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THE CAUCHY PROBLEM FOR A MODIFIED CAMASSA-HOLM EQUATION WITH ANALYTIC INITIAL DATA

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Abstract. We show that the periodic Cauchy problem for a modified Camassa-Holm equation with analytic initial data is analytic in the space variable x for time near zero. By differentiating the equation and the initial condition with respect to x we obtain a sequence of initial-value problems of KdV-type equations. These, written in the form of integral equations, define a mapping on a Banach space whose elements are sequences of functions equipped with a norm expressing the Cauchy estimates in terms of the KdV norms of the components introduced in the works of Bourgain, Kenig, Ponce, Vega, and others. By proving appropriate bilinear estimates we show that this mapping is a contraction, and therefore we obtain a solution whose derivatives in the space variable satisfy the Cauchy estimates.

1. Introduction

We consider the periodic initial-value problem for the following modified Camassa-Holm (mCH) equation with analytic initial data:

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x\big(u^2\big) + \big(1-\partial_x^2\big)^{-1}\partial_x\big[u^2 + \frac{1}{2}(\partial_x u)^2\big] = 0, \tag{1.1}$$

$$u(x,0) = \varphi(x) \in \mathcal{C}^\omega(\mathbb{T}), \quad t \in \mathbb{R}, \quad x \in \mathbb{T}. \tag{1.2}$$

If we drop the dispersive term $\partial_x^3 u$ from this equation, then we obtain the recently discovered CH equation by Camassa and Holm [5], and by Fuchssteiner and Fokas [14],

$$\partial_t u + \frac{1}{2}\partial_x\big(u^2\big) + \big(1-\partial_x^2\big)^{-1}\partial_x\big[u^2 + \frac{1}{2}(\partial_x u)^2\big] = 0. \tag{1.3}$$

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If we drop the nonlocal and nonlinear term, then we obtain the well-known KdV equation

$$\partial_t u + \partial_x^3 u + \tfrac{1}{2}\partial_x\big(u^2\big) = 0. \tag{1.4}$$

In [3] Bourgain has shown that the KdV equation is both locally and globally well-posed in $H^s(\mathbb{T})$ for $s \geq 0$. Then, in [22] Kenig, Ponce, and Vega have shown that the KdV equation is locally well-posed for $s \geq -1/2$, and in [6] Colliander, Keel, Staffilani, Takaoka, and Tao have shown that it is globally well-posed for $s \geq -1/2$. Following Bourgain's approach, it has been shown in [4] that the mCH equation is well-posed in $H^s(\mathbb{T})$ for $s > 1/2$, while following the approach of Kenig, Ponce, and Vega, this result has been extended in [15] to $s = 1/2$ for small initial data. Concerning the CH equation, it is known to be locally well-posed in $H^s(\mathbb{T})$ for $s > 3/2$ (see [17] and [24]). Global existence, as well as blowup in finite time, for the CH equation may occur (see [8] and [9]). For the well-posedness of CH in Besov spaces we refer the reader to Danchin [13]. First results on the well-posedness of CH in the periodic case were published in [8] for initial data in H^4, and in [9] for initial data in H^3 (see also [10] and [12]). In the nonperiodic case, the local well-posedness in $H^3(\mathbb{R})$ and global existence results were obtained in [11], while the local well-posedness in $H^s(\mathbb{R})$ for $s > 3/2$ was proved by Li and Olver [23] using a regularization technique of Bona and Smith [2]. Finally, for traveling-wave solutions to the CH equation we refer the reader to Beals, Sattinger, and Szmigielski [1].

Concerning analytic regularity, it has been shown in [18] that the solution to the CH Cauchy problem with analytic initial data is analytic in both the space and time variables, globally in space and locally in time. In contrast, it is well known that for analytic initial data KdV solutions are analytic in the space variable for all time (see Trubowitz [25]) but are not analytic in the time variable (see Byers and Himonas [4]). In this paper we show that, like the KdV equation, the solution $u(x,t)$ of the mCH initial-value problem with analytic initial data is analytic in x. More precisely, we prove the following result.

Theorem 1.1. *The Cauchy problem for the mCH equation,* (1.1)–(1.2), *with analytic initial data has a unique solution* $u(x,t)$ *such that* $u(\cdot,t) \in \mathcal{C}^\omega(\mathbb{T})$ *for* t *near zero.*

Existence and uniqueness of the solution follows from [4], since the initial data are in $\mathcal{C}^\omega(\mathbb{T})$ and therefore in $H^s(\mathbb{T})$ for any s. Also there, global well-posedness was proved in $H^1(\mathbb{T})$ by using the fact that the H^1 norm of a solution to the mCH equation is conserved.

Below, in Section 2, by differentiating our initial-value problem (1.1)–(1.2) k times and writing the resulting equations in an integral form, we obtain an integral equation for the sequence $\{u_k \doteq \partial_x^k u\}$. Then, using the bilinear estimates in Proposition 2.2, we prove that the mapping defined by this integral equation is a contraction on a Banach space whose elements are sequences of analytic functions equipped with a norm expressing the Cauchy estimates in terms of the KdV norms of the components introduced in the works of Bourgain, Kenig, Ponce, Vega, and others. Thus we obtain a unique solution which is analytic in the space variable x. Finally, in Section 3 we prove the fundamental estimates of Proposition 2.2 by using the basic bilinear estimate for the mCH equation proved by Himonas and Misiołek [4].

2. Proof of Theorem 1.1

We shall consider only the case of mean-zero data; that is, we shall assume that

$$\widehat{\varphi}(0) = \int_{\mathbb{T}} \varphi(x)dx = 0. \tag{2.1}$$

The general case can be reduced to this one by replacing u with $u - \frac{1}{2\pi}\widehat{\varphi}(0)$ (see [4]).

Differentiating equation (1.1) k times with respect to x yields the following initial-value problem:

$$\partial_t(\partial_x^k u) + \partial_x^3(\partial_x^k u) + \tfrac{1}{2}\partial_x[\partial_x^k(u^2)] + (1-\partial_x^2)^{-1}\partial_x[\partial_x^k(u^2) + \tfrac{1}{2}\partial_x^k((\partial_x u)^2)] = 0,$$

$$\partial_x^k u(x,0) = \partial_x^k \varphi(x) \in \mathcal{C}^\omega(\mathbb{T}), \quad k \in \mathbb{N}_0,$$

where $\mathbb{N}_0 \doteq \{0,1,2,\dots\}$. Defining

$$u_k \doteq \partial_x^k u, \quad \text{and} \quad \varphi_k \doteq \partial_x^k \varphi, \tag{2.2}$$

we obtain the system

$$\partial_t u_k + \partial_x^3 u_k + \tfrac{1}{2}\partial_x\left[\partial_x^k(u\cdot u)\right] + (1-\partial_x^2)^{-1}\partial_x\left[\partial_x^k(u\cdot u) + \tfrac{1}{2}\partial_x^k(\partial_x u \cdot \partial_x u)\right] = 0, \tag{2.3}$$

$$u_k(x,0) = \varphi_k(x) \in \mathcal{C}^\omega(\mathbb{T}), \quad k \in \mathbb{N}_0. \tag{2.4}$$

Since the integral of a solution to mCH is conserved, observe that the mean-zero initial-data assumption (2.1) implies that

$$\widehat{u}_k(0,t) = \int_{\mathbb{T}} u_k(x,t)dx = 0, \quad k \in \mathbb{N}_0. \tag{2.5}$$

Letting

$$B_k(f,g) \doteq \partial_x[\partial_x^k(f\cdot g)], \tag{2.6}$$

and using the Leibniz formula we have that

$$B_k(f,g) = \partial_x \sum_{j=0}^{k} \binom{k}{j} \partial_x^{k-j} f \; \partial_x^j g = \sum_{j=0}^{k} \binom{k}{j} \partial_x \big(\partial_x^{k-j} f \; \partial_x^j g\big).$$

From this,

$$\partial_x \Big[\partial_x^k \big(u \cdot u\big)\Big] = B_k(u,u) = \sum_{j=0}^{k} \binom{k}{j} \partial_x \big(u_{k-j} \cdot u_j\big),$$

$$\partial_x \Big[\partial_x^k \big(\partial_x u \cdot \partial_x u\big)\Big] = B_k(\partial_x u, \partial_x u) = \sum_{j=0}^{k} \binom{k}{j} \partial_x \big(\partial_x u_{k-j} \cdot \partial_x u_j\big),$$

and system (2.3)–(2.4) becomes

$$\partial_t u_k + \partial_x^3 u_k = -F_k(u,u), \tag{2.7}$$

$$u_k(x,0) = \varphi_k(x), \quad k \in \mathbb{N}_0, \tag{2.8}$$

where

$$F_k(u,u) \doteq \Big[\frac{1}{2} + \big(1-\partial_x^2\big)^{-1}\Big] B_k(u,u) + \frac{1}{2}\big(1-\partial_x^2\big)^{-1} B_k(\partial_x u, \partial_x u). \tag{2.9}$$

Next, using Duhamel's formula, we write this system in the integral form

$$u_k(x,t) = W(t)\varphi_k(x) - \int_0^t W(t-\tau) F_k(u,u)(x,\tau) d\tau, \quad k \in \mathbb{N}_0, \tag{2.10}$$

where $W(t) \doteq e^{-t\partial_x^3}$. We now localize in t by multiplying (2.10) by a cut-off function $\psi(t) \in C_0^\infty(-1,1)$ with $0 \le \psi \le 1$ and such that $\psi(t) \equiv 1$ for $|t| < 1/2$. Thus, we obtain

$$\psi(t) u_k(x,t) = T_k(u_0, u_1, \dots, u_k), \quad k \in \mathbb{N}_0, \tag{2.11}$$

where

$$T_k(u_0, u_1, \dots, u_k) \doteq \psi(t) W(t) \varphi_k(x) - \psi(t) \int_0^t W(t-\tau) F_k(u,u)(x,\tau) d\tau. \tag{2.12}$$

In order to define our space for proving analyticity, we need the space X^s for $s \ge 0$, defined by $X^s \doteq \big\{u \in L^2(\mathbb{T}\times\mathbb{R}) : |||u|||_{X^s} < \infty\big\}$, where

$$|||u|||_{X^s} \doteq \Big(\sum_{n\in\mathbb{Z}} |n|^{2s} \int_{\mathbb{R}} (1 + |\lambda - n^3|) |\widehat{u}(n,\lambda)|^2 d\lambda\Big)^{1/2}.$$

These spaces have been used by Bourgain [3], Kenig, Ponce, and Vega [20], [21], [22], and others. We now define a subspace of the space X^s by

$$Y^s \doteq \big\{u \in L^2(\mathbb{T}\times\mathbb{R}) : |||u|||_{Y^s} < \infty\big\}, \tag{2.13}$$

where

$$|||u|||_{Y^s} \doteq |||u|||_{X^s} + \Big(\sum_{n\in\mathbb{Z}} |n|^{2s} \big(\int_{\mathbb{R}} |\widehat{u}(n,\lambda)| d\lambda \big)^2 \Big)^{1/2}. \tag{2.14}$$

From this definition, it follows that Y^s is a Banach space. To incorporate our mean-zero data assumption (2.5) into our space, we define the following Banach subspace of Y^s:

$$\dot{Y}^s \doteq \big\{ u \in Y^s : \widehat{u}(0,t) = 0 \big\}. \tag{2.15}$$

The space Y^s in (2.13) was first introduced by Colliander, Keel, Staffilani, Takaoka, and Tao [6] (see also [7]) for studying global well-posedness of the KdV.

Before explaining how we will go about proving analyticity in x, let us prove the following lemma, which gives us an estimate for the Y^s norm of the multiplication operator by use of a cut-off function $\psi(t)$.

Lemma 2.1. *There is a constant $c = c(\psi) > 0$ such that*

$$|||\psi u|||_{Y^s} \le c|||u|||_{Y^s},$$

for all $u \in \dot{Y}^s$.

To avoid confusion when working with functions of two variables, we shall use the notation $\widehat{\ }^x$ and $\widehat{\ }^t$ to denote the Fourier transform with respect to x and t, respectively. We shall also use the standard notation "$f \lesssim g$" and "$f \simeq g$" to denote "$f \le cg$" and "$f = cg$," respectively, where c is a constant. This notation will be used in the proof of Lemma 2.1, as well as throughout this paper.

Proof of Lemma 2.1. Letting $f(x,t) = \psi(t)u(x,t)$, we have that the Fourier transform of f with respect to x is $\widehat{f}^x(n,t) = \widehat{u}^x(n,t)\psi(t)$, and hence, by the definition of the convolution,

$$\widehat{f}(n,\lambda) \simeq \int_{\mathbb{R}} \widehat{u}(n,\lambda-\lambda_1)\widehat{\psi}(\lambda_1)d\lambda_1.$$

Taking the Y^s norm of $\psi(t)u(x,t)$ gives us

$$|||\psi(t)u(x,t)|||_{Y^s} \simeq$$

$$\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \int_{\mathbb{R}} (1+|\lambda-n^3|) \Big(\Big| \int_{\mathbb{R}} \widehat{u}(n,\lambda-\lambda_1)\widehat{\psi}(\lambda_1)d\lambda_1 \Big| \Big)^2 d\lambda \Big)^{1/2} \tag{2.16}$$

$$+ \Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \Big(\int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \widehat{u}(n,\lambda-\lambda_1)\widehat{\psi}(\lambda_1)d\lambda_1 \Big| d\lambda \Big)^2 \Big)^{1/2}. \tag{2.17}$$

First, let us estimate (2.16). Using that

$$(1+|\lambda-n^3|)\leq(1+|\lambda_1|)(1+|\lambda-\lambda_1-n^3|),$$

and passing the absolute-value sign inside the integral we get that (2.16) is bounded above by

$$\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\big|(1+|\lambda_1|)^{\frac{1}{2}}(1+|\lambda-\lambda_1-n^3|)^{\frac{1}{2}}\widehat{u}(n,\lambda-\lambda_1)\widehat{\psi}(\lambda_1)\big|d\lambda_1\Big)^2 d\lambda\Big)^{\frac{1}{2}}.$$

Applying Minkowski's inequality yields

$$\begin{aligned}(2.16)\quad\leq\quad&\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\big|(1+|\lambda_1|)^{\frac{1}{2}}(1+|\lambda-\lambda_1-n^3|)^{\frac{1}{2}}\times\\&\widehat{u}(n,\lambda-\lambda_1)\widehat{\psi}(\lambda_1)\big|^2 d\lambda\Big)^{1/2}d\lambda_1\Big)^2\Big)^{1/2}.\end{aligned}$$

Rearranging terms and making the change of variable $\lambda'=\lambda-\lambda_1$, we obtain

$$\begin{aligned}(2.16)\quad\leq\quad&\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{\mathbb{R}}(1+|\lambda_1|)^{\frac{1}{2}}\big|\widehat{\psi}(\lambda_1)\big|\times\\&\Big(\int_{\mathbb{R}}(1+|\lambda-\lambda_1-n^3|)\big|\widehat{u}(n,\lambda-\lambda_1)\big|^2 d\lambda\Big)^{1/2}d\lambda_1\Big)^2\Big)^{1/2}\\=\quad&\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\Big(\int_{\mathbb{R}}(1+|\lambda_1|)^{\frac{1}{2}}\big|\widehat{\psi}(\lambda_1)\big|d\lambda_1\Big)\times\\&\Big(\int_{\mathbb{R}}(1+|\lambda'-n^3|)\big|\widehat{u}(n,\lambda')\big|^2 d\lambda'\Big)^{1/2}\Big)^2\Big)^{1/2}\\\leq\quad&c\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{\mathbb{R}}(1+|\lambda'-n^3|)\big|\widehat{u}(n,\lambda')\big|^2 d\lambda'\Big)^{1/2}=c|||u|||_{X^s},\end{aligned}$$

where $c=c(\psi)$. Now let us estimate (2.17). Taking the absolute-value sign inside the integral and making the change of variable $\lambda'=\lambda-\lambda_1$ gives us

$$\begin{aligned}(2.17)\quad\leq\quad&\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{\mathbb{R}}|\widehat{\psi}(\lambda_1)|d\lambda_1\int_{\mathbb{R}}|\widehat{u}(n,\lambda')|d\lambda'\Big)^2\Big)^{1/2}\\\leq\quad&c\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{\mathbb{R}}|\widehat{u}(n,\lambda')|d\lambda'\Big)^2\Big)^{1/2},\end{aligned}$$

where $c=c(\psi)$. Combining the estimates for (2.16) and (2.17) gives that

$$|||\psi(t)u(x,t)|||_{Y^s}\leq c|||u|||_{Y^s},$$

where $c=c(\psi)$. This completes the proof of Lemma 2.1. □

We now turn to our strategy for proving analyticity in x. For this we shall assume that

$$s \geq 1, \tag{2.18}$$

considering that the KdV-type bilinear estimates hold for $s \geq 1/2$. In terms of the H^s norm, our hypothesis that $\varphi \in \mathcal{C}^\omega(\mathbb{T})$ reads that

$$\|\varphi_k\|_{H^s(\mathbb{T})} \leq M_0 \big(\frac{1}{2C_0}\big)^k k!, \quad k \in \mathbb{N}_0, \tag{2.19}$$

for some positive constants M_0 and C_0. Using this, we define a finite norm for our sequence of initial conditions in the system (2.7)–(2.8). Letting

$$\{\varphi_k\} \doteq (\varphi_0, \varphi_1, \varphi_2, \dots), \tag{2.20}$$

we define

$$\|\{\varphi_k\}\|_s \doteq \sum_{k=0}^{\infty} \frac{C_0^k}{k!} \|\varphi_k\|_{H^s(\mathbb{T})}, \tag{2.21}$$

and observe that from (2.19) we have

$$\|\{\varphi_k\}\|_s \leq \sum_{k=0}^{\infty} \frac{C_0^k}{k!} M_0 \Big(\frac{1}{2C_0}\Big)^k k! = \sum_{k=0}^{\infty} M_0 \Big(\frac{1}{2}\Big)^k = 2M_0 < \infty. \tag{2.22}$$

Next we define a space whose elements are sequences of functions in the space $\dot{Y}^s$ (see (2.15)). It seems that a natural space for our analyticity problem is the following space:

$$\mathcal{A}(\dot{Y}^s) \doteq \big\{(u_0, u_1, u_2, \dots) \doteq \{u_k\} : u_j \in \dot{Y}^s,\ j \in \mathbb{N}_0,\ \text{and}\ |||\{u_k\}||| < \infty\big\}, \tag{2.23}$$

where

$$|||\{u_k\}||| \doteq \sum_{k=0}^{\infty} \frac{C_0^k}{k!} |||u_k|||_{Y^s}. \tag{2.24}$$

It can be proved that $\mathcal{A}(\dot{Y}^s)$ is a Banach space. Norms similar to (2.24) have been used by Kato and Ogawa [19] for the KdV equation in the nonperiodic case.

Our desire is to show that the sequence $\{u_k\}$, made up of the solutions u_k of (2.11), satisfies the estimate

$$|||\{u_k\}||| < \infty. \tag{2.25}$$

By definition (2.24), this will imply that there exists $M > 0$ such that

$$\frac{C_0^k}{k!} |||u_k|||_{Y^s} \leq M, \quad k \in \mathbb{N}_0;$$

i.e.,

$$|||u_k|||_{Y^s} \le M\Big(\frac{1}{C_0}\Big)^k k!, \quad k \in \mathbb{N}_0.$$

Since we will show later (see Lemma 2.5) that for $u \in Y^s$,

$$\|u(\cdot,t)\|_{H^s(\mathbb{T})} \lesssim |||u|||_{Y^s} \quad \text{for every } t,$$

the last inequality implies that

$$\|u_k(\cdot,t)\|_{H^s(\mathbb{T})} \lesssim |||u_k|||_{Y^s} \le M\Big(\frac{1}{C_0}\Big)^k k!, \quad k \in \mathbb{N}_0.$$

Since $u \doteq u_0$ is the solution to (1.1)–(1.2) and $u_k = \partial_x^k u$ for t near zero, we have that

$$\|\partial_x^k u(\cdot,t)\|_{H^s(\mathbb{T})} = \|u_k(\cdot,t)\|_{H^s(\mathbb{T})} \lesssim |||u_k|||_{Y^s} \le M\Big(\frac{1}{C_0}\Big)^k k!, \quad k \in \mathbb{N}_0.$$

This means that $u(\cdot,t) \in \mathcal{C}^\omega(\mathbb{T})$ for t near zero, and hence Theorem 1.1 will be proved.

Let us now prove that our solutions u_k satisfy (2.25), namely, that $|||\{u_k\}||| < \infty$. We will do this by showing that the system (2.11) has a unique solution in $\mathcal{A}(\dot{Y}^s)$, and therefore, by definition (2.23) of $\mathcal{A}(\dot{Y}^s)$, we have that $|||\{u_k\}||| < \infty$.

For $0 < \delta < 1$, we define $\psi_\delta(t) \doteq \psi(t/\delta)$, as well as the map

$$T_k^\delta(u_0, u_1, \ldots, u_k) \doteq T_k(\psi_\delta u_0, \psi_\delta u_1, \ldots, \psi_\delta u_k). \tag{2.26}$$

We shall estimate $|||T_k^\delta|||_{Y^s}$ using the following proposition, whose proof can be found in Section 3.

Proposition 2.2. *If $s > 1/2$, then there is a constant $c > 0$ such that*

$$|||T_k^\delta(u_0, u_1, \ldots, u_k)|||_{Y^s} \le c\delta^{1/24} \sum_{j=0}^{k} \binom{k}{j} |||u_{k-j}|||_{Y^s} |||u_j|||_{Y^s} + \|\varphi_k\|_{H^s}, \tag{2.27}$$

$$|||T_k^\delta(u_0, u_1, \ldots, u_k) - T_k^\delta(v_0, v_1, \ldots, v_k)|||_{Y^s} \le c\delta^{1/24} \sum_{j=0}^{k} \binom{k}{j} |||u_{k-j} + v_{k-j}|||_{Y^s} |||u_j - v_j|||_{Y^s}, \tag{2.28}$$

for all $u_0, u_1, \ldots, u_k; v_0, v_1, \ldots, v_k \in \dot{Y}^s$.

Now we are able to define the map T^δ on $\mathcal{A}(\dot{Y}^s)$ by

$$\{u_k\}_{k=0}^\infty \longmapsto T^\delta\big(\{u_k\}_{k=0}^\infty\big) \doteq \big(T_0^\delta(u_0), T_1^\delta(u_0, u_1), \ldots\big). \tag{2.29}$$

In the next lemma, we state the basic estimate for T^δ.

Lemma 2.3. *If $s > 1/2$, then there is a constant $c > 0$ such that*

$$|||T^\delta(\{u_k\})||| \le c\delta^{1/24}|||\{u_k\}|||^2 + c\|\{\varphi_k\}\|_s, \tag{2.30}$$

and

$$|||T^\delta(\{u_k\}) - T^\delta(\{v_k\})||| \le c\delta^{1/24}|||\{u_k\} + \{v_k\}||| \cdot |||\{u_k\} - \{v_k\}|||, \tag{2.31}$$

for all $\{u_k\}$ and $\{v_k\}$ in $\mathcal{A}(\dot{Y}^s)$.

Proof of Lemma 2.3. Applying the definition of the $|||\cdot|||$ norm (see (2.24)) and estimate (2.27) in Proposition 2.2 gives

$$|||T^\delta(\{u_k\})||| = |||(T_0^\delta(u_0), T_1^\delta(u_0,u_1),\dots)||| = \sum_{k=0}^{\infty}\frac{C_0^k}{k!}|||T_k^\delta(u_0,u_1,\dots,u_k)|||_{Y^s}$$

$$\le c\sum_{k=0}^{\infty}\frac{C_0^k}{k!}\left(\delta^{1/24}\sum_{j=0}^{k}\binom{k}{j}|||u_{k-j}|||_{Y^s}|||u_j|||_{Y^s} + \|\varphi_k\|_{H^s}\right)$$

$$= c\delta^{1/24}\sum_{k=0}^{\infty}\frac{C_0^k}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}|||u_{k-j}|||_{Y^s}|||u_j|||_{Y^s} + c\sum_{k=0}^{\infty}\frac{C_0^k}{k!}\|\varphi_k\|_{H^s}.$$

Using (2.21) and that $C_0^k = C_0^j C_0^{k-j}$, and interchanging the summations, we have

$$|||T^\delta(\{u_k\})||| \le c\delta^{1/24}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{C_0^j}{j!}|||u_j|||_{Y^s}\frac{C_0^{k-j}}{(k-j)!}|||u_{k-j}|||_{Y^s} + c\|\{\varphi_k\}\|_s$$

$$= c\delta^{1/24}\sum_{j=0}^{\infty}\frac{C_0^j}{j!}|||u_j|||_{Y^s}\sum_{k=j}^{\infty}\frac{C_0^{k-j}}{(k-j)!}|||u_{k-j}|||_{Y^s} + c\|\{\varphi_k\}\|_s$$

$$= c\delta^{1/24}|||\{u_k\}|||^2 + c\|\{\varphi_k\}\|_s.$$

Thus (2.30) is proved, and we now show (2.31). Applying the definitions of T^δ (see (2.29)), the $|||\cdot|||$ norm, and T_k^δ (see (2.26)), we obtain

$$\begin{aligned}
&|||T^\delta(\{u_k\}) - T^\delta(\{v_k\})|||\\
&= |||(T_0^\delta(u_0), T_1^\delta(u_0,u_1),\dots) - (T_0^\delta(v_0), T_1^\delta(v_0,v_1),\dots)|||\\
&= |||(T_0^\delta(u_0) - T_0^\delta(v_0), T_1^\delta(u_0,u_1) - T_1^\delta(v_0,v_1),\dots)|||\\
&= \sum_{k=0}^{\infty}\frac{C_0^k}{k!}|||T_k^\delta(u_0,u_1,\dots,u_k) - T_k^\delta(v_0,v_1,\dots,v_k)|||_{Y^s}.
\end{aligned}$$

From (2.28) in Proposition 2.2,

$$
\begin{aligned}
&|||T^{\delta}(\{u_k\}) - T^{\delta}(\{v_k\})||| \\
&\le c\delta^{1/24}\sum_{k=0}^{\infty}\frac{C_0^k}{k!}\sum_{j=0}^{k}\binom{k}{j}|||u_{k-j}+v_{k-j}|||_{Y^s}|||u_j - v_j|||_{Y^s} \\
&\le c\delta^{1/24}\sum_{k=0}^{\infty}\frac{C_0^k}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}|||u_{k-j}+v_{k-j}|||_{Y^s}|||u_j - v_j|||_{Y^s}.
\end{aligned}
$$

As in the proof of (2.30), we obtain

$$
\begin{aligned}
&|||T^{\delta}(\{u_k\}) - T^{\delta}(\{v_k\})||| \\
&\le c\delta^{1/24}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{C_0^j}{j!}|||u_j - v_j|||_{Y^s}\cdot\frac{C_0^{k-j}}{(k-j)!}|||u_{k-j}+v_{k-j}|||_{Y^s} \\
&= c\delta^{1/24}\sum_{j=0}^{\infty}\frac{C_0^j}{j!}|||u_j - v_j|||_{Y^s}\sum_{k=j}^{\infty}\frac{C_0^{k-j}}{(k-j)!}|||u_{k-j}+v_{k-j}|||_{Y^s} \\
&= c\delta^{1/24}|||\{u_k\}-\{v_k\}|||\cdot|||\{u_k\}+\{v_k\}|||.
\end{aligned}
$$

Hence, Lemma 2.3 is proved. □

Proposition 2.4. *If $s > 1/2$ and we choose $r = 2c\|\{\varphi_k\}\|_s$, $0 < \delta < (8c^2\|\{\varphi_k\}\|_s)^{-24}$ and $B(0,r) = \big\{\{u_k\} \in \mathcal{A}(\dot{Y}^s) : |||\{u_k\}||| \le r\big\}$, then $T^{\delta} : B(0,r) \to B(0,r)$ and it is a contraction. More precisely, we have*

$$|||T^{\delta}(\{u_k\})||| \le r \tag{2.32}$$

and

$$|||T^{\delta}(\{u_k\}) - T^{\delta}(\{v_k\})||| \le \frac{1}{2}|||\{u_k\}-\{v_k\}||| \tag{2.33}$$

for all $\{u_k\}$ and $\{v_k\}$ in $B(0,r)$.

Proof of Proposition 2.4. First note that by (2.22), our radius r is finite. Let $\{u_k\}, \{v_k\} \in B(0,r)$. From Lemma 2.3 and our hypotheses on r and δ, we obtain

$$
\begin{aligned}
|||T^{\delta}(\{u_k\})||| &\le c\delta^{1/24}|||\{u_k\}|||^2 + c\|\{\varphi_k\}\|_s \\
&\le c\frac{(2c\|\{\varphi_k\}\|_s)^2}{8c^2\|\{\varphi_k\}\|_s} + c\|\{\varphi_k\}\|_s < 2c\|\{\varphi_k\}\|_s = r.
\end{aligned}
$$

Hence, (2.32) holds. Applying estimate (2.31) in Lemma 2.3 yields

$$|||T^{\delta}(\{u_k\}) - T^{\delta}(\{v_k\})||| \le c\delta^{1/24}|||\{u_k\}+\{v_k\}|||\cdot|||\{u_k\}-\{v_k\}|||$$

$$\le c\delta^{1/24}\big(|||\{u_k\}||| + |||\{v_k\}|||\big)\cdot |||\{u_k\}-\{v_k\}|||$$
$$< c\frac{4c\|\{\varphi_k\}\|_s}{8c^2\|\{\varphi_k\}\|_s}|||\{u_k\}-\{v_k\}||| = \frac{1}{2}|||\{u_k\}-\{v_k\}|||.$$

Therefore, we have (2.33) and hence Proposition 2.4. □

The following lemma will complete the proof of the Theorem 1.1 .

Lemma 2.5. *If $u \in \dot{Y}^s$, then $\|u(\cdot,t)\|_{H^s(\mathbb{T})} \lesssim |||u|||_{Y^s}$, for all $t \in \mathbb{R}$.*

Proof of Lemma 2.5. By definition of the Y^s norm (see (2.14)), we have

$$\|u(\cdot,t)\|_{H^s} \lesssim \Big(\sum_{n\in\mathbb{Z}}|n|^{2s}|\widehat{u}^x(n,t)|^2\Big)^{1/2} = \Big(\sum_{n\in\mathbb{Z}}|n|^{2s}\Big|\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda t}\widehat{u}(n,\lambda)d\lambda\Big|^2\Big)^{1/2}$$
$$\lesssim \Big(\sum_{n\in\mathbb{Z}}|n|^{2s}\Big(\int_{\mathbb{R}}|\widehat{u}(n,\lambda)|d\lambda\Big)^2\Big)^{1/2} \le |||u|||_{Y^s},$$

which proves Lemma 2.5. □

End of proof of Theorem 1.1. By Proposition 2.4, we have that T^δ is a contraction on a closed ball in $\mathcal{A}(\dot{Y}^s)$. By the contraction mapping principle, T^δ has a fixed point, $\{u_k\}$, in $B(0,r)$, and hence, system (2.11) has a unique solution for $|t| < \delta/2$. Since $\{u_k\} \in \mathcal{A}(\dot{Y}^s)$, we have (2.25); i.e., we have that $|||\{u_k\}||| < \infty$. Using definition (2.24) we obtain that

$$|||u_k|||_{Y^s} \le M\big(\frac{1}{C_0}\big)^k k!, \quad k \in \mathbb{N}_0.$$

Combining this with Lemma 2.5 implies that

$$\|u_k(\cdot,t)\|_{H^s(\mathbb{T})} \lesssim |||u_k|||_{Y^s} \le M\big(\frac{1}{C_0}\big)^k k!, \quad k \in \mathbb{N}_0.$$

Since $u \doteq u_0$ is the solution to (1.1)–(1.2) and $u_k = \partial_x^k u$ for $|t| < \delta/2$, we have that

$$\|\partial_x^k u(\cdot,t)\|_{H^s(\mathbb{T})} = \|u_k(\cdot,t)\|_{H^s(\mathbb{T})} \lesssim |||u_k|||_{Y^s} \le M\big(\frac{1}{C_0}\big)^k k!, \quad k \in \mathbb{N}_0.$$

Therefore, $u(\cdot,t) \in \mathcal{C}^\omega(\mathbb{T})$ for $|t| < \delta/2$. We conclude that the solution to our mCH initial-value problem is analytic in x for $|t| < \delta/2$. This completes the proof of Theorem 1.1. □

3. PROOF OF PROPOSITION 2.2

Proof of (2.27). We introduce the following convenient notation:

$$F_k^\delta(u,u) \doteq F_k(\psi_\delta u, \psi_\delta u) = \Big[\frac{1}{2} + \big(1-\partial_x^2\big)^{-1}\Big]\sum_{j=0}^{k}\binom{k}{j}\partial_x(\psi_\delta u_{k-j}\cdot\psi_\delta u_j)$$

$$+\frac{1}{2}\left(1-\partial_x^2\right)^{-1}\sum_{j=0}^{k}\binom{k}{j}\partial_x(\partial_x\psi_\delta u_{k-j}\cdot\partial_x\psi_\delta u_j), \tag{3.1}$$

where definitions (2.6) and (2.9) were used in the expansion of $F_k(\psi_\delta u, \psi_\delta u)$. Recalling system (2.11), we have that

$$\psi(t)u_k(x,t) = T_k(u_0, u_1, \ldots, u_k),$$

where by (2.12)

$$T_k(u_0, u_1, \ldots, u_k) \doteq \psi(t)W(t)\varphi_k(x) - \psi(t)\int_0^t W(t-\tau)F_k(u,u)(x,\tau)d\tau.$$

Replacing each u_j with $\psi_\delta u_j$ yields

$$\psi(t)\cdot\psi_\delta(t)u_k(x,t) = T_k^\delta(u_0, u_1, \ldots, u_k)$$

$$= \psi(t)W(t)\varphi_k(x) - \psi(t)\int_0^t W(t-\tau)F_k(\psi_\delta u, \psi_\delta u)(x,\tau)d\tau$$

$$= \psi(t)\sum_{n\in\dot{\mathbb{Z}}}\widehat{\varphi}_k(n)e^{i(nx+n^3t)} \tag{3.2}$$

$$+ i\sum_{\ell=1}^{\infty}\frac{i^\ell}{\ell!}t^\ell\psi(t)\sum_{n\in\dot{\mathbb{Z}}}e^{i(nx+n^3t)}\int_{-\infty}^{\infty}\psi(\lambda-n^3)(\lambda-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda)d\lambda \tag{3.3}$$

$$+ i\psi(t)\sum_{n\in\dot{\mathbb{Z}}}e^{inx}\int_{-\infty}^{\infty}\frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\, e^{i\lambda t}\widehat{F_k^\delta}(n,\lambda)d\lambda \tag{3.4}$$

$$- i\psi(t)\sum_{n\in\dot{\mathbb{Z}}}e^{i(nx+n^3t)}\int_{-\infty}^{\infty}\frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\widehat{F_k^\delta}(n,\lambda)d\lambda. \tag{3.5}$$

The last equality follows from the definition of $W(t)$ and application of the Fourier transform. Note that by the mean-zero data assumption, we can assume $n \in \dot{\mathbb{Z}}$, where $\dot{\mathbb{Z}} \doteq \mathbb{Z} - \{0\}$.

Estimate for (3.2). For the estimate of $|||(3.2)|||_{Y^s}$, let us first find the Fourier transform of the function

$$f_k(x,t) = \psi(t)\sum_{n\in\dot{\mathbb{Z}}}\widehat{\varphi}_k(n)e^{i(nx+n^3t)}.$$

We see that $\widehat{f_k^x}(n,t) = \widehat{\varphi}_k(n)e^{in^3t}\psi(t)$, and therefore,

$$\widehat{f_k}(n,\lambda) = \widehat{\varphi}_k(n)\int_{t\in\mathbb{R}}e^{-i\lambda t}\psi(t)e^{in^3t}dt = \widehat{\varphi}_k(n)\widehat{\psi}(\lambda-n^3).$$

Taking the Y^s norm of (3.2) and making the change of variable $\lambda' = \lambda - n^3$, we get

$$\begin{aligned}|||(3.2)|||_{Y^s} &= \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{\lambda\in\mathbb{R}}(1+|\lambda-n^3|)\big|\widehat{\varphi}_k(n)\widehat{\psi}(\lambda-n^3)\big|^2 d\lambda\Big)^{1/2}\\ &\quad+\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{\lambda\in\mathbb{R}}\big|\widehat{\varphi}_k(n)\widehat{\psi}(\lambda-n^3)\big|d\lambda\Big)^2\Big)^{1/2}\\ &= \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\big|\widehat{\varphi}_k(n)\big|^2\Big(\int_{\lambda'\in\mathbb{R}}(1+|\lambda'|)\big|\widehat{\psi}(\lambda')\big|^2 d\lambda'\Big)\Big)^{1/2}\\ &\quad+\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\big|\widehat{\varphi}_k(n)\big|^2\Big(\int_{\lambda'\in\mathbb{R}}\big|\widehat{\psi}(\lambda')\big|d\lambda'\Big)^2\Big)^{1/2}.\end{aligned}$$

Hence,

$$|||(3.2)|||_{Y^s} \lesssim \|\varphi_k\|_{H^s}. \tag{3.6}$$

Estimate for (3.3). Beginning with the Fourier transform of the function

$$f_{k,\ell}(x,t) = t^\ell\psi(t)\sum_{n\in\dot{\mathbb{Z}}}e^{i(nx+n^3t)}\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1,$$

we have

$$\widehat{f}^x_{k,\ell}(n,t) = t^\ell\psi(t)e^{in^3t}\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1,$$

and therefore,

$$\begin{aligned}\widehat{f}_{k,\ell}(n,\lambda) &= \Big(\int_{t\in\mathbb{R}}e^{-i\lambda t}t^\ell\psi(t)e^{in^3t}dt\Big)\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\\ &= \Big(\widehat{t^\ell\psi}(\lambda-n^3)\Big)\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1.\end{aligned}$$

Using the above computation of $\widehat{f}_{k,\ell}(n,\lambda)$, we now estimate (3.3).

$$\begin{aligned}|||(3.3)|||_{Y^s} &= |||\sum_{\ell=1}^\infty\frac{i^{\ell+1}}{\ell!}f_{k,\ell}(x,t)|||_{Y^s} \le \sum_{\ell=1}^\infty\frac{1}{\ell!}|||f_{k,\ell}(x,t)|||_{Y^s}\\ &= \sum_{\ell=1}^\infty\frac{1}{\ell!}\Big\{\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{\lambda\in\mathbb{R}}(1+|\lambda-n^3|)\big|\widehat{t^\ell\psi}(\lambda-n^3)\big|^2\\ &\times\Big|\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|^2 d\lambda\Big)^{1/2}\end{aligned}$$

$$+\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{\lambda\in\mathbb{R}}\big|\widehat{t^\ell\psi}(\lambda-n^3)\big|$$
$$\times\Big|\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|d\lambda\Big)^2\Big)^{1/2}\Big\}.$$

With the change of variable $\lambda'=\lambda-n^3$, we have

$$|||(3.3)|||_{Y^s}\le\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\Big\{\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{\lambda'\in\mathbb{R}}(1+|\lambda'|)\big|\widehat{t^\ell\psi}(\lambda')\big|^2$$
$$\times\Big|\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|^2d\lambda'\Big)^{1/2}$$
$$+\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{\lambda'\in\mathbb{R}}\big|\widehat{t^\ell\psi}(\lambda')\big|$$
$$\times\Big|\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|d\lambda'\Big)^2\Big)^{1/2}\Big\}$$
$$=\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\Big\{\Big\|(1+|\lambda|)\big|\widehat{t^\ell\psi}(\lambda)\big|^2\Big\|_{L^1}^{1/2}$$
$$\times\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big|\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|^2\Big)^{1/2}$$
$$+\big\|\widehat{t^\ell\psi}(\lambda)\big\|_{L^1}\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big|\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|^2\Big)^{1/2}\Big\}$$
$$\le\Big(\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\Big(\Big\|(1+|\lambda|)\big|\widehat{t^\ell\psi}(\lambda)\big|^2\Big\|_{L^1}^{1/2}+\big\|\widehat{t^\ell\psi}(\lambda)\big\|_{L^1}\Big)\Big)$$
$$\times\sup_{\ell\ge1}\Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big|\int_{\lambda_1\in\mathbb{R}}\psi(\lambda_1-n^3)(\lambda_1-n^3)^{\ell-1}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|^2\Big)^{1/2}.$$

Using that $1+|\lambda|\le2(1+|\lambda|^2)$ and $\psi(t),\psi'(t)\in C_0^\infty(-1,1)$ yields

$$\Big\|(1+|\lambda|)\big|\widehat{t^\ell\psi}(\lambda)\big|^2\Big\|_{L^1}^{1/2}\lesssim\Big(\int_{\lambda\in\mathbb{R}}(1+|\lambda|^2)\big|\widehat{t^\ell\psi}(\lambda)\big|^2d\lambda\Big)^{1/2}=\|t^\ell\psi\|_{H^1}$$
$$\le\|t^\ell\psi\|_{L^2}+\|\ell t^{\ell-1}\psi+t^\ell\psi'\|_{L^2}\le\|t^\ell\psi\|_{L^2}+\|\ell t^{\ell-1}\psi\|_{L^2}+\|t^\ell\psi'\|_{L^2}$$
$$\le\|\psi\|_{L^2}+\ell\|\psi\|_{L^2}+\|\psi'\|_{L^2}\le\ell\big(\|\psi\|_{L^2}+\|\psi\|_{L^2}+\|\psi'\|_{L^2}\big)=c\cdot\ell,$$

where $c = c(\psi)$. Applying the Cauchy-Schwarz inequality and that $(1+|\lambda|)^2 \le 3(1+|\lambda|^2)$, we obtain

$$\begin{aligned}\|\widehat{t^\ell\psi}(\lambda)\|_{L^1} &= \int_{\lambda\in\mathbb{R}} |\widehat{t^\ell\psi}(\lambda)| \frac{1+|\lambda|}{1+|\lambda|} d\lambda \\ &\le \Big(\int_{\lambda\in\mathbb{R}} |\widehat{t^\ell\psi}(\lambda)|^2 (1+|\lambda|)^2 d\lambda\Big)^{1/2} \Big(\int_{\lambda\in\mathbb{R}} \frac{1}{(1+|\lambda|)^2} d\lambda\Big)^{1/2}, \\ &\lesssim \Big(\int_{\lambda\in\mathbb{R}} |\widehat{t^\ell\psi}(\lambda)|^2 (1+|\lambda|^2) d\lambda\Big)^{1/2} = \|t^\ell\psi\|_{H^1} \lesssim \ell,\end{aligned}$$

where we bound $\|t^\ell\psi\|_{H^1}$ as above. Hence, the infinite series

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell!}\Big(\Big\|(1+|\lambda|)|\widehat{t^\ell\psi}(\lambda)|^2\Big\|_{L^1}^{1/2} + \|\widehat{t^\ell\psi}(\lambda)\|_{L^1}\Big) \le c\sum_{\ell=1}^{\infty} \frac{\ell}{\ell!} < \infty. \tag{3.7}$$

Combining estimate (3.7) and the fact that $\psi(t) \in C_0^\infty(-1,1)$, we have

$$\begin{aligned}&|||(3.3)|||_{Y^s} \\ &\lesssim \sup_{\ell\ge1}\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big(\int_{|\lambda_1-n^3|\le1} |\psi(\lambda_1-n^3)||(\lambda_1-n^3)|^{\ell-1}|\widehat{F_k^\delta}(n,\lambda_1)|d\lambda_1\Big)^2\Big)^{1/2} \\ &\le \Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big(\int_{|\lambda_1-n^3|\le1} |\widehat{F_k^\delta}(n,\lambda_1)|d\lambda_1\Big)^2\Big)^{1/2}.\end{aligned}$$

Since $|\lambda_1 - n^3| \le 1$, we now have

$$\begin{aligned}|||(3.3)|||_{Y^s} &\lesssim 2\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big(\int_{|\lambda_1-n^3|\le1} \frac{|\widehat{F_k^\delta}(n,\lambda_1)|}{1+1} d\lambda_1\Big)^2\Big)^{1/2} \\ &\lesssim \Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big(\int_{|\lambda_1-n^3|\le1} \frac{|\widehat{F_k^\delta}(n,\lambda_1)|}{1+|\lambda_1-n^3|} d\lambda_1\Big)^2\Big)^{1/2},\end{aligned}$$

and hence,

$$|||(3.3)|||_{Y^s} \lesssim \Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big(\int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n,\lambda)|}{1+|\lambda-n^3|} d\lambda\Big)^2\Big)^{1/2}. \tag{3.8}$$

Estimate for (3.4). First, computing the Fourier transform with respect to t of

$$f_k(x,t) = \sum_{n\in\dot{\mathbb{Z}}} e^{inx} \int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\ e^{i\lambda_1 t}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1$$

yields

$$\widehat{f}_k^t(x,\lambda) = \sum_{n\in\dot{\mathbb{Z}}} e^{inx}\frac{1-\psi(\lambda-n^3)}{\lambda-n^3}\widehat{F_k^\delta}(n,\lambda),$$

and therefore,

$$\widehat{f}_k(n,\lambda) = \frac{1-\psi(\lambda-n^3)}{\lambda-n^3}\widehat{F_k^\delta}(n,\lambda).$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
&|||(3.4)|||_{Y^s} = |||i\psi(t)f_k(x,t)|||_{Y^s} \le c|||f_k(x,t)|||_{Y^s} \\
&\simeq \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{\lambda\in\mathbb{R}}(1+|\lambda-n^3|)\Big|\frac{1-\psi(\lambda-n^3)}{\lambda-n^3}\widehat{F_k^\delta}(n,\lambda)\Big|^2 d\lambda\Big)^{1/2} \\
&\qquad + \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{\lambda\in\mathbb{R}}\Big|\frac{1-\psi(\lambda-n^3)}{\lambda-n^3}\widehat{F_k^\delta}(n,\lambda)\Big|d\lambda\Big)^2\Big)^{1/2} \\
&= \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{|\lambda-n^3|\ge\frac{1}{2}}(1+|\lambda-n^3|)\frac{\big(1-\psi(\lambda-n^3)\big)^2}{|\lambda-n^3|^2}|\widehat{F_k^\delta}(n,\lambda)|^2 d\lambda\Big)^{1/2} \\
&\qquad + \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{|\lambda-n^3|\ge\frac{1}{2}}\frac{|1-\psi(\lambda-n^3)|}{|\lambda-n^3|}|\widehat{F_k^\delta}(n,\lambda)|d\lambda\Big)^2\Big)^{1/2},
\end{aligned}$$

where the last equality holds since $\psi(t)\equiv 1$ for $|t|<1/2$. Using that $1+|\lambda-n^3|\le 3|\lambda-n^3|$ and $1-\psi(\lambda-n^3)\le 1$, we have that

$$\begin{aligned}
&|||(3.4)|||_{Y^s} \\
&\lesssim \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{|\lambda-n^3|\ge\frac{1}{2}}(1+|\lambda-n^3|)\frac{\big(1-\psi(\lambda-n^3)\big)^2}{(1+|\lambda-n^3|)^2}|\widehat{F_k^\delta}(n,\lambda)|^2 d\lambda\Big)^{1/2} \\
&\qquad + \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{|\lambda-n^3|\ge\frac{1}{2}}\frac{|1-\psi(\lambda-n^3)|}{1+|\lambda-n^3|}|\widehat{F_k^\delta}(n,\lambda)|d\lambda\Big)^2\Big)^{1/2} \\
&\le \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\int_{|\lambda-n^3|\ge\frac{1}{2}}\frac{|\widehat{F_k^\delta}(n,\lambda)|^2}{1+|\lambda-n^3|}d\lambda\Big)^{1/2} \\
&\qquad + \Big(\sum_{n\in\dot{\mathbb{Z}}}|n|^{2s}\Big(\int_{|\lambda-n^3|\ge\frac{1}{2}}\frac{|\widehat{F_k^\delta}(n,\lambda)|}{1+|\lambda-n^3|}d\lambda\Big)^2\Big)^{1/2}.
\end{aligned}$$

Therefore,

$$|||(3.4)|||_{Y^s} \lesssim \Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n,\lambda)|^2}{1+|\lambda-n^3|}d\lambda\Big)^{1/2}$$
$$+\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big(\int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n,\lambda)|}{1+|\lambda-n^3|}d\lambda\Big)^2\Big)^{1/2}. \qquad (3.9)$$

Estimate for (3.5). To estimate $|||(3.5)|||_{Y^s}$, first we shall find the Fourier transform of the function

$$f_k(x,t) = \psi(t)\sum_{n\in\dot{\mathbb{Z}}} e^{i(nx+n^3t)}\int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1.$$

Clearly,

$$\widehat{f_k^x}(n,t) = \psi(t)e^{in^3t}\int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1,$$

and hence

$$\begin{aligned}\widehat{f_k}(n,\lambda) &= \Big(\int_{t\in\mathbb{R}} e^{-i\lambda t}\psi(t)e^{in^3t}dt\Big)\int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\\ &= \widehat{\psi}(\lambda-n^3)\int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1.\end{aligned}$$

Applying the definition of the Y^s norm yields

$$|||(3.5)|||_{Y^s} = \Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\int_{\lambda\in\mathbb{R}}(1+|\lambda-n^3|)$$
$$\times\Big|\widehat{\psi}(\lambda-n^3)\int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|^2 d\lambda\Big)^{1/2}$$
$$+\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big(\int_{\lambda\in\mathbb{R}}\Big|\widehat{\psi}(\lambda-n^3)\Big|$$
$$\times\Big|\int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|d\lambda\Big)^2\Big)^{1/2}$$
$$\le c_1\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big|\int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|^2\Big)^{1/2}$$
$$+\ c_2\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s}\Big|\int_{\lambda_1\in\mathbb{R}} \frac{1-\psi(\lambda_1-n^3)}{\lambda_1-n^3}\widehat{F_k^\delta}(n,\lambda_1)d\lambda_1\Big|^2\Big)^{1/2},$$

where c_1 and c_2 are constants that depend on $\psi(t)$. Since $\psi(\lambda - n^3) \equiv 1$ for $|\lambda - n^3| < \frac{1}{2}$, and $|1 - \psi(\lambda - n^3)| \leq 1$ for $|\lambda - n^3| \geq \frac{1}{2}$, we have that

$$|||(3.5)|||_{Y^s} \lesssim \Big(\sum_{n \in \dot{\mathbb{Z}}} |n|^{2s} \Big(\int_{|\lambda - n^3| \geq \frac{1}{2}} \frac{|1 - \psi(\lambda - n^3)|}{|\lambda - n^3|} |\widehat{F_k^\delta}(n, \lambda)| d\lambda \Big)^2 \Big)^{1/2}$$

$$\leq \Big(\sum_{n \in \dot{\mathbb{Z}}} |n|^{2s} \Big(\int_{|\lambda - n^3| \geq \frac{1}{2}} \frac{|\widehat{F_k^\delta}(n, \lambda)|}{|\lambda - n^3|} d\lambda \Big)^2 \Big)^{1/2}.$$

Therefore,

$$|||(3.5)|||_{Y^s} \lesssim \Big(\sum_{n \in \dot{\mathbb{Z}}} |n|^{2s} \Big(\int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n, \lambda)|}{1 + |\lambda - n^3|} d\lambda \Big)^2 \Big)^{1/2}, \tag{3.10}$$

where we have used that $1 + |\lambda - n^3| \leq 3|\lambda - n^3|$.

Thus far we have shown

$$\begin{aligned} &|||T_k^\delta(u_0, \dots, u_k)|||_{Y^s} = |||(3.2) + (3.3) + (3.4) + (3.5)|||_{Y^s} \\ &\leq |||(3.2)|||_{Y^s} + |||(3.3)|||_{Y^s} + |||(3.4)|||_{Y^s} + |||(3.5)|||_{Y^s} \\ &\leq c\Big(\sum_{n \in \dot{\mathbb{Z}}} |n|^{2s} \int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n, \lambda)|^2}{1 + |\lambda - n^3|} d\lambda \Big)^{1/2} \\ &\qquad + c\Big(\sum_{n \in \dot{\mathbb{Z}}} |n|^{2s} \Big(\int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n, \lambda)|}{1 + |\lambda - n^3|} d\lambda \Big)^2 \Big)^{1/2} + c\|\varphi_k\|_{H^s}, \end{aligned}$$

where $c = c(\psi)$. This is stated in the following proposition.

Proposition 3.1. *For all $u_0, \dots, u_k \in \dot{Y}^s$, we have*

$$\begin{aligned} |||T_k^\delta(u_0, \dots, u_k)|||_{Y^s} \leq\ & c\Big(\sum_{n \in \dot{\mathbb{Z}}} |n|^{2s} \int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n, \lambda)|^2}{1 + |\lambda - n^3|} d\lambda \Big)^{1/2} \\ & + \ c\Big(\sum_{n \in \dot{\mathbb{Z}}} |n|^{2s} \Big(\int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n, \lambda)|}{1 + |\lambda - n^3|} d\lambda \Big)^2 \Big)^{1/2} + c\|\varphi_k\|_{H^s}, \end{aligned} \tag{3.11}$$

where $c = c(\psi)$.

By Proposition 3.1, it now suffices to prove the following lemma in order to complete the proof of estimate (2.27).

Lemma 3.2. *For all $u_0, \dots, u_k \in \dot{Y}^s$, we have*

$$\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n,\lambda)|^2}{1+|\lambda-n^3|}\, d\lambda\Big)^{1/2} \lesssim \delta^{1/24} \sum_{j=0}^{k} \binom{k}{j} |||u_{k-j}|||_{Y^s} |||u_j|||_{Y^s}, \tag{3.12}$$

$$\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \Big(\int_{\mathbb{R}} \frac{|\widehat{F_k^\delta}(n,\lambda)|}{1+|\lambda-n^3|}\, d\lambda\Big)^2\Big)^{1/2} \lesssim \delta^{1/24} \sum_{j=0}^{k} \binom{k}{j} |||u_{k-j}|||_{Y^s} |||u_j|||_{Y^s}. \tag{3.13}$$

Proof of Lemma 3.2. The proof follows from the bilinear estimates below (see Himonas and Misiołek [4]).

Proposition 3.3. *For all $f, g \in \dot{X}^s$ with t-support in the interval $[-\delta, \delta]$ we have*

$$\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \int_{\mathbb{R}} \frac{|\widehat{w}_{fg}(n,\lambda)|^2}{1+|\lambda-n^3|}\, d\lambda\Big)^{1/2} \lesssim \delta^{1/12} |||f|||_{X^s} \cdot |||g|||_{X^s}, \tag{3.14}$$

$$\Big(\sum_{n\in\dot{\mathbb{Z}}} |n|^{2s} \Big(\int_{\mathbb{R}} \frac{|\widehat{w}_{fg}(n,\lambda)|}{1+|\lambda-n^3|}\, d\lambda\Big)^2\Big)^{1/2} \lesssim \delta^{1/12} |||f|||_{X^s} \cdot |||g|||_{X^s}, \tag{3.15}$$

where $\widehat{w}_{fg}$ is defined by

$$\widehat{w}_{fg}(n,\lambda) \simeq \Big(n + \frac{2n}{1+n^2}\Big) \widehat{f} * \widehat{g}(n,\lambda) + \frac{n}{1+n^2} \widehat{\partial_x f} * \widehat{\partial_x g}(n,\lambda). \tag{3.16}$$

Note that here, as before, we define $\dot{X}^s \doteq \big\{u \in X^s : \widehat{u}(0,t) = 0\big\}$. Let us see in detail how to obtain estimates (3.12) and (3.13) from Proposition 3.3. In [4], it sufficed to prove Proposition 3.3 separately for

$$\widehat{w}_1(n,\lambda) \doteq \Big(n + \frac{2n}{1+n^2}\Big) \widehat{f} * \widehat{g}(n,\lambda)$$

and

$$\widehat{w}_2(n,\lambda) \doteq \frac{n}{1+n^2} \widehat{\partial_x f} * \widehat{\partial_x g}(n,\lambda).$$

By definition of F_k^δ (see (3.1)) we have

$$\begin{aligned} F_k^\delta(u,u) \;&=\; \Big[\frac{1}{2} + \big(1-\partial_x^2\big)^{-1}\Big] \sum_{j=0}^{k} \binom{k}{j} \partial_x(\psi_\delta u_{k-j} \cdot \psi_\delta u_j) \\ &\quad + \frac{1}{2}\big(1-\partial_x^2\big)^{-1} \sum_{j=0}^{k} \binom{k}{j} \partial_x(\partial_x \psi_\delta u_{k-j} \cdot \partial_x \psi_\delta u_j), \end{aligned}$$

and therefore,

$$\widehat{F_k^\delta(u,u)} \simeq \sum_{j=0}^{k} \binom{k}{j} \Big[\frac{1}{2} n\widehat{\psi_\delta u}_{k-j} * \widehat{\psi_\delta u}_j + \frac{n}{1+n^2}\widehat{\psi_\delta u}_{k-j} * \widehat{\psi_\delta u}_j$$
$$+\frac{1}{2}\cdot\frac{n}{1+n^2}\widehat{\partial_x\psi_\delta u}_{k-j} * \widehat{\partial_x \psi_\delta u}_j\Big].$$

Observe that the left-hand side of (3.12) is the L^2 norm in n and λ of $\frac{n^s \widehat{F_k^\delta}(n,\lambda)}{(1+|\lambda-n^3|)^{1/2}}$. We can use Minkowski's inequality and then apply Proposition 3.3 to the L^2 norm of each term in $\frac{n^s \widehat{F_k^\delta}(n,\lambda)}{(1+|\lambda-n^3|)^{1/2}}$ since each term in $\widehat{F_k^\delta}$ is of the form $\widehat{w}_1$ or $\widehat{w}_2$. Note that we can apply Proposition 3.3, since for $j\in\{0,1,\ldots,k\}$ we have that $\psi_\delta u_j \in \dot{Y}^s \subset \dot{X}^s$. Similarly, observing that the the left-hand side of (3.13) is the L^2 norm in n of $\int_{\mathbb{R}} \frac{n^s|\widehat{F_k^\delta}(n,\lambda)|}{1+|\lambda-n^3|}d\lambda$, we apply Proposition 3.3. Hence, we have that the left-hand sides of (3.12) and (3.13) are bounded by a universal constant $c=c(\psi)$ times

$$\delta^{1/12}\sum_{j=0}^{k}\binom{k}{j}|||\psi_\delta u_{k-j}|||_{X^s}|||\psi_\delta u_j|||_{X^s}.$$

Now by applying the following lemma, proved in [4], we arrive at our desired result.

Lemma 3.4. *For any $\varepsilon>0$ there is C_ε such that*

$$|||\psi_\delta u|||_{X^s} \le C_\varepsilon \delta^{-\varepsilon}|||u|||_{X^s}, \ \textit{for all } u\in\dot{X}^s. \tag{3.17}$$

Lemma 3.4 yields

$$\delta^{1/12}\sum_{j=0}^{k}\binom{k}{j}|||\psi_\delta u_{k-j}|||_{X^s}|||\psi_\delta u_j|||_{X^s}$$
$$\lesssim \delta^{1/12}C_\varepsilon^2\delta^{-2\varepsilon}\sum_{j=0}^{k}\binom{k}{j}|||u_{k-j}|||_{X^s}|||u_j|||_{X^s}.$$

This gives us Lemma 3.2 after choosing $\varepsilon=1/48$ and noting that $|||u|||_{X^s}\le |||u|||_{Y^s}$. ☐

Proof of (2.28). By definition of T_k^δ and (2.12), we have

$$|||T_k^\delta(u_0,u_1,\ldots,u_k)-T_k^\delta(v_0,v_1,\ldots,v_k)|||_{Y^s}$$
$$= |||T_k(\psi_\delta u_0,\psi_\delta u_1,\ldots,\psi_\delta u_k)-T_k(\psi_\delta v_0,\psi_\delta v_1,\ldots,\psi_\delta v_k)|||_{Y^s}$$

$$= |||\psi(t)\int_0^t W(t-\tau)[F_k(\psi_\delta u,\psi_\delta u)-F_k(\psi_\delta v,\psi_\delta v)](x,\tau)d\tau|||_{Y^s}.$$

As in the proof of (2.27), we will let

$$F_k^\delta(u,u)-F_k^\delta(v,v)\doteq F_k(\psi_\delta u,\psi_\delta u)-F_k(\psi_\delta v,\psi_\delta v),$$

where, by definition of F_k,

$$\begin{aligned}
&F_k(\psi_\delta u,\psi_\delta u)-F_k(\psi_\delta v,\psi_\delta v)\\
&=\Big[\frac{1}{2}+\big(1-\partial_x^2\big)^{-1}\Big]\sum_{j=0}^k\binom{k}{j}\partial_x\big[(\psi_\delta u_{k-j}+\psi_\delta v_{k-j})(\psi_\delta u_j-\psi_\delta v_j)\big]\\
&+\frac{1}{2}\big(1-\partial_x^2\big)^{-1}\sum_{j=0}^k\binom{k}{j}\partial_x\big[(\partial_x\psi_\delta u_{k-j}+\partial_x\psi_\delta v_{k-j})(\partial_x\psi_\delta u_j-\partial_x\psi_\delta v_j)\big]\\
&=\Big[\frac{1}{2}+\big(1-\partial_x^2\big)^{-1}\Big]\sum_{j=0}^k\binom{k}{j}\partial_x\big[(\psi_\delta(u_{k-j}+v_{k-j}))\cdot(\psi_\delta(u_j-v_j))\big]\\
&+\frac{1}{2}\big(1-\partial_x^2\big)^{-1}\sum_{j=0}^k\binom{k}{j}\partial_x\big[(\partial_x\psi_\delta(u_{k-j}+v_{k-j}))\cdot(\partial_x\psi_\delta(u_j-v_j))\big].
\end{aligned}$$

The remainder of the proof is analogous to that of (2.27), where we replace u_{k-j} and u_j with $u_{k-j}+v_{k-j}$ and u_j-v_j, respectively. This completes the proof of Proposition 2.2. □

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