Differential and Integral Equations
Volume 17, Numbers 11-12, Pages 1273–1292
November/December 2004

GLOBAL EXISTENCE OF POSITIVE SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS IN A HALF-SPACE

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(Submitted by: J.A. Goldstein)

Abstract. We prove the global existence of continuous solutions of the semilinear parabolic problem $\Delta u-\frac{\partial}{\partial t}u+Vu^p=0$ in $\mathbb{R}^n_+\times(0,\infty)$, where $\mathbb{R}^n_+$ is a half-space in $\mathbb{R}^n$, $n\geq 3$. The potential V is in some functional class $\mathcal{K}^\infty$. Our approach uses the Shauder fixed-point theorem.

1. Introduction

In [5], Zhang studied the following semilinear parabolic problem:

$$\begin{cases}\Delta u-\dfrac{\partial}{\partial t}u+Vu^p=0 \quad \text{in} \quad \mathbb{R}^n\times(0,\infty),\\ u>0 \quad \text{in} \quad \mathbb{R}^n\times(0,\infty),\\ u(x,0)=u_0(x), \quad x\in\mathbb{R}^n,\end{cases} \tag{1.1}$$

where $n\geq 3$, $p>1$, Δ is the Laplacian operator, and V is in some functional class P^∞. Then he proved that for all $M>1$, there exists b_0 such that for any nonnegative $u_0\in C^2(\mathbb{R}^n)$ with $\|u_0\|_\infty\leq b_0$, the problem (1.1) has a global continuous solution u satisfying

$$\frac{1}{M}\int_{\mathbb{R}^n}G_0(x,t;y,0)u_0(y)dy\leq u(x,t)\leq M\int_{\mathbb{R}^n}G_0(x,t;y,0)u_0(y)dy.$$

Here G_0 is the fundamental solution of $\Delta-\dfrac{\partial}{\partial t}$ on $\mathbb{R}^n\times(0,\infty)$. The elliptic problem,

$$\begin{cases}\Delta u+Vu^p=0 \quad \text{in} \quad D,\\ u>0 \quad \text{in} \quad D,\\ u=0 \quad \text{on} \quad \partial D,\end{cases} \tag{1.2}$$

Accepted for publication: May 2004.
AMS Subject Classifications: 35J60, 35K55.

is widely studied (see Kenig and Ni [1], Lin [2], Ni [3], and Zhao [6]). In particular, Zhao showed in [6] that (1.2) has infinitely many bounded continuous solutions when D is an unbounded domain in $\mathbb{R}^n$ ($n \geq 3$) with compact Lipschitz boundary, and V is in the class of Green tight functions which encompasses all the cases discussed by Kenig, Lin, and Ni.

In this paper we consider the semilinear parabolic problem in the case of the half-space $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ x_n > 0\}$ which has no compact boundary. More precisely, we shall study the following problem:

$$\begin{cases} \Delta u - \dfrac{\partial}{\partial t} u + V u^p = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0,\infty), \\ u > 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n_+, \\ u = 0 \quad \text{on} \quad \partial\mathbb{R}^n_+ \times (0,\infty), \end{cases} \tag{1.3}$$

where $n \geq 3$ and $p > 1$. Solutions of these problems are understood in the distribution sense.

The first purpose is to introduce a new and general functional class denoted by $\mathcal{K}^\infty$. This class properly contains the class P^∞ introduced by Zhang [5], and consequently it contains the class of Green tight functions introduced in [6]. The second is to prove the global existence of bounded continuous solutions of (1.3). Note that the elliptic counterpart with $V \equiv 0$,

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n_+, \\ u = 0 \quad \text{on} \quad \partial\mathbb{R}^n_+, \end{cases}$$

has no bounded solutions.

Let us introduce the condition on the potential V.
Let $V = V(x,t)$ be a Borel-measurable function on $\mathbb{R}^n_+ \times \mathbb{R}$. For $h > 0$ and $c > 0$, we put

$$\begin{aligned} M^c_h(V) = \sup_{x,t} \int_{t-h}^{t} \int_{(|x-y|<\sqrt{h})\cap\mathbb{R}^n_+} \min\Big(1, \frac{y_n}{\sqrt{t-s}}\Big)\Gamma_c(x,t;y,s)|V(y,s)|\,dy\,ds \\ + \sup_{y,s} \int_{s}^{s+h} \int_{(|x-y|<\sqrt{h})\cap\mathbb{R}^n_+} \min\Big(1, \frac{x_n}{\sqrt{t-s}}\Big)\Gamma_c(x,t;y,s)|V(x,t)|\,dx\,dt. \end{aligned}$$

Here and always $\Gamma_c(x,t;y,s) = \frac{1}{(t-s)^{n/2}} \exp(-c\frac{|x-y|^2}{t-s})$.

Definition 1.1. We say that V is in the class $\mathcal{K}^\infty$ if it satisfies, for all $c > 0$,

$$\lim_{h\to 0} M^c_h(V) = 0 \text{ and } M^c_\infty(V) \equiv \lim_{h\to\infty} M^c_h(V) < \infty.$$

Our main result is the next.

Theorem 1.1. *Let $p > 1$ and $V \in \mathcal{K}^\infty$. For $M > 1$, there is a constant $b_0 > 0$ such that for each nonnegative $u_0 \in C^2(\mathbb{R}^n_+)$ satisfying $\|u_0\|_\infty \leq b_0$, there exists a positive and continuous solution u of* (1.3) *such that*

$$\frac{1}{M}\int_{\mathbb{R}^n_+} G(x,t;y,0)u_0(y)dy \leq u(x,t) \leq M\int_{\mathbb{R}^n_+} G(x,t;y,0)u_0(y)dy,$$

for all $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$. Here G is the Green's function of $\Delta - \frac{\partial}{\partial t}$ on $\mathbb{R}^n_+ \times (0,\infty)$.

The outline of this paper is as follows. In Section 2, we study some properties of the class $\mathcal{K}^\infty$, and we give typical examples of their functions. In Section 3, we prove some estimates on the Green's function G that will be used to prove the existence result. In Section 4, we prove the main result. Our approach is based on a fixed-point argument.

In the sequel, a point $x \in \mathbb{R}^n_+$ will be denoted by (x', x_n) with $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$, when needed. The letters k, C, and c denote generic constants which may vary in value from line to line.

2. The class $\mathcal{K}^\infty$

Proposition 2.1. *Let $M > 0$. For $0 < \lambda < 2$ and $0 < \varepsilon < 1$, the function*

$$\varphi_{\lambda,\varepsilon}(x) = \begin{cases} \frac{1}{x_n^\lambda |x|^{\frac{2-\lambda}{2}}} & \text{if } x_n \leq M \\ \frac{1}{x_n^2 |x|^\varepsilon} & \text{if } x_n \geq M \end{cases}$$

is in the class $\mathcal{K}^\infty$.

Proof. We first prove $\lim_{h\to\infty} M_h^c(\varphi_{\lambda,\varepsilon}) = 0$. We have

$$\begin{aligned}
&\int_{t-h}^t \int_{B(x,\sqrt{h})} |\varphi_{\lambda,\varepsilon}(y)| \min(1, \frac{y_n}{\sqrt{t-s}})\Gamma_c(x,t;y,s)dy\,ds \\
&\leq \int_{\mathbb{R}^n_+ \cap \{y_n \leq M\} \cap B(x,\sqrt{h})} \frac{1}{y_n^\lambda |y|^{1-\lambda/2}} \min(1, \frac{y_n}{|x-y|}) \frac{1}{|x-y|^{n-2}} dy \\
&+ \int_{\mathbb{R}^n_+ \cap \{y_n \geq M\} \cap B(x,\sqrt{h})} \frac{1}{y_n^2 |y|^\varepsilon} \min(1, \frac{y_n}{|x-y|}) \frac{1}{|x-y|^{n-2}} dy \equiv I_1 + I_2. \quad (2.1)
\end{aligned}$$

We estimate I_1. We have

$$\begin{aligned}
I_1 &\leq 2\int_{\mathbb{R}^n_+ \cap \{y_n \leq M\} \cap B(x,\sqrt{h})} \frac{1}{|y|^{1-\lambda/2}} \frac{y_n^{1-\lambda}}{(y_n + |x-y|)} \frac{1}{|x-y|^{n-2}} dy \\
&\leq 2\int_{\mathbb{R}^n_+ \cap \{y_n \leq M\} \cap B(x,\sqrt{h})} \frac{y_n^{-\lambda/2}}{(y_n + |x-y|)|x-y|^{n-2}} dy
\end{aligned}$$

$$\leq 2\int_{|x'-y'|\leq\sqrt{h}}\int_{0<y_n\leq M}\frac{y_n^{-\lambda/2}}{(y_n+|x'-y'|)|x'-y'|^{n-2}}dy_n\,dy'$$
$$= C\int_0^{\sqrt{h}}\int_0^M\frac{t^{-\lambda/2}}{t+r}dt\,dr = C\int_0^{\sqrt{h}}r^{-\lambda/2}\int_0^{M/r}\frac{s^{-\lambda/2}}{1+s}ds\,dr$$
$$\leq C\int_0^{\sqrt{h}}r^{-\lambda/2}dr\int_0^{\infty}\frac{s^{-\lambda/2}}{1+s}ds = C(\sqrt{h})^{1-\lambda/2}. \tag{2.2}$$

Now we estimate I_2.

$$I_2 \leq 2\int_{\{y_n\geq M\}\cap B(x,\sqrt{h})}\frac{1}{y_n|y|^{\varepsilon}(y_n+|x-y|)|x-y|^{n-2}}dy$$
$$= 2\Big(\int_{\{y_n\geq M\}\cap B(x,\sqrt{h})\cap\{|y|\geq|x-y|\}}\dots dy + \int_{\{y_n\geq M\}\cap B(x,\sqrt{h})\cap\{|y|\leq|x-y|\}}\dots dy\Big)$$
$$\leq 2\Big(\int_{\{y_n\geq M\}\cap\{|x'-y'|\leq\sqrt{h}\}}\frac{1}{y_n(y_n+|x'-y'|)|x'-y'|^{n-2+\varepsilon}}dy_n\,dy'$$
$$+ \int_{\{y_n\geq M\}\cap\{|y'|\leq\sqrt{h}\}}\frac{1}{y_n(y_n+|y'|)|y'|^{n-2+\varepsilon}}dy_n\,dy'\Big)$$
$$= C\int_0^{\sqrt{h}}r^{-\varepsilon}\int_M^{\infty}\frac{1}{t(t+r)}dt\,dr$$
$$\leq C\int_0^{\sqrt{h}}r^{-\varepsilon}dr\int_M^{\infty}\frac{1}{t^2}dt = \frac{2}{M(1-\varepsilon)}(\sqrt{h})^{1-\varepsilon}. \tag{2.3}$$

Combining (2.1)–(2.3), we obtain

$$M_h^c(\varphi_{\lambda,\varepsilon}) \leq C\Big((\sqrt{h})^{1-\lambda/2} + (\sqrt{h})^{1-\varepsilon}\Big).$$

Now we prove that $M_\infty^c(\varphi_{\lambda,\varepsilon}) < \infty$. We have

$$\int_{-\infty}^t\int_{\mathbb{R}^n_+}|\varphi_{\lambda,\varepsilon}(y)|\min(1,\frac{y_n}{\sqrt{t-s}})\Gamma_c(x,t;y,s)dy\,ds$$
$$= \int_{\mathbb{R}^n_+}|\varphi_{\lambda,\varepsilon}(y)|\min(1,\frac{y_n}{|x-y|})\frac{1}{|x-y|^{n-2}}dy$$
$$= 2\Big(\int_{\mathbb{R}^n_+\cap\{y_n\leq M\}}\dots dy + \int_{\mathbb{R}^n_+\cap\{y_n\geq M\}}\dots dy\Big) \equiv 2(I_3+I_4). \tag{2.4}$$

We estimate I_3. We have

$$I_3 \leq 2\int_{\mathbb{R}^n_+\cap\{y_n\leq M\}}\frac{y_n^{1-\lambda}}{|y|^{1-\lambda/2}(y_n+|x-y|)}\frac{1}{|x-y|^{n-2}}dy$$

$$
\begin{aligned}
&\leq 2\int_0^M \int_{\mathbb{R}^{n-1}} \frac{1}{|y'|^{1-\lambda/2}} \frac{y_n^{1-\lambda}}{(y_n+|x'-y'|)} \frac{1}{|x'-y'|^{n-2}} dy' dy_n \\
&= 4\Big(\int_0^M\int_{\mathbb{R}^{n-1}\cap\{|y'|\geq|x'-y'|\}} \ldots dy'\,dy_n + \int_0^M\int_{\mathbb{R}^{n-1}\cap\{|y'|\leq|x'-y'|\}} \ldots dy'\,dy_n\Big) \\
&\leq 4\Big(\int_0^M \int_{\mathbb{R}^{n-1}} \frac{1}{|x'-y'|^{1-\lambda/2}} \frac{y_n^{1-\lambda}}{(y_n+|x'-y'|)} \frac{1}{|x'-y'|^{n-2}} dy'\,dy_n \\
&+ \int_0^M \int_{\mathbb{R}^{n-1}} \frac{1}{|y'|^{1-\lambda/2}} \frac{y_n^{1-\lambda}}{(y_n+|y'|)} \frac{1}{|y'|^{n-2}} dy' dy_n\Big) \\
&\leq C\int_0^M t^{1-\lambda}\int_0^\infty \frac{1}{r^{1-\lambda/2}}\frac{1}{(t+r)} dr dt \\
&= C\int_0^M t^{-\lambda/2} dt \int_0^\infty \frac{1}{s^{1-\lambda/2}} \frac{1}{(1+s)} ds < \infty. \qquad (2.5)
\end{aligned}
$$

Now we estimate I_4. We have

$$
\begin{aligned}
I_4 &\leq 2\int_{\mathbb{R}^n_+\cap\{y_n\geq M\}} \frac{1}{|y|^\varepsilon y_n(y_n+|x-y|)|x-y|^{n-2}} dy \\
&\leq 2\int_M^\infty \int_{\mathbb{R}^{n-1}} \frac{1}{|y'|^\varepsilon} \frac{1}{y_n(y_n+|x'-y'|)|x'-y'|^{n-2}} dy'\,dy_n \\
&= 2\Big(\int_M^\infty \int_{\{|y'|\leq|x'-y'|\}} \ldots dy'\,dy_n + \int_M^\infty \int_{\{|y'|\geq|x'-y'|\}} \ldots dy'\,dy_n\Big) \\
&\leq 2\Big(\int_M^\infty \int_{\{|y'|\leq|x'-y'|\}} \frac{1}{|y'|^\varepsilon} \frac{1}{y_n(y_n+|y'|)|y'|^{n-2}} dy'\,dy_n \qquad (2.6) \\
&+ \int_M^\infty \int_{\{|y'|\geq|x'-y'|\}} \frac{1}{|x'-y'|^\varepsilon} \frac{1}{y_n(y_n+|x'-y'|)|y'|^{n-2}} dy'\,dy_n\Big) \\
&= C\int_M^\infty \frac{1}{t}\int_0^\infty \frac{1}{r^\varepsilon(t+r)} dr\,dt = C\int_M^\infty \frac{1}{t^{1+\varepsilon}} dt \int_0^\infty \frac{1}{s^\varepsilon(1+s)} ds < \infty.
\end{aligned}
$$

Combining (2.4)–(2.6), we obtain $M^c_\infty(\varphi_{\lambda,\varepsilon}) < \infty$. □

Now we shall compare the classes $\mathcal{K}^\infty$ and P^∞. For the reader's convenience, we recall the definition of the class P^∞ introduced by Zhang in [5].

Definition 2.1. We say that a Borel measurable function $V = V(x,t)$ on $\mathbb{R}^{n+1}$ is in the class P^∞ if it satisfies, for all $c > 0$,

$$\lim_{h\to 0} N^c_h(V) = 0 \text{ and } N^c_\infty(V) \equiv \lim_{h\to\infty} N^c_h(V) < \infty,$$

where

$$N_h^c(V) = \sup_{x,t} \int_{t-h}^{t} \int_{|x-y|<\sqrt{h}} \Gamma_c(x,t;y,s)|V(y,s)|dy\,ds$$
$$+ \sup_{y,s} \int_{s}^{s+h} \int_{|x-y|<\sqrt{h}} \Gamma_c(x,t;y,s)|V(x,t)|dx\,dt.$$

The class P^∞ contains the time-independent class of Green tight functions introduced by Zhao in [6]. We recall that a Borel-measurable function $V = V(x)$ on $\mathbb{R}^n$, $n \geq 3$, is said to be a Green tight function if it satisfies

$$\lim_{h\to 0} \sup_x \int_{|x-y|<h} \frac{|V(y)|}{|x-y|^{n-2}} dy = \lim_{M\to\infty} \sup_x \int_{|y|\geq M} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0.$$

Proposition 2.2. *The class $\mathcal{K}^\infty$ properly contains the class P^∞.*

Proof. From the definitions it is clear that $P^\infty \subset \mathcal{K}^\infty \cap L^1_{\text{loc}}(\overline{\mathbb{R}^n_+} \times (0,\infty))$. On the other hand, for $1 < \lambda < 2$ and $0 < \varepsilon < 1$, the function $\varphi_{\lambda,\varepsilon} \notin L^1_{\text{loc}}(\overline{\mathbb{R}^n_+} \times (0,\infty))$. The conclusion follows from Proposition 2.1. $\square$

For $c > 0$, we denote

$$G_c(x,t;y,s) = \Gamma_c(x,t;y,s) - \Gamma_c(x,t;\tilde{y},s), \tag{2.7}$$

where $\tilde{y} = (y', -y_n)$. Note that $(\frac{c}{\pi})^{n/2} G_c$ is the Green's function of $\frac{1}{4c}\Delta_x - \frac{\partial}{\partial t}$ on $\mathbb{R}^n_+ \times \mathbb{R}$ with Dirichlet boundary condition.

Lemma 2.3. *i) For all $x, y \in \mathbb{R}^n_+$ and $s < t$, we have*

$$\frac{1}{2}\min(1, \frac{x_n y_n}{t-s})\Gamma_c(x,t;y,s) \leq G_c(x,t;y,s) \leq \min(1, \frac{x_n y_n}{t-s})\Gamma_c(x,t;y,s).$$

ii) For all $c' : 0 < c' < c$, we have

$$G_c(x,t;y,s) \leq (1 + \frac{2}{\sqrt{c-c'}}) \min(1, \frac{x_n}{\sqrt{t-s}}) \min(1, \frac{y_n}{\sqrt{t-s}})\Gamma_{c'}(x,t;y,s).$$

iii) There exists a constant $k > 0$ such that

$$|\nabla_x G_c(x,t;y,s)| \leq k \min(1, \frac{y_n}{\sqrt{t-s}}) \frac{\exp(-\frac{c}{2}\frac{|x-y|^2}{t-s})}{(t-s)^{(n+1)/2}}.$$

iv) There exists a constant $k > 0$ such that

$$|\frac{\partial G_c}{\partial t}(x,t;y,s)| \leq k \min(1, \frac{y_n}{\sqrt{t-s}}) \frac{\exp(-\frac{c}{2}\frac{|x-y|^2}{t-s})}{(t-s)^{n/2+1}}.$$

Proof. i) follows from the fact that

$$G_c(x,t;y,s) = \Big(1-\exp(-4c\frac{x_ny_n}{t-s})\Big)\frac{\exp(-c\frac{|x-y|^2}{t-s})}{(t-s)^{n/2}},$$

and

$$\frac{1}{2}\min(1,a) \le 1-e^{-a} \le \min(1,a), \text{ for all } a>0.$$

We will prove ii). Let $c' \in (0,c)$. We will show that

$$\min(1,\frac{x_ny_n}{t-s}) \le \min(1,\frac{x_n}{\sqrt{t-s}})\min(1,\frac{y_n}{\sqrt{t-s}})\Big(1+\frac{|x_n-y_n|}{\sqrt{t-s}}\Big).$$

It suffices to check this inequality for $0 < \frac{x_n}{\sqrt{t-s}} \le 1$ and $\frac{y_n}{\sqrt{t-s}} \ge 1$, since in the other cases, it is clear. We have

$$\frac{x_ny_n}{t-s} \le \frac{x_n(x_n+|x_n-y_n|)}{t-s} \le \frac{x_n}{\sqrt{t-s}}\Big(1+\frac{|x-y|}{\sqrt{t-s}}\Big).$$

Hence, by using the inequality $\big(1+\frac{|x-y|}{\sqrt{t-s}}\big)e^{(c'-c)\frac{|x-y|^2}{t-s}} \le 1+\frac{2}{\sqrt{c-c'}}$, we obtain the inequality in ii).

Now we prove iii). We have $G_c(x,t;y,s) = \Gamma_c(x,t;y,s) - \Gamma_c(x,t;\tilde{y},s)$. By a simple computation, we have

$$|\nabla_x G_c(x,t;y,s)| \le k\frac{\exp(-\frac{c}{2}\frac{|x-y|^2}{t-s})}{(t-s)^{(n+1)/2}}. \tag{2.8}$$

From the reproducing property, (2.8), and ii), we then have

$$\begin{aligned}|\nabla_x G_c(x,t;y,s)| &\le \int_{\mathbb{R}^n_+} |\nabla_x G_c(x,t;\xi,\frac{t+s}{2})|G_c(\xi,\frac{t+s}{2};y,s)d\xi \\ &\le k\int_{\mathbb{R}^n} \frac{\exp(-c\frac{|x-\xi|^2}{t-s})}{(t-s)^{(n+1)/2}}\min(1,\frac{y_n}{\sqrt{t-s}})\frac{\exp(-c\frac{|\xi-y|^2}{t-s})}{(t-s)^{n/2}}d\xi \\ &= k'\min(1,\frac{y_n}{\sqrt{t-s}})\frac{\exp(-\frac{c}{2}\frac{|x-y|^2}{t-s})}{(t-s)^{(n+1)/2}}.\end{aligned}$$

We will prove iv). By a simple computation, we have

$$|\frac{\partial G_c}{\partial t}(x,t;y,s)| = |\frac{1}{4c}\Delta_x G_c(x,t;y,s)| \le k\frac{\exp(-\frac{c}{2}\frac{|x-y|^2}{t-s})}{(t-s)^{n/2+1}}. \tag{2.9}$$

From the reproducing property, (2.9), and ii), we then have

$$|\frac{\partial G_c}{\partial t}(x,t;y,s)| = |\frac{1}{4c}\Delta_x G_c(x,t;y,s)|$$

$$\leq \frac{1}{4c}\int_{\mathbb{R}^n_+} |\Delta_x G_c(x,t;\xi,\frac{t+s}{2})| G_c(\xi,\frac{t+s}{2};y,s)d\xi$$

$$\leq k\int_{\mathbb{R}^n} \frac{\exp(-c\frac{|x-\xi|^2}{t-s})}{(t-s)^{n/2+1}}\min(1,\frac{y_n}{\sqrt{t-s}})\frac{\exp(-c\frac{|\xi-y|^2}{t-s})}{(t-s)^{n/2}}d\xi$$

$$= k'\min(1,\frac{y_n}{\sqrt{t-s}})\frac{\exp(-\frac{c}{2}\frac{|x-y|^2}{t-s})}{(t-s)^{n/2+1}}.$$

The key in proving the existence theorem is the following lemma.

Lemma 2.4. *For $0 < a < b$, there exist positive constants $C_{a,b}$ and c depending only on a and b such that*

$$i) \int_s^t \int_{\mathbb{R}^n_+} G_a(x,t;z,\tau)|V(z,\tau)|G_b(z,\tau;y,s)dz\,d\tau \leq C_{a,b}M^c_\infty(V)G_a(x,t;y,s).$$

$$ii) \int_s^t \int_{\mathbb{R}^n_+} G_b(x,t;z,\tau)|V(z,\tau)|G_a(z,\tau;y,s)dz\,d\tau \leq C_{a,b}M^c_\infty(V)G_a(x,t;y,s).$$

Proof. We will prove only i), since the proof of ii) is similar. Clearly we can assume $s = 0$. For simplicity we write

$$J(x,t;y,0) = \int_0^t \int_{\mathbb{R}^n_+} G_a(x,t;z,\tau)|V(z,\tau)|G_b(z,\tau;y,0)dz\,d\tau.$$

Let $\rho \in (0,1)$ be chosen later. We have

$$J = \int_0^{\rho t}\int_{\mathbb{R}^n_+} \ldots dz\,d\tau + \int_{\rho t}^t \int_{\mathbb{R}^n_+} \ldots dz\,d\tau \equiv J_1 + J_2.$$

We first estimate J_1. By Lemma 2.3, we have

$$J_1 \leq k\int_0^{\rho t}\int_{\mathbb{R}^n_+} \omega(z,\tau)\frac{\exp(-a\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}}|V(z,\tau)|\frac{\exp(-(\frac{a+b}{2})\frac{|z-y|^2}{\tau})}{\tau^{n/2}}dz\,d\tau,$$

where $\omega(z,\tau) = \min(1,\frac{x_n z_n}{t-\tau})\min(1,\frac{y_n}{\sqrt{\tau}})\min(1,\frac{z_n}{\sqrt{\tau}})$. Using the inequality

$$\frac{|x-z|^2}{t-\tau} + \frac{|z-y|^2}{\tau} \geq \frac{|x-y|^2}{t},\ \forall \tau \in (0,t),$$

it follows that

$$J_1 \leq k\int_0^{\rho t}\int_{\mathbb{R}^n_+} \omega(z,\tau)\frac{e^{-a(\frac{|x-z|^2}{t-\tau}+\frac{|z-y|^2}{\tau})}}{(t-\tau)^{n/2}}|V(z,\tau)|\frac{e^{-(\frac{b-a}{2})\frac{|z-y|^2}{\tau}}}{\tau^{n/2}}dz\,d\tau$$

$$\le k\frac{\exp(-a\frac{|x-y|^2}{t})}{((1-\rho)t)^{n/2}}\int_0^{\rho t}\int_{\mathbb{R}^n_+}\omega(z,\tau)|V(z,\tau)|\frac{e^{-(\frac{b-a}{2})\frac{|z-y|^2}{\tau}}}{\tau^{n/2}}dz\,d\tau. \tag{2.10}$$

On the other hand,

$$\int_0^{\rho t}\int_{\mathbb{R}^n_+}\omega(z,\tau)|V(z,\tau)|\frac{\exp-(\frac{b-a}{2})\frac{|z-y|^2}{\tau})}{\tau^{n/2}}dz\,d\tau$$

$$=\int_0^{\rho t}\int_{z_n\le 2y_n}\dots dz\,d\tau+\int_0^{\rho t}\int_{z_n\ge 2y_n}\dots dz\,d\tau\equiv J_{11}+J_{12}. \tag{2.11}$$

If $z_n\le 2y_n$ and $\tau\in(0,\rho t)$, then we have

$$\omega(z,\tau)\le\frac{2}{1-\rho}\min(1,\frac{x_ny_n}{t})\min(1,\frac{z_n}{\sqrt{\tau}}).$$

If $z_n\ge 2y_n$ and $\tau\in(0,\rho t)$, then we have

$$\begin{aligned}\omega(z,\tau)&\le\frac{1}{1-\rho}\min(1,\frac{x_ny_n}{t})\min(1,\frac{z_n}{\sqrt{\tau}})\frac{z_n}{\sqrt{\tau}}\\&\le\frac{2}{1-\rho}\min(1,\frac{x_ny_n}{t})\min(1,\frac{z_n}{\sqrt{\tau}})\frac{|z-y|}{\sqrt{\tau}}.\end{aligned}$$

It follows that for all $t>0$, we have

$$J_{11}\le\frac{2}{1-\rho}\min(1,\frac{x_ny_n}{t})M_\infty^{\frac{b-a}{2}}(V) \tag{2.12}$$

and

$$\begin{aligned}J_{12}&\le\frac{2}{1-\rho}\min(1,\frac{x_ny_n}{t})\\&\times\int_0^{\rho t}\int_{z_n\ge 2y_n}\min(1,\frac{z_n}{\sqrt{\tau}})\frac{|z-y|}{\sqrt{\tau}}|V(z,\tau)|\frac{\exp(-(\frac{b-a}{2})\frac{|z-y|^2}{\tau})}{\tau^{n/2}}dz\,d\tau\\&\le\frac{2(b-a)^{-1/2}}{1-\rho}\min(1,\frac{x_ny_n}{t})\\&\times\int_0^{\rho t}\int_{z_n\ge 2y_n}\min(1,\frac{z_n}{\sqrt{\tau}})|V(z,\tau)|\frac{\exp(-(\frac{b-a}{4})\frac{|z-y|^2}{\tau})}{\tau^{n/2}}dz\,d\tau\\&\le\frac{2(b-a)^{-1/2}}{1-\rho}\min(1,\frac{x_ny_n}{t})M_\infty^{\frac{b-a}{4}}(V).\end{aligned} \tag{2.13}$$

Combining (2.10)–(2.13), we obtain

$$J_1\le\frac{C_{a,b}}{(1-\rho)^{n/2+1}}M_\infty^{\frac{b-a}{4}}(V)G_a(x,t;y,0), \tag{2.14}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n_+$.

We next estimate J_2. We have

$$\begin{aligned} J_2 &\le k \int_{\rho t}^{t} \int_{\mathbb{R}^n_+} u(z,\tau) \frac{\exp(-\frac{a}{2}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}} |V(z,\tau)| \frac{\exp(-b\frac{|z-y|^2}{\tau})}{\tau^{n/2}} dz\, d\tau \\ &= k\Big(\int_{\rho t}^{t} \int_{|z-y| \ge (\frac{a}{b})^{1/2}|x-y|} \dots dz\, d\tau + \int_{\rho t}^{t} \int_{|z-y| \le (\frac{a}{b})^{1/2}|x-y|} \dots dz\, d\tau \Big) \\ &\equiv k(J_{21} + J_{22}), \end{aligned} \tag{2.15}$$

where $u(z,\tau) = \min(1, \frac{x_n}{\sqrt{t-\tau}}) \min(1, \frac{z_n}{\sqrt{t-\tau}}) \min(1, \frac{z_n y_n}{\tau})$. When $|z-y| \ge (\frac{a}{b})^{1/2}|x-y|$ and $\tau \in (\rho t, t)$, we have

$$\frac{\exp(-b\frac{|z-y|^2}{\tau})}{\tau^{n/2}} \le \frac{\exp(-a\frac{|x-y|^2}{t})}{(\rho t)^{n/2}}.$$

Therefore,

$$J_{21} \le \frac{\exp(-a\frac{|x-y|^2}{t})}{(\rho t)^{n/2}} \int_{\rho t}^{t} \int_{\mathbb{R}^n_+} u(z,\tau) \frac{\exp(-\frac{a}{2}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}} |V(z,\tau)| dz\, d\tau. \tag{2.16}$$

Now we estimate J_{22}. We have

$$\begin{aligned} J_{22} &= \int_{\rho t}^{t} \int_{|z-y| \le (\frac{a}{b})^{1/2}|x-y|} u(z,\tau) \frac{\exp(-\frac{a}{2}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}} |V(z,\tau)| \frac{\exp(-b\frac{|z-y|^2}{\tau})}{\tau^{n/2}} dz\, d\tau \\ &\le \frac{1}{(\rho t)^{n/2}} \int_{\rho t}^{t} \int_{|z-y| \le (\frac{a}{b})^{1/2}|x-y|} u(z,\tau) \frac{\exp(-\frac{a}{2}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}} |V(z,\tau)| dz\, d\tau. \end{aligned} \tag{2.17}$$

If $|z-y| \le (\frac{a}{b})^{1/2}|x-y|$, then $|x-z| \ge |x-y| - |z-y| \ge (1-(\frac{a}{b})^{1/2})|x-y|$, and hence

$$\begin{aligned} \exp(-\frac{a}{2}\frac{|x-z|^2}{t-\tau}) &\le \exp(-\frac{a}{4}\frac{|x-z|^2}{t-\tau}) \exp\Big[-\frac{a}{4}\frac{|x-y|^2}{t-\tau}(1-(\frac{a}{b})^{1/2})^2\Big] \\ &\le \exp(-\frac{a}{4}\frac{|x-z|^2}{t-\tau}) \exp\Big[-\frac{a}{4}\frac{|x-y|^2}{(1-\rho)t}(1-(\frac{a}{b})^{1/2})^2\Big]. \end{aligned}$$

Now, taking ρ so that $\frac{(1-(\frac{a}{b})^{1/2})^2}{4(1-\rho)} = 1$, we obtain

$$\exp(-\frac{a}{2}\frac{|x-z|^2}{t-\tau}) \le \exp(-\frac{a}{4}\frac{|x-z|^2}{t-\tau}) \exp(-a\frac{|x-y|^2}{t}). \tag{2.18}$$

Combining (2.17) and (2.18), we have

$$J_{22} \le \frac{\exp(-a\frac{|x-y|^2}{t})}{(\rho t)^{n/2}} \int_{\rho t}^{t}\int_{\mathbb{R}^n_+} u(z,\tau)\frac{\exp(-\frac{a}{4}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}}|V(z,\tau)|dz\,d\tau. \tag{2.19}$$

From (2.15), (2.16), and (2.19), it follows that

$$J_2 \le 2k\frac{\exp(-a\frac{|x-y|^2}{t})}{(\rho t)^{n/2}} \int_{\rho t}^{t}\int_{\mathbb{R}^n_+} u(z,\tau)\frac{\exp(-\frac{a}{4}\frac{|x-z|^2}{t-\tau})}{(t-\tau)^{n/2}}|V(z,\tau)|dz\,d\tau. \tag{2.20}$$

Note that the inequality (2.20), is similar to (2.10). Then by the same method used to prove (2.14), we obtain, for all $t>0$ and $x, y \in \mathbb{R}^n_+$,

$$J_2 \le \frac{C_a}{\rho^{n/2+1}} M_\infty^{a/8}(V) G_a(x,t;y,0). \tag{2.21}$$

Combining (2.14) and (2.21) and using the fact that $\frac{(1-(\frac{a}{b})^{1/2})^2}{4(1-\rho)} = 1$, we obtain

$$J(x,t;y,0) \le C_{a,b} M_\infty^c(V) G_a(x,t;y,0),$$

where $c = \min\left(\frac{a}{8}, \frac{b-a}{4}\right)$, which completes the proof of the lemma. □

3. Modulus of continuity

In this section we prove the $\frac{1}{2}$-Hölder-continuous property of the Green's function G and the associated potentials. This will be used in the next section to prove the main result.

Recall that

$$G(x,t;y,s) = (4\pi)^{-n/2} G_{1/4}(x,t;y,s) \tag{3.1}$$

is the Green's function of $\Delta_x - \partial/\partial t$ on $\mathbb{R}^n_+ \times (0,\infty)$.

Lemma 3.1. *There exists a constant $k>0$ such that for $R \in (0,1)$, $|x - x_0| \le R$, $|x_0 - y| \ge 2(R|x_0-x|)^{1/2}$, and $t_0 - s \le 4R|x_0-x|$, we have*

$$|G(x_0,t_0;y,s) - G(x,t_0;y,s)|$$
$$\le k\Big(\frac{|x-x_0|}{R}\Big)^{1/2} \min(1, \frac{y_n}{\sqrt{t_0-s}}) \frac{\exp(-\frac{|x_0-y|^2}{32(t_0-s)})}{(t_0-s)^{n/2}}.$$

Proof. We have

$$|G(x_0,t_0;y,s) - G(x,t_0;y,s)| \le |x-x_0| \sup_{z\in[x_0,x]} |\nabla_z G(z,t_0;y,s)|.$$

On the other hand, by (3.1) and iii) of Lemma 2.3, we have

$$|\nabla_z G(z,t_0;y,s)| \leq k\min(1,\frac{y_n}{\sqrt{t_0-s}})\frac{\exp(-\frac{|z-y|^2}{8(t_0-s)})}{(t_0-s)^{\frac{n+1}{2}}}.$$

Hence,

$$\begin{aligned}&|G(x_0,t_0;y,s)-G(x,t_0;y,s)|\\ &\leq k|x-x_0|\min(1,\frac{y_n}{\sqrt{t_0-s}})\sup_{z\in[x_0,x]}\frac{\exp(-\frac{|z-y|^2}{8(t_0-s)})}{(t_0-s)^{\frac{n+1}{2}}}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}|z-y| &\geq |x_0-y|-|x_0-z| \geq |x_0-y|-|x_0-x|\\ &\geq |x_0-y|-\sqrt{R|x_0-x|} \geq \frac{1}{2}|x_0-y| \geq \sqrt{R|x_0-x|}.\end{aligned}$$

Since $\theta e^{-\frac{\theta^2}{16}} \leq 2$, for all $\theta \geq 0$ and $|x_0-y| \geq 2(R|x_0-x|)^{1/2}$, then

$$\begin{aligned}&|G(x_0,t_0;y,s)-G(x,t_0;y,s)|\\ &\leq k\frac{|x-x_0|}{\sqrt{R|x_0-x|}}\min(1,\frac{y_n}{\sqrt{t_0-s}})\sup_{z\in[x_0,x]}\frac{|z-y|}{\sqrt{t_0-s}}\frac{\exp(-\frac{|z-y|^2}{8(t_0-s)})}{(t_0-s)^{n/2}}\\ &\leq k\Big(\frac{|x_0-x|}{R}\Big)^{1/2}\min(1,\frac{y_n}{\sqrt{16(t_0-s)}})\sup_{z\in[x_0,x]}\frac{\exp(-\frac{|z-y|^2}{16(t_0-s)})}{(t_0-s)^{n/2}}\\ &\leq k\Big(\frac{|x_0-x|}{R}\Big)^{1/2}\min(1,\frac{y_n}{\sqrt{t_0-s}})\frac{\exp(-\frac{|x_0-y|^2}{32(t_0-s)})}{(t_0-s)^{n/2}}.\end{aligned}$$

Lemma 3.2. *There exists a constant $k>0$ such that for $R\in(0,1)$, $|x-x_0|\leq R$ and $t_0-s\geq 4R|x_0-x|$, we have*

$$\begin{aligned}&|G(x_0,t_0;y,s)-G(x,t_0;y,s)|\\ &\leq k\Big(\frac{|x_0-x|}{R}\Big)^{1/2}\min(1,\frac{y_n}{\sqrt{t_0-s}})\frac{\exp(-\frac{|x_0-y|^2}{32(t_0-s)})}{(t_0-s)^{n/2}}.\end{aligned}$$

Proof. As in the proof of Lemma 3.1, we have

$$\begin{aligned}&|G(x_0,t_0;y,s)-G(x,t_0;y,s)|\\ &\leq k|x-x_0|\min(1,\frac{y_n}{\sqrt{t_0-s}})\sup_{z\in[x_0,x]}\frac{\exp(-\frac{|z-y|^2}{8(t_0-s)})}{(t_0-s)^{\frac{n+1}{2}}}.\end{aligned}$$

Now recall that $t_0 - s \geq 4R|x_0 - x|$, and using the fact that

$$|z-y|^2 \geq \frac{1}{2}|x_0-y|^2 - |x_0-z|^2 \geq \frac{1}{2}|x_0-y|^2 - |x_0-x|^2$$
$$\geq \frac{1}{2}|x_0-y|^2 - R|x_0-x| \geq \frac{1}{2}|x_0-y|^2 - \frac{t_0-s}{4},$$

we obtain

$$|G(x_0,t_0;y,s) - G(x,t_0;y,s)| \leq k\Big(\frac{|x_0-x|}{R}\Big)^{\frac{1}{2}} \min(1, \frac{y_n}{\sqrt{t_0-s}}) \frac{\exp(-\frac{|x_0-y|^2}{32(t_0-s)})}{(t_0-s)^{n/2}}.$$

Lemma 3.3. *There exists a constant $k > 0$ such that for $R \in (0,1)$, $0 \leq t_0 - t \leq R^2$, $|x_0 - y| \geq 2(R|t-t_0|^{1/2})^{1/2}$, and $t_0 - s \leq 4R|t-t_0|^{1/2}$, we have*

$$|G(x_0,t_0;y,s) - G(x_0,t;y,s)| \leq k\Big(\frac{|t_0-t|}{R^2}\Big)^{\frac{1}{2}} \min(1, \frac{y_n}{\sqrt{t_0-s}}) \frac{\exp(-\frac{|x_0-y|^2}{16(t_0-s)})}{(t_0-s)^{n/2}}.$$

Proof. We know that there exists $\theta \in (t, t_0)$ such that

$$|G(x_0,t_0;y,s) - G(x_0,t;y,s)| \leq |t_0-t|\Big|\frac{\partial G}{\partial t}(x_0,\theta;y,s)\Big|.$$

Moreover, by (3.1) and iv) of Lemma 2.3, we have

$$\Big|\frac{\partial G}{\partial t}(x_0,\theta;y,s)\Big| \leq k\min(1, \frac{y_n}{\sqrt{\theta-s}}) \frac{\exp(-\frac{|x_0-y|^2}{8(\theta-s)})}{(\theta-s)^{n/2+1}}.$$

Hence,

$$|G(x_0,t_0;y,s) - G(x_0,t;y,s)| \leq k|t_0-t|\min(\sqrt{\theta-s}, y_n) \frac{\exp(-\frac{|x_0-y|^2}{8(\theta-s)})}{(\theta-s)^{\frac{n+3}{2}}}$$

$$\leq k\frac{|t_0-t|}{|x_0-y|^2}\min(\sqrt{\theta-s}, y_n) \frac{\exp(-\frac{|x_0-y|^2}{16(\theta-s)})}{|x_0-y|^{n+1}}.$$

Using the fact that $\theta - s < t_0 - s$ and $|x_0 - y|^2 \geq 4R|t-t_0|^{\frac{1}{2}} \geq t_0 - s$, it follows that

$$|G(x_0,t_0;y,s) - G(x_0,t;y,s)| \leq k\Big(\frac{|t_0-t|}{R^2}\Big)^{\frac{1}{2}} \min(1, \frac{y_n}{\sqrt{t_0-s}}) \frac{\exp(-\frac{|x_0-y|^2}{16(t_0-s)})}{(t_0-s)^{n/2}}.$$

Lemma 3.4. *There exists a constant $k > 0$ such that for $R \in (0,1)$, $0 \leq t_0 - t \leq R^2$, and $t_0 - s \geq 4R|t-t_0|^{1/2}$, we have*

$$|G(x_0,t_0;y,s) - G(x_0,t;y,s)| \leq k\Big(\frac{|t_0-t|}{R^2}\Big)^{\frac{1}{2}} \min(1, \frac{y_n}{\sqrt{t_0-s}}) \frac{\exp(-\frac{|x_0-y|^2}{8(t_0-s)})}{(t_0-s)^{n/2}}.$$

Proof. As in the proof of Lemma 3.3, we have, for some $\theta \in (t, t_0)$,

$$|G(x_0,t_0;y,s) - G(x_0,t;y,s)| \leq k|t_0 - t| \min(1, \frac{y_n}{\sqrt{\theta - s}}) \frac{\exp(-\frac{|x_0-y|^2}{8(\theta-s)})}{(\theta - s)^{n/2+1}}.$$

From the hypothesis, we have

$$\begin{aligned} t_0 - s > \theta - s > t - s > t_0 - s - |t_0 - t| \\ \geq t_0 - s - \frac{(t_0-s)^2}{16R^2} \geq t_0 - s - \frac{t_0 - s}{16} = \frac{15}{16}(t_0 - s) \end{aligned}$$

and $t_0 - s \geq 4R|t - t_0|^{1/2}$; then we obtain

$$|G(x_0,t_0;y,s) - G(x_0,t;y,s)| \leq k\frac{|t_0 - t|}{t_0 - s} \min(1, \frac{y_n}{\sqrt{t_0 - s}}) \frac{\exp(-\frac{|x_0-y|^2}{8(t_0-s)})}{(t_0 - s)^{n/2}}$$

$$\leq k\Big(\frac{|t_0 - t|}{R^2}\Big)^{1/2} \min(1, \frac{y_n}{\sqrt{t_0 - s}}) \frac{\exp(-\frac{|x_0-y|^2}{8(t_0-s)})}{(t_0 - s)^{n/2}}.$$

Proposition 3.5. *Let $V \in \mathcal{K}^\infty$. Then the family of functions*

$$\Big\{(x,t) \to \int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)W(y,s)dy\,ds, \quad |W| \leq |V|\Big\}$$

is equicontinuous in $C(\overline{\mathbb{R}^n_+} \times (0,\infty))$.

Proof. Let $R \in (0,1)$, x, $x_0 \in \overline{\mathbb{R}^n_+}$ and $t, t_0 \in (0,\infty)$ such that $|t - t_0| \leq R^2$ and $|x - x_0| \leq R$. We may assume that $t_0 \leq t$. We have

$$\begin{aligned} &| \int_0^{t_0} \int_{\mathbb{R}^n_+} G(x_0,t_0;y,s)W(y,s)dy\,ds - \int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)W(y,s)dy\,ds| \\ &\leq \int_0^{t_0} \int_{\mathbb{R}^n_+} |G(x_0,t_0;y,s) - G(x,t_0;y,s)||W(y,s)|dy\,ds \\ &+ \int_0^{t_0} \int_{\mathbb{R}^n_+} |G(x,t_0;y,s) - G(x,t;y,s)||W(y,s)|dy\,ds \\ &+ \int_{t_0}^t \int_{\mathbb{R}^n_+} G(x,t;y,s)|W(y,s)|dy\,ds \equiv I_1 + I_2 + I_3. \end{aligned} \tag{3.2}$$

We estimate I_1. Since $|W| \leq |V|$, we have

$$I_1 \leq \int_0^{t_0} \int_{\mathbb{R}^n_+} |G(x_0,t_0;y,s) - G(x,t_0;y,s)||V(y,s)|dy\,ds$$

$$= \int_0^{t_0-4R|x-x_0|} \int_{\mathbb{R}_+^n} \dots dy\, ds + \int_{t_0-4R|x-x_0|}^{t_0} \int_{|y-x_0|\geq 2(R|x-x_0|)^{1/2}} \dots dy\, ds$$

$$+ \int_{t_0-4R|x-x_0|}^{t_0} \int_{|y-x_0|\leq 2(R|x-x_0|)^{1/2}} \dots dy\, ds \equiv I_{1,1} + I_{1,2} + I_{1,3}. \tag{3.3}$$

By Lemma 3.2, we have

$$I_{1,1} \leq k(|x-x_0|/R)^{1/2} M_\infty^{1/32}(V). \tag{3.4}$$

By Lemma 3.1, we have

$$I_{1,2} \leq k(|x-x_0|/R)^{1/2} M_\infty^{1/32}(V). \tag{3.5}$$

Moreover from (3.1) and ii) of Lemma 2.3, we have

$$\begin{aligned} I_{1,3} \leq & \int_{t_0-4R|x-x_0|}^{t_0} \int_{|y-x_0|\leq 2(R|x-x_0|)^{1/2}} G(x_0,t_0;y,s)|V(y,s)|dy\, ds \\ & + \int_{t_0-4R|x-x_0|}^{t_0} \int_{|y-x_0|\leq 2(R|x-x_0|)^{1/2}} G(x,t_0;y,s)|V(y,s)|dy\, ds \\ & \leq 2k M_{4R|x-x_0|}^{1/32}(V). \end{aligned} \tag{3.6}$$

Combining (3.3)–(3.6), we obtain

$$I_1 \leq 2k\Big((|x-x_0|/R)^{1/2} M_\infty^{1/32}(V) + M_{4R|x-x_0|}^{1/32}(V)\Big). \tag{3.7}$$

Now we estimate I_2.

$$I_2 \leq \int_0^{t_0} \int_{\mathbb{R}_+^n} |G(x,t_0;y,s) - G(x,t;y,s)||V(y,s)|dy\, ds$$

$$= \int_0^{t_0-4R|t-t_0|^{1/2}} \int_{\mathbb{R}_+^n} \dots dy\, ds + \int_{t_0-4R|t-t_0|^{1/2}}^{t_0} \int_{|y-x|\geq 2(R|t-t_0|^{1/2})^{1/2}} \dots dy\, ds$$

$$+ \int_{t_0-4R\sqrt{|t-t_0|}}^{t_0} \int_{|y-x|\leq 2(R|t-t_0|^{1/2})^{1/2}} \dots dy\, ds \equiv I_{2,1} + I_{2,2} + I_{2,3}. \tag{3.8}$$

By Lemma 3.4 and Lemma 3.3, we obtain

$$I_{2,1} + I_{2,2} \leq 2k(|t-t_0|/R)^{1/2} M_\infty^{1/32}(V). \tag{3.9}$$

Moreover, since $|t-t_0| \leq R^2/2$, then we have

$$t - 5R|t-t_0|^{\frac{1}{2}} \leq t_0 + R^2 - 4R|t-t_0|^{1/2} - R|t-t_0|^{\frac{1}{2}} \leq t_0 - 4R|t-t_0|^{\frac{1}{2}}.$$

It follows from (3.1) and ii) of Lemma 2.3 that

$$\begin{aligned}
I_{2,3} &\leq \int_{t_0-4R|t-t_0|^{1/2}}^{t_0} \int_{|y-x|\leq 2(R|t-t_0|^{1/2})^{1/2}} G(x,t_0;y,s)|V(y,s)|dy\,ds \\
&+ \int_{t-5R|t-t_0|^{1/2}}^{t} \int_{|y-x|\leq (5R|t-t_0|^{1/2})^{1/2}} G(x,t;y,s)|V(y,s)|dy\,ds \\
&\leq 2kM^{1/32}_{5R|t-t_0|^{1/2}}(V). \qquad (3.10)
\end{aligned}$$

Combining (3.8)–(3.10), we obtain

$$I_2 \leq 2k\Big((|t-t_0|/R)^{1/2}M^{1/32}_{\infty}(V) + M^{1/32}_{5R|t-t_0|^{1/2}}(V)\Big). \qquad (3.11)$$

To estimate I_3, we see that $G(x,t_0;y,s)=0$, for $s \geq t_0$, and then

$$\begin{aligned}
I_3 &\leq \int_{t_0}^{t}\int_{\mathbb{R}^n_+} G(x,t;y,s)|V(y,s)|dy\,ds \\
&= \int_{t_0}^{t}\int_{\mathbb{R}^n_+} |G(x,t;y,s)-G(x,t_0;y,s)||V(y,s)|dy\,ds \\
&\leq \int_{0}^{t}\int_{\mathbb{R}^n_+} |G(x,t;y,s)-G(x,t_0;y,s)||V(y,s)|dy\,ds.
\end{aligned}$$

Hence, by the same method used to estimate I_2, we obtain

$$I_3 \leq 2k\Big((|t-t_0|/R)^{1/2}M^{1/32}_{\infty}(V) + M^{1/32}_{5R|t-t_0|^{1/2}}(V)\Big). \qquad (3.12)$$

Combining (3.2), (3.7), (3.11), and (3.12), we find

$$\begin{aligned}
&|\int_0^{t_0}\int_{\mathbb{R}^n_+} G(x_0,t_0;y,s)W(y,s)dy\,ds - \int_0^{t}\int_{\mathbb{R}^n_+} G(x,t;y,s)W(y,s)dy\,ds| \\
&\leq 4k\Big(\Big[(|x-x_0|/R)^{1/2} + (|t-t_0|/R)^{1/2}\Big]M^{1/32}_{\infty}(V) \\
&+ M^{1/32}_{4R|x-x_0|}(V) + M^{1/32}_{5R|t-t_0|^{1/2}}(V)\Big),
\end{aligned}$$

which tends to 0 as $|x-x_0| \to 0$ and $|t-t_0| \to 0$, uniformly with respect to W. □

We will also need the following lemma. Its proof is similar to that of Lemma 5.2 in [5].

Lemma 3.6. *Let*

$$h(x,t) = \int_{\mathbb{R}^n_+} G(x,t;y,0)u_0(y)dy,$$

where u_0 is a bounded nonnegative function on $\mathbb{R}^n_+$. Then, we have the following conclusions:

(a) *For $p > 1$, there exists a constant $C(p) > 0$ such that*

$$h^p(x,t) \le C(p)\|u_0\|_\infty^{p-1} h(x,t), \qquad \textit{for all } t > 0.$$

(b) *If $\lim_{|x|\to\infty} u_0(x) = 0$, then $\lim_{|x|\to\infty} h(x,t) = 0$ uniformly with respect to $t > 0$.*

(c) *For $p > 1$ and $\varepsilon \in (0,1)$, there exists a constant $C(p,\varepsilon) > 0$ such that*

$$h^p(x,t) \le C(p,\varepsilon)\|u_0\|_\infty^{p-1} h(x,t/p\varepsilon), \qquad \textit{for all } t > 0.$$

4. Proof of Theorem 1.1

We would like to show that for $M > 1$, there exists a constant $b_0 > 0$ such that for each nonnegative $u_0 \in C^2(\mathbb{R}^n_+)$ satisfying $\|u_0\|_\infty \le b_0$, the following integral equation,

$$u(x,t) = \int_{\mathbb{R}^n_+} G(x,t;y,0)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)V(y,s)u^p(y,s)dy\,ds,$$

has a positive and continuous solution u on $\mathbb{R}^n_+ \times (0,\infty)$ satisfying for all $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$,

$$\frac{1}{M}\int_{\mathbb{R}^n_+} G(x,t;y,0)u_0(y)dy \le u(x,t) \le M\int_{\mathbb{R}^n_+} G(x,t;y,0)u_0(y)dy.$$

To achieve this, we will use the Schauder fixed-point theorem. We divide the proof into two steps.

Step 1. In this step we assume, moreover, that $\lim\limits_{|x|\to\infty} u_0(x) = 0$. Let $d > 1$ and put $C_0(\overline{\mathbb{R}^n_+} \times [0,d]) = \big\{u \in C(\overline{\mathbb{R}^n_+} \times [0,d]) : \lim_{|x|\to\infty} \sup_{t\in[0,d]} |u(x,t)| = 0\big\}$. Then $C_0(\overline{\mathbb{R}^n_+} \times [0,d])$ is a Banach space endowed with the uniform norm

$$\|u\|_\infty = \sup_{(x,t)\in\mathbb{R}^n_+\times[0,d]} |u(x,t)|.$$

Let $M > 1$ and consider the set $\mathcal{A}_d = \big\{u \in C(\overline{\mathbb{R}^n_+} \times [0,d]) : M^{-1}h \le u \le Mh\big\}$, where h is given in Lemma 3.6.

$\mathcal{A}_d$ is a nonempty, closed, bounded, and convex set. By Lemma 3.6, $\mathcal{A}_d$ is contained in $C_0(\overline{\mathbb{R}^n_+} \times [0,d])$. Define the integral operator T on $\mathcal{A}_d$ by

$$Tu(x,t) = h(x,t) + \int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)V(y,s)u^p(y,s)dy\,ds.$$

First we shall prove that $T\mathcal{A}_d \subset \mathcal{A}_d$. Let $u \in \mathcal{A}_d$. Since $\|u\|_\infty \le M\|u_0\|_\infty$, by Proposition 3.5, Tu is continuous on $\overline{\mathbb{R}^n_+} \times [0,d]$. Moreover, we have $u \le Mh$, and hence by Lemma 3.6, for $\varepsilon \in (0,1)$, we have

$$u^p(y,s) \le kM^p\|u_0\|_\infty^{p-1} h(y, s/p\varepsilon).$$

Now we fix $\varepsilon < 1$ so that $p\varepsilon > 1$; then

$$u^p(y,s) \le kM^p\|u_0\|_\infty^{p-1} \int_{\mathbb{R}^n_+} G(y, s/p\varepsilon; z, 0) u_0(z) dz.$$

This yields

$$\begin{aligned}&|Tu(x,t) - h(x,t)|\\ &\le kM^p\|u_0\|_\infty^{p-1} \int_{\mathbb{R}^n_+} \Big(\int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)|V(y,s)|G(y,s/p\varepsilon;z,0) dy\, ds \Big) u_0(z) dz.\end{aligned}$$

From (3.1) and (2.7) we know that $G(x,t;y,s) = (4\pi)^{-n/2} G_{1/4}(x,t;y,s)$ and $G(y, s/p\varepsilon; z, 0) = (4\pi)^{-n/2} G_{p\varepsilon/4}(y,s;z,0)$. Thus,

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)|V(y,s)|G(y,s/p\varepsilon;z,0) dy\, ds\\ &= (4\pi)^{-n} \int_0^t \int_{\mathbb{R}^n_+} G_{1/4}(x,t;y,s)|V(y,s)|G_{p\varepsilon/4}(y,s;z,0) dy\, ds.\end{aligned}$$

Applying Lemma 2.4 with $a = 1/4$ and $b = p\varepsilon/4$, we have, for a $C_{a,b} > 0$,

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)|V(y,s)|G_{p\varepsilon/4}(y,s;z,0) dy\, ds\\ &\le (4\pi)^{-n} C_{a,b} M_\infty^c(V) G_{1/4}(x,t;z,0) = (4\pi)^{-n/2} C_{a,b} M_\infty^c(V) G(x,t;z,0).\end{aligned}$$

It follows that

$$\begin{aligned}&|Tu(x,t) - h(x,t)|\\ &\le kM^p\|u_0\|_\infty^{p-1} (4\pi)^{-n/2} C_{a,b} M_\infty^c(V) \int_{\mathbb{R}^n_+} G(x,t;z,0) u_0(z) dz,\end{aligned}$$

which yields

$$|Tu(x,t) - h(x,t)| \le kM^p\|u_0\|_\infty^{p-1} (4\pi)^{-n/2} C_{a,b} M_\infty^c(V) h(x,t).$$

By taking $\|u_0\|_\infty \le b_0$ with b_0 sufficiently small we get

$$M^{-1} h(x,t) \le Tu(x,t) \le Mh(x,t). \tag{4.1}$$

Next we will prove that $T\mathcal{A}_d$ is relatively compact in $C_0(\overline{\mathbb{R}^n_+} \times [0,d])$. By (4.1), $T\mathcal{A}_d$ is uniformly bounded, and by Proposition 3.5 it is equicontinuous. Moreover, since $\lim_{|x|\to\infty} u_0(x) = 0$, by (4.1) and Lemma 3.6 we have

$$\lim_{|x|\to\infty} \sup_{t\in[0,d]} |Tu(x,t)| = 0,$$

uniformly for all $u \in \mathcal{A}_d$. Hence, in view of the Ascoli-Arzela theorem, $T\mathcal{A}_d$ is a relatively compact subset in $C_0(\overline{\mathbb{R}^n_+} \times [0,d])$.

It remains to show that T is continuous. For $u_1, u_2 \in \mathcal{A}_d$, we have

$$\|Tu_1 - Tu_2\|_\infty \leq M^c_\infty(V)\|u_1^p - u_2^p\|_\infty,$$

which proves the continuity of T. Therefore, by the Shauder fixed-point theorem, there exists $u_d \in \mathcal{A}_d$ such that $Tu_d = u_d$. Now, define

$$U_d(x,t) = \begin{cases} u_d(x,t) & \text{if } t \leq d \\ u_d(x,d) & \text{if } t > d. \end{cases}$$

Since $M^{-1}h \leq u_d \leq Mh$, $\{U_d\}_{d>1}$ is uniformly bounded, and by Proposition 3.5 it is equicontinuous. Hence there is a subsequence $\{U_{d_m}\}_{m\geq 1}$ which converges uniformly to a function u in any compact subset of $\overline{\mathbb{R}^n_+} \times (0,\infty)$.

For $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$ and m sufficiently large, we have

$$U_{d_m}(x,t) = h(x,t) + \int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)V(y,s)U^p_{d_m}(y,s)dy\,ds.$$

Then by the dominated convergence theorem, we have

$$u(x,t) = h(x,t) + \int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)V(y,s)u^p(y,s)dy\,ds,$$

for all $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$. This shows that u is a solution of (1.3). Moreover, $M^{-1}h \leq U_{d_m} \leq Mh$ yields $M^{-1}h \leq u \leq Mh$.

Step 2. We only assume $u_0 \in C^2(\mathbb{R}^n_+)$, nonnegative, and $\|u_0\|_\infty \leq b_0$, where b_0 is the given number in Step 1.

Let $(\omega_k)_k \subset C^2(\mathbb{R}^n_+)$, $\omega_k \geq 0$ such that $\lim_{|x|\to\infty} \omega(x) = 0$, $\|\omega_k\|_\infty \leq b_0$, and $\lim_{k\to\infty} \omega_k = u_0$ pointwise. Put

$$I_k(x,t) = \int_{\mathbb{R}^n_+} G(x,t;y,0)\omega_k(y)dy.$$

Clearly,

$$\lim_{k\to\infty} I_k(x,t) = \int_{\mathbb{R}^n_+} G(x,t;y,0)u_0(y)dy = h(x,t).$$

From Step 1, for each $k = 1, 2, \ldots$, there exists W_k satisfying

$$W_k(x,t) = I_k(x,t) + \int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)V(y,s)W_k^p(y,s)dy\,ds,$$

for all $(x,t) \in \mathbb{R}^n_+ \times [0,\infty)$.

Again $\{W_k\}_k$ is uniformly bounded and equicontinuous. Then there is a subsequence, still called $(W_k)_k$, which converges uniformly to a function W in any compact subset in $\mathbb{R}^n_+ \times [0,\infty)$. The dominated convergence theorem implies

$$W(x,t) = h(x,t) + \int_0^t \int_{\mathbb{R}^n_+} G(x,t;y,s)V(y,s)W^p(y,s)dy\,ds,$$

for all $(x,t) \in \mathbb{R}^n_+ \times [0,\infty)$. This shows that (1.3) has a global positive solution.

Remark. By means of the Green's function bounds proved in [4], we extend the estimates in Section 3 to the Green's function of $\operatorname{div}(A(x,t)\nabla_x) - \partial/\partial t$, where A is a positive-definite matrix with Lipschitz-continuous coefficients on $\mathbb{R}^n_+ \times [0,T]$, where $0 < T < \infty$. Hence, replacing the Laplacian operator in (1.3) by $\operatorname{div}(A(x,t)\nabla_x)$ and assuming that $V \geq 0$, we may prove in the same way the existence of positive continuous solutions for the following problem:

$$\begin{cases} \operatorname{div}(A(x,t)\nabla_x u) - \dfrac{\partial}{\partial t}u + Vu^p = 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n_+, \\ u = 0 \quad \text{on} \quad \partial\mathbb{R}^n_+ \times (0,T), \end{cases}$$

where $n \geq 3$ and $p > 1$.

REFERENCES

[1] C. Kenig and W.M. Ni, *An exterior Dirichlet problem with applications to some equations arising in geometry,* Amer. J. Math., 106 (1984), 689–702.

[2] F.H. Lin, *On the elliptic equation* $D_i(a_{ij}(x)D_jU) - k(x)U^p = 0$, Proc. Amer. Math. Soc., 95 (1985), 219–266.

[3] W.M. Ni, *On the elliptic equation* $\Delta v + K(x)u^{(\frac{n+2}{n-2})} = 0$, *its generalizations and applications in geometry,* Indiana Univ. Math. J., 31 (1982), 493–529.

[4] L. Riahi, *Green function bounds and parabolic potentials on a half-space,* Potential Analysis, 15 (2001), 133–150.

[5] Q.S. Zhang, *Global existence and local continuity of solutions for semilinear parabolic equations,* Comm. Part. Diff. Eq., 22 (1997), 1529–1557.

[6] Z. Zhao, *On the existence of positive solutions of nonlinear elliptic equations—A probabilistic potential theory approach,* Duke Math. J., 69 (1993), 247–258.