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ON THE CRITICAL EXPONENT FOR THE SCHRÖDINGER EQUATION WITH A NONLINEAR BOUNDARY CONDITION

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Dedicated to Howard A. Levine in honor of his 60th birthday

Abstract. We study the Schrödinger equation: $iu_t + u_{xx} = 0$, $x \in \mathbf{R}_+$, $t > 0$ with a nonlinear boundary condition $-u_x(0,t) = |u(0,t)|^{p-1}u(0,t)$, $t > 0$. We show that if $1 < p < 3$, every solution is global in $H^1(\mathbf{R}_+)$, while if $p \geq 3$, then nonglobal solutions exist.

1. Introduction

In this paper, we study the following initial-/boundary-value problem for the Schrödinger equation:

$$\begin{aligned} &iu_t + u_{xx} = 0, && 0 < x < \infty, \quad t > 0, \\ &-u_x(0,t) = |u(0,t)|^{p-1}u(0,t), && t > 0, \\ &u(x,0) = u_0(x), && 0 < x < \infty, \end{aligned} \tag{1.1}$$

where $p > 1$ is a real number.

In order to motivate the main results for problem (1.1), we recall some old results for three related problems. First, several authors have studied the initial-value problem for a nonlinear Schrödinger equation,

$$\begin{aligned} &iu_t + u_{xx} + |u|^{p-1}u = 0, && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = u_0(x), && -\infty < x < \infty. \end{aligned} \tag{1.2}$$

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They showed that if $1 < p < 5$, every (weak) solution of (1.2) which is initially in $H^1(\mathbf{R})$ is global and remains in this Sobolev space for all $t > 0$ (see [9, 10]); if $p \geq 5$ and the potential energy for u_0 is

$$E(u_0) \equiv \frac{1}{2}\int_{-\infty}^{\infty} |u_0'(x)|^2 dx - \frac{1}{p+1}\int_{-\infty}^{\infty} |u_0(x)|^{p+1} dx < 0,$$

the solution blows up in finite time (see [11, 14, 18]). Accordingly, the number $p_c = 5$ is called the critical exponent for (1.2). For more details, see [4, 13, 17] and the references cited therein.

It is also worth mentioning the following initial-value problem for the Schrödinger equation with strongly concentrated nonlinearity:

$$\begin{aligned} &iu_t + u_{xx} - \sum_{j=1}^{m} \gamma_j \delta(x - x_j)|u|^{2\sigma_j} u = 0, && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = u_0(x), && -\infty < x < \infty, \end{aligned} \tag{1.3}$$

where $\gamma_j \in \mathbf{R}$, $\sigma_j \in \mathbf{R}_+$, and the delta function $\delta(x - x_j)$ acts to localize the nonlinear effect at a specified point $x = x_j$. Global existence and blow-up results have been established for (1.3) in [3].

Secondly, consider the initial-/boundary-value problem for the heat equation

$$\begin{aligned} &u_t = u_{xx}, && 0 < x < \infty, \quad t > 0, \\ &-u_x(0,t) = u^p(0,t), && t > 0, \\ &u(x,0) = u_0(x), && 0 < x < \infty. \end{aligned} \tag{1.4}$$

In [7, 8] it was shown that if $1 < p \leq 2$, all nonnegative solutions of (1.4) blow up in finite time, while if $p > 2$, there are nontrivial global solutions. Thus for (1.4) $p_c = 2$.

Thirdly, we investigated the initial-/boundary-value problem for the wave equation:

$$\begin{aligned} &u_{tt} = u_{xx}, && 0 < x < \infty, \quad t > 0, \\ &-u_x(0,t) = |u(0,t)|^p, && t > 0, \\ &u(x,0) = f(x), u_t(x,0) = g(x), && 0 < x < \infty. \end{aligned} \tag{1.5}$$

In [1] we proved that if $0 < p \leq 1$, all (mild) solutions of (1.5) are global, while if $p > 1$ and

$$\int_0^{\infty} f(x)dx + \int_0^{\infty}\int_0^{y} g(x)dx\,dy > 0,$$

then the solution blows up in finite time. Hence for (1.5) $p_c = 1$.

Motivated by these results, in this paper we will determine the critical exponent for problem (1.1). However, due to the distinct nature of solutions on the half space, arguments used for (1.2) cannot apply to (1.1), and therefore certain different arguments will be employed in the sequel.

The plan of the paper is as follows: In Section 2 we establish the local existence and uniqueness result, and in Section 3 we show the existence and nonexistence of global solutions.

2. LOCAL EXISTENCE AND UNIQUENESS

We begin this section by noting that due to different representation formulas, all arguments used in [5, 9, 12] for (1.2) are not applicable to (1.1). For that reason, we first consider the following initial-/boundary-value problem

$$\begin{aligned} & iu_t + u_{xx} = 0, && 0 < x < \infty, \quad t > 0, \\ & -u_x(0,t) = h(t), && \qquad\qquad\quad t > 0, \\ & u(x,0) = u_0(x), && 0 < x < \infty. \end{aligned} \tag{2.1}$$

Theorem 2.1. *If $h(t) \in C[0,T)$ with $h'(t) \in C(0,T) \cap L^1(0,T)$ for some $T > 0$ and $u_0 \in H^2(\mathbf{R}_+)$ with $-u_0'(0) = h(0)$, then there exists a sequence $\{u^k\}_{k=1}^{\infty} \in C([0,T); H^2(\mathbf{R}_+)) \cap C^1((0,T); L^2(\mathbf{R}_+))$ such that $u^k(\cdot,t)$ converges in $H^1(\mathbf{R}_+)$ to the unique solution $u \in C([0,T); H^1(\mathbf{R}_+))$ of problem* (2.1).

Proof. Let $v(x,t) = u(x,t) - e^{-x}h(t)$. Then v satisfies

$$\begin{aligned} & iv_t + v_{xx} + F(x,t) = 0, && 0 < x < \infty, \quad t > 0, \\ & -v_x(0,t) = 0, && \qquad\qquad\quad t > 0, \\ & v(x,0) = v_0(x), && 0 < x < \infty, \end{aligned} \tag{2.2}$$

where $F(x,t) = e^{-x}(h(t) + ih'(t))$ and $v_0(x) = u_0(x) - h(0)e^{-x}$. We then introduce a sequence $\{v^k\}_{k=1}^{\infty}$ as follows:

$$\begin{aligned} & iv_t^k + v_{xx}^k + F^k(x,t) = 0, && 0 < x < \infty, \quad t > 0, \\ & -v_x^k(0,t) = 0, && \qquad\qquad\quad t > 0, \\ & v^k(x,0) = v_0(x), && 0 < x < \infty, \end{aligned} \tag{2.3}$$

where $F^k(x,t) = f^k(x)(h(t) + ih'(t))$ with $f^k(x)$ defined by

$$f^k(x) = \begin{cases} \Big[\big(\frac{k^2}{4} + \frac{k^3}{2}\big)x^4 - \big(\frac{k}{3} + k^2\big)x^3 + \big(1 + \frac{1}{2k} + \frac{1}{12k^2}\big)\Big]e^{-\frac{1}{k}}, & \\ & \text{if } 0 \le x \le \frac{1}{k}, \\ e^{-x}, & \text{if } \frac{1}{k} < x < \infty. \end{cases}$$

Denote by $\tilde{f}^k(x)$ the even extension of $f^k(x)$ in $-\infty < x < \infty$. It is easily seen that $\tilde{f}^k(x) \in H^3(\mathbf{R})$ and $\tilde{f}^k$ converges in $H^1(\mathbf{R})$ to $e^{-|x|}$. Define $\tilde{F}^k(x,t) = \tilde{f}^k(x)(h(t)+ih'(t))$ on $\mathbf{R}\times(0,T)$, $\tilde{v}_0(x) = v_0(-x)$ for $-\infty < x < 0$ and $\tilde{v}_0(x) = v_0(x)$ for $0 < x < \infty$. The function v^k then further satisfies the initial-value problem

$$\begin{aligned} &iv_t^k + v_{xx}^k + \tilde{F}^k(x,t) = 0, \qquad && -\infty < x < \infty, \quad t > 0, \\ &v^k(x,0) = \tilde{v}_0(x), && -\infty < x < \infty. \end{aligned} \tag{2.4}$$

Clearly, $\tilde{F}^k \in L^1((0,T);L^2(\mathbf{R}))$ is continuous on $(0,T)$, $\tilde{F}^k(\cdot,t) \in H^2(\mathbf{R})$ for $0 < t < T$, and $\tilde{F}_{xx}^k \in L^1((0,T);L^2(\mathbf{R}))$. Since $\tilde{v}_0 \in H^2(\mathbf{R})$, by Corollary 4.2.6 of [16], $v^k \in C([0,T);H^2(\mathbf{R})) \cap C^1((0,T);L^2(\mathbf{R}))$.

Let $u^k(x,t) = v^k(x,t) + e^{-x}h(t)$. Then

$$u^k \in C([0,T);H^2(\mathbf{R}_+)) \cap C^1((0,T);L^2(\mathbf{R}_+)),$$

and it satisfies

$$\begin{aligned} &iu_t^k + u_{xx}^k = (e^{-x} - f^k(x))(h(t) + ih'(t)), \quad && 0 < x < \infty, \quad t > 0, \\ &-u_x^k(0,t) = h(t), && t > 0, \\ &u^k(x,0) = u_0(x), && 0 < x < \infty. \end{aligned} \tag{2.5}$$

We consider $w(x,t) = u^j(x,t) - u^k(x,t)$. Then w satisfies

$$\begin{aligned} &iw_t + w_{xx} = (f^k(x) - f^j(x))(h(t) + ih'(t)), \qquad && 0 < x < \infty, \quad t > 0, \\ &-w_x(0,t) = 0, && t > 0, \\ &w(x,0) = 0, && 0 < x < \infty. \end{aligned}$$

Multiplying the above equation by $-i\overline{w}$ and using integration by parts, we have

$$\begin{aligned} \frac{d}{dt}\int_0^\infty w\overline{w}\,dx &= \int_0^\infty w_t\overline{w}\,dx + \int_0^\infty w\overline{w}_t\,dx \\ &= \int_0^\infty i\overline{w}w_{xx}\,dx - \int_0^\infty iw\overline{w}_{xx}\,dx \\ &\quad + i\int_0^\infty (f^k - f^j)[\overline{w}(h(t) + ih'(t)) - w(\overline{h}(t) - i\overline{h}'(t))]dx \\ &= i\int_0^\infty (f^k - f^j)[\overline{w}(h(t) + ih'(t)) - w(\overline{h}(t) - i\overline{h}'(t))]dx. \end{aligned} \tag{2.6}$$

Integrating (2.6) from 0 to t, we further have

$$\|w(\cdot,t)\|_{L^2}^2 = i\int_0^t\int_0^\infty (f^k - f^j)[\overline{w}(h(\tau) + ih'(\tau)) - w(\overline{h}(\tau) - i\overline{h}'(\tau))]dx\,d\tau$$

$$\leq 2\|f^j - f^k\|_{L^2} \int_0^t (|h(\tau)| + |h'(\tau)|)\|w(\cdot,\tau)\|_{L^2}\, d\tau. \tag{2.7}$$

Let $\Phi(t) = \big(2\|f^j - f^k\|_{L^2} \int_0^t (|h(\tau)| + |h'(\tau)|)\|w(\cdot,\tau)\|_{L^2} d\tau\big)^{1/2}$. Then Φ satisfies

$$\Phi'(t) \leq \|f^j - f^k\|_{L^2}(|h(t)| + |h'(t)|) \quad \text{and} \quad \Phi(0) = 0,$$

which combined with (2.7) yields

$$\|u^j(\cdot,t) - u^k(\cdot,t)\|_{L^2} \leq \|f^j - f^k\|_{L^2} \int_0^t (|h(\tau)| + |h'(\tau)|) d\tau. \tag{2.8}$$

We then consider $u_x^j(x,t) - u_x^k(x,t)$. Since $\tilde{f}^j(x) - \tilde{f}^k(x)$, the even extension of $f^j(x) - f^k(x)$ in $-\infty < x < \infty$ belongs to $H^3(\mathbf{R})$, $(u_x^j - u_x^k) \in C([0,T); H^2(\mathbf{R}_+)) \cap C^1((0,T); L^2(\mathbf{R}_+))$. Proceeding analogously, we find

$$\|u_x^j(\cdot,t) - u_x^k(\cdot,t)\|_{L^2} \leq \|(f^j)' - (f^k)'\|_{L^2} \int_0^t (|h(\tau)| + |h'(\tau)|) d\tau. \tag{2.9}$$

A combination of (2.8) and (2.9) then leads to

$$\|u^j(\cdot,t) - u^k(\cdot,t)\|_{H^1} \leq \|f^j - f^k\|_{H^1} \int_0^t |(|h(\tau)| + |h'(\tau)|) d\tau,$$

which means that $\{u^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $H^1(\mathbf{R}_+)$, and hence it converges in H^1 to the unique solution u of problem (2.1).

We are now in a position to establish the local existence and uniqueness result for problem (1.1).

Theorem 2.2. *Suppose that $u_0 \in H^3(\mathbf{R}_+)$. Then problem (1.1) has a unique solution $u \in C([0,T_0); H^1(\mathbf{R}_+))$ for some $T_0 > 0$.*

Proof. From the representation formula we have

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{4\pi t i}} \int_0^\infty \Big[\exp\Big(\frac{i(x-y)^2}{4t}\Big) + \exp\Big(\frac{i(x+y)^2}{4t}\Big)\Big] u_0(y) dy \\ + i \int_0^t \frac{1}{\sqrt{\pi(t-\tau)i}} \exp\Big(\frac{ix^2}{4(t-\tau)}\Big) |u(0,\tau)|^{p-1} u(0,\tau) d\tau. \end{aligned} \tag{2.10}$$

In particular,

$$u(0,t) = \frac{1}{\sqrt{\pi t i}} \int_0^\infty \exp\Big(\frac{iy^2}{4t}\Big) u_0(y) dy + i \int_0^t \frac{1}{\sqrt{\pi(t-\tau)i}} |u(0,\tau)|^{p-1} u(0,\tau) d\tau. \tag{2.11}$$

Noticing the previous theorem, it suffices to show that (2.11) has a unique solution $u(0,t) \in C[0,T_0) \cap C^1(0,T_0)$ for some $T_0 > 0$ and $\frac{d}{dt}(|u(0,t)|^{p-1} u(0,t))$

$\in C(0,T_0)\cap L^1(0,T_0)$. Equation (2.11) is a Volterra-type integral equation. However, the theory of [6] does not apply, and therefore we will employ the contraction-mapping argument. For simplicity, let $u(t)=u(0,t)$ and $\phi(x,t)=(4\pi ti)^{-1/2}\int_0^\infty[\exp(i(x-y)^2/4t)+\exp(i(x+y)^2/4t)]u_0(y)dy$. Note that ϕ satisfies the initial-/boundary-value problem

$$\begin{array}{lll} i\phi_t+\phi_{xx}=0, & 0<x<\infty, & t>0,\\ -\phi_x(0,t)=0, & & t>0,\\ \phi(x,0)=u_0(x), & 0<x<\infty. & \end{array}$$

We may use a sequence $\{\phi^k\}_{k=1}^\infty\in C([0,\infty);H^2(\mathbf{R}_+))\cap C^1((0,\infty);L^2(\mathbf{R}_+))$ to approximate ϕ as follows:

$$\begin{array}{lll} i\phi^k_t+\phi^k_{xx}=0, & 0<x<\infty, & t>0,\\ -\phi^k_x(0,t)=0, & & t>0,\\ \phi^k(x,0)=u^k_0(x), & 0<x<\infty, & \end{array}$$

where $\frac{d}{dx}(u_0^k)\in H_0^1(\mathbf{R}_+)$ and u_0^k converges in H^1 to u_0. Thus, $\phi\in C([0,\infty);H^1(\mathbf{R}_+))$. Moreover, it is easily seen that $\|\phi(\cdot,t)\|_{H^1}=\|u_0\|_{H^1}$. Hence by the Sobolev imbedding theorem (cf. Theorem 5.4 of [2]) $H^1(\mathbf{R}_+)\hookrightarrow C_B(\mathbf{R}_+)$, we find

$$|\phi(0,t)|\le\sup_{\mathbf{R}_+}|\phi(x,t)|\le C\|\phi(\cdot,t)\|_{H^1}=C\|u_0\|_{H^1}=\frac{m}{2}. \tag{2.12}$$

Rewriting (2.11)

$$u(t)=\mathcal{T}u(t)\equiv\phi(0,t)+i\int_0^t\frac{1}{\sqrt{\pi(t-\tau)i}}|u(\tau)|^{p-1}u(\tau)d\tau, \tag{2.13}$$

we show that $\mathcal{T}$ is a contraction mapping over $\mathcal{B}=\{u\in L^\infty(0,T_0):\|u\|_{L^\infty}\le m\}$ for some $T_0>0$. If $u\in\mathcal{B}$,

$$|\mathcal{T}u|\le|\phi(0,t)|+\int_0^t\frac{1}{\sqrt{\pi(t-\tau)}}m^p d\tau\le\frac{m}{2}+\frac{2m^p}{\sqrt{\pi}}\sqrt{T_0}\le m,$$

provided $T_0\le(\pi/16)m^{2(1-p)}$. On the other hand, since

$$\max\left(\left|\frac{\partial}{\partial z}(|z|^{p-1}z)\right|,\left|\frac{\partial}{\partial\overline{z}}(|z|^{p-1}z)\right|\right)\le((p+1)/2)|z|^{p-1},$$

letting $z=v+s(u-v)$ $(0\le s\le 1)$, we find

$$|u|^{p-1}u-|v|^{p-1}v=\int_0^1\frac{d}{ds}\left(|z|^{p-1}z\right)ds$$

$$= \int_0^1 \Big(\frac{\partial}{\partial z}(|z|^{p-1}z)(u-v) + \frac{\partial}{\partial \overline{z}}(|z|^{p-1}z)(\overline{u}-\overline{v})\Big)ds. \tag{2.14}$$

If $u, v \in \mathcal{B}$, since $|z| = |su + (1-s)v| \le m$, it follows from (2.14) that

$$||u|^{p-1}u - |v|^{p-1}v| \le (p+1)m^{p-1}|u-v|.$$

Therefore,

$$\|\mathcal{T}u - \mathcal{T}v\|_{L^\infty} \le \int_0^t \frac{p+1}{\sqrt{\pi(t-\tau)}} m^{p-1}\|u-v\|_{L^\infty}d\tau \le \alpha\|u-v\|_{L^\infty},$$

where $0 < \alpha < 1$ provided $T_0 < (\pi/4(p+1)^2)m^{2(1-p)}$. Hence (2.11) admits a unique solution $u(t)$ for $0 \le t < T_0$. Since $\phi(0,t) \in C[0,T_0]$, $u(t) \in C[0,T_0)$.

To show that $u(t) \in C^1(0,T_0)$, we introduce the formula

$$\begin{aligned}\frac{d}{dt}\left(|z(t)|^{p-1}z(t)\right) &= \frac{\partial}{\partial z}\left(|z|^{p-1}z\right)\frac{dz}{dt} + \frac{\partial}{\partial \overline{z}}\left(|z|^{p-1}z\right)\frac{d\overline{z}}{dt} \\ &= \frac{p+1}{2}|z|^{p-1}\frac{dz}{dt} + \frac{p-1}{2}|z|^{p-1}\left(\frac{z}{\overline{z}}\right)\frac{d\overline{z}}{dt}.\end{aligned} \tag{2.15}$$

Then by (2.13) and (2.15) we have

$$\begin{aligned}u(t) &= \phi(0,t) + \sqrt{\frac{4ti}{\pi}}|u(0)|^{p-1}u(0) \\ &+ \sqrt{\frac{4i}{\pi}}\int_0^t \sqrt{t-\tau}\Big(\frac{p+1}{2}|u(\tau)|^{p-1}u'(\tau) + \frac{p-1}{2}|u(\tau)|^{p-1}(\frac{u(\tau)}{\overline{u}(\tau)})\overline{u}'(\tau)\Big)d\tau.\end{aligned} \tag{2.16}$$

Differentiating (2.16) we obtain

$$\begin{aligned}u'(t) &= \phi_t(0,t) + \sqrt{\frac{i}{\pi t}}|u(0)|^{p-1}u(0) \\ &+ \sqrt{\frac{i}{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\Big(\frac{p+1}{2}|u(\tau)|^{p-1}u'(\tau) + \frac{p-1}{2}|u(\tau)|^{p-1}(\frac{u(\tau)}{\overline{u}(\tau)})\overline{u}'(\tau)\Big)d\tau.\end{aligned}$$

Let $\psi(x,t) = \phi_t(x,t)$. Then ψ satisfies

$$\begin{aligned}&i\psi_t + \psi_{xx} = 0, && 0 < x < \infty, \quad t > 0,\\ &-\psi_x(0,t) = 0, && t > 0,\\ &\psi(x,0) = iu_0''(x), && 0 < x < \infty.\end{aligned}$$

Thus, $\psi \in C([0,\infty); H^1(\mathbf{R}_+))$ with $\|\psi(\cdot,t)\|_{H^1} = \|u_0''\|_{H^1} \le \|u_0\|_{H^3}$, which by the Sobolev imbedding theorem $H^1(\mathbf{R}_+) \hookrightarrow C_B(\mathbf{R}_+)$ implies that $|\phi_t(0,t)| < \infty$. Hence $u'(t) \in C(0,T_0) \cap L^1(0,T_0)$.

Finally, by means of (2.15), we find that

$$\frac{d}{dt}\left(|u(t)|^{p-1}u(t)\right) \in C(0,T_0) \cap L^1(0,T_0).$$

3. EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS

As is well known, for solutions of problem (1.2), the L^2 norm of the solution is constant in time. Moreover, the potential energy

$$E(u) \equiv \frac{1}{2}\int_{-\infty}^{\infty} |u_x|^2 dx - \frac{1}{p+1}\int_{-\infty}^{\infty} |u|^{p+1} dx$$

is also constant. We show that similar results hold for problem (1.1). To this end, define

$$\mathcal{E}(u) \equiv \frac{1}{2}\int_0^{\infty} |u_x|^2 dx - \frac{1}{p+1}|u(0,t)|^{p+1}. \tag{3.1}$$

Lemma 3.1. *Let* $u \in C([0,T);H^1(\mathbf{R}_+))$ *be the solution of problem* (1.1). *Then the following hold:*

$$\|u(\cdot,t)\|_{L^2} \equiv \|u_0\|_{L^2}; \tag{3.2}$$

$$\mathcal{E}(u) \equiv \mathcal{E}(u_0). \tag{3.3}$$

Proof. Making use of the sequence $\{u^k\}_{k=1}^{\infty}$ given in (2.5), we first show (3.2). By the Sobolev imbedding theorem, $H^2(\mathbf{R}_+) \hookrightarrow C_B^1(\mathbf{R}_+)$, and hence $u^k \in C([0,T);C_B^1(\mathbf{R}_+))$. Multiplying the equation in (2.5) by $-i\overline{u}^k$ and using integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^{\infty} u^k\overline{u}^k dx = 2Re\int_0^{\infty} \overline{u}^k u_t^k dx \\
&= 2Re\int_0^{\infty} i\overline{u}^k u_{xx}^k dx + 2Im\int_0^{\infty} \overline{u}^k(e^{-x} - f^k(x))(h(t)+ih'(t))dx \\
&= -2Im\int_0^{\infty} \overline{u}^k u_{xx}^k dx + 2Im\int_0^{\infty} \overline{u}^k(e^{-x} - f^k(x))(h(t)+ih'(t))dx \\
&= 2Im(\overline{u}^k(0,t)u_x^k(0,t)) + 2Im\int_0^{\infty} \overline{u}_x^k u_x^k dx \\
&\quad + 2Im\int_0^{\infty} \overline{u}^k(e^{-x} - f^k(x))(h(t)+ih'(t))dx \\
&= -2Im(\overline{u}^k(0,t)u(0,t)|u(0,t)|^{p-1}) \\
&\qquad + 2Im\int_0^{\infty} \overline{u}^k(e^{-x} - f^k(x))(h(t)+ih'(t))dx.
\end{aligned} \tag{3.4}$$

Integrating (3.4) from 0 to t and passing to the limit, we obtain (3.2), since $Im(\overline{u}(0,t)\ u(0,t)|u(0,t)|^{p-1}) = 0$.

To show (3.3), we need to establish the following identity:

$$\begin{aligned}
&\int_0^\infty e^{-x}u\overline{h}\,dx + \int_0^\infty e^{-x}\overline{u}h\,dx - \int_0^\infty e^{-x}u_0\overline{h}(0)dx - \int_0^\infty e^{-x}\overline{u}_0h(0)dx \\
&= i\int_0^t\int_0^\infty e^{-x}u\overline{h}\,dx\,d\tau - i\int_0^t\int_0^\infty e^{-x}\overline{u}h\,dx\,d\tau \\
&\qquad + \int_0^t\int_0^\infty e^{-x}u\overline{h}'\,dx\,d\tau + \int_0^t\int_0^\infty e^{-x}\overline{u}h'\,dx\,d\tau. \qquad (3.5)
\end{aligned}$$

Making use of the sequence $\{v^k\}_{k=1}^\infty$ given in (2.3), we have

$$\begin{aligned}
\frac{d}{dt}\int_0^\infty v^k\overline{v}^k dx &= \int_0^\infty v_t^k\overline{v}^k dx + \int_0^\infty v^k\overline{v}_t^k dx \\
&= \int_0^\infty i\overline{v}^k v_{xx}^k dx - \int_0^\infty iv^k\overline{v}_{xx}^k dx \\
&\quad + \int_0^\infty i\overline{v}^k f^k(h+ih')dx - \int_0^\infty iv^k f^k(\overline{h}-i\overline{h}')dx \\
&= \int_0^\infty i\overline{v}^k f^k(h+ih')dx - \int_0^\infty iv^k f^k(\overline{h}-i\overline{h}')dx. \qquad (3.6)
\end{aligned}$$

Integrating (3.6) over $(0,t)$ and letting $k\to\infty$, we further have

$$\begin{aligned}
&\int_0^\infty v\overline{v}\,dx - \int_0^\infty v_0\overline{v}_0\,dx \\
&= i\int_0^t\int_0^\infty e^{-x}\overline{v}(h+ih')dx\,d\tau - i\int_0^t\int_0^\infty e^{-x}v(\overline{h}-i\overline{h}')dx\,d\tau,
\end{aligned}$$

which, by the transformation $v(x,t) = u(x,t) - e^{-x}h(t)$, leads to (3.5).

We now show (3.3). Letting $V^k(t) \equiv (1/2)\int_0^\infty v_x^k\overline{v}_x^k dx$, we find

$$\begin{aligned}
&V^k(t+\eta) - V^k(t) = \frac{1}{2}\int_0^\infty [v_x^k(x,t+\eta)\overline{v}_x^k(x,t+\eta) - v_x^k(x,t)\overline{v}_x^k(x,t)]dx \\
&= \frac{1}{2}\int_0^\infty [v_x^k(x,t+\eta) - v_x^k(x,t)]\overline{v}_x^k(x,t+\eta)dx \\
&\quad + \frac{1}{2}\int_0^\infty [\overline{v}_x^k(x,t+\eta) - \overline{v}_x^k(x,t)]v_x^k(x,t)dx \\
&= -\frac{1}{2}\int_0^\infty [v^k(x,t+\eta) - v^k(x,t)]\overline{v}_{xx}^k(x,t+\eta)dx
\end{aligned}$$

$$- \frac{1}{2}\int_0^\infty [\overline{v}^k(x,t+\eta) - \overline{v}^k(x,t)]v_{xx}^k(x,t)dx.$$

Hence we have

$$\begin{aligned}
\frac{d}{dt}\Big[\frac{1}{2}\int_0^\infty v_x^k \overline{v}_x^k\,dx\Big] &= \lim_{\eta\to 0}\frac{V^k(t+\eta)-V^k(t)}{\eta}\\
&= -\frac{1}{2}\int_0^\infty v_t^k \overline{v}_{xx}^k\,dx - \frac{1}{2}\int_0^\infty \overline{v}_t^k v_{xx}^k\,dx\\
&= -\frac{1}{2}\int_0^\infty [if^k\overline{v}_{xx}^k(h+ih') - if^k v_{xx}^k(\overline{h}-i\overline{h}')]dx\\
&= \frac{i}{2}\int_0^\infty [(f^k)'\overline{v}_x^k(h+ih') - (f^k)'v_x^k(\overline{h}-i\overline{h}')]dx. \qquad (3.7)
\end{aligned}$$

Integrating (3.7) from 0 to t and letting $k\to\infty$, we then find

$$\frac{1}{2}\int_0^\infty v_x\overline{v}_x\,dx - \frac{1}{2}\int_0^\infty v_0'\overline{v}_0'\,dx = -\frac{i}{2}\int_0^t\int_0^\infty e^{-x}[\overline{v}_x(h+ih') - v_x(\overline{h}-i\overline{h}')]dx\,d\tau,$$

which, by the transformation $v_x = u_x + e^{-x}h$, gives

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty u_x\overline{u}_x\,dx + \frac{1}{2}\int_0^\infty e^{-x}u_x\overline{h}\,dx + \frac{1}{2}\int_0^\infty e^{-x}\overline{u}_x h\,dx\\
&- \frac{1}{2}\int_0^\infty u_0'\overline{u}_0'\,dx - \frac{1}{2}\int_0^\infty e^{-x}u_0'\overline{h}(0)dx - \frac{1}{2}\int_0^\infty e^{-x}\overline{u}_0' h(0)dx\\
&\quad = \frac{i}{2}\int_0^t\int_0^\infty e^{-x}u_x\overline{h}\,dx\,d\tau - \frac{i}{2}\int_0^t\int_0^\infty e^{-x}\overline{u}_x h\,dx\,d\tau\\
&+ \frac{1}{2}\int_0^t\int_0^\infty e^{-x}u_x\overline{h}'\,dx\,d\tau + \frac{1}{2}\int_0^t\int_0^\infty e^{-x}\overline{u}_x h'\,dx\,d\tau.
\end{aligned}$$

Since $h(t) = |u(0,t)|^{p-1}u(0,t)$, using integration by parts and recalling (3.5) we have

$$\begin{aligned}
\frac{1}{2}\int_0^\infty u_x\overline{u}_x\,dx - \frac{1}{2}\int_0^\infty u_0'\overline{u}_0'\,dx &= \frac{1}{2}\int_0^t h(\tau)\overline{u}_t(0,\tau)d\tau + \frac{1}{2}\int_0^t \overline{h}(\tau)u_t(0,\tau)d\tau\\
&= \frac{1}{p+1}|u(0,t)|^{p+1} - \frac{1}{p+1}|u_0(0)|^{p+1}.
\end{aligned}$$

We then present the global existence result.

Theorem 3.2. *If $1 < p < 3$, every solution $u \in C([0,T);H^1(\mathbf{R}_+))$ is global.*

Proof. By the Cauchy-Schwartz inequality and (3.2), we have

$$|u(0,t)|^2 = -\int_0^\infty (u\overline{u}_x + u_x\overline{u})dx \tag{3.8}$$
$$\leq 2\Big(\int_0^\infty |u|^2 dx\Big)^{\frac{1}{2}}\Big(\int_0^\infty |u_x|^2 dx\Big)^{\frac{1}{2}} = 2\|u_0\|_{L^2}\|u_x\|_{L^2}.$$

Let $\sigma = (p+1)/4$, $0 < \sigma < 1$. Combining (3.3) and (3.8), we then find

$$\|u_x\|_{L^2}^2 = 2\mathcal{E}(u_0) + \frac{2}{p+1}|u(0,t)|^{p+1} \leq 2\mathcal{E}(u_0) + \frac{2^{\frac{p+3}{2}}}{p+1}\|u_0\|_{L^2}^{\frac{p+1}{2}}\|u_x\|_{L^2}^{2\sigma}. \tag{3.9}$$

This inequality can be rewritten as $f(r) \equiv r - br^\sigma \leq 2\mathcal{E}(u_0)$, where $r = ||u_x||_{L^2}^2$ and $b = \big(2^{(p+3)/2}/(p+1)\big)\,||u_0||_{L^2}^{(p+1)/2}$. Since $f(r)$ is convex and $f(0) = f(r_0) = 0$ with $r_0 = b^{1/(1-\sigma)}$, we further find

$$(1-\sigma)(r - r_0) = f'(r_0)(r - r_0) \leq f(r) \leq 2\mathcal{E}(u_0),$$

which implies $\|u_x\|_{L^2}^2 \leq M$ for some $M = M(u_0,\sigma) > 0$. As a consequence, we obtain an a priori bound on $|u(0,t)|$,

$$\begin{aligned} |u(0,t)| &= \left(\tfrac{p+1}{2}\right)^{\frac{1}{p+1}}\left(\|u_x\|_{L^2}^2 - 2\mathcal{E}(u_0)\right)^{\frac{1}{p+1}} \\ &\leq \left(\tfrac{p+1}{2}\right)^{\frac{1}{p+1}}\left(M + 2|\mathcal{E}(u_0)|\right)^{\frac{1}{p+1}} \qquad \text{for } t \geq 0. \end{aligned}$$

Hence, by Theorem 2.1 $u \in C([0,\infty); H^1(\mathbf{R}_+))$.

For $\sigma = 1$, we see that if $||u_0||_{L^2}$ is so small that

$$\frac{2^{\frac{p+3}{2}}}{p+1}||u_0||_{L^2}^{\frac{p+1}{2}} < 1,$$

by virtue of (3.9), we again obtain an a priori time-independent bound on the H^1 norm. Thus we have the following result.

Corollary 3.3 *If $p = 3$ and $||u_0||_{L^2}$ is sufficiently small, the solution of problem* (1.1) *is global.*

Motivated by [15], we then establish the finite-time blow-up result for problem (1.1). Introduce a function $\psi(x) = (x^2 + x)e^{-x}$ on $[0,\infty)$, and let $\psi_\epsilon(x) = (1/\epsilon)\psi(\epsilon x)$, where ϵ is a small positive constant to be determined. Clearly $\psi_\epsilon(0) = 0$, $\psi_\epsilon'(0) = 1$, and $\psi_\epsilon''(0) = 0$. Moreover,

$$\left|\frac{d^k}{dx^k}\psi_\epsilon(x)\right| \leq \epsilon^{k-1}C_k, \qquad \text{for } k = 0,1,2,3. \tag{3.10}$$

In particular, $|\psi'_\epsilon(x)| \le 1$ for $0 \le x < \infty$ since

$$|\psi'(x)| = \begin{cases} (-x^2 + x + 1)e^{-x}, & \text{if } 0 \le x \le \frac{1+\sqrt{5}}{2}, \\ (x^2 - x - 1)e^{-x}, & \text{if } \frac{1+\sqrt{5}}{2} < x < \infty. \end{cases}$$

Theorem 3.4. *If $p \ge 3$ and $\mathcal{E}(u_0) < 0$, the solution of problem* (1.1) *blows up in finite time.*

Proof. Assume to the contrary that the solution $u \in C([0,\infty); H^1(\mathbf{R}_+))$. Setting $W^k(t) = -Im\int_0^\infty \psi_\epsilon u^k \overline{u}_x^k\,dx$, we have

$$\begin{aligned} &W^k(t+\eta) - W^k(t) \\ &= -Im\int_0^\infty [\psi_\epsilon(x)u^k(x,t+\eta)\overline{u}_x^k(x,t+\eta) - \psi_\epsilon(x)u^k(x,t)\overline{u}_x^k(x,t)]dx \\ &= -Im\int_0^\infty \psi_\epsilon(x)[u^k(x,t+\eta) - u^k(x,t)]\overline{u}_x^k(x,t+\eta)dx \\ &\quad - Im\int_0^\infty \psi_\epsilon(x)[\overline{u}_x^k(x,t+\eta) - \overline{u}_x^k(x,t)]u^k(x,t)dx \\ &= -Im\int_0^\infty \psi_\epsilon(x)[u^k(x,t+\eta) - u(x,t)]\overline{u}_x^k(x,t+\eta)dx \\ &\quad + Im\int_0^\infty [\overline{u}^k(x,t+\eta) - \overline{u}^k(x,t)](\psi_\epsilon(x)u^k(x,t))_x\,dx. \end{aligned}$$

Therefore,

$$(W^k)'(t) = -Im\int_0^\infty \psi_\epsilon u_t^k \overline{u}_x^k\,dx + Im\int_0^\infty \overline{u}_t^k(\psi_\epsilon u^k)_x\,dx \tag{3.11}$$

$$\begin{aligned} &= -Im\int_0^\infty \psi_\epsilon(iu_{xx}^k)\overline{u}_x^k\,dx + Im\int_0^\infty(-i\overline{u}_{xx}^k)(\psi_\epsilon u^k)_x\,dx + \delta^k \\ &= -Re\int_0^\infty \psi_\epsilon u_{xx}^k\overline{u}_x^k\,dx - Re\int_0^\infty \overline{u}_{xx}^k(\psi'_\epsilon u^k + \psi_\epsilon u_x^k)dx + \delta^k \\ &= -Re\int_0^\infty \psi_\epsilon(u_x^k\overline{u}_x^k)_x\,dx - |u(0,t)|^{p-1}\overline{u}(0,t)u^k(0,t) \\ &\quad + Re\int_0^\infty \overline{u}_x^k(\psi''_\epsilon u^k + \psi'_\epsilon u_x^k)dx + \delta^k \\ &= 2\int_0^\infty \psi'_\epsilon u_x^k\overline{u}_x^k\,dx - |u(0,t)|^{p-1}\overline{u}(0,t)u^k(0,t) + \frac{1}{2}Re\int_0^\infty (u^k\overline{u}^k)_x\psi''_\epsilon\,dx + \delta^k \\ &= 2\int_0^\infty \psi'_\epsilon|u_x^k|^2\,dx - |u(0,t)|^{p-1}\overline{u}(0,t)u^k(0,t) - \frac{1}{2}\int_0^\infty \psi'''_\epsilon|u^k|^2\,dx + \delta^k, \end{aligned}$$

where

$$\delta^k = -Re \int_0^\infty (e^{-x} - f^k)[\psi_\epsilon \overline{u}_x^k (h + ih') + (\psi_\epsilon u^k)_x (\overline{h} - i\overline{h}')]dx,$$

and $\delta^k \to 0$ as $k \to \infty$. Integrating (3.11) and letting $k \to \infty$ then yields

$$\begin{aligned} &- Im \int_0^\infty \psi_\epsilon u \overline{u}_x \, dx + Im \int_0^\infty \psi_\epsilon u_0 \overline{u}_0' \, dx \\ &= \int_0^t \Big[2\int_0^\infty \psi_\epsilon' |u_x|^2 \, dx - |u(0,\tau)|^{p+1} - \frac{1}{2} \int_0^\infty \psi_\epsilon''' |u|^2 \, dx \Big] d\tau. \end{aligned} \tag{3.12}$$

Since $|\psi_\epsilon'(x)| \le 1$ and $|\psi_\epsilon'''(x)| \le \epsilon^2 C_3$, for a sufficiently small ϵ we find that if $p \ge 3$, then

$$\begin{aligned} &2\int_0^\infty \psi_\epsilon' |u_x|^2 \, dx - |u(0,t)|^{p+1} - \frac{1}{2} \int_0^\infty \psi_\epsilon''' |u|^2 \, dx \\ &\le 2\int_0^\infty |u_x|^2 \, dx - |u(0,t)|^{p+1} + \frac{1}{2}\epsilon^2 C_3 \int_0^\infty |u|^2 \, dx \\ &= 4\mathcal{E}(u_0) - \frac{p-3}{p+1} |u(0,t)|^{p+1} + \frac{1}{2}\epsilon^2 C_3 \|u_0\|_{L^2}^2 \le 2\mathcal{E}(u_0), \end{aligned}$$

which, in conjunction with (3.12), then leads to

$$-Im \int_0^\infty \psi_\epsilon u \overline{u}_x \, dx \le -Im \int_0^\infty \psi_\epsilon u_0 \overline{u}_0' \, dx + 2\mathcal{E}(u_0) t \le \lambda - \mu t, \tag{3.13}$$

where λ and μ are positive constants.

Set $\Psi(x) = \int_0^x \psi_\epsilon(y) dy$. Then $\Psi(0) = 0$ and $\Psi(x) > 0$ for $0 < x < \infty$, and $\Psi \in C_B^1[0,\infty)$. Upon manipulation and integration by parts, we find

$$\begin{aligned} &\frac{d}{dt} \int_0^\infty \Psi |u^k|^2 \, dx = 2Re \int_0^\infty \Psi u^k \overline{u}_t^k \, dx = 2Im \int_0^\infty i\Psi u^k \overline{u}_t^k \, dx \\ &= 2Im \int_0^\infty \Psi u^k \overline{u}_{xx}^k \, dx + 2Im \int_0^\infty \Psi u^k (e^{-x} - f^k)(h + ih') \, dx \\ &= -2Im \int_0^\infty \overline{u}_x^k (\Psi' u^k + \Psi u_x^k) \, dx + 2Im \int_0^\infty \Psi u^k (e^{-x} - f^k)(h + ih') dx \\ &= -2Im \int_0^\infty \psi_\epsilon u^k \overline{u}_x^k \, dx + 2Im \int_0^\infty \Psi u^k (e^{-x} - f^k)(h + ih') dx. \end{aligned} \tag{3.14}$$

Integrating (3.14) over $(0,t)$ and passing to the limit, we have

$$\int_0^\infty \Psi |u|^2 \, dx - \int_0^\infty \Psi |u_0|^2 \, dx = -2Im \int_0^t \int_0^\infty \psi_\epsilon u \overline{u}_x \, dx,$$

which, together with (3.13), then yields

$$0 \leq \int_0^\infty \Psi |u|^2\,dx \leq -\mu t^2 + 2\lambda t + \int_0^\infty \Psi |u_0|^2\,dx \leq -\mu t^2 + 2\lambda t + \sup_{[0,\infty)} \Psi ||u_0||_{L^2}^2. \tag{3.15}$$

For large values of t, the right-hand side of (3.15) becomes negative, which is a contradiction, and hence the solution u cannot be global. The proof is completed.

Remark 1. Theorem 3.4 is optimal because if $p \geq 3$ and $\mathcal{E}(u_0) \geq 0$, problem (1.1) may have a global solution. For example, consider $u(x,t) = \exp(-x + it)$. Clearly, u satisfies (1.1) with $u_0(x) = e^{-x}$, and moreover $\mathcal{E}(u_0) = (p-3)/4(p+1) \geq 0$ for $p \geq 3$. A similar phenomenon has been observed for problem (1.2) (see [18, 19]).

Remark 2. The higher-dimensional version of problem (1.1) takes the form

$$\begin{aligned} &iu_t + \Delta u = 0, && x \in \mathbf{R}_+^n, \quad t > 0,\\ &-u_{x_1}(0,x',t) = |u(0,x',t)|^{p-1}u(0,x',t), && t > 0,\\ &u(x,0) = u_0(x), && x \in \mathbf{R}_+^n, \end{aligned} \tag{3.16}$$

where $\mathbf{R}_+^n = \{(x_1, x') : x_1 > 0,\ x' \in \mathbf{R}^{n-1}\}$. Although our arguments appear not to be extendable for $n \geq 2$, noticing the fact that for problem (1.2) in n-dimensional spaces $p_c = 1 + 4/n$, we conjecture the following: The critical exponent for problem (3.16) is $p_c = 1 + 2/n$.

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