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ROBUSTNESS OF ASYMPTOTIC PROPERTIES OF EVOLUTION FAMILIES UNDER PERTURBATIONS

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Abstract. In this paper, we present some conditions under which asymptotic properties of an evolution family $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$ persist under perturbations by a family $\mathcal{B} := (B(t), D(B(t))_{t\in\mathbb{R}_+}$ of linear operators on a Banach space X satisfying suitable conditions. Our results concern asymptotic properties like boundedness, periodicity, and asymptotic almost periodicity (even in the sense of Eberlein). An application of the abstract results to nonautonomous partial differential equations with delay is given.

1. Introduction

Consider the perturbed nonautonomous Cauchy problem

$$(PCP) \qquad \begin{cases} u'(t) = A(t)u(t) + B(t)u(t), & t \geq s \geq 0, \\ u(s) = x, \end{cases}$$

where $(A(t), D(A(t)))_{t\in\mathbb{R}_+}$ generates an evolution family $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$ of bounded linear operators on a Banach space X, and $(B(t), D(B(t)))_{t\in\mathbb{R}_+}$ a family of (possibly unbounded) operators. It is well-known that, under suitable assumptions, there exists a unique evolution family$\mathcal{U}_B$:=$(U_B(t,s))_{t\geq s\geq 0}$, related to $\mathcal{U}$ by a variation-of-constants formula and a Dyson-Phillips series (see [18, 19, 23, 24]). Our aim in this paper is to give sufficient conditions so that asymptotic properties of the evolution family $\mathcal{U}$ persist under unbounded perturbations. The properties considered here are, for instance, uniform boundedness, periodicity, and almost periodicity (even in the sense of Eberlein).

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For evolution families the exponential dichotomy is the asymptotic property which is extensively studied (see [10, 16, 17, 28, 29]), in contrast to the asymptotic properties considered here, which are not much studied. Recently, in the first part of [9], the authors considered nonautonomous perturbations of a semigroup, and in the second part they considered nonautonomous bounded perturbations of evolution families, by using evolution semigroups. This method requires assuming that the family of operators $(B(t), D(B(t)))_{t\in\mathbb{R}_+}$ has the same asymptotic behaviour as the original evolution family $\mathcal{U}$. In the first part of [9] (see also [8] for the autonomous case) the authors used a result of C. J. Batty and R. Chill [5], showing that the convolution product between a strongly continuous function L from $\mathbb{R}_+$ to the Banach space $\mathcal{L}(X)$ of all linear, bounded operators on X and a map $g \in L^1(\mathbb{R}_+, X)$ preserves some asymptotic properties of L. In this paper, we prove a nonautonomous version of the Batty-Chill result [5, Lemma 3.4]. That is, we extend this result for evolution families. This result (Lemma 3.1) will allow us to avoid the evolution semigroups method and then reach our aim without assumptions on the asymptotic behaviour of the family $\mathcal{B}$.

The same question has also been recently studied by Batty-Chill [6] for parabolic evolution families.

In Section 2, we fix some notations and define some asymptotic properties. In Section 3, we consider an evolution family $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$, perturbed by a family $\mathcal{B}$ of closed linear operators $(B(t), D(B(t)))_{t\geq 0}$ on X, fulfilling suitable Miyadera conditions, and prove that the perturbed evolution family $\mathcal{U}_B = (U_B(t,s))_{t\geq s\geq 0}$ inherits the asymptotic behaviour of $\mathcal{U}$. More specifically, we assume that the map $t \longmapsto U(t+s,s)x$ belongs, for every $x \in X$, and $s \geq 0$, to some closed subspace $\mathcal{E}$ of the Banach space $\mathrm{BUC}(\mathbb{R}_+, X)$, the space of uniformly continuous and bounded functions (e.g., the space of asymptotically almost-periodic functions or the space of Eberlein weakly almost-periodic functions); then the same is true for the evolution family $(U_B(t,s))_{t\geq s\geq 0}$.

In Section 4 we apply our abstract results to the following nonautonomous partial differential equation with delay

$$(DEs) \qquad \begin{cases} u'(t) = A(t)u(t) + L(t)u_t, & t \geq s \geq 0, \\ u(s) = x, \ \ u_s = f, \end{cases}$$

where $(A(t), D(A(t)))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ on X, $x \in X$, $f \in L^p([-1,0], X)$ for some $1 \leq p < \infty$, and $\big(L(t)\big)_{t\geq 0}$ is a family of bounded linear operators from $L^p([-1,0], X)$ to X. We first prove that for every initial value there is a unique mild solution of (DEs). Then we study, using results of Section 3, the asymptotic behaviour of these solutions.

For the autonomous version of the equation (DEs) the reader is referred to [4, 8, 26]. We also mention [12] for the inhomogeneous case. For the general theory of partial differential equations with delay we refer to [13, 31].

2. Preliminaries

In this section we recall some definitions and fix notations which will be used in the sequel.

Let X be a Banach space, and denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. If $\mathcal{L}(X)$ is endowed with the strong topology, it will be denoted by $\mathcal{L}_s(X)$.

A family of linear bounded operators $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$ on X is called an *evolution family* if

(1) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = Id$ for all $t \geq r \geq s \geq 0$,
(2) the mapping $\{(t,s) \in \mathbb{R}^2_+ : t \geq s\} \ni (t,s) \mapsto U(t,s)$ is strongly continuous.

For evolution families and well-posedness of the nonautonomous Cauchy problems we refer to [11, 21, 22, 28].

Let $\mathrm{BC}(\mathbb{R}_+, X)$ be the Banach space of all bounded, continuous functions from $\mathbb{R}_+$ to X, endowed with the uniform norm. The closed subspace of uniformly continuous functions will be denoted by $\mathrm{BUC}(\mathbb{R}_+, X)$.

If $f : \mathbb{R}_+ \to X$, the set of all translates, called the hull of f, is $H(f) := \{f(\cdot + t) : t \in \mathbb{R}_+\}$.

A function $f \in \mathrm{BC}(\mathbb{R}_+, X)$ is said to be *asymptotically almost periodic* if $H(f)$ is relatively compact in $\mathrm{BC}(\mathbb{R}_+, X)$.

If $H(f)$ is weakly relatively compact in $\mathrm{BC}(\mathbb{R}_+, X)$, the bounded continuous function $f : \mathbb{R}_+ \to X$ is called *Eberlein weakly asymptotically almost periodic*.

We recall also that a closed subspace $\mathcal{E}$ of $\mathrm{BUC}(\mathbb{R}_+, X)$ is said to be *translation bi-invariant* if for all $t \geq 0$, $f \in \mathcal{E} \iff f(\cdot + t) \in \mathcal{E}$, and *operator invariant* if $M \circ f \in \mathcal{E}$ for every $f \in \mathcal{E}$ and $M \in \mathcal{L}(X)$, where $M \circ f$ is defined by $(M \circ f)(t) = M(f(t))$, $t \geq 0$. A closed subspace $\mathcal{E}$ of $\mathrm{BUC}(\mathbb{R}_+, X)$ is said to be *homogeneous* if it is translation bi-invariant and operator invariant.

From [5] the following classes of X-valued functions are closed homogeneous subspaces of $\mathrm{BUC}(\mathbb{R}_+, X)$:

- the space $C_0(\mathbb{R}_+, X)$ of all continuous functions vanishing at infinity;
- the space $AAP(\mathbb{R}_+, X)$ of asymptotically almost-periodic functions;

- the space $W(\mathbb{R}_+, X)$ of Eberlein weakly asymptotically almost-periodic functions.

Observe that, if $f \in BC(\mathbb{R}_+, X)$, then f is uniformly continuous on compact sets of $\mathbb{R}_+$, but not necessarily on the whole set $\mathbb{R}_+$. Thus the fact that f belongs to $\mathrm{BUC}(\mathbb{R}_+, X)$ defines a particular type of asymptotic behaviour.

For more details on almost-periodic functions, we refer to [1, 14, 15]. For the almost periodicity of solutions of Cauchy problems, see, e.g., [2, 3, 7, 15, 27].

3. Asymptotic behaviour of perturbed evolution families

Let $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$ be an evolution family of bounded linear operators on X generated by the family of operators $(A(t),\, D(A(t)))_{t\geq 0}$.

Take a family $\mathcal{B} := (B(t), D(B(t)))_{t\geq 0}$ of linear operators, such that the following assumptions are satisfied:
(M1) For any $t \geq 0$, $B(t)$ is closed; for all $s \geq 0$ and $x \in X$, $U(t,s)x \in D(B(t))$, for almost every $t \geq s$, the map $B(\cdot + s)U(s + \cdot, s)x$ is measurable on $[0, +\infty)$, and

$$\int_s^{s+\alpha} ||B(\sigma)U(\sigma, s)x|| d\sigma \leq q||x||$$

for some constants $\alpha > 0$ and $0 \leq q < 1$;
(M2) there is $s_0 \geq 0$ such that for all $x \in X$ and $s \geq s_0$,

$$\int_0^{+\infty} ||B(s+\sigma)U(s+\sigma, s)x|| d\sigma \leq q||x||.$$

It follows from [24, Theorem 3.4 c)] that, under the assumption **(M1)**, there exists a unique evolution family $\mathcal{U}_B = (U_B(t,s))_{t\geq s\geq 0}$ such that, for every $x \in X$ and $s \in \mathbb{R}_+$, $U_B(t,s)x \in D(B(t))$ for almost every $t \geq s$, $B(\cdot)U_B(\cdot, s)x$ is locally integrable, and the following variation-of-constants formula holds:

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\sigma)B(\sigma)U_B(\sigma,s)x d\sigma\,, \quad t \geq s \geq 0. \tag{3.1}$$

We assume in this section that every orbit $t \mapsto U(t+s, s)x$, $x \in X$, $s \geq 0$, of the evolution family $\mathcal{U}$ belongs to a homogeneous subspace $\mathcal{E}$ of the space $\mathrm{BUC}(\mathbb{R}_+, X)$, and we prove that under the hypotheses **(M1)** and **(M2)** the trajectories of the perturbed evolution family $\mathcal{U}_B$ belong to the same subspace $\mathcal{E}$.

For this purpose we need to show the nonautonomous version of the Batty-Chill result [5, Lemma 7.2].

Lemma 3.1. *Let $(U(t,s))_{t\geq s\geq 0}$ be a bounded, strongly continuous evolution family of bounded linear operators on X. Let $\mathcal{E}$ be a closed homogeneous subspace of $BUC(\mathbb{R}_+,X)$. Assume that the function $\mathbb{R}_+ \ni t \mapsto U(t+s,s)x \in X$ belongs to $\mathcal{E}$ for every $x \in X$ and $s \geq 0$. If $g \in L^1(\mathbb{R}_+ ,X)$, then $\mathbb{R}_+ \ni t \mapsto U_s * g(t) := \int_0^t U(t+s,s+\sigma)g(\sigma)d\sigma$ belongs to $\mathcal{E}$ for all $s \geq 0$.*

Proof. Let $s \geq 0$. We can easily see that the map $g \longmapsto U_s * g$ is bounded from $L^1(\mathbb{R}_+,X)$ into $BC(\mathbb{R}_+,X)$. For a simple function $g := 1_{(a,b)} \otimes x$, $a \leq b \in \mathbb{R}_+$, $x \in X$, and $t \geq 0$,

$$U_s * g(t+b) = \int_0^{t+b} U(t+b+s,s+\sigma)g(\sigma)\,d\sigma = \int_a^b U(t+b+s,s+\sigma)x\,d\sigma$$
$$= U(t+b+s,b+s)\int_a^b U(b+s,s+\sigma)x\,d\sigma.$$

Hence, since $t \longmapsto U(t+s+b,s+b)x$ belongs to $\mathcal{E}$ for every $x \in X$, and $\mathcal{E}$ is translation bi-invariant, we can conclude that the function $\int_0^\cdot U(\cdot+s,s+\sigma)g(\sigma)\,d\sigma$ belongs to $\mathcal{E}$. By linearity, density, and the closedness of $\mathcal{E}$ we obtain the result. □

For the perturbed evolution family we have the following result.

Proposition 3.2. *Assume that* **(M1)** *and* **(M2)** *hold. For all $x \in X$ and $s \geq s_0$, we have*

$$\int_0^\infty ||B(\sigma+s)U_B(\sigma+s,s)x||d\sigma \leq \frac{q}{1-q}\|x\|.$$

Proof. Let $x \in X$ and $s \geq s_0$. From [24, Theorem 3.4 c)] the map $B(\cdot+s)U_B(\cdot+s,s)x$ is locally integrable and hence measurable. Now from the variation-of-constants formula (3.1), the closedness of the operators $B(t)$, $t \geq 0$, and the assumption **(M2)**, we have

$$\int_0^t ||B(\sigma+s)U_B(\sigma+s,s)x||d\sigma \leq \int_0^t ||B(s+\sigma)U(s+\sigma,s)x||d\sigma$$
$$+\int_0^t\int_s^{s+\sigma} ||B(\sigma+s)U(\sigma+s,\tau)B(\tau)U_B(\tau,s)x||d\tau\,d\sigma$$
$$\leq q\|x\| + \int_s^{s+t}\int_0^{t+s-\tau} ||B(\sigma+\tau)U(\sigma+\tau,\tau)B(\tau)U_B(\tau,s)x||d\sigma\,d\tau$$
$$\leq q||x|| + q\int_0^t ||B(\tau+s)U_B(\tau+s,s)x||d\tau \quad \text{for all} \quad t \geq 0.$$

Hence, since $0 \leq q < 1$, it follows that

$$\int_0^\infty ||B(\sigma+s)U_B(\sigma+s,s)x||d\sigma \leq \frac{q}{1-q}\|x\|. \qquad \square$$

We can now state our main abstract result.

Theorem 3.3. *Let $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$ be a bounded, strongly continuous evolution family of bounded linear operators on X, and let $(B(t), D(B(t)))_{t\geq 0}$ be a family of linear operators satisfying assumptions* **(M1)** *and* **(M2)**.

If $\mathcal{E}$ is a closed homogeneous subspace of $BUC(\mathbb{R}_+, X)$ and if the function $t \mapsto U(t+s,s)x$ belongs to $\mathcal{E}$ for all $x \in X$ and $s \geq 0$, then the function $t \mapsto U_B(t+s,s)x$ belongs to $\mathcal{E}$ for all $x \in X$ and $s \geq 0$.

Proof. For $x \in X$, $t \geq 0$, and $s \geq s_0$, by the formula (3.1), we have

$$\begin{aligned} U_B(t+s,s)x &= U(t+s,s)x + \int_s^{t+s} U(t+s,\sigma)B(\sigma)U_B(\sigma,s)x\,d\sigma, \\ &= U(t+s,s)x + \int_0^t U(t+s,\sigma+s)B(\sigma+s)U_B(\sigma+s,s)x\,d\sigma\,. \end{aligned}$$

Let the map f, from $\mathbb{R}_+$ to X, be defined as

$$f(t) := \int_0^t U(t+s,\sigma+s)B(\sigma+s)U_B(\sigma+s,s)x\,d\sigma, \quad t \geq 0.$$

Since the function $t \mapsto U(t+s,s)x$ belongs to $\mathcal{E}$, it suffices to show that the map f belongs to $\mathcal{E}$ as well. By Lemma 3.1, it is sufficient to show that the function $g(\cdot) := B(\cdot+s)U_B(\cdot+s,s)x$ belongs to $L^1(\mathbb{R}_+, X)$, and this follows from Proposition 3.2.

We have shown that the function $\mathbb{R}_+ \ni t \longmapsto U_B(t+s,s)x$ belongs to $\mathcal{E}$ for $s \geq s_0$ and $x \in X$. For $0 \leq s \leq s_0$ and $t \geq 0$, one can write

$$U_B(t+s_0+s,s)x = U_B(t+s+s_0, s+s_0)U_B(s+s_0,s)x.$$

As $s+s_0 \geq s_0$, then as shown above $t \longmapsto U_B(t+s_0+s,s)x$ belongs to $\mathcal{E}$, and by the translation bi-invariance of $\mathcal{E}$, $t \longmapsto U_B(t+s,s)x$ belongs to $\mathcal{E}$. This achieves the proof. $\square$

Remark 3.4. (i) The assumption **(M2)** holds, for example, if the family $\mathcal{U}$ is bounded and $B(t),\ t \geq 0$, are bounded linear operators such that

$$\int_0^\infty \|B(s)\|ds < \infty.$$

The assumption **(M2)** holds also if $\|U(t,s)\| \leq Me^{-\omega(t-s)},\ \omega > 0$, $\|B\|_\infty := \sup_{t\geq 0}\|B(t)\| < \infty$, and $M\|B\|_\infty < \omega$.

(ii) The condition **(M2)** has appeared in [8] for the autonomous case. In [9], the authors have assumed that **(M2)** holds for all $s \geq 0$, which does not include the case $\int_0^\infty \|B(s)\| ds < \infty$.

(iii) The Miyadera condition **(M1)** appeared for the first time in [20, 30] for the autonomous case and in [24] for the nonautonomous case.

4. Nonautonomous partial differential equations with delay

Let $(A(t), D(A(t)))_{t\geq 0}$ be a family of closed operators on a Banach space X generating an evolution family $(U(t,s))_{t\geq s\geq 0}$. Consider the following nonautonomous differential equation with delay

$$(DEs) \qquad \begin{cases} u'(t) = A(t)u(t) + L(t)u_t, & t \geq s \geq 0, \\ u(s) = x, \;\; u_s = f, \end{cases}$$

where

- $x \in X$, $f \in L^p([-1,0], X)$, for some $1 \leq p < \infty$,
- $\big(L(t)\big)_{t\geq 0}$ is a family of bounded linear operators from $L^p([-1,0],X)$ to X such that $L(\cdot) \in L^\infty_{loc}(\mathbb{R}_+, \mathcal{L}_s(L^p([-1,0],X),X))$,
- $u : [s-1,\infty) \to X$ and $u_t : [-1,0] \to X$, $t \geq s$, is defined by $u_t(\sigma) := u(t+\sigma)$ for all $\sigma \in [-1,0]$.

We will say that a function $u : [s-1,\infty) \longrightarrow X$ is a *mild solution* of the delay equation (DEs) if the map $r \mapsto L(r)u_r$ is locally integrable on $[s,\infty)$ and

$$u(t) = \begin{cases} U(t,s)x + \int_s^t U(t,r)L(r)u_r\, dr & \text{for all } t \geq s, \\ f(t-s) & \text{for a.e. } t \in [s-1,s). \end{cases} \tag{4.1}$$

In particular, we note that u is continuous for $t \geq s$.

In order to show the existence of mild solutions of (DEs) and their asymptotic behaviour, we proceed in a way similar to that of [4, 9] and consider the Banach space $\mathcal{X} := X \times L^p([-1,0],X)$, and the family of operators

$$\mathcal{A}(t) := \begin{pmatrix} A(t) & L(t) \\ 0 & \frac{d}{d\tau} \end{pmatrix}, \; t \geq 0,$$

with domains

$$D(\mathcal{A}(t)) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A(t)) \times W^{1,p}([-1,0],X) \;:\; f(0) = x \right\}, \quad t \geq 0 .$$

For each $t \geq 0$, the operator $\mathcal{A}(t)$ can be written as

$$\mathcal{A}(t) = \mathcal{A}_0(t) + \mathcal{B}(t) := \begin{pmatrix} A(t) & 0 \\ 0 & \frac{d}{d\tau} \end{pmatrix} + \begin{pmatrix} 0 & L(t) \\ 0 & 0 \end{pmatrix} .$$

We have the following intermediate result; see [4] for an analogous one.

Lemma 4.1. *The family of operators* $(\mathcal{A}_0(t))_{t\geq 0}$ *generates an evolution family* $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ *on* $\mathcal{X}$, *given by*

$$\mathcal{U}(t,s) := \begin{pmatrix} U(t,s) & 0 \\ U_{t,s} & T_0(t-s) \end{pmatrix},$$

where $(T_0(t))_{t\geq 0}$ *is the nilpotent left-translation semigroup on* $L^p([-1,0],X)$ *and* $U_{t,s} : X \to L^p([-1,0],X)$ *is defined as*

$$(U_{t,s}x)(\sigma) := \begin{cases} U(t+\sigma,s)x, & t+\sigma \geq s, \\ 0 & \text{otherwise} \end{cases}$$

for all $t \geq s \geq 0$.

Observe that $\big(\mathcal{B}(t)\big)_{t\geq 0}$ defines a family of bounded linear operators on $\mathcal{X}$, since

$$||\mathcal{B}(t)\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big)||_{\mathcal{X}} = ||\big(\begin{smallmatrix} L(t)f \\ 0 \end{smallmatrix}\big)||_{\mathcal{X}} = ||L(t)f||_X \leq ||L(t)|| \cdot ||f||_{L^p([-1,0],X)}.$$

According to the fact that

$$L(\cdot) \in L^\infty_{loc}(\mathbb{R}_+, \mathcal{L}_s(L^p([-1,0],X),X))$$

and following [25, Theorem 2.1] there is a unique "perturbed" evolution family $\big(\mathcal{U}_B(t,s)\big)_{t\geq s\geq 0}$ on $\mathcal{X}$, related to $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ by the variation-of-constants formula (3.1).

In the following proposition we relate the solutions of the delay differential equations (DEs), $s \geq 0$, to the perturbed evolution family $\big(\mathcal{U}_B(t,s)\big)_{t\geq s\geq 0}$. Let $\pi_1 : \mathcal{X} \longrightarrow X$ be the projection with respect to the first variable, $\pi_1\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big) := x$. The proof is similar to the one given in [9].

Proposition 4.2. *Let* $\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big) \in \mathcal{X}$, *and let* $u : [s-1,\infty) \to X$ *be the function defined by*

$$u(t) := \begin{cases} \pi_1(\mathcal{U}_B(t,s)\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big)), & t \geq s, \\ f(t-s) & \textit{for a.e. } t \in [s-1,s). \end{cases}$$

Then u *is a mild solution of equation* (DEs).

Conversely, if $u : [s-1,\infty) \to X$ *is a mild solution of equation* (DEs), *then for each* $\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big) \in \mathcal{X}$ *we have*

$$\mathcal{U}_B(t,s)\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big) = \big(\begin{smallmatrix} u(t) \\ u_t \end{smallmatrix}\big), \; t \geq s \geq 0.$$

In particular, we have existence and uniqueness of the mild solution of Equation (DEs) *for every* $s \geq 0$.

We will now show that the asymptotic behaviour of the undelayed equation, which is given by that of the evolution family $(U(t,s))_{t\geq s\geq 0}$, provides us the asymptotic behaviour of mild solutions of the delay equation (DEs), by means of the method illustrated in Section 3.

Theorem 4.3. *Assume that* $\big(U(t,s)\big)_{t\geq s\geq 0}$ *is bounded, and there exist* $s_0 \geq 0$ *and a constant* $0 \leq q < 1$ *such that*

$$\int_0^\infty \|L(s+\sigma)(U_{s+\sigma,s}\,x + T_0(\sigma)\,f)\|\,d\sigma \leq q\,\left\|\binom{x}{f}\right\| \tag{4.2}$$

for all $s \geq s_0$ *and* $\binom{x}{f} \in \mathcal{X}$. *If for every* $s \geq 0$ *and* $x \in X$ *the map* $\mathbb{R}_+ \ni t \mapsto U(t+s,s)x$ *belongs to a closed homogeneous subspace* $\mathcal{E}$ *of* $BUC(\mathbb{R}_+, X)$, *then the solutions of* (DEs), $s \geq 0$, $x \in X$, $f \in L^p([-1,0],X)$, $\mathbb{R}_+ \ni t \mapsto u(t+s)$ *belong to* $\mathcal{E}$.

Proof. Let $t \geq 0$ and $s \geq s_0$. By the definition of a mild solution of (DEs) (see (4.1)), we have

$$u(t+s) = U(t+s,s)x + \int_0^t U(t+s,r+s)L(r+s)u_{r+s}\,dr.$$

Hence, as $\mathbb{R}_+ \ni t \mapsto U(t+s,s)x$ belongs to $\mathcal{E}$, via Lemma 3.1 we need only to show that the function $g_s(\cdot) := L(\cdot+s)u_{\cdot+s}$ belongs to $L^1(\mathbb{R}_+, X)$. Now, from the definition of $\mathcal{U}$ and $\mathcal{B}$, we have

$$\int_0^\infty ||\mathcal{B}(t+s)\mathcal{U}(t+s,s)\binom{x}{f}||\,dt = \int_0^\infty ||L(t+s)(U_{s+t,s}\,x + T_0(t)f)||\,dt \leq q\,\|\binom{x}{f}\|$$

for all $s \geq s_0$. Then $\mathcal{U}$ and $\big(\mathcal{B}(t)\big)_{t\geq 0}$ satisfy the assumption **(M2)**. Hence, by Propositions 4.2 and 3.2 we have

$$\int_0^\infty ||g(r)||dr = \int_0^\infty ||\mathcal{B}(r+s)\mathcal{U}_B(r+s,s)\binom{x}{f}||\,dr \leq \frac{q}{1-q}\,\left\|\binom{x}{f}\right\|,$$

which proves the claim, and then $\mathbb{R}_+ \ni t \mapsto u(t+s)$ belongs to $\mathcal{E}$, for all $x \in X$, $f \in L^p([-1,0],X)$, and $s \geq s_0$. The case $s \leq s_0$ can be done using Proposition 4.2, and the same argument as for abstract results in Section 3. □

Example 4.4. Consider an operator $(A, D(A))$ generating a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X, e.g., the Dirichlet Laplacian Δ on $X := L^p(\Omega)$. Let α be a positive scalar function such that its integral is asymptotically almost periodic. It is known that the family $(A(t), D(A(t)))_{t\geq 0}$, $A(t) := \alpha(t)A$, generates the evolution family $(U(t,s))_{t\geq s\geq 0}$,

$$U(t,s) = T(\int_s^t \alpha(\sigma)\, d\sigma) \quad \text{for} \;\; t \geq s \geq 0.$$

One can see easily that the trajectories $t \longmapsto U(t+s,s)x,\ s \geq 0,\ x \in X$, are asymptotically almost periodic.

Consider also the family $(L(t))_{t\geq 0}$ given by

$$L(t)f := \int_{-1}^0 k(t,\sigma) f(\sigma)\, d\sigma,$$

where $k \in L^\infty_{loc}(\mathbb{R}_+ \times [-1,0], \mathcal{L}_s(X))$ and $\int_0^\infty \int_{-1}^0 \|k(t,\theta)\|\, d\theta\, dt < \infty$. This last condition implies $\int_0^\infty \|L(t)\|\, dt < \infty$, and then the assumption (4.2) is satisfied, via Remark 3.4 (i). Consequently, Theorem 4.3 allows us to conclude that the solutions of the partial differential equation with delay

$$\begin{cases} u'(t) = \alpha(t)Au(t) + \int_{-1}^0 k(t,\sigma)u(t+\sigma)\, d\sigma, \quad t \geq s \geq 0, \\ u(s) = x, \quad u_s = f, \end{cases}$$

are asymptotically almost periodic.

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