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# REGULARITY RESULTS FOR THE BLOW-UP TIME AS A FUNCTION OF THE INITIAL DATA

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**Abstract.** We study the dependence of the finite blow-up time with respect to the initial data for solutions of the equation $u_t = \Delta u^m + u^p$. We obtain Lipschitz continuity for a certain class of initial data and Hölder regularity for wider classes.

## Introduction

In this paper we study the dependence with respect to the initial data of the blow-up time for solutions of the following problem,

$$\begin{cases} u_t = \Delta(u^m) + u^p, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $m \geq 1$ and $p > 1$. On the initial data we assume that $u_0$ is compactly supported or positive in the whole of $\mathbb{R}^N$ and smooth.

A remarkable fact is that the solutions of parabolic problems may develop singularities in finite time, no matter how smooth the initial data are. It is well known that for many differential equations or systems the solutions can become unbounded in finite time (a phenomenon that is known as blow-up). The study of blow-up solutions has attracted a considerable attention in recent years; see [8], [13], [15], and the references therein.

For our problem, if the solution is defined on a maximal time interval, $[0,T)$ with $T < +\infty$, then $\lim_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty} = +\infty$. We say that $T$ is the blow-up time. The existence of blowing-up solutions for (1.1) has been proved in [6] and [15]. In [9] it is proved that when $1 < p \leq m + 2/N$

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every nontrivial solution blows up in finite time, while if $p > m + 2/N$ there are blowing-up solutions for initial data large enough and also global solutions for small initial data. The speed at which solutions blow up (blow-up rate) and the spatial structure of the set in which the solution becomes unbounded (blow-up sets) have been studied in this problem. Under adequate hypotheses, the blow-up rate is given by $\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\alpha}$ with $\alpha = 1/(p-1)$. Concerning the blow-up set, there are three different cases according to whether $p < m$, $p = m$, or $p > m$. In fact, the blow-up set is, typically, the whole of $\mathbb{R}^N$ if $p < m$ (global blow-up), a bounded subset if $p = m$ (regional blow-up), and a single point if $p > m$ (single-point blow-up). For all these results see [2], [3], [15], and references therein.

Our main interest here is to investigate the dependence of the blow-up time with respect to the initial data. We assume that we are dealing with an initial datum $u_0$ which produces a solution $u$ that blows up at time $T = T(u_0)$ and an arbitrary small perturbation $h(x)$ such that the solution $u_h$ with initial datum $u_{0,h} = u_0(x) + h(x)$ also blows up in finite time, which we call $T_h = T(u_{0,h})$. Our concern is to obtain bounds for $|T_h - T|$ in terms of $h$.

For the semilinear case, $m = 1$, it is known that the blow-up time is continuous with respect to the initial data in $L^\infty$ when $1 < p < (N+2)/(N-2)$ (see [1], [12], [14]), if $\Omega$ is bounded (with Dirichlet boundary conditions), and [5] if $\Omega = \mathbb{R}^N$. That is, $T_h \to T$ as $\|h\|_{L^\infty} \to 0$. Remark that the restriction on $p$ is not technical. Indeed, if it does not hold, then the blow-up time may not even be continuous as a function of the initial data; see [7]. Moreover, for subcritical $p$, it is proved in [10] that $T$ is almost Lipschitz in the following sense: $|T_h - T| \leq C\|h\|_{L^\infty} |\ln(\|h\|_{L^\infty})|^{\frac{N+2}{2}+\varepsilon}$. The one-dimensional case was treated in [11], where it was shown that $T$ is Lipschitz for some special initial data and some particular perturbations $h$. For general $m \geq 1$, continuity of the blow-up time follows by adapting the ideas in [14], which combines the blow-up rate (when known) with the property of continuous dependence of the solution on the initial data. In fact, assume that $\|u(t)\|_{L^\infty} \sim (T-t)^{-\alpha}$ and fix $t_0 < T$ such that $(T - t_0) \leq \varepsilon$ and $(\|u(t_0)\|_{L^\infty} - 1)^{-1/\alpha} \leq C\varepsilon$; by continuity with respect to the initial data, if $u_{0,h}$ is close to $u_0$, then $u_h(\cdot, t_0)$ is close to $u(\cdot, t_0)$; therefore,

$$|T - T_h| \leq |T - t_0| + |T_h - t_0| \leq \varepsilon + C\|u_h(t_0)\|_{L^\infty}^{-1/\alpha} \leq C\varepsilon.$$

Here we improve the above results in two ways. On the one hand, we show that, under certain conditions on $u_0$, the blow-up time $T$ is Lipschitz without any restriction on the perturbation $h$. On the other hand, the main ideas of the previous works [1], [10], [12], and [14] heavily rely on the linearity of the Laplace operator, while our approach allows us to deal with nonlinear operators such as the porous medium equation ($m > 1$). For this equation, to the best of our knowledge, this is the first analysis of the dependence of the blow-up time with respect to the initial data.

One of the tools involved in the analysis relies on the natural scaling invariance of the problem. We mean the following: if $u(x,t)$ is a solution of (1.1), then $u_\lambda(x,t) = \lambda^{-\alpha}u(\lambda^{-\beta}x, \lambda^{-1}t)$ is also a solution if we choose $\alpha = 1/(p-1)$ and $\beta = (p-m)/2(p-1)$. This solution $u_\lambda$ blows up at time $T_\lambda = \lambda T$ and has initial data $u_\lambda(x,0) = \lambda^{-\alpha}u_0(\lambda^{-\beta}x)$. From the explicit form of $T_\lambda$ and $u_\lambda(x,0)$ it is not difficult to derive Lipschitz continuity in this case. The result for perturbations of a similar kind and standard comparison arguments allow us to handle a general perturbation $h$.

**Theorem 1.1.** *Assume that $u_0$ satisfies*

$$-\frac{\partial(u_\lambda(x,0))}{\partial\lambda}\Big|_{\lambda=1} = \alpha u_0(x) + \beta x\nabla u_0(x) \geq c > 0, \;\; x \in supp(u_0). \qquad (1.2)$$

*Then, the blow-up time is Lispchitz with respect to the initial data in the following sense: there exists a positive constant $C$ such that*

$$|T - T_h| \leq C dist(u_0; u_{0,h}),$$

*for every $u_{0,h}$ with $dist(u_0, u_{0,h}) = (\|u_0 - u_{0,h}\|_{L^\infty} + |a - a_h|)$ small. Here $a$ and $a_h$ denote the corresponding interfaces for $u_0$ and $u_{0,h}$ and $|a - a_h|$ stands for the usual distance between sets.*

We note that the Lipschitz constant cannot be uniform. This follows just by looking at the ODE $u_t = u^p$. We want to add some comments on the hypotheses that appear in Theorem 1.1. On the one hand, the distance involves a term related to the interfaces, which is natural since solutions of the porous-medium equation have finite speed of propagation of their supports. Obviously, if $u_0$ is positive, this term does not appear. Concerning our assumption (1.2), we first note that due to the spatial translation invariance of the problem (keeping the same blow-up time), one can check it on any translation of $u_0$ if convenient. Using this fact, one can see that, after an appropriate spatial translation, (1.2) holds for a wide class of initial data, including positive or compactly supported bell-shaped ones when $1 < p < m$. On the other hand, it is important to note that in this range of exponents, the blow-up behavior is described by means of a self-similar solution with a profile $f(\xi)$ satisfying (1.2). Therefore, one can expect that solutions satisfy

(1.2) for times close to their blow-up time. For instance, this fact can be proved in the one-dimensional case, where both, uniqueness of the profile and the asymptotic behaviour of the solutions are known in great detail, [15], [4]. If $p \geq m$, since $\beta \geq 0$, hypothesis (1.2) becomes more restrictive and implies that $u_0$ must be positive. However, also in this case there exists a class of initial data with suitable behavior at $\infty$ satisfying (1.2).

By means of a similar approach, we also state Hölder-continuity results for wider classes of solutions. It is done by imposing higher-order conditions on $u_0$.

**Theorem 1.2.** *Let $u_0$ be such that*

$$\frac{\partial^k (u_\lambda(x,0))}{\partial \lambda^k}\Big|_{\lambda=1} \neq 0,$$

*when $x$ satisfies $\partial^j (u_\lambda(x,0))/\partial\lambda^j = 0$ for $1 \leq j \leq k-1$. Assume that $T$ is continuous at $u_0$; then $T$ is Hölder continuous with exponent $1/k$; that is,*

$$|T - T_h| \leq C(dist(u_0; u_{0,h}))^{1/k}.$$

We note that this result improves continuity to Hölder continuity when (1.2) does not hold. Remark that for certain initial data even continuity may fail, [7]. As a particular application, we note that Theorem 1.2 provides us with Hölder 1/2 continuity for a wide and significant class of initial data, including for instance concave, compactly supported ones in the one-dimensional case.

The same ideas apply when considering other scalings leaving invariant the equation. For instance, the family of solutions obtained by time translation, $u_\lambda(x,t) = u(x, t+\lambda)$, allows us to derive Lipschitz continuity for every initial data satisfying the condition $u_t > 0$. We will comment on these extensions after the proof of Theorem 1.2.

Finally, we remark that the same approach can also be used to deal with equations involving other operators and/or source terms like $u_t = \Delta u + e^u$, $u_t = div(|\nabla u|^{q-2}\nabla u) + u^p$, etc. We only need a scaling invariance law together with a comparison result.

## PROOF OF THE RESULTS

**Proof of Theorem 1.1**. To begin the analysis, we prove the result for the one-parameter family of initial data obtained from the scaling invariance of the equation. As we mentioned in the introduction, if $u(x,t)$ is a solution of

$u_t = \Delta u^m + u^p$ a straightforward calculation shows that

$$u_\lambda(x,t) = \lambda^{-\alpha} u(\lambda^{-\beta} x, \lambda^{-1} t)$$

is also a solution when $\alpha$ and $\beta$ are the self-similar exponents associated to the problem under consideration, namely, $\alpha = \frac{1}{p-1}$ and $\beta = \frac{p-m}{2(p-1)}$. The solution $u_\lambda$ has initial datum $u_\lambda(x,0) = \lambda^{-\alpha} u_0(\lambda^{-\beta} x)$ and blow-up time $T_\lambda = T\lambda$. Therefore, we get $|T - T_\lambda| = T|1-\lambda|$ and

$$dist(u_0; u_{0,\lambda}) = |1-\lambda| \left( \|\alpha u_0(x) + \beta x \nabla u_0(x)\|_{L^\infty} + |\beta||a_0| \right) + o(|1-\lambda|).$$

Hence, $|T - T_\lambda| \le C dist(u_0; u_{0,\lambda})$, where the constant $C$ can be chosen as

$$C = \frac{T}{(\|\alpha u_0(x) + \beta x \nabla u_0(x)\|_{L^\infty} + |\beta||a_0|)} + o(1)$$

for $|1-\lambda|$ small, and Theorem 1.1 follows for the special family $\mathcal{F} = \{u_{0,\lambda}, \lambda \in \mathbb{R}\}$.

In order to deal with a general perturbation, we use standard comparison arguments between $u_h$ (the solution with initial datum $u_{0,h}$) and suitable elements $u_\lambda^+$ and $u_\lambda^-$ of $\mathcal{F}$ .

Let $u_{0,h}$ be an arbitrary initial data with $dist(u_0, u_{0,h})$ small and define $\lambda^+ = \sup\{\lambda < 1;\ u_{0,\lambda}(x) \ge u_{0,h}(x)\}$ and $\lambda_- = \inf\{\lambda > 1;\ u_{0,\lambda}(x) \le u_{0,h}(x)\}$. Note that both $\lambda^+$ and $\lambda_-$ are well defined by hypothesis (1.2).

From the definition of $\lambda^+$ and $\lambda_-$ it is clear that

$$u_{0,\lambda_-}(x) \le u_{0,h}(x) \le u_{0,\lambda^+}(x),$$

and from the maximum principle we get

$$u_{\lambda_-}(x,t) \le u_h(x,t) \le u_{\lambda^+}(x,t).$$

Then, if we denote by $T_{\lambda^+}$, $T_h$, and $T_{\lambda_-}$ the blow-up times for $u_{\lambda^+}$, $u_h$, and $u_{\lambda_-}$ respectively, we obtain $T_{\lambda^+} \le T_h \le T_{\lambda_-}$. Therefore,

$$|T - T_h| \le \max\{T_{\lambda_-} - T\ ;\ T - T_{\lambda^+}\}.$$

Our goal is to obtain bounds on $T_{\lambda_-} - T$ and $T - T_{\lambda^+}$ in terms of $dist(u_0, u_{0,h})$. We derive in detail the bound for $T_{\lambda_-} - T$. The bound for $T - T_{\lambda^+}$ can be handled in a similar way, although some technical differences appear. We add the details when appropriate.

To begin with, we get from the stated result for the family $u_\lambda$ that

$$T_{\lambda_-} - T \le C dist(u_0; u_{0,\lambda_-}).$$

In order to estimate the distance $dist(u_0; u_{0,\lambda_-})$ in terms of $dist(u_0; u_{0,h})$ we remark that $u_{0,h}$ and $u_{0,\lambda_-}$ must have at least a contact point, $x_{\lambda_-} \in supp(u_0)$. Then

$$dist(u_0, u_{0,h}) \geq \|u_0 - u_{0,h}\|_{L^\infty} \geq |u_0(x_{\lambda_-}) - u_{0,h}(x_{\lambda_-})|$$
$$= |u_0(x_{\lambda_-}) - u_{0,\lambda_-}(x_{\lambda_-})| \geq C|1 - \lambda_-|,$$

where we can take $C = \inf |\alpha u_0 + \beta x \nabla u_0| + o(1)$. Remark that $C > 0$ by our hypothesis on the initial data $u_0$, (1.2). If $u_0 > 0$ and no contact points are available, the above argument still holds thanks to the fact that in this case there exists a sequence $x_n$ such that $|u_{0,h}(x_n) - u_{0,\lambda_-}(x_n)| \to 0$. On the other hand, $|1 - \lambda_-| \geq Cdist(u_0, u_{0,\lambda_-})$, and collecting these bounds we get

$$T_{\lambda_-} - T \leq Cdist(u_0, u_{0,h}).$$

When we deal with the estimate for $T - T_{\lambda^+}$, the only difference in the arguments appears if $u_0$ is compactly supported. Also in this case $u_{0,h}$ and $u_{0,\lambda^+}$ must have a contact point, $x_{\lambda^+}$. However $x_{\lambda^+}$ must not necessarily belong to the support of $u_0$. First, if $x_{\lambda^+} \in supp(u_0)$, the analysis is analogous to the previous one and can be handled in a similar fashion. It remains to consider the case when $x_{\lambda^+} \notin supp(u_0)$. If $u_{0,h}(x_{\lambda^+}) \geq |1 - \lambda^+|$ we get

$$dist(u_0, u_{0,h}) \geq \|u_0 - u_{0,h}\|_{L^\infty} \geq |u_0(x_{\lambda^+}) - u_{0,h}(x_{\lambda^+})|$$
$$= |u_{0,\lambda^+}(x_{\lambda^+})| \geq |1 - \lambda^+|,$$

and we finish the argument as above. Finally, if $u_{0,h}(x_{\lambda^+}) < |1 - \lambda^+|$, the condition involving the interfaces comes into play and we have, using our hypothesis on $u_0$, (1.2),

$$dist(u_0, u_{0,h}) \geq |a - a_h| \geq C|1 - \lambda^+|.$$

This ends the proof of Theorem 1.1. □

**Proof of Theorem 1.2**. Next we show how to obtain Hölder regularity when the initial data satisfies higher-order conditions. Let us analyze in detail the case $k = 2$. Assume that $u_0$ satisfies $\partial^2(u_\lambda(x, 0))/\partial\lambda^2 \neq 0$ for $x$ where (1.2) fails and that $T$ is continuous at $u_0$. Under these assumptions, by using appropriate modifications of the ideas above, we get that $T$ is Hölder continuous with exponent 1/2. It is convenient to deal first with regular perturbations of the form $u^+_{0,\varepsilon} = u_0 + \varepsilon$ if $x \in supp(u_0)$ compactly supported and such that $dist(u_0, u^+_{0,\varepsilon}) \leq 2\varepsilon$ and $u^-_{0,\varepsilon} = (u_0 - \varepsilon)_+$. This special family of initial data plays in this case the same role as the family $\mathcal{F}$ in the analysis of general perturbations performed above. Once we prove Hölder continuity for the family $u_{0,\varepsilon}$, the general result follows by applying the maximum principle to suitable elements of this special family.

In order to get the Hölder 1/2 regularity for this special perturbation, we use again the family $u_\lambda$ obtained by scaling. For a given $\varepsilon$, let us consider $u_\varepsilon^+$, the solution with initial data $u_{0,\varepsilon}^+$, and $T_\varepsilon^+$ its blow-up time. We select the only value $\lambda^*$ such that $T_{\lambda^*} = \lambda^* T = T_\varepsilon^+$. From this choice, it is clear that both data $u_{0,\varepsilon}^+$ and $u_\lambda^*$ must have an intersection point $x^*$ (otherwise they will be ordered and their blow-up times will be different). If $x^* \in supp(u_0)$, at this point we have, for $\varepsilon$ small (since the continuity of the blow-up time implies $|1-\lambda^*|$ small),

$$\varepsilon = u_{\lambda^*}(x^*,0) - u_0(x^*) = \frac{\partial u_\lambda(x^*,0)}{\partial \lambda}\Big|_{\lambda=1}(\lambda^*-1) + \frac{\partial^2 u_\lambda(x^*,0)}{\partial \lambda^2}\Big|_{\lambda=1}(\lambda^*-1)^2 + o((\lambda^*-1)^2).$$

Therefore,

$$(dist(u_0, u_{0,\varepsilon}^+))^{1/2} = (\varepsilon)^{1/2} \geq C|1-\lambda^*|,$$

and hence

$$|T_\varepsilon^+ - T| = |T_{\lambda^*} - T| = T|1-\lambda^*| \leq C(dist(u_0, u_{0,\varepsilon}^+))^{1/2}.$$

The same procedure gives an analogous estimate when considering $u_{0,\varepsilon}^-$. On the other hand, if $x$ does not belong to the support of $u_0$, the condition on the interface appears by using arguments similar to those in Theorem 1.1.

For a general $u_{0,h}$ we take a value of $\varepsilon$ of order $(dist(u_0, u_{0,h}))$ such that $u_{0,\varepsilon}^- \leq u_{0,h} \leq u_{0,\varepsilon}^+$ and proceed as before. Indeed, from $T_\varepsilon^+ \leq T_h \leq T_\varepsilon^-$ we obtain

$$|T_h - T| \leq \max\{T - T_\varepsilon^+,\ T_\varepsilon^- - T\} \leq C(\varepsilon)^{1/2} \leq C(dist(u_0, u_{0,h}))^{1/2}.$$

In a similar way we deal with higher-order conditions,

$$\frac{\partial^k (u_\lambda(x,0))}{\partial \lambda^k} \neq 0,$$

at points where the first $k-1$ derivatives vanish, obtaining Hölder continuity with exponent $1/k$. □

**Further results**. Finally, we consider the family $u_\lambda$ given by time translations, $u_\lambda(x,t) = u(x,t+\lambda)$. Assume that $u_0$ is an initial data such that the solution $u(x,t)$ is defined for $t \in (-a,a)$ and that $u_t(x,0) \geq c > 0$. Let $u_{0,h}$ be a perturbation of $u_0$, and assume that $T_h \leq T$. Define $\tau^+ = \inf\{\tau > 0;\ u_{0,h}(x) \leq u(x,\tau)\}$. This $\tau^+$ is well defined due to our assumption $u_t > 0$.

We have $T - T_h \leq T - (T - \tau^+) = \tau^+$. Moreover, as $u(x, \tau^+)$ and $u_{0,h}$ must have a contact point, we get, as before,

$$dist(u_0, u_{0,h}) \geq C\tau^+.$$

Therefore, we obtain a Lipschitz estimate.

If $T_h > T$, we use $\tau_- = \sup\{\tau < 0;\ u_{0,h}(x) \geq u(x, \tau)\}$. As above, $\tau_-$ is well defined due to our assumption $u_t > 0$ and the fact that the initial datum $u_0$ corresponds to a solution defined for negative small times. We have $T_h - T \leq (T - \tau_-) - T = -\tau_-$. Moreover, as $u(x, \tau_-)$ and $u_{0,h}$ must have a contact point, we get, as before,

$$dist(u_0, u_{0,h}) \geq C(-\tau_-).$$

Therefore we get the result.

Also in this case, Hölder regularity results can be derived for wider classes of initial data if higher-order conditions are imposed.

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