for proving the existence of a global solution with low regularity is based on an observation made by Grillakis, which is that the initial data of (0.1), $\psi_0 \in L^2$, $\phi_0 \in H^{\frac{1}{2}}$, and $\phi_1 \in H^{-\frac{1}{2}}$, is the right space for the existence of an invariant measure; see [1] and [12], which resulted from the DKG equations.

1. Solution representation

In what follows, we denote by (t, x) the time-space variables and by (τ, ξ) the dual variables with respect to the Fourier transform of a given function. We will use $\alpha = \frac{1}{4} - \epsilon$ throughout the paper. We will also often skip the constant in the inequalities. For convenience, we denote the multipliers by

$$\widehat{E}(\tau, \xi) = |\tau| + |\xi| + 1, \tag{1.1a}$$

$$\widehat{S}(\tau, \xi) = \Big||\tau| - |\xi|\Big| + 1, \tag{1.1b}$$

$$\widehat{W}(\tau, \xi) = \tau^2 - |\xi|^2, \tag{1.1c}$$

$$\widehat{D}(\tau, \xi) = \gamma^0\tau + \gamma^1\xi, \tag{1.1d}$$

$$\widehat{M}(\xi) = |\xi| + 1. \tag{1.1e}$$

Notice that $\widehat{W}$ and $\widehat{D}$ are the symbols of the wave and Dirac operators respectively.

Consider the Dirac equation,

$$\begin{cases} \mathcal{D}\psi = G, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^1, \\ \psi(0) = \psi_0. \end{cases} \tag{1.2}$$

First taking the Fourier transform on (1.2) over the space variable and solving the resulting ODE, we can formally write down the solution as follows:

$$\widetilde{\psi}(t, \xi) = \frac{e^{it|\xi|}}{2|\xi|}\widehat{D}(|\xi|, \xi)\gamma^0\widehat{\psi}_0(\xi) + \frac{e^{-it|\xi|}}{2|\xi|}\widehat{D}(|\xi|, -\xi)\gamma^0\widehat{\psi}_0(\xi) \tag{1.3}$$
$$+ \int_0^t \frac{e^{i(t-s)|\xi|}}{2|\xi|}\widehat{D}(|\xi|, \xi)i\widetilde{G}(s, \xi)\, ds + \int_0^t \frac{e^{-i(t-s)|\xi|}}{2|\xi|}\widehat{D}(|\xi|, -\xi)i\widetilde{G}(s, \xi)\, ds.$$

Rewriting the inhomogeneous terms in (1.3) gives

$$\widetilde{\psi}(t,\xi)=\Big[\frac{e^{it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,\xi)+\frac{e^{-it|\xi|}}{2|\xi|}\widehat{D}(|\xi|,-\xi)\Big]\gamma^0\widehat{\psi}_0(\xi)$$
$$+\int\Big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}(|\xi|,\xi)+\frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{D}(|\xi|,-\xi)\Big]\widehat{G}(\tau,\xi)d\tau. \qquad (1.4)$$

Now we split the function $\widehat{G}$ into several parts in the following manner. Consider $\widehat{a}(\tau)$, a cut-off function which equals 1 if $|\tau|\le\frac{1}{2}$ and equals 0 if $|\tau|\ge 1$, and denote by $h(\tau)$ the Heaviside function. For simplicity, let us write

$$\widehat{G}_\pm(\tau,\xi):=h(\pm\tau)\widehat{a}(\tau\mp|\xi|)\widehat{G}(\tau,\xi), \qquad (1.5a)$$
$$\widehat{G}_f(\tau,\xi):=\widehat{G}(\tau,\xi)-\big(\widehat{G}_+(\tau,\xi)+\widehat{G}_-(\tau,\xi)\big), \qquad (1.5b)$$
$$\widehat{D}_\pm:=\widehat{D}(|\xi|,\pm\xi). \qquad (1.5c)$$

Notice that $\widehat{G}_\pm$ are supported in the regions $\{(\tau,\xi):\pm\tau>0,|\tau\mp|\xi||\le 1\}$ respectively. Using the decomposition of the forcing term $\widehat{G}=\widehat{G}_f+\widehat{G}_++\widehat{G}_-$, the inhomogeneous term in (1.4) can be written as

$$\int\Big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}(|\xi|,\xi)+\frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{D}(|\xi|,-\xi)\Big]\widehat{G}_f(\tau,\xi)d\tau \qquad (1.6a)$$
$$=\int e^{it\tau}\frac{\widehat{D}(\tau,\xi)}{\tau^2-|\xi|^2}\widehat{G}_f\,d\tau-e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int\frac{\widehat{G}_f}{\tau-|\xi|}d\tau-e^{-it|\xi|}\frac{\widehat{D}_-}{2|\xi|}\int\frac{\widehat{G}_f}{\tau+|\xi|}d\tau,$$

$$\int\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(\tau-|\xi|)}\widehat{D}_+(\widehat{G}_++\widehat{G}_-)d\tau$$
$$=\ e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int\frac{e^{it(\tau-|\xi|)}-1}{\tau-|\xi|}(\widehat{G}_++\widehat{a}_6(\tau)\widehat{G}_-)d\tau$$
$$+\int e^{it\tau}\frac{(1-\widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}d\tau-e^{it|\xi|}\frac{\widehat{D}_+}{2|\xi|}\int\frac{(1-\widehat{a}_6(\tau))\widehat{G}_-}{\tau-|\xi|}d\tau, \qquad (1.6b)$$

$$\int\frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{D}_-(\widehat{G}_++\widehat{G}_-)d\tau$$
$$=\ e^{-it|\xi|}\frac{\widehat{D}_-}{2|\xi|}\int\frac{e^{it(\tau+|\xi|)}-1}{\tau+|\xi|}(\widehat{a}_6(\tau)\widehat{G}_++\widehat{G}_-)d\tau+$$

$$+\int e^{it\tau}\frac{(1-\widehat{a}_6(\tau))\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}d\tau - e^{-it|\xi|}\frac{\widehat{D}_-}{2|\xi|}\int\frac{(1-\widehat{a}_6(\tau))\widehat{G}_+}{\tau+|\xi|}d\tau, \quad (1.6c)$$

where $\widehat{a}_6(\tau)=\widehat{a}(\frac{\tau}{6})$ and $\widehat{a}$ is the cut-off function defined previously. Recall the power expansion

$$e^{it(\tau\pm|\xi|)}-1=\sum_{k=1}^{\infty}\frac{1}{k!}(it)^k(\tau\pm|\xi|)^k. \quad (1.7)$$

Combining (1.4)–(1.7), we can give a formula for $\widehat{\psi}$, namely

$$\widehat{\psi}(\tau,\xi)=\sum_{k=0}^{\infty}\Big(\delta_+^{(k)}(\tau,\xi)\widehat{A}_{+,k}(\xi)+\delta_-^{(k)}(\tau,\xi)\widehat{A}_{-,k}(\xi)\Big)+\widehat{K}(\tau,\xi), \quad (1.8)$$

where $\delta_\pm(\tau,\xi)$ are the delta functions supported on $\{\tau=\pm|\xi|\}$ respectively, $\delta^{(k)}$ the mean derivatives of the delta function, and

$$\widehat{K}(\tau,\xi):=\frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f+\frac{(1-\widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}+\frac{(1-\widehat{a}_6(\tau))\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}, \quad (1.9a)$$

$$\widehat{A}_{\pm,0}(\xi):=\frac{\widehat{D}_\pm}{2|\xi|}\Big[\gamma^0\widehat{\psi}_0-\int\frac{\widehat{G}_f+(1-\widehat{a}_6(\lambda))\widehat{G}_\mp}{\lambda\mp|\xi|}d\lambda\Big], \quad (1.9b)$$

$$\widehat{A}_{\pm,k}(\xi):=\frac{\widehat{D}_\pm(-1)^k}{2|\xi|k!}\int(\lambda\mp|\xi|)^{k-1}\big[\widehat{G}_\pm+\widehat{a}_6(\lambda)\widehat{G}_\mp\big]d\lambda. \quad (1.9c)$$

Consider the wave equation,

$$\begin{cases}\Box\phi=F, \quad (t,x)\in\mathbb{R}^1\times\mathbb{R}^1,\\ \phi(0)=\phi_0, \ \ \phi_t(0)=\phi_1.\end{cases} \quad (1.10)$$

Taking the Fourier transform in (1.13) and solving the resulting ODE gives

$$\widetilde{\phi}(t,\xi)=\cos t|\xi|\widehat{\phi}_0(\xi)+\frac{\sin t|\xi|}{|\xi|}\widehat{\phi}_1(\xi)-\int_0^t\frac{\sin{(t-s)|\xi|}}{|\xi|}\widetilde{F}(s,\xi)ds. \quad (1.11)$$

$$\begin{aligned}\widetilde{\phi}(t,\xi)&=\frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\widehat{\phi}_0(\xi)+\frac{e^{it|\xi|}-e^{-it|\xi|}}{2i|\xi|}\widehat{\phi}_1(\xi)\\ &-\int\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)}\widehat{F}(\tau,\xi)d\tau-\int\frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(\tau+|\xi|)}\widehat{F}(\tau,\xi)d\tau.\end{aligned} \quad (1.12)$$

For the homogeneous part, we rewrite it as

$$\frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\widehat{\phi}_0(\xi)+\frac{e^{it|\xi|}-e^{-it|\xi|}}{2i|\xi|}\widehat{\phi}_1(\xi)=\frac{e^{it|\xi|}}{2|\xi|}\widehat{\phi}_++\frac{e^{-it|\xi|}}{2|\xi|}\widehat{\phi}_-, \quad (1.13)$$

where

$$\widehat{\phi}_\pm = |\xi|\widehat{\phi}_0 \mp i\widehat{\phi}_1. \tag{1.14}$$

Now we split $\widehat{F}$ in the same manner as we did $\widehat{G}$. Let us write

$$\widehat{F}_\pm(\tau,\xi) := h(\pm\tau)\widehat{a}(\tau \mp |\xi|)\widehat{F}(\tau,\xi), \tag{1.16a}$$

$$\widehat{F}_f(\tau,\xi) := \widehat{F}(\tau,\xi) - \big(\widehat{F}_+(\tau,\xi) + \widehat{F}_-(\tau,\xi)\big). \tag{1.16b}$$

For the inhomogeneous part, we obtain

$$\begin{aligned}&\int \Big[\frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)} + \frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(|\xi|+\tau)}\Big]\widehat{F}_f(\tau,\xi)d\tau\\ &= \int e^{it\tau}\frac{\widehat{F}_f}{|\xi|^2-\tau^2}\,d\tau - \frac{e^{it|\xi|}}{2|\xi|}\int \frac{\widehat{F}_f}{|\xi|-\tau}d\tau - \frac{e^{-it|\xi|}}{2|\xi|}\int\frac{\widehat{F}_f}{|\xi|+\tau}d\tau,\end{aligned} \tag{1.17a}$$

$$\begin{aligned}&\int \frac{e^{it\tau}-e^{it|\xi|}}{2|\xi|(|\xi|-\tau)}(\widehat{F}_+ + \widehat{F}_-)d\tau = \frac{e^{it|\xi|}}{2|\xi|}\int \frac{e^{it(\tau-|\xi|)}-1}{|\xi|-\tau}(\widehat{F}_+ + \widehat{a}_6\widehat{F}_-)d\tau +\\ &\int e^{it\tau}\frac{(1-\widehat{a}_6)\widehat{F}_-}{2|\xi|(|\xi|-\tau)}d\tau - \frac{e^{it|\xi|}}{2|\xi|}\int\frac{(1-\widehat{a}_6)\widehat{F}_-}{|\xi|-\tau}d\tau,\end{aligned} \tag{1.17b}$$

$$\begin{aligned}&\int \frac{e^{it\tau}-e^{-it|\xi|}}{2|\xi|(|\xi|+\tau)}(\widehat{F}_+ + \widehat{F}_-)d\tau = \frac{e^{-it|\xi|}}{2|\xi|}\int \frac{e^{it(\tau+|\xi|)}-1}{|\xi|+\tau}(\widehat{a}_6\widehat{F}_+ + \widehat{F}_-)d\tau +\\ &\int e^{it\tau}\frac{(1-\widehat{a}_6)\widehat{F}_+}{2|\xi|(|\xi|+\tau)}d\tau - \frac{e^{-it|\xi|}}{2|\xi|}\int\frac{(1-\widehat{a}_6)\widehat{F}_+}{|\xi|+\tau}d\tau.\end{aligned} \tag{1.17c}$$

Combining (1.17a)–(1.17c), we can give a formula for $\widehat{\phi}$, namely

$$\widehat{\phi}(\tau,\xi) = \sum_{k=0}^{\infty}\Big(\delta_+^{(k)}(\tau,\xi)\widehat{B}_{+,k}(\xi) + \delta_-^{(k)}(\tau,\xi)\widehat{B}_{-,k}(\xi)\Big) + \widehat{L}(\tau,\xi), \tag{1.18}$$

where $\delta_\pm(\tau,\xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ the mean derivatives of the delta function, and

$$\widehat{L}(\tau,\xi) := \frac{\widehat{F}_f}{\widehat{W}(\tau,\xi)} - \frac{(1-\widehat{a}_6(\tau))\widehat{F}_-}{2|\xi|(|\xi|-\tau)} - \frac{(1-\widehat{a}_6(\tau))\widehat{F}_+}{2|\xi|(|\xi|+\tau)}, \tag{1.19a}$$

$$\widehat{B}_{\pm,0}(\xi) := \frac{1}{2|\xi|}\Big[\widehat{\phi}_\pm + \int \frac{\widehat{F}_f + (1-\widehat{a}_6(\lambda))\widehat{F}_\mp}{|\xi|\mp\lambda}d\lambda\Big], \tag{1.19b}$$

$$\widehat{B}_{\pm,k}(\xi) := \frac{\pm(-1)^k}{2|\xi|k!}\int(\lambda\mp|\xi|)^{k-1}\big[\widehat{F}_\pm + \widehat{a}_6(\lambda)\widehat{F}_\mp\big]d\lambda. \tag{1.19c}$$

Remark. We need to localize the solutions for the Dirac equation and wave equation due to the presence of the delta function.

2. Estimates

To localize the solution in time, let $b(t)$ be a cut-off function such that $b(t)$ equals 1 if $|t| \leq \frac{1}{2}$, and equals 0 if $|t| > 1$, and $b_T(t) = b(t/T)$. For an arbitrary function $f(t,x)$, we have

$$\|\widehat{b}_T * \widehat{f}\|_{L^2} = \|b_T f\|_{L^2} \leq \|b_T\|_{L^\infty}\|f\|_{L^2}. \tag{2.1}$$

Lemma 2.1. *If $\psi_0 \in H^{-\alpha}$, then we have*

$$\Big\|\widehat{b}_T * [\widehat{M}^{-\alpha}\widehat{S}^{\frac{3}{4}}\widehat{\psi}]\Big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^1)} \leq C\Big(\|\psi_0\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big). \tag{2.2}$$

Proof. Without loss of generality, we prove the special case.

$$\Big\|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\widehat{\psi}]\Big\|_{L^2(\mathbb{R}^1\times\mathbb{R}^1)} \leq C\Big(\|\psi_0\|_{L^2} + \Big\|\frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big). \tag{2.3}$$

To estimate $\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\widehat{\psi}]$, we apply formulae (1.8) and the (1.9)s. First we compute

$$\begin{aligned}
&\|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\widehat{K}]\|_{L^2} \leq \|\widehat{S}^{\frac{3}{4}}\widehat{K}\|_{L^2} \leq \Big\|\widehat{S}^{\frac{3}{4}}\frac{\widehat{D}}{\widehat{W}}\widehat{G}_f\Big\|_{L^2} \\
&+ \Big\|\widehat{S}^{\frac{3}{4}}\frac{(1-\widehat{a}_6)\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau-|\xi|)}\Big\|_{L^2} + \Big\|\widehat{S}^{\frac{3}{4}}\frac{(1-\widehat{a}_6)\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau+|\xi|)}\Big\|_{L^2} \leq C\Big\|\frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}.
\end{aligned} \tag{2.4}$$

For the term $\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\delta_+^{(k)}\widehat{A}_{+,k}]$, we can mollify $\widehat{S}(\tau,\xi)$ without loss of generality such that $\partial_\tau^k\widehat{S}(\pm|\xi|,\xi) = 0$ if $k \geq 1$. Thus we can compute

$$\begin{aligned}
&\|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}}\delta_+^{(k)}](\xi)\|^2_{L^2(d\tau)} \sim \int\Big(\int \widehat{b}_T(\tau-\lambda)\widehat{S}(\lambda,\xi)^{\frac{3}{4}}\delta^{(k)}(\lambda-|\xi|)d\lambda\Big)^2 d\tau \\
&\sim \int\Big(\frac{\partial^k}{\partial\lambda^k}\big(\widehat{b}_T(\tau-\lambda)\widehat{S}(\lambda,\xi)^{\frac{3}{4}}\big)\Big|_{\lambda=|\xi|}\Big)^2 d\tau \\
&\leq \int\Big(T^{k+1}\widehat{b}^{(k)}\big(T(\tau-|\xi|)\big)\Big)^2 d\tau \leq T^{2k+1}\|t^k b\|_{L^2} \leq CT^{2k+1}.
\end{aligned} \tag{2.5}$$

Then we calculate

$$\|\widehat{A}_{+,0}\|_{L^2(d\xi)} \leq \|\psi_0\|_{L^2} + \Big(\int\Big(\int\frac{\widehat{G}_f + \big(1-\widehat{a}_6(\tau)\big)\widehat{G}_-}{\tau-|\xi|}d\tau\Big)^2 d\xi\Big)^{\frac{1}{2}}$$

$$\leq \|\psi_0\|_{L^2} + \Big\| \frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \tag{2.6}$$

and

$$\begin{aligned} \|\widehat{A}_{+,k}\|_{L^2(d\xi)} \leq & \frac{1}{k!}\Big(\int \Big(\int (\tau - |\xi|)^{k-1} \big[\widehat{G}_+ + \widehat{a}_6 \widehat{G}_- \big](\tau,\xi) d\tau \Big)^2 d\xi \Big)^{\frac{1}{2}} \\ \leq & \frac{2^k}{k!}\Big(\int\int \Big| \widehat{G}_+ + \widehat{a}_6 \widehat{G}_- \Big|^2 (\tau,\xi) d\tau\, d\xi \Big)^{\frac{1}{2}} \leq \frac{2^k}{k!} \Big\| \frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}. \end{aligned} \tag{2.7}$$

Therefore, we have

$$\begin{aligned} \|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}} \delta_+ \widehat{A}_{+,0}]\|_{L^2} \leq & T^{\frac{1}{2}} \Big(\|\psi_0\|_{L^2} + \Big\| \frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big), \\ \|\widehat{b}_T * [\widehat{S}^{\frac{3}{4}} \delta_+^{(k)} \widehat{A}_{+,k}]\|_{L^2} \leq & T^{k+\frac{1}{2}} \frac{2^k}{k!} \Big\| \frac{\widehat{G}}{\widehat{S}^{\frac{1}{4}}} \Big\|_{L^2}. \end{aligned} \tag{2.8}$$

The calculation for the term $\widehat{b}_T * [\widehat{S}^{\frac{3}{4}} \delta_-^{(k)} \widehat{A}_{-,k}]$ is analogous. Combine the above results we complete the proof. $\square$

Consider two Dirac equations,

$$\begin{cases} \mathcal{D}\psi_j = G_j, \quad j = 1, 2 \\ \psi_j(0) = \psi_{0j}. \end{cases} \tag{2.9}$$

For the solutions of (2.9), we have the following key estimate, whose proof will be presented in the last section.

Lemma 2.2. (Null Form Estimate) *Let $\alpha = \frac{1}{4} - \epsilon$, $\epsilon > 0$, and let ψ_1 and ψ_2 be the solutions for* (2.9). *If $\psi_{0j} \in H^{-\alpha}$, we have*

$$\Big\| \frac{\widehat{(b_T \overline{\psi}_1 \psi_2)}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \leq C(T) \Big(\|\psi_{01}\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}_1}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big) \Big(\|\psi_{02}\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}_2}{\widehat{M}^\alpha \widehat{S}^{\frac{1}{4}}} \Big\|_{L^2} \Big). \tag{2.10}$$

For the wave equation (1.10), we have the following estimate.

Lemma 2.3. *Let ϕ be the solution of* (1.10). *If $\phi_0 \in H^{1-2\alpha}$ and $\phi_1 \in H^{-2\alpha}$, then*

$$\Big\| \widehat{b}_T * \Big[\widehat{M}^{-\alpha} (\widehat{E}\widehat{S})^{1-\alpha} \widehat{\phi} \Big] \Big\|_{L^2} \leq C \Big(\|\phi_0\|_{H^{1-2\alpha}} + \|\phi_1\|_{H^{-2\alpha}} + \Big\| \frac{\widehat{F}}{\widehat{M}^\alpha \widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \Big). \tag{2.11}$$

Proof. Without loss of generality, we show the following special case:

$$\big\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi}\big]\big\|_{L^2} \leq C\Big(\|\phi_0\|_{H^{1-\alpha}} + \|\phi_1\|_{H^{-\alpha}} + \Big\|\frac{\widehat{F}}{\widehat{E}^\alpha \widehat{S}^\alpha}\Big\|_{L^2}\Big). \quad (2.12)$$

To estimate $\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi}\big]$ in the L^2 norm, we invoke the formulae (1.18) and (1.19). First we compute

$$\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\widehat{L}\big]\|_{L^2} \leq \|(\widehat{E}\widehat{S})^{1-\alpha}\widehat{L}\|_{L^2} \leq \Big\|\frac{(\widehat{E}\widehat{S})^{1-\alpha}\widehat{F}_f}{\widehat{W}}\Big\|_{L^2} +$$
$$\Big\|\frac{(\widehat{E}\widehat{S})^{1-\alpha}(1-\widehat{a}_6)\widehat{F}_-}{2|\xi|(|\xi|-\tau)}\Big\|_{L^2} + \Big\|\frac{(\widehat{E}\widehat{S})^{1-\alpha}(1-\widehat{a}_6)\widehat{F}_+}{2|\xi|(|\xi|+\tau)}\Big\|_{L^2} \leq \Big\|\frac{\widehat{F}}{\widehat{E}^\alpha \widehat{S}^\alpha}\Big\|_{L^2}. \quad (2.13)$$

For the term $\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_+^{(k)}\widehat{B}_{+,k}\big]$, we can mollify $\widehat{E}\widehat{S}(\tau,\xi)$ without loss of generality such that $\partial_\tau^k \widehat{S}(\pm|\xi|,\xi) = 0$ if $k \geq 1$. Thus we compute

$$\begin{aligned}
&\big\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_+^{(k)}\big](\xi)\big\|^2_{L^2(d\tau)} \\
= &\int \Big|\int \widehat{b}_T(\tau-\lambda)(\widehat{E}\widehat{S})^{1-\alpha}(\lambda,\xi)\delta^{(k)}(\lambda-|\xi|)d\lambda\Big|^2 d\tau \\
= &\int \Big|\frac{\partial^k}{\partial\lambda^k}\Big(\widehat{b}_T(\tau-\lambda)(\widehat{E}\widehat{S})^{1-\alpha}(\lambda,\xi)\Big)\Big|_{\lambda=|\xi|}\Big|^2 d\tau \\
\sim &\int \Big|T^{k+1}\widehat{b}^{(k)}\big(T(\tau-|\xi|)\big)\Big|^2(|\xi|+1)^{2-2\alpha}d\tau \\
\leq &\; T^{2k+1}\|t^k b\|_{L^2}(|\xi|+1)^{2-2\alpha} \leq \; CT^{2k+1}(|\xi|+1)^{2-2\alpha}, \quad (2.14)
\end{aligned}$$

which implies

$$\big\|\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}\delta_+^{(k)}\widehat{B}_{+,k}\big]\big\|_{L^2} \leq CT^{k+\frac{1}{2}}\Big(\int (|\xi|+1)^{2-2\alpha}\big|\widehat{B}_{+,k}(\xi)\big|^2 d\xi\Big)^{\frac{1}{2}}. \quad (2.15)$$

To estimate the above integral, we first focus on the region where $|\xi| > 1$. Due to the observation that on the supports of $(1-\widehat{a}_6)\widehat{F}_-$ and $\widehat{F}_f$, the inequality

$$\widehat{E}^{2\alpha}\widehat{S}^{2\alpha} = \big(|\lambda|+|\xi|+1\big)^{2\alpha}\big(\big||\lambda|-|\xi|\big|+1\big)^{2\alpha} \leq (|\xi|+1)^{2\alpha}\big|\lambda-|\xi|\big|^{4\alpha} \quad (2.16)$$

holds, we have the following bounds:

$$\int \Big|\int \frac{\widehat{F}_f(\lambda,\xi)}{(|\xi|+1)^\alpha\big||\xi|-\lambda\big|}d\lambda\Big|^2 d\xi \quad (2.17)$$

$$\leq \iint_{||\xi|-\lambda|\geq\frac{1}{2}} \frac{1}{\big||\xi|-\lambda\big|^{1+4\epsilon}} d\lambda \int \frac{|\widehat{F}_f(\lambda,\xi)|^2}{(|\xi|+1)^{2\alpha}\big||\xi|-\lambda\big|^{1-4\epsilon}} d\lambda\, d\xi \leq C \Big\| \frac{\widehat{F}_f}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2}^2,$$

and in the same vein

$$\int \Big| \int \frac{\big(1-\widehat{a}_6(\lambda)\big)\widehat{F}_-(\lambda,\xi)}{(|\xi|+1)^\alpha \big||\xi|-\lambda\big|} d\lambda \Big|^2 d\xi \leq C \Big\| \frac{\widehat{F}_-}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2}^2. \tag{2.18}$$

Hence we get

$$\begin{aligned} &\big\| \widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha} \delta_+ \widehat{B}_{+,0}\big] \big\|_{L^2(L^2(|\xi|>1))} \\ &\leq C T^{\frac{1}{2}} \Big(\|\phi_0\|_{H^{1-\alpha}} + \|\phi_1\|_{H^{-\alpha}} + \Big\| \frac{\widehat{F}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2} \Big) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\big\| \widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha} \delta_+^{(k)} \widehat{B}_{+,k}\big] \big\|_{L^2(L^2(|\xi|>1))} \\ &\leq C T^{k+\frac{1}{2}} \frac{c^k}{k!} \Big(\int\int \frac{\big|\widehat{F}_+ + \widehat{a}_6 \widehat{F}_-\big|^2(\tau,\xi)}{(|\xi|+1)^{2\alpha}} d\lambda\, d\xi \Big)^{\frac{1}{2}} \leq C T^{k+\frac{1}{2}} \frac{c^k}{k!} \Big\| \frac{\widehat{F}}{\widehat{E}^\alpha \widehat{S}^\alpha} \Big\|_{L^2}. \end{aligned} \tag{2.20}$$

The calculation for the term $\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha} \delta_-^{(k)} \widehat{B}_{-,k}\big]$ is analogous.

For the region $|\xi| \leq 1$, we consider $\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}(\delta_+^{(k)} \widehat{B}_{+,k} + \delta_-^{(k)} \widehat{B}_{-,k})\big]$. This is clear from the derivation of the solution representation, which indicates that the solution is actually not singular along the cones.

$$\begin{aligned} &\widehat{b}_T * \big[(\widehat{E}\widehat{S})^{1-\alpha}(\delta_+^{(k)} \widehat{B}_{+,k} + \delta_-^{(k)} \widehat{B}_{-,k})\big](\tau,\xi) \\ &\sim T^{k+1}\big(|\xi|+1\big)^{1-\alpha}\big[\widehat{t^k b}\big(T(\tau-|\xi|)\big)\widehat{B}_{+,k}(\xi) + \widehat{t^k b}\big(T(\tau+|\xi|)\big)\widehat{B}_{-,k}(\xi)\big] \\ &= T^{k+1}\big(|\xi|+1\big)^{1-\alpha}\big[\widehat{t^k b}\big(T(\tau-|\xi|)\big) - \widehat{t^k b}\big(T(\tau+|\xi|)\big)\big]\widehat{B}_{+,k}(\xi) + \\ &\quad T^{k+1}\big(|\xi|+1\big)^{1-\alpha}\widehat{t^k b}\big(T(\tau+|\xi|)\big)\big[\widehat{B}_{+,k}(\xi) + \widehat{B}_{-,k}(\xi)\big]. \end{aligned} \tag{2.21}$$

Under the restriction of $|\xi| \leq 1$, we have

$$\widehat{t^k b}\big(T(\tau-|\xi|)\big) - \widehat{t^k b}\big(T(\tau+|\xi|)\big) \sim T\widehat{t^{k+1} b}\big(T(\tau-(1-2\theta)|\xi|)\big)|\xi|, \tag{2.23}$$

$$\widehat{B}_{+,0}(\xi) + \widehat{B}_{-,0}(\xi) \sim \widehat{\phi}_0 + \int \frac{\widehat{F}_f}{|\xi|^2-\lambda^2}\, d\lambda, \tag{2.24}$$

$$\widehat{B}_{+,k}(\xi) + \widehat{B}_{-,k}(\xi) \sim \frac{1}{(k-1)!} \int \big(\lambda-(1-2\theta)|\xi|\big)^{k-2}(\widehat{F}_+ + \widehat{a}_6 \widehat{F}_-)\, d\lambda. \tag{2.25}$$

Combining the above results, we complete the proof. □

We will also need some technical lemmas.

Lemma 2.4. (Hardy-Littlewood-Polya) *Let* $r = 2 - \frac{1}{p} - \frac{1}{q}$. *Then we have*

$$\int_{\mathbb{R}^1\times\mathbb{R}^1} \frac{f(s)g(t)}{|s-t|^r} ds\, dt \leq C\|f\|_{L^p}\|g\|_{L^q}. \tag{2.26}$$

Lemma 2.5. *Let* $f(t,x)$ *and* $g(t,x)$ *be any functions such that* $f \in L^q(L^2(\mathbb{R}))$ *and* $\widehat{S}^\beta \widehat{g} \in L^2(L^2(\mathbb{R}))$. *Assume that* $\delta \geq 0$, $q = \frac{8}{5-4\delta}$, $\frac{1}{r} = \frac{1}{2} - \beta$, *and* $2 \leq r < \infty$. *Then we have*

$$\Big\|\frac{\widehat{b_T * f}}{\widehat{S}^{\frac{1}{4}-\delta}}\Big\|_{L^2} \leq C\|b_T f\|_{L^q(L^2)}, \tag{2.27}$$

$$\|g\|_{L^r(L^2)} \leq C\|\widehat{S}^\beta \widehat{g}\|_{L^2(L^2)}. \tag{2.28}$$

Proof. The proofs for (2.27) and (2.28) are analogous. Therefore we will prove only the case of (2.28).

Taking the inverse Fourier transform in the time variable over the identity

$$\widehat{g} = \frac{1}{\widehat{S}^\beta}\widehat{S}^\beta\widehat{g} \quad \text{gives} \quad \widetilde{g}(t,\xi) = \int \frac{e^{\pm i(t-s)|\xi|}}{|t-s|^{1-\beta}} \mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s,\xi)\, ds. \tag{2.29}$$

Then we use duality and the Hardy-Littlewood-Polya inequality to compute

$$\begin{aligned}\Big|\Big\langle g,\varphi\Big\rangle\Big| =& \Big|\Big\langle \widetilde{g},\widetilde{\varphi}\Big\rangle\Big| = \Big|\iiint \frac{e^{\pm i(t-s)|\xi|}}{|t-s|^{1-\beta}}\mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s,\xi)\, ds\overline{\widetilde{\varphi}}(t,\xi)\, dt\, d\xi\Big| \\ &\leq \int \frac{\|\mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})(s)\|_{L^2}\|\widetilde{\varphi}(t)\|_{L^2}}{|t-s|^{1-\beta}} ds\, dt \\ &\leq C\|\mathcal{F}_\tau^{-1}(\widehat{S}^\beta\widehat{g})\|_{L^2}\|\widetilde{\varphi}\|_{L^{r'}(L^2)} = C\|\widehat{S}^\beta\widehat{g}\|_{L^2}\|\varphi\|_{L^{r'}(L^2)}.\end{aligned} \tag{2.30}$$

This completes the proof of (2.28). □

3. LOCAL EXISTENCE

Now we are ready to prove the local existence for the (DKG) equations.

Proof of Theorem 0.1. Consider the DKG problem

$$\mathcal{D}\psi = b_T\phi\psi; \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^1 \tag{3.1a}$$

$$\Box\phi = b_T\overline{\psi}\psi; \tag{3.1b}$$

$$\psi(0,x) := \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x). \tag{3.1c}$$

An iteration scheme induces a map $\mathcal{T}$ defined by

$$\mathcal{T}(\psi^k,\phi^k) = (\psi^{k+1},\phi^{k+1}). \tag{3.2a}$$

We want to show that $\mathcal{T}$ is a contraction under the norm

$$\mathcal{N}(\psi,\phi)=\big\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi}\big\|_{L^2}+\big\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi}\big\|_{L^2}. \tag{3.2b}$$

For convenience, we call

$$J(0)=\|\phi_0\|_{H^{1-2\alpha}}+\|\phi_1\|_{H^{-2\alpha}}+\|\psi_0\|^2_{H^{-\alpha}}+1. \tag{3.3}$$

First we apply (2.11), (2.10), and (2.27) to compute

$$\begin{aligned}\big\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\mathcal{T}\phi}\big\|_{L^2}&\le C\Big(J(0)+\Big\|\frac{\widehat{b_T\overline{\psi}\psi}}{\widehat{M}^{\alpha}\widehat{E}^{\alpha}\widehat{S}^{\alpha}}\Big\|_{L^2}\Big)\\ &\le C\Big(J(0)+\Big\|\frac{\widehat{b_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\Big\|^2_{L^2}\Big)\le C\Big(J(0)+\Big\|\frac{b_T\widetilde{\phi\psi}}{\widehat{M}^{\alpha}}\Big\|^2_{L^{\frac85}([0,T],L^2)}\Big)\\ &\le C\Big(J(0)+T^{\frac18}\Big\|\frac{b_T\widetilde{\phi\psi}}{\widehat{M}^{\alpha}}\Big\|^2_{L^2([0,T],L^2)}\Big).\end{aligned} \tag{3.4}$$

To bound the term above, we first compute

$$\begin{aligned}&\Big\|\widehat{M}^{-\alpha}\widetilde{\phi\psi}(t)\Big\|_{L^2}\sim\|G_\alpha*(\phi\psi)(t)\|_{L^2}\\ \le&\|\phi(t)\|_{L^\infty}\|G_\alpha*\psi(t)\|_{L^2}\le\|\phi(t)\|_{H^{1-2\alpha}}\|\psi(t)\|_{H^{-\alpha}},\end{aligned} \tag{3.5}$$

where $G_\alpha(x)$ is an L^1 function with the following property:

$$\widehat{G_\alpha}(\xi)\sim(1+|\xi|)^{-\alpha} \tag{3.6}$$

(see [S]); then we invoke (2.27) and (2.28) to obtain

$$\Big\|b_T\widehat{M}^{-\alpha}\widetilde{\phi\psi}\Big\|_{L^2}\le C\|\phi\|_{L^4([0,T],H^{1-2\alpha})}\|\psi\|_{L^4([0,T],H^{-\alpha})} \tag{3.7}$$

$$\le C\big\|\widehat{S}^{\frac14}\widehat{M}^{1-2\alpha}\widehat{\phi}\big\|_{L^2}\big\|\widehat{S}^{\frac14}\widehat{M}^{-\alpha}\widehat{\psi}\big\|_{L^2}\le C\big\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi}\big\|_{L^2}\big\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi}\big\|_{L^2}.$$

Next we want to bound the term involved with $\widehat{\psi}$. The estimate (2.2) implies that

$$\Big\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\mathcal{T}\psi}\Big\|_{L^2}\le C\Big(\|\psi_0\|_{H^{-\alpha}}+\Big\|\frac{\widehat{b_T\phi\psi}}{\widehat{M}^{\alpha}\widehat{S}^{\frac14}}\Big\|_{L^2}\Big). \tag{3.9}$$

Hence, using (3.4), (3.7), and (3.9), we have

$$\mathcal{N}(\mathcal{T}(\psi,\phi))\le C\big(J(0)+T^{\frac18}N^4(\psi,\phi)\big). \tag{3.10}$$

Choosing sufficiently large L, for suitable T, we have

$$\mathcal{N}(\psi,\phi)\le L\Longrightarrow N\big(\mathcal{T}(\psi,\phi)\big)\le L, \tag{3.11}$$

provided that

$$C(J(0)+T^{\frac18}L^4)\le L. \tag{3.12}$$

Now we consider the difference $\mathcal{T}(\psi,\phi)-\mathcal{T}(\psi',\phi')$, based on the observations

$$\overline{\psi}\psi-\overline{\psi'}\psi'=\frac{1}{2}(\overline{\psi-\psi'})(\psi+\psi')+\frac{1}{2}(\overline{\psi+\psi'})(\psi-\psi'), \tag{3.13a}$$

$$\phi\psi-\phi'\psi'=\frac{1}{2}(\phi-\phi')(\psi+\psi')+\frac{1}{2}(\phi+\phi')(\psi-\psi'). \tag{3.13b}$$

Employing (2.11), (2.10), and (3.13), we first calculate

$$\begin{aligned}
&\big\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\mathcal{F}(\mathcal{T}\phi-\mathcal{T}\phi')\big\|_{L^2}\\
\le& C\Big(\|\frac{\mathcal{F}(b_T(\overline{\psi-\psi'})(\psi+\psi'))}{\widehat{M}^\alpha\widehat{E}^\alpha\widehat{S}^\alpha}\|_{L^2}+\|\frac{\mathcal{F}(b_T(\overline{\psi+\psi'})(\psi-\psi'))}{\widehat{M}^\alpha\widehat{E}^\alpha\widehat{S}^\alpha}\|_{L^2}\Big)\\
\le& C\Big(\Big\|\frac{\mathcal{F}(b_T(\phi-\phi')(\psi+\psi'))}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}+\Big\|\frac{\mathcal{F}(b_T(\phi+\phi')(\psi-\psi'))}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\Big)\cdot\\
&\Big(I.D.+\Big\|\frac{\mathcal{F}(b_T(\phi\psi+\phi'\psi'))}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\Big)\\
\le& CT^{\frac18}\big(\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi-\psi'}\|_{L^2}\big)L(I.D.+L^2)\\
\le& CT^{\frac18}L^3\big(\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi-\psi'}\|_{L^2}\big).
\end{aligned} \tag{3.15}$$

Analogously, we get

$$\begin{aligned}
&\big\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\mathcal{F}(\mathcal{T}\psi-\mathcal{T}\psi')\big\|_{L^2}\\
\le& CT^{\frac18}L\Big(\|\widehat{M}^{-\alpha}(\widehat{E}\widehat{S})^{1-\alpha}\widehat{\phi-\phi'}\|_{L^2}+\|\widehat{M}^{-\alpha}\widehat{S}^{\frac34}\widehat{\psi-\psi'}\|_{L^2}\Big).
\end{aligned} \tag{3.16}$$

Combining (3.15)and (3.16), we have

$$\mathcal{N}\big(\mathcal{T}(\psi-\psi',\phi-\phi')\big)\le CT^{\frac18}L^3N(\psi-\psi',\phi-\phi'). \tag{3.17}$$

Therefore, for suitable T we obtain

$$\mathcal{N}\big(\mathcal{T}(\psi-\psi',\phi-\phi')\big)\le \frac{1}{2}N(\psi-\psi',\phi-\phi'), \tag{3.18}$$

provided that $CT^{\frac18}L^3\le\frac12$. We can conclude that the map $\mathcal{T}$ is indeed a contraction with respect to the norm $\mathcal{N}$; thus, it has a unique fixed point. □

We now prove global existence.

Proof of Theorem 0.2. From the law of conservation of charge, we have

$$\sup_{[0,T]}\|\psi(t)\|_{L^2}=\|\psi_0\|_{L^2}. \tag{3.20}$$

To bound ϕ we apply the following formula:

$$2\phi(t,x) = \phi_0(x+t) + \phi_0(x-t) + \int_{x-t}^{x+t} \phi_1(y)\,dy + \int_0^t \int_{x-t+s}^{x+t-s} \overline{\psi}\psi(s,y)dy\,ds. \tag{3.21}$$

First we write $\phi = \phi_L + \phi_N$, the homogeneous and inhomogeneous parts of the solution, then we obtain

$$\|\phi_L(t)\|_{L^\infty} \le \|\phi_L(t)\|_{H^{\frac{1}{2}+\delta}} \le \|\phi_0\|_{H^{\frac{1}{2}+\delta}} + \|\phi_1\|_{H^{-\frac{1}{2}+\delta}} \le J(0), \tag{3.22}$$

and

$$\|\phi_N(t)\|_{L^\infty} \le \int_0^t \int_{x-t+s}^{x+t-s} \left|\overline{\psi}\psi(s,y)\right| dy\,ds \le CT\|\psi_0\|_{L^2}^2. \tag{3.23}$$

Combining (3.22) and (3.23), we get

$$\|\phi(t)\|_{L^\infty} \le C(T, J(0)). \tag{3.24}$$

Taking Fourier transform of the solution $\phi(t)$, we have

$$\widetilde{\phi}(t,\xi) = \cos t|\xi|\widehat{\phi}_0(\xi) + \frac{\sin t|\xi|}{|\xi|}\widehat{\phi}_1(\xi) + \int_0^t \frac{\sin(t-s)|\xi|}{|\xi|}\widetilde{\overline{\psi}\psi}(s,\xi)ds. \tag{3.25}$$

Then we invoke (3.21), (2.10) (for $\alpha = 0$), (2.27), and (3.24) to compute

$$\begin{aligned} \|\phi(t)\|_{H^{\frac{1}{2}+\delta}} \le& \|\phi_0\|_{H^{\frac{1}{2}+\delta}} + \|\phi_1\|_{H^{-\frac{1}{2}+\delta}} + \int_0^t \|b_T\overline{\psi}\psi(s)\|_{H^{-\frac{1}{2}+\delta}}\,ds \\ \le& J(0) + T^{\frac{1}{2}} \left\| \frac{\widehat{b_T\overline{\psi}\psi}}{\widehat{M}^{\frac{1}{2}-\delta}} \right\|_{L^2} \le J(0) + T^{\frac{1}{2}} \|\widehat{b_T\overline{\psi}\psi}\|_{L^2} \\ \le& J(0) + T^{\frac{1}{2}} \left\| \frac{\widehat{b_T\phi\psi}}{\widehat{S}^{\frac{1}{4}}} \right\|_{L^2}^2 \le J(0) + T^\rho \|b_T\phi\psi\|_{L^2}^2 \\ \le& J(0) + T^\rho \int_0^T \|\phi(t)\|_{L^\infty}^2 \|\psi(t)\|_{L^2}^2\,dt \le C\big(T, J(0)\big), \end{aligned} \tag{3.26}$$

where ρ is some positive number. The calculation for $\|\phi_t(t)\|_{H^{-\frac{1}{2}+\delta}}$ is analogous. Thus the above bounds ensure that we may proceed to the construction of a solution beyond T. $\square$

4. NULL-FORM ESTIMATE

In this section, we demonstrate the proof of the key estimate.

Lemma 2.2 (Null-Form Estimate) *Let $\alpha = \frac{1}{4} - \epsilon$, $\epsilon > 0$, and let ψ_1 and ψ_2 be the solutions for the Dirac equations* (2.9). *If the initial data $\psi_{0j} \in L^2$, $j = 1, 2$, then we have*

$$\Big\|\frac{\widehat{b_T\overline{\psi}_1\psi_2}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \le C(T)\Big(\|\psi_{01}\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}_1}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big)\Big(\|\psi_{02}\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}_2}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big). \tag{4.1}$$

The proof for the estimate is based on the duality argument, and it will be given in a number of steps. Without loss of generality, we assume that $\psi_1 = \psi_2$, and prove that if ψ is a solution of the Dirac equation (1.2), then

$$\Big\|\frac{\widehat{b_T\overline{\psi}\psi}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \le C(T)\Big(\|\psi_0\|_{H^{-\alpha}} + \Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac{1}{4}}}\Big\|_{L^2}\Big)^2. \tag{4.2}$$

Recall the following notation:

$$\widehat{E}(\tau,\xi) := |\tau| + |\xi| + 1, \qquad \widehat{S}(\tau,\xi) := \Big||\tau| - |\xi|\Big| + 1, \tag{4.3a}$$

$$\widehat{W}(\tau,\xi) := \tau^2 - |\xi|^2, \qquad \widehat{D}(\tau,\xi) := \gamma^0\tau + \gamma^1\xi, \tag{4.3b}$$

$$\widehat{D}_+ := \widehat{D}(|\xi|, +\xi), \qquad \widehat{D}_- := \widehat{D}(|\xi|, -\xi). \tag{4.3c}$$

The formula for $\widehat{\psi}$, as in (1.8), for the Dirac equation (1.2) is given by

$$\widehat{\psi}(\tau,\xi) = \sum_{k=0}^{\infty}\Big(\delta_+^{(k)}(\tau,\xi)\widehat{A}_{+,k}(\xi) + \delta_-^{(k)}(\tau,\xi)\widehat{A}_{-,k}(\xi)\Big) + \widehat{K}(\tau,\xi), \tag{4.4}$$

where $\delta_\pm(\tau,\xi)$ are the delta functions supported on $\{\tau = \pm|\xi|\}$ respectively, $\delta^{(k)}$ are mean derivatives of the delta function, and

$$\widehat{K}(\tau,\xi) := \frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f + \frac{(1-\widehat{a}_6(\tau))\widehat{D}_+\widehat{G}_-}{2|\xi|(\tau - |\xi|)} + \frac{(1-\widehat{a}_6(\tau))\widehat{D}_-\widehat{G}_+}{2|\xi|(\tau + |\xi|)}, \tag{4.5a}$$

$$\widehat{A}_{\pm,0}(\xi) := \frac{\widehat{D}_\pm}{2|\xi|}\Big[\gamma^0\widehat{\psi}_0 - \int\frac{\widehat{G}_f + (1-\widehat{a}_6(\lambda))\widehat{G}_\mp}{\lambda \mp |\xi|}d\lambda\Big], \tag{4.5b}$$

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_\pm(-1)^k}{2|\xi|k!}\int(\lambda \mp |\xi|)^{k-1}\big[\widehat{G}_\pm + \widehat{a}_6(\lambda)\widehat{G}_\mp\big]d\lambda. \tag{4.5c}$$

Moreover, we write

$$\widehat{A}_{\pm,k}(\xi) := \frac{\widehat{D}_{\pm}}{2|\xi|}\widehat{f}_{\pm,k}(\xi), \tag{4.6}$$

and split $\widehat{K} = \widehat{K}_1 + \widehat{K}_2$, where $\widehat{K}_1 := \frac{\widehat{D}(\tau,\xi)}{\widehat{W}(\tau,\xi)}\widehat{G}_f$, $\widehat{K}_2 := \frac{b_1\widehat{D}_+\widehat{G}_- + b_2\widehat{D}_-\widehat{G}_+}{\widehat{E}\widehat{S}}$, and b_1 and b_2 are bounded functions. The Fourier transform of the quadratic expression $\widehat{\overline{\psi}\psi} = \widehat{\overline{\psi}} * \widehat{\psi}$ can be written as the sum of the following terms:

$$\sum_{k,l} \big(\delta_{\mp}^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_{\pm}^{(l)}\widehat{A}_{\pm,l}\big), \tag{4.8a}$$

$$\sum_{k,l} \big(\delta_{\mp}^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_{\mp}^{(l)}\widehat{A}_{\mp,l}\big), \tag{4.8b}$$

$$\sum_{k} \big(\delta_{\mp}^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\widehat{K}_1 + \widehat{K}_2\big) + \big(\widehat{\overline{K}}_1 + \widehat{\overline{K}}_2\big) * \sum_{k} \big(\delta_{\pm}^{(k)}\widehat{A}_{\pm,k}\big), \tag{4.8c}$$

$$\widehat{\overline{K}}_1 * \widehat{K}_1 + \widehat{\overline{K}}_1 * \widehat{K}_2 + \widehat{\overline{K}}_2 * \widehat{K}_1 + \widehat{\overline{K}}_2 * \widehat{K}_2. \tag{4.8d}$$

Notice that

$$\widehat{A^{\dagger}_{\pm,k}}(\xi) = \widehat{A}^{\dagger}_{\pm,k}(-\xi), \qquad \widehat{f^{+}_{\pm,k}}(\xi) = \widehat{f}^{\dagger}_{\pm,k}(-\xi), \tag{4.9a}$$

$$\widehat{\overline{A}_{\pm,k}}(\xi) = \widehat{f}^{\dagger}_{\pm,k}(-\xi)\frac{\widehat{D}_{\pm}}{|\xi|}\gamma^0, \qquad \widehat{\overline{K}}(\tau,\xi) = \widehat{K}^{\dagger}(-\tau,-\xi)\gamma^0, \tag{4.9b}$$

and

$$\widehat{\overline{\psi}}(\tau,\xi) = \sum_{k=0}^{\infty}\Big(\delta_{-}^{(k)}(\tau,\xi)\widehat{\overline{A}}_{+,k}(\xi) + \delta_{+}^{(k)}(\tau,\xi)\widehat{\overline{A}}_{-,k}(\xi)\Big) + \widehat{\overline{K}}(\tau,\xi). \tag{4.10}$$

Lemma 4.1. *Let $\alpha < \frac{1}{4}$. The following estimate holds:*

$$\left\|\frac{\widehat{b}_T * \big(\delta_{\mp}^{(k)}\widehat{\overline{A}}_{\pm,k}\big) * \big(\delta_{\mp}^{(l)}\widehat{A}_{\mp,l}\big)}{\widehat{E}^{\alpha}\widehat{S}^{\alpha}}\right\|_{L^2} \le C(k+l+1)T^{k+l-2\alpha}\|f_{\pm,k}\|_{H^{-\alpha}}\|f_{\mp,l}\|_{H^{-\alpha}}. \tag{4.11}$$

Proof. Let us call

$$\widehat{Z}_{\pm,k} \equiv \delta_{\pm}^{(k)}\widehat{A}_{\pm,k} = \delta_{\pm}^{(k)}\frac{\widehat{D}_{\pm}}{|\xi|}\widehat{f}_{\pm,k}. \tag{4.12}$$

Using duality, we demonstrate the case $(-,+)$, the case $(+,-)$ being similar. We first compute the fractional term

$$\frac{\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|} = \begin{cases} 0, & \text{if } \xi\eta > 0, \\ 2\gamma^0 \pm 2\gamma^1, & \text{if } \xi\eta < 0, \end{cases} \tag{4.13}$$

and observe that, for $\xi\eta < 0$,

$$\big||\xi| + |\eta|\big| + |\xi+\eta| + 1 \sim \max\{|\xi|,|\eta|\} + 1, \tag{4.14a}$$

$$\Big|\big||\xi|+|\eta|\big| - |\xi+\eta|\Big| + 1 \sim \min\{|\xi|,|\eta|\} + 1. \tag{4.14b}$$

Thus,

$$\begin{aligned}
&\Big|\left\langle b_T \overline{Z}_{-,k} Z_{+,l}, g\right\rangle\Big| \\
=&\Big|\int \widehat{f}^{\dagger}_{-,k}(-\xi)\frac{\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|}\widehat{f}_{+,l}(\eta)\overline{\widehat{t^{k+l}b_T g}}(|\xi|+|\eta|,\xi+\eta)\,d\xi\,d\eta\Big| \\
\le& C\|f_{-,k}\|_{H^{-\alpha}}\|f_{+,l}\|_{H^{-\alpha}}\cdot \\
&\Big(\int(|\xi|+1)^{2\alpha}(|\eta|+1)^{2\alpha}\big|\widehat{t^{k+l}b_T g}(|\xi|+|\eta|,\xi+\eta)\big|^2\,d\xi\,d\eta\Big)^{\frac{1}{2}} \\
\le& C\|f_{-,k}\|_{H^{-\alpha}}\|f_{+,l}\|_{H^{-\alpha}}\|\widehat{E}^{\alpha}\widehat{S}^{\alpha}\widehat{t^{k+l}b_T g}\|_{L^2}.
\end{aligned} \tag{4.15}$$

Through some computation, we have

$$\big\|\widehat{t^{k+l}b_T}\big\|_{L^1} \le C(k+l)T^{k+l}\|b\|_{H^1}, \tag{4.16a}$$

$$\big\||\tau|^{2\alpha}\widehat{t^{k+l}b_T}\big\|_{L^1} \le C(k+l)T^{k+l-2\alpha}\|b\|_{H^1}, \tag{4.16b}$$

provided that $\alpha < \frac{1}{4}$. With the aid of the above and the observation

$$\widehat{E}(\tau,\xi) \le |\tau-\lambda| + \widehat{E}(\lambda,\xi), \quad \widehat{S}(\tau,\xi) \le |\tau-\lambda| + \widehat{S}(\lambda,\xi), \tag{4.16c}$$

we can estimate

$$\begin{aligned}
\|\widehat{E}^{\alpha}\widehat{S}^{\alpha}\widehat{t^{k+l}b_T g}\|_{L^2} \le& \Big(\big\|\widehat{t^{k+l}b_T}\big\|_{L^1} + \big\||\tau|^{2\alpha}\widehat{t^{k+l}b_T}\big\|_{L^1}\Big)\|\widehat{E}^{\alpha}\widehat{S}^{\alpha}\widehat{g}\|_{L^2} \\
\le& C(k+l+1)T^{k+l-2\alpha}\|\widehat{E}^{\alpha}\widehat{S}^{\alpha}\widehat{g}\|_{L^2}.
\end{aligned} \tag{4.17}$$

This completes the proof. □

Lemma 4.2 *Let $\alpha < \frac{1}{4}$. The following estimate holds:*

$$\Big\|\frac{\widehat{b}_T * \big(\delta^{(k)}_{\mp}\widehat{\widehat{A}}_{\pm,k}\big) * \big(\delta^{(l)}_{\pm}\widehat{A}_{\pm,l}\big)}{\widehat{E}^{\alpha}\widehat{S}^{\alpha}}\Big\|_{L^2} \le C(k+l+1)T^{k+l-2\alpha}\|f_{\pm,k}\|_{H^{-\alpha}}\|f_{\pm,l}\|_{H^{-\alpha}}. \tag{4.18}$$

Proof. Using duality, we demonstrate the case $(+,+)$, the case $(-,-)$ being similar. We first compute the fractional term

$$\frac{\widehat{D}(|\xi|,\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|} = \begin{cases} 0, & \text{if } \xi\eta < 0, \\ 2\gamma^0 \mp 2\gamma^1, & \text{if } \xi\eta > 0, \end{cases} \tag{4.19a}$$

and observe that, for $\xi\eta > 0$,

$$\big| -|\xi| + |\eta| \big| + |\xi+\eta| + 1 \sim \max\{|\xi|,|\eta|\} + 1, \tag{4.19b}$$

$$\Big| \big| -|\xi| + |\eta| \big| - |\xi+\eta| \Big| + 1 \sim \min\{|\xi|,|\eta|\} + 1. \tag{4.19c}$$

Thus, in the same manner we have

$$\begin{aligned}
&\Big| \left\langle b_T \overline{Z}_{+,k} Z_{+,l}, g \right\rangle \Big| \\
=&\Big| \int \widehat{f}^{\dagger}_{+,k}(-\xi) \frac{\widehat{D}(|\xi|,-\xi)\gamma^0\widehat{D}(|\eta|,\eta)}{|\xi||\eta|} \widehat{f}_{+,l}(\eta) \overline{t^{\widehat{k+l}b_T}g}(-|\xi|+|\eta|,\xi+\eta)\, d\xi\, d\eta \Big| \\
\le& C\|f_{+,k}\|_{H^{-\alpha}} \|f_{+,l}\|_{H^{-\alpha}} \cdot \\
&\Big(\int (|\xi|+1)^{2\alpha} (|\eta|+1)^{2\alpha} \big| \widehat{t^{k+l}b_T}g(-|\xi|+|\eta|,\xi+\eta) \big|^2\, d\xi\, d\eta \Big)^{\frac{1}{2}} \\
\le& C\|f_{+,k}\|_{H^{-\alpha}} \|f_{+,l}\|_{H^{-\alpha}} \|\widehat{E}^{\alpha}\widehat{S}^{\alpha}\widehat{t^{k+l}b_T g}\|_{L^2} \\
\le& C(k+l+1)T^{k+l-2\alpha} \|f_{+,k}\|_{H^{-\alpha}} \|f_{+,l}\|_{H^{-\alpha}} \|\widehat{E}^{\alpha}\widehat{S}^{\alpha}\widehat{g}\|_{L^2}.
\end{aligned} \tag{4.20}$$

This completes the proof. □

Lemma 4.3. *Let $\delta > 0$. The following estimates hold:*

$$\|f_{\pm,0}\|_{H^{-\alpha}} \le C\Big(\|\psi_0\|_{H^{-\alpha}} + \Big\| \frac{\widehat{G}}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{2}-\delta}} \Big\|_{L^2} \Big), \tag{4.21a}$$

$$\|f_{\pm,k}\|_{H^{-\alpha}} \le C\frac{1}{k!} \| \frac{\widehat{G}_{\pm}}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{2}-\delta}} \|_{L^2}. \tag{4.21b}$$

The proof for Lemma 4.3 is straightforward, so we skip it. Notice that, in (4.21b), $\widehat{S} \sim 1$ on the support of $\widehat{G}_{\pm}$.

Lemma 4.4. *With the notation above, the following estimate holds:*

$$\Big\| \frac{\widehat{b}_T * \widehat{\overline{K}}_1 * \widehat{K}_1}{\widehat{E}^{\alpha}\widehat{S}^{\alpha}} \Big\|_{L^2} \le C \Big\| \frac{\widehat{G}_f}{\widehat{M}^{\alpha}\widehat{S}^{\frac{1}{4}}} \Big\|^2_{L^2}. \tag{4.22}$$

Proof. For simplicity, we write $\widehat{G} := \widehat{G}_f$ and $\widehat{K} := \widehat{K}_1$. We use dyadic decomposition to handle this case. Assume that

$$\widehat{G} = \sum_{k=1}^{\infty} \widehat{G}_{\pm,\pm,k}, \tag{4.23}$$

where $\widehat{G}_{\pm,\pm,k}(\tau,\xi)$ is supported in one of the following types of regions:

$$\Sigma_{+,+} := \{(\tau,\xi) : \tau > 0, +2^{k-1} < \tau - |\xi| < +2^{k+1}\}, \tag{4.24a}$$

$$\Sigma_{+,-} := \{(\tau,\xi) : \tau > 0, -2^{k+1} < \tau - |\xi| < -2^{k-1}\}, \tag{4.24b}$$

$$\Sigma_{-,+} := \{(\tau,\xi) : \tau < 0, +2^{k-1} < \tau + |\xi| < +2^{k+1}\}, \tag{4.24c}$$

$$\Sigma_{-,-} := \{(\tau,\xi) : \tau < 0, -2^{k+1} < \tau + |\xi| < -2^{k-1}\}. \tag{4.24d}$$

The decomposition of $\widehat{G}$ induces a decomposition for $\widehat{K}$, namely

$$\widehat{K}_{\pm,\pm,k} = \frac{\widehat{D}}{\widehat{W}} \widehat{G}_{\pm,\pm,k}. \tag{4.25a}$$

To compute the convolution in (4.22),

$$\begin{aligned} \widehat{\overline{K}}_{\pm,\pm,k} * \widehat{K}_{\pm,\pm,l}(-\tau,-\xi) &= \int \widehat{\overline{K}}_{\pm,\pm,k}(-\tau-\sigma,-\xi-\eta)\widehat{K}_{\pm,\pm,l}(\sigma,\eta)\, d\sigma\, d\eta \\ &= \int \widehat{K}^{\dagger}_{\pm,\pm,k}(\tau+\sigma,\xi+\eta)\gamma^0 \widehat{K}_{\pm,\pm,l}(\sigma,\eta)\, d\sigma\, d\eta, \end{aligned} \tag{4.25b}$$

we have 16 cases resulting from (4.24a–d) and (4.25b) as follows:

$$\{(\tau,\sigma,\xi,\eta) \colon \tau+\sigma>0, \sigma>0, \tau+\sigma-|\xi+\eta| \sim \pm 2^k, \sigma-|\eta| \sim \pm 2^l\} \tag{4.26a}$$

$$\{(\tau,\sigma,\xi,\eta) \colon \tau+\sigma<0, \sigma<0, \tau+\sigma+|\xi+\eta| \sim \pm 2^k, \sigma+|\eta| \sim \pm 2^l\} \tag{4.26b}$$

$$\{(\tau,\sigma,\xi,\eta) \colon \tau+\sigma<0, \sigma>0, \tau+\sigma+|\xi+\eta| \sim \pm 2^k, \sigma-|\eta| \sim \pm 2^l\} \tag{4.26c}$$

$$\{(\tau,\sigma,\xi,\eta) \colon \tau+\sigma>0, \sigma<0, \tau+\sigma-|\xi+\eta| \sim \pm 2^k, \sigma+|\eta| \sim \pm 2^l\} \tag{4.26d}$$

We label them as

$$\Sigma_{k,l}[(\pm,\pm);(\pm,\pm)], \tag{4.27}$$

and denote by $\Sigma_{k,l}$ without specifying which one precisely. We also use $\widehat{K}_k$ for an abbreviation of $\widehat{K}_{\pm,\pm,k}$ and $\widehat{G}_k$ for $\widehat{G}_{\pm,\pm,k}$.

Let g be an arbitrary function. We first compute

$$\begin{aligned} &\big[\gamma^0(\tau+\sigma) - \gamma^1(\xi+\eta)\big]\gamma^0\big[\gamma^0\sigma + \gamma^1\eta\big] \\ &= \gamma^0\big[(\tau+\sigma)\sigma - (\xi+\eta)\eta\big] + \gamma^1\big[(\tau+\sigma)\eta - \sigma(\xi+\eta)\big]. \end{aligned} \tag{4.28}$$

Thus, we have

$$\begin{aligned}
&\Big|\Big\langle \widehat{\overline{K}}_k * \widehat{K}_l, \widehat{g}\Big\rangle\Big| \\
=&\Big|\int \widehat{G}_k^\dagger(\tau+\sigma,\xi+\eta)\frac{\gamma^0(\tau+\sigma)-\gamma^1(\xi+\eta)}{(\tau+\sigma)^2-(\xi+\eta)^2}\gamma^0\frac{\gamma^0\sigma+\gamma^1\eta}{\sigma^2-\eta^2}\widehat{G}_l(\sigma,\eta)\cdot \\
&\widehat{\overline{g}}(-\tau,-\xi)d\sigma\, d\eta\, d\tau\, d\xi\Big| \\
\le& C\|\frac{\widehat{G}_k}{\widehat{M}^\alpha}\|_{L^2}\|\frac{\widehat{G}_l}{\widehat{M}^\alpha}\|_{L^2}\Big(\int I_{k,l}(\tau,\xi)\big|\widehat{g}(-\tau,-\xi)\big|^2 d\tau\, d\xi\Big)^{\frac{1}{2}}, \qquad (4.29a)
\end{aligned}$$

where $I_{k,l}(\tau,\xi)$ is given by

$$I_{k,l}(\tau,\xi) := \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)Q(\tau,\sigma,\xi,\eta)}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)}d\sigma\, d\eta, \qquad (4.29b)$$

and Q is given by the expression

$$Q(\tau,\sigma,\xi,\eta) := \big[(\tau+\sigma)\sigma-(\xi+\eta)\eta\big]^2+\big[(\tau+\sigma)\eta-\sigma(\xi+\eta)\big]^2, \qquad (4.29c)$$

and $D_{k,l}(\tau,\xi)$ is a slice of $\Sigma_{k,l}$ for fixed (τ,ξ); i.e.

$$D_{k,l}(\tau,\xi) := \{(\sigma,\eta) : (\tau,\sigma,\xi,\eta)\in\Sigma_{k,l}\}. \qquad (4.29d)$$

We distinguish the cases into two sets,

$$\Sigma_{k,l}[(\pm,\cdot);(\pm,\cdot)] \quad \text{and} \quad \Sigma_{k,l}[(\pm,\cdot);(\mp,\cdot)], \qquad (4.30)$$

due to the fact that the computation for the 8 cases in each set is similar. For simplicity, we will assume $k \ge l$, while the other case is similar.

Cases H: We have the following estimate:

$$\Big\|\frac{\widehat{\overline{K}}_{+,\cdot,k} * \widehat{K}_{+,\cdot,l}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac{1}{2}-\alpha)k}}\Big\|\frac{\widehat{G}_{+,\cdot,k}}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_{+,\cdot,l}}{\widehat{M}^\alpha}\Big\|_{L^2}, \qquad (4.31a)$$

$$\Big\|\frac{\widehat{\overline{K}}_{-,\cdot,k} * \widehat{K}_{-,\cdot,l}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac{1}{2}-\alpha)k}}\Big\|\frac{\widehat{G}_{-,\cdot,k}}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_{-,\cdot,l}}{\widehat{M}^\alpha}\Big\|_{L^2}. \qquad (4.31b)$$

In these cases, we have $(\tau+\sigma)\sigma > 0$. Through some algebraic manipulation, the expression Q can be written as

$$\begin{aligned}
2Q =&(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2+(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2+ \\
&8(\tau+\sigma)\sigma\big[|\xi+\eta||\eta|-(\xi+\eta)\eta\big]. \qquad (4.32)
\end{aligned}$$

Take the case of $\widehat{\overline{K}}_{+,+,k} * \widehat{K}_{+,+,l}$, as an example and in which $D_{k,l} = \{(\eta,\sigma) : \tau+\sigma-|\xi+\eta| \sim 2^k, \sigma-|\eta| \sim 2^l, (\tau,\sigma,\xi,\eta)\in\Sigma_{k,l}[(+,+);(+,+)]\}$. In this case

$\tau+\sigma>0$ and $\sigma>0$. In the $\eta\sigma$-plane, this is the region of the intersection of two forward cones. One has the thickness of 2^k and the translation of $(-\xi,-\tau)$, while the other has thickness of 2^l. It is mostly bounded, except for the extreme case which is when one cone moves along the other cone such that the intersection region is unbounded. Denote the set $\widetilde{D}_{kl}$ to be the projection of the set $D_{k,l}$ onto the η-axis. When the set $D_{k,l}$ is bounded, two facts, $|\xi|=|\xi+\eta|+|\eta|$ and $|\widetilde{D}_{kl}|\le C2^k$, are available and will be used in the following estimates.

For the first part, we have

$$\begin{aligned}
I^1_{k,l}(\tau,\xi) &:= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)}d\sigma\,d\eta \\
&= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2}d\sigma\,d\eta \\
&\le \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^k+|\xi+\eta|)^2}d\eta. \qquad (4.33a)
\end{aligned}$$

Consider the case when $|\xi+\eta|\ge|\eta|$; we get

$$\begin{aligned}
I^1_{k,l}(\tau,\xi) &\le \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^k+|\xi+\eta|)^2}d\eta \qquad (4.33b)\\
&\le \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(1+|\eta|)^{2\alpha}}{(2^k+|\xi+\eta|)^2}d\eta\,\widehat{E}^{2\alpha}(\tau,\xi) \le \frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}.
\end{aligned}$$

The extreme case, when one of the cones moves along the other, say down right, will not cause any trouble. Here, the region $D_{k,l}$ is unbounded. For the case $|\xi+\eta|\le|\eta|$, we get

$$\begin{aligned}
I^1_{k,l}(\tau,\xi) &\le \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^k+|\xi+\eta|)^2}d\eta \qquad (4.33c)\\
&\le \frac{C}{2^l}\int_{\widetilde{D}_{k,l}} \frac{(1+|\xi+\eta|)^{2\alpha}}{(2^k+|\xi+\eta|)^2}d\eta\,\widehat{E}^{2\alpha}(\tau,\xi) \le \frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}.
\end{aligned}$$

Again the extreme case, when one of the cones moves along the other cone, say up right, will not cause trouble. Here, the region $D_{k,l}$ is unbounded.

For the second part, we obtain

$$I^2_{k,l}(\tau,\xi) := \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma+|\xi+\eta|)^2(\sigma-|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)}d\sigma\,d\eta$$

$$= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma\, d\eta$$

$$\leq \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^l+|\eta|)^2} d\eta. \tag{4.34a}$$

Consider the case when $|\xi+\eta| \geq |\eta|$; we get

$$I^2_{k,l}(\tau,\xi) \leq \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^l+|\eta|)^2} d\eta$$

$$\leq \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{(|\eta|+1)^{2\alpha}}{(2^l+|\eta|)^2} d\eta\, \widehat{E}^{2\alpha} \leq \frac{C}{2^k} \frac{1}{2^{(1-2\alpha)l}} \widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \tag{4.34b}$$

For the case $|\xi+\eta| \leq |\eta|$, we get

$$I^2_{k,l}(\tau,\xi) \leq \frac{C}{2^{2k-l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(2^k+|\xi+\eta|)^2} d\eta \tag{4.34c}$$

$$\leq \frac{C}{2^l} \int_{\widetilde{D}_{k,l}} \frac{(1+|\xi+\eta|)^{2\alpha}}{(2^l+|\eta|)^2} d\eta\, \widehat{E}^{2\alpha}(\tau,\xi) \leq \frac{C}{2^k} \frac{1}{2^{(1-2\alpha)l}} \widehat{E}^{2\alpha}\widehat{S}^{2\alpha}.$$

For the third part, we get

$$I^3_{k,l}(\tau,\xi) := \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma)\sigma\big[|\xi+\eta||\eta| - (\xi+\eta)\eta\big]}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma\, d\eta$$

$$\leq \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma)\sigma|\xi+\eta||\eta|}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma\, d\eta$$

$$\leq \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}(\tau+\sigma)\sigma|\xi+\eta||\eta|}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma\, d\eta$$

$$\leq \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)} d\eta. \tag{4.35a}$$

Consider the case when $|\xi+\eta| \geq |\eta|$; we have

$$I^3_{k,l}(\tau,\xi) \leq \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)} d\eta$$

$$\leq \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} \frac{(|\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)} d\eta\, \widehat{E}^{2\alpha}$$

$$\leq \frac{C}{2^{2k+l}} \int_{\widetilde{D}_{k,l}} (|\eta|+1)^{2\alpha} d\eta\, \widehat{E}^{2\alpha} \leq \frac{C}{2^l} \frac{1}{2^{(1-2\alpha)k}} \widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \tag{4.35b}$$

The extreme case will not cause trouble since $\xi+\eta$ and η are of the same sign except on a bounded region; i.e., $\big[|\xi+\eta||\eta|-(\xi+\eta)\eta\big]=0$ except on a bounded region. For the case $|\xi+\eta|\le|\eta|$, we get

$$\begin{aligned} I^3_{k,l}(\tau,\xi) \le& \frac{C}{2^{2k+l}}\int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)}\,d\eta \\ \le& \frac{C}{2^{2k+l}}\int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}|\xi+\eta||\eta|}{(2^k+|\xi+\eta|)(2^l+|\eta|)}\,d\eta\,\widehat{E}^{2\alpha} \\ \le& \frac{C}{2^{2k+l}}\int_{\widetilde{D}_{k,l}} (|\xi+\eta|+1)^{2\alpha}\,d\eta\,\widehat{E}^{2\alpha} \le \frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \end{aligned} \tag{4.35c}$$

The extreme case will not cause trouble since $\xi+\eta$ and η are of the same sign except on a bounded region; i.e., $\big[|\xi+\eta||\eta|-(\xi+\eta)\eta\big]=0$ except on a bounded region.

Cases E: We have the following estimate:

$$\Big\|\frac{\widehat{\widehat{K}}_{-,\cdot,k}*\widehat{K}_{+,\cdot,l}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac12-\alpha)k}}\Big\|\frac{\widehat{G}_{-,\cdot,k}}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_{+,\cdot,l}}{\widehat{M}^\alpha}\Big\|_{L^2}, \tag{4.36a}$$

$$\Big\|\frac{\widehat{\widehat{K}}_{+,\cdot,k}*\widehat{K}_{-,\cdot,l}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2} \le \frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac12-\alpha)k}}\Big\|\frac{\widehat{G}_{+,\cdot,k}}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_{-,\cdot,l}}{\widehat{M}^\alpha}\Big\|_{L^2}. \tag{4.36b}$$

In these cases, we have $(\tau+\sigma)\sigma<0$. Through some algebraic manipulation, the expression Q can be written as

$$\begin{aligned} 2Q =& (\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2+(\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2 \\ &-8(\tau+\sigma)\sigma\big[|\xi+\eta||\eta|+(\xi+\eta)\eta\big]. \end{aligned} \tag{4.37}$$

Take the case of $\widehat{\widehat{K}}_{-,+,k}*\widehat{K}_{+,+,l}$, as an example and in which $D_{k,l}=\{(\eta,\sigma):\tau+\sigma+|\xi+\eta|\sim 2^k,\sigma-|\eta|\sim 2^l,(\tau,\sigma,\xi,\eta)\in\Sigma_{k,l}[(-,+);(+,+)]\}$. In this case $\tau+\sigma<0$ and $\sigma>0$. In $\eta\sigma$-plane, this is the region of the intersection of a truncated backward cone with a forward cone. One has the thickness of 2^k and the translation of $(-\xi,-\tau)$, while the other has thickness of 2^l. It is bounded for all cases. We still have the extreme case, which is when one cone moves along the other cone; though the region of intersection can be as large as possible, nevertheless it is bounded.

Again for the first part, we can estimate

$$I^1_{k,l}(\tau,\xi):=\int_{D_{k,l}}\frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)}d\sigma\,d\eta$$

$$= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2} d\sigma\, d\eta$$

$$\leq \frac{C}{2^{2l}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(\tau+\sigma-|\xi+\eta|)^2} d\sigma\, d\eta. \tag{4.38a}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \geq |\eta|$, $|\xi+\eta| \leq |\eta|$, and the extreme case. Through some calculations, in each case, we have

$$I^1_{k,l}(\tau,\xi) \leq \frac{C}{2^l} \frac{1}{2^{(1-2\alpha)k}} \widehat{E}^{2\alpha} \widehat{S}^{2\alpha}. \tag{4.38b}$$

For the second part, we derive

$$I^2_{k,l}(\tau,\xi) := \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)(\tau+\sigma-|\xi+\eta|)^2(\sigma-|\eta|)^2}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma\, d\eta$$

$$= \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)}{(\tau+\sigma+|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma\, d\eta$$

$$\leq \frac{C}{2^{2k}} \int_{D_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}}{(\sigma+|\eta|)^2} d\sigma d\eta$$

$$\leq \frac{C2^l}{2^{2k}} \int_{\widetilde{D}_{k,l}} \frac{(|\xi+\eta|+1)^{2\alpha}}{(2^l+|\eta|)^{2+2\alpha}} d\eta. \tag{4.39a}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \geq |\eta|$, $|\xi+\eta| \leq |\eta|$, and the extreme case. Through some calculations, in each case, we have

$$I^2_{k,l}(\tau,\xi) \leq \frac{C}{2^l} \frac{1}{2^{(1-2\alpha)k}} \widehat{E}^{2\alpha} \widehat{S}^{2\alpha}. \tag{4.39b}$$

For the third part, we have

$$I^3_{k,l}(\tau,\xi) := \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)|\tau+\sigma|\sigma\big[|\xi+\eta||\eta|+(\xi+\eta)\eta\big]}{\widehat{W}^2(\tau+\sigma,\xi+\eta)\widehat{W}^2(\sigma,\eta)} d\sigma\, d\eta$$

$$\leq \frac{C}{2^{2k+2l}} \int_{D_{k,l}} \frac{\widehat{M}^{2\alpha}(\xi+\eta)\widehat{M}^{2\alpha}(\eta)|\tau+\sigma|\sigma|\xi+\eta||\eta|}{(\tau+\sigma-|\xi+\eta|)^2(\sigma+|\eta|)^2} d\sigma\, d\eta$$

$$\leq \frac{C}{2^{2k+2l}} \int_{D_{k,l}} (|\xi+\eta|+1)^{2\alpha}(|\eta|+1)^{2\alpha}\, d\sigma\, d\eta. \tag{4.40a}$$

To estimate the above integral, we separate the cases for $|\xi+\eta| \geq |\eta|$, $|\xi+\eta| \leq |\eta|$, and the extreme case. Notice that for the extreme case, we have $|\xi+\eta||\eta|+(\xi+\eta)\eta=0$ except on a small part of the region of the intersection. Through some calculations, in each case, we have

$$I^3_{k,l}(\tau,\xi)\le\frac{C}{2^l}\frac{1}{2^{(1-2\alpha)k}}\widehat{E}^{2\alpha}\widehat{S}^{2\alpha}. \tag{4.40b}$$

Now we return to the proof of (4.22). Combining (4.31) and (4.36), we get

$$\begin{aligned}\Big|\big\langle\overline{K}_kK_l,g\big\rangle\Big| &\le C\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha}\Big\|_{L^2}\Big(\int I_{k,l}(\tau,\xi)\big|\widehat{g}(-\tau,-\xi)\big|^2d\tau\,d\xi\Big)^{\frac12}\\ &\le\frac{C}{2^{\frac{l}{2}}}\frac{1}{2^{(\frac12-\alpha)k}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha}\Big\|_{L^2}\|\widehat{E}^\alpha\widehat{S}^\alpha\widehat{g}\|_{L^2}\\ &\le\frac{C}{2^{(\frac14-\alpha)k+\frac{l}{4}}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\|\widehat{E}^\alpha\widehat{S}^\alpha\widehat{g}\|_{L^2}.\end{aligned} \tag{4.41}$$

Finally, we have

$$\begin{aligned}&\Big\|\frac{\widehat{\overline{K}}*\widehat{K}}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\le\sum_{k,l}\Big\|\frac{\widehat{\overline{K}}_k*\widehat{K}_l}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\\ &\le\sum_{k,l}\frac{C}{2^{(\frac14-\alpha)k+\frac{l}{4}}}\Big\|\frac{\widehat{G}_k}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\Big\|\frac{\widehat{G}_l}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}\le C\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|^2_{L^2}.\end{aligned} \tag{4.42}$$

This completes the proof. □

The estimates for the remaining cases are given in the following lemma.

Lemma 4.5. *For $j=1,2$ and $k=0,1,2,\dots$. The following estimates hold:*

$$\Big\|\frac{\widehat{b}_T*\big(\delta^{(k)}_{\mp}\widehat{\overline{A}}_{\pm,k}\big)*\big(\widehat{K}_j\big)}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\le C(k+1)T^{k-\frac12}\|f_{\pm,k}\|_{H^{-\alpha}}\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}, \tag{4.43a}$$

$$\Big\|\frac{\widehat{b}_T\widehat{\overline{K}}_j*\big(\delta^{(k)}_{\pm}\widehat{A}_{\pm,k}\big)}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\le C(k+1)T^{k-\frac12}\|f_{\pm,k}\|_{H^{-\alpha}}\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|_{L^2}, \tag{4.43b}$$

$$\Big\|\frac{\widehat{b}_T*\widehat{\overline{K}}_1*\widehat{K}_2}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\le C\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|^2_{L^2}, \tag{4.43c}$$

$$\Big\|\frac{\widehat{b}_T*\widehat{\overline{K}}_2*\widehat{K}_j}{\widehat{E}^\alpha\widehat{S}^\alpha}\Big\|_{L^2}\le C\Big\|\frac{\widehat{G}}{\widehat{M}^\alpha\widehat{S}^{\frac14}}\Big\|^2_{L^2}. \tag{4.43d}$$

The proof of Lemma 4.5 is a repetition of the arguments presented in Lemmas 4.1, 4.2, and 4.4, so we omit it.

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