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MULTIPLE POSITIVE SOLUTIONS FOR CLASSES OF P-LAPLACIAN EQUATIONS

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Abstract. We study positive $C^1(\bar{\Omega})$ solutions to classes of boundary-value problems of the form

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where Δ_p denotes the p-Laplacian operator defined by

$$\Delta_p z := \operatorname{div}(|\nabla z|^{p-2}\nabla z);\ p > 1,\ \lambda > 0$$

is a parameter and Ω is a bounded domain in R^N; $N \geq 2$ with $\partial\Omega$ of class C^2 and connected. (If $N = 1$, we assume that Ω is a bounded open interval.) In particular, we establish existence of three positive solutions for classes of nondecreasing, p-sublinear functions f belonging to $C^1([0,\infty))$. Our proofs are based on sub-supersolution techniques.

1. Introduction

We consider weak solutions to classes of boundary-value problems of the form

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

where Δ_p denotes the p-Laplacian operator defined by

$$\Delta_p z := \operatorname{div}(|\nabla z|^{p-2}\nabla z); \quad p > 1,$$

λ is a positive parameter and Ω is a bounded domain in R^N; $N \geq 2$ with $\partial\Omega$ in class C^2 and connected. (If $N = 1$, we assume that Ω is a bounded open

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interval.) By a weak solution of (1.1), we mean a function $u \in W_0^{1,p}(\Omega)$ that satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \omega = \int_\Omega \lambda f(u)\,\omega,\ \forall\, \omega \in C_0^\infty(\Omega).$$

However, in this paper, we in fact study the existence and multiplicity of $C^1(\bar{\Omega})$ solutions that are strictly positive in Ω. Throughout this paper our classes of functions f satisfy

(A1) $f \in C^1([0,\infty))$ is a nondecreasing function such that $f(0) > 0$ and $\lim_{v\to\infty} \frac{f(v)}{v^{p-1}} = 0$ (p-sublinear).

For such positone p-sublinear nonlinearities, it is easy to establish that there is a positive solution for every $\lambda > 0$. Further, when $\frac{v^{p-1}}{f(v)}$ is nondecreasing, uniqueness of the positive solution for every λ follows from [5]. In this paper we will consider the case when $\frac{v^{p-1}}{f(v)}$ is not monotonic. In particular, we consider f for which there exists a and b such that $0 < a < b$ and

$$Q(a,b) := (\frac{a^{p-1}}{f(a)})\,/\,(\frac{b^{p-1}}{f(b)})$$

is sufficiently large. For such classes of nonlinearities we discuss the existence of three positive solutions for a certain range of λ. Our work extends the multiplicity result of [2], where the authors study S-shaped bifurcation curves for the Laplacian case ($p = 2$). In [2] the Green's function played a crucial role in the proof. However, here in the p-Laplacian case new ideas are required to overcome the nonavailability of the Green's function.

We now state our main result.

Theorem 1.1. *There exists a positive constant $C = C(p, N, \Omega)$ such that if $Q(a,b) > C$ for some points a and b, $a < b$, then the equation* (1.1) *has at least three positive solutions for a certain range of λ.*

Remark. Recently in [3], the authors study a multiplicity result for a class of positone p-sublinear problems via the antimaximum principle. However, this requires rather restrictive assumptions on f for small u and also does not extend the work in [2] in a natural way.

We establish Theorem 1.1 by the method of sub-supersolutions. By a supersolution ϕ we mean a function $\in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that $\phi = 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla \phi|^{p-2}\nabla \phi \cdot \nabla \omega \geq \int_\Omega \lambda f(\phi)\,\omega,\ \forall\, \omega \in W \tag{1.2}$$

and by a subsolution ψ we mean a function $\in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ such that $\psi = 0$ on $\partial\Omega$ and

$$\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \omega \leq \int_{\Omega} \lambda f(\psi)\, \omega, \ \forall\, \omega \in W, \tag{1.3}$$

where $W = \{v \in C_0^{\infty}(\Omega) : v \geq 0 \text{ in } \Omega\}$. Then by the weak comparison principle (see [6] or [4]), if there exist sub- and supersolutions ψ and ϕ respectively such that $\psi \leq \phi$ in Ω, then (1.1) has a $C^1(\bar{\Omega})$ solution u such that $\psi \leq u \leq \phi$.

We prove multiplicity by a sub-supersolution result for the p-Laplacian case discussed in [3]. This result extends the corresponding result for the Laplacian (p=2) case proved in [1] and [7]. The result is as follows:

Lemma 1.1. *Let f be nonnegative and nondecreasing, and suppose there exist a subsolution ψ_1, a strict supersolution ϕ_1, a strict subsolution ψ_2, and a supersolution ϕ_2 for* (1.1) *such that $\psi_1 < \phi_1 < \phi_2$, $\psi_1 < \psi_2 < \phi_2$, and $\psi_2 \not\leq \phi_1$. Then* (1.1) *has at least three distinct solutions u_i $(i = 1, 2, 3)$ such that $\psi_1 \leq u_1 < u_2 < u_3 \leq \phi_2$.*

We will prove Theorem 1.1 for the case when Ω is a ball in Section 2. Here the proof depends heavily on the construction of a crucial positive subsolution. In Section 3, we extend the theorem for general domains, by using a simple variant of this subsolution. Finally, in Section 4 we will discuss a popular example arising in combustion theory .

2. Case when Ω is a ball

In this section we shall prove the theorem in the case when Ω is a ball B_R of radius R.

Lemma 2.1. *There exists a positive constant $C_1 = C_1(p, N, R)$ such that for any positive number b if $\lambda > C_1 \frac{b^{p-1}}{f(b)}$, then there exists a subsolution ψ of the equation* (1.1) *on B_R, with $\|\psi\|_{\infty} \geq b$.*

Proof of Lemma 2.1. Let us define, for some $\alpha, \beta > 1$ and $\varepsilon > 0$,

$$v(r) := \begin{cases} 1 & r \leq \varepsilon \\ 1 - (1 - (\frac{R-r}{R-\varepsilon})^{\beta})^{\alpha} & \varepsilon < r \leq R \end{cases}$$

and let $\tilde{v}(r) = b\, v(r)$. Denoting

$$\mu_1(r) = \frac{R-r}{R-\varepsilon}, \quad \mu_2(r) = 1 - (\frac{R-r}{R-\varepsilon})^{\beta}$$

we have that for $\varepsilon < r < R$,

$$-\tilde{v}'(r) \;=\; b\frac{\alpha\beta}{R-\varepsilon}(\mu_2(r))^{\alpha-1}(\mu_1(r))^{\beta-1},$$

and hence

$$|\tilde{v}'(r)| \;\leq\; b\frac{\alpha\beta}{R-\varepsilon}.$$

Then define ψ as the radially symmetric solution of

$$\begin{aligned} -\Delta_p\,\psi(x) \;&=\; \lambda f(\tilde{v}(|x|)) \text{ in } B_R \\ \psi \;&=\; 0 \text{ on } \partial B_R. \end{aligned} \tag{2.1}$$

Then ψ satisfies

$$-(r^{N-1}G(\psi'(r)))' = \lambda r^{N-1}f(\tilde{v}(r)), \quad \psi'(0) = 0;\ \psi(R) = 0,$$

where for any real t, $G(t) = |t|^{p-2}t$. Integrating once, we get for $0 < r < R$,

$$-G(\psi'(r)) \;=\frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}f(\tilde{v}(s))\;ds\;.$$

Observe that G being monotone, G^{-1} also is continuous and monotone. Hence,

$$-\psi'(r) = G^{-1}\{\;\frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}f(\tilde{v}(s))\;ds\}. \tag{2.2}$$

We claim that

$$\psi(r) \geq \tilde{v}(r)\;\forall\;0 \leq r \leq R. \tag{2.3}$$

Then from 2.1 it follows that $\psi(r)$ is a subsolution since f is monotone. Both ψ and $\tilde{v}$ vanish at R. Thus, in order to show 2.3, it is enough to show that

$$\psi'(r) \leq \tilde{v}'(r)\;\forall\;0 \leq r \leq R. \tag{2.4}$$

Since $f > 0$, from equation (2.2) $\psi'(r) \leq 0$, while $\tilde{v}'(r) = 0$ for $0 \leq r \leq \varepsilon$, it is enough to verify equation (2.4) in the range $\varepsilon \leq r \leq R$. For $r \geq \varepsilon$ we have

$$\int_0^r s^{N-1}f(\tilde{v}(s))ds \;\geq\; \int_0^\varepsilon s^{N-1}f(\tilde{v}(s)) \;\geq\; f(b)\frac{\varepsilon^N}{N},$$

and hence from (2.2), using the monotonicity of G^{-1},

$$-\psi'(r) \;\geq\; G^{-1}\{\;\frac{\lambda}{R^{N-1}}f(b)\frac{\varepsilon^N}{N}\}.$$

Thus (2.4) will hold also for all $\varepsilon \leq r \leq R$, if

$$G^{-1}(\frac{\lambda}{R^{N-1}}f(b)\frac{\varepsilon^N}{N}) \;\geq\; \frac{\alpha\beta b}{R-\varepsilon},$$

which is the same as

$$\frac{\lambda}{R^{N-1}}f(b)\frac{\varepsilon^N}{N} \geq G(\frac{\alpha\beta b}{R-\varepsilon}) = (\frac{\alpha\beta b}{R-\varepsilon})^{p-1}.$$

Thus, if

$$\lambda \geq \frac{b^{p-1}}{f(b)}R^{N-1}\frac{N}{\varepsilon^N}(\frac{\alpha\beta}{R-\varepsilon})^{p-1},$$

then 2.4 holds. To get the best possible lower bound for λ, we can take $\alpha = \beta = 1$, since v is C^1 for any $\alpha = \beta = 1 + \delta$. Define

$$C_1 := \inf_{\varepsilon} \frac{N}{\varepsilon^N}(\frac{1}{R-\varepsilon})^{p-1}R^{N-1}.$$

Then

$$C_1 \ = \ \frac{N}{R^p}\inf_t \frac{1}{(1-t)^{p-1}t^N}.$$

This infimum is attained for the value $t = \frac{N}{N+p-1}$, which gives $C_1(p, N, R)$. This proves the lemma.

Theorem 2.1. *There exists a positive constant $C = C(p, N, R)$ such that if $Q(a, b) > C$ for some points a and b, $a < b$, then the equation* (1.1) *on B_R has at least three positive solutions for a certain range of λ.*

Proof. We shall construct supersolutions ϕ_1 and ϕ_2 and subsolutions ψ_1 and ψ_2 as in Lemma 1.1. Clearly $\psi_1 = 0$ is a subsolution for every $\lambda > 0$ since $f(0) > 0$. Let $\phi_1 = a\frac{e}{\|e\|_\infty}$, where $e \in C^1(\bar{\Omega})$ is the solution of $-\Delta_p e = 1$ in Ω, $e = 0$ on $\partial\Omega$. Then $-\Delta_p \phi_1 = (\frac{a}{\|e\|_\infty})^{p-1} \geq \lambda f(a) \geq \lambda f(\phi_1)$, and hence a supersolution if $\lambda \leq (\frac{a^{p-1}}{f(a)})(\frac{1}{(\|e\|_\infty)^{p-1}}) = B$ (say). Note that $\|\phi_1\|_\infty = a$. Next let $\psi_2 = \psi$. Then by Lemma 2.1, ψ_2 is a subsolution such that $\|\psi_2\|_\infty \geq b$ if $\lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A$ (say). Note that if $Q(a, b) > C$ where $C = C(p, N, R) = C_1(\|e\|_\infty)^{p-1}$, then $A < B$. Finally, let $\phi_2 = M(\lambda)\frac{e}{\|e\|_\infty}$. Then $-\Delta_p \phi_2 = (\frac{M(\lambda)}{\|e\|_\infty})^{p-1} \geq \lambda f(M(\lambda)) \geq \lambda f(\phi_2)$, and hence a supersolution for any given λ, if $M(\lambda)$ is chosen sufficiently large so that $\frac{(M(\lambda))^{p-1}}{f(M(\lambda))} \geq \lambda(\|e\|_\infty)^{p-1}$. This is possible since the function f is p-sublinear. Here since $\frac{\partial e}{\partial n} < 0$ on $\partial\Omega$, we can also choose $M(\lambda)$ large enough so that $\phi_2 > \psi_2$ and $\phi_2 > \phi_1$. Hence there exist three positive solutions for λ in $[A, B]$.

3. Proof of Theorem 1.1

In this section we will prove Theorem 1.1 in general domains.

First we construct a positive subsolution $z(x)$ in Ω with $\|z\|_\infty \geq b$. Let B_R be the largest inscribed ball in Ω and $C_1(p, N, R)$ be as in Lemma 2.1. Assume $Q(a, b) > C_1$ and $\lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A$, and let $\psi(r)$ be the subsolution constructed in B_R in Lemma 2.1. Now define $z(x) = \psi(|x|)$ if $x \in B_R$ and $z(x) = 0$ if $x \in \Omega - B_R$. Then $z \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ and $z = 0$ on $\partial\Omega$. Further, on B_R we have $-\Delta_p z(x) = -\Delta_p \psi(|x|) \leq \lambda f(\psi(|x|)) = \lambda f(z(x))$, while outside B_R we have $-\Delta_p z(x) = 0 < \lambda f(0) = \lambda f(z(x))$ (since $f(0) > 0$). Hence $z(x)$ is a subsolution in Ω for $\lambda \geq C_1 \frac{b^{p-1}}{f(b)} = A$ with $\|z\|_\infty \geq b$.

The rest of the proof of Theorem 1.1 is identical to the proof of Theorem 2.1, except that here we define $\psi_2 = z$.

4. Application in combustion theory

Here we consider the example

$$\begin{aligned} -\Delta_p u &= \lambda \exp[\frac{\alpha u}{\alpha + u}] \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The nonlinearity $f(u) = \exp[\frac{\alpha u}{\alpha+u}]$ arises in the theory of combustion, and it was discussed in [2] and many references cited within for the case when $p = 2$ (Laplacian case). In [2] the authors prove that a necessary condition for multiple positive solutions is $\alpha > 4$. Further, they prove that if α is large enough then there are at least three positive solutions for a certain range of λ. Here we will establish similar results for the p-Laplacian case.

Clearly f satisfies hypothesis $(A1)$. Also a simple calculation shows that $\frac{u^{p-1}}{f(u)}$ is nondecreasing if $\alpha \leq 4(p-1)$. Hence a necessary condition for multiplicity is $\alpha > 4(p-1)$. Further choosing $a = 1$ and $b = \alpha$ we have

$$Q(a, b) := (\frac{a^{p-1}}{f(a)}) \,/\, (\frac{b^{p-1}}{f(b)}) \; = (\alpha)^{1-p} \exp[\frac{\alpha}{2} - \frac{\alpha}{\alpha + 1}].$$

Hence, given any positive constant $C = C(p, N, \Omega)$, for α large we have $Q(1, \alpha) > C$, and thus there exists at least three positive solutions for a certain range of λ by Theorem 1.1.

References

[1] H. Amman, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., 18 (1976), 620–709.

[2] K.J. Brown, M.M.A. Ibrahim, and R. Shivaji, *S-shaped bifurcation curves*, J. of Nonlinear Analysis, TMA, 5 (1981), 475–486.

[3] C. Maya, S. Oruganti, and R. Shivaji, *Positive solutions for classes of p-Laplacian equations,* to appear in Diff. Int. Eqns.

[4] P. Drábek, P. Krejčí, and P. Takáč, "Nonlinear Differential Equations," Chapman & Hall/CRC, 1999.

[5] J.I. Díaz and J.E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, Comptes Rendus Acad. Sc. Paris, Série I, 305 (1983), 521–524.

[6] J. Fleckinger and P. Takáč, *Uniqueness of positive solutions for nonlinear cooperative systems with the p-Laplacian*, Indiana Univ. Math. J., 43 (1994), 1227–1253.

[7] R. Shivaji, *A remark on the existence of three solutions via sub-super solutions*, "Nonlinear Analysis and Application," Lecture Notes in Pure and Applied Mathematics, 109 (1987), 561–566, V. Lakshmikantham, ed.