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# LIMIT AT INFINITY AND NONEXISTENCE RESULTS FOR SONIC TRAVELLING WAVES IN THE GROSS-PITAEVSKII EQUATION

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**Abstract.** We study the limit at infinity of sonic travelling waves of finite energy in the Gross-Pitaevskii equation in dimension $N \geq 2$ and prove the nonexistence of nonconstant sonic travelling waves of finite energy in dimension two.

## Introduction

In this article, we focus on the travelling waves of speed $c > 0$ in the Gross-Pitaevskii equation

$$i\partial_t u = \Delta u + u(1 - |u|^2), \tag{0.1}$$

which are of the form $u(t,x) = v(x_1 - ct, \dots, x_N)$. The equation for $v$, which we will study now, is

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{0.2}$$

The Gross-Pitaevskii equation is a physical model for Bose-Einstein condensation. It is formally associated to the energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |v|^2)^2 = \int_{\mathbb{R}^N} e(v), \tag{0.3}$$

and to the vectorial momentum

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N} i\nabla v \cdot v. \tag{0.4}$$

Equation (0.1) presents a hydrodynamic form. Indeed, if we make use of the Madelung transform [15] $v = \sqrt{\rho}e^{i\theta}$ (which is only meaningful where $\rho$ does

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not vanish), and if we denote $\mathrm{v} = 2\nabla\theta$, we compute

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathrm{v}) = 0 \\ \rho(\partial_t \mathrm{v} + \mathrm{v} \cdot \nabla \mathrm{v}) + \nabla \rho^2 = \rho \nabla\big(\frac{\Delta\rho}{\rho} - \frac{|\nabla\rho|^2}{2\rho^2}\big). \end{cases} \tag{0.5}$$

Equations (0.5) look like Euler equations for an irrotational ideal fluid with pressure $p(\rho) = \rho^2$ (see [3, 4] for more details). In particular, the sound speed of this fluid near the constant solution $u = 1$ is $c_s = \sqrt{2}$. The nonconstant travelling waves of finite energy play a great role in the long-time dynamics of general solutions. This motivates their study by C.A. Jones, S.J. Putterman, and P.H. Roberts [12, 13]. In particular, they conjectured that they can only exist when $0 < c < \sqrt{2}$; i.e., they are subsonic. F. Bethuel and J.C. Saut [1] first studied mathematically this conjecture. In dimension $N \geq 2$, they proved that all the travelling waves of finite energy and of speed $c = 0$ are constant. On the other hand, we proved in [10] the nonexistence of nonconstant travelling waves of finite energy and of speed $c > \sqrt{2}$ in dimension $N \geq 2$. Thus, the nonexistence conjecture of C.A. Jones, S.J. Putterman, and P.H. Roberts remains an open problem only in the case $c = \sqrt{2}$. That is the reason why we focus here on the sonic travelling waves of finite energy; i.e., we assume $c = \sqrt{2}$. In particular, we will prove their conjecture in dimension two.

**Theorem 1.** *In dimension two, a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $c = \sqrt{2}$ is constant.*

**Remarks.** 1. Theorem 1 holds also in dimension one, but its proof is fairly elementary. Indeed, equation (0.2) is entirely integrable in dimension one. If $c \geq \sqrt{2}$, the solutions $v$ of equation (0.2) are constant functions of modulus one. Instead, if $0 < c < \sqrt{2}$, up to a multiplication by a constant of modulus one and a translation, the solutions $v$ of equation (0.2) are equal either to the constant function 1 or to the function

$$\begin{aligned} v(x) = &\sqrt{1 - \frac{2 - c^2}{2\mathrm{ch}^2(\frac{\sqrt{2-c^2}}{2}x)}} \\ &\times \exp\Big(i \arctan\big(\frac{e^{\sqrt{2-c^2}x} + c^2 - 1}{c\sqrt{2-c^2}}\big) - i \arctan\big(\frac{c}{\sqrt{2-c^2}}\big)\Big). \end{aligned}$$

We refer to the appendix for more details (see also the article of M. Maris [16]).

2. In dimension two, F. Béthuel and J.C. Saut [1, 2] showed the existence of travelling waves of finite energy when $c$ is small and for a sequence of values of $c$ tending to $\sqrt{2}$.

In dimension $N \geq 3$, Theorem 1 is still open. We believe that a positive answer to the nonexistence of nonconstant sonic travelling waves of finite energy would be an important step towards another fundamental conjecture: the nonexistence of nonconstant travelling waves of small energy.[1] Indeed, if the speed $c = \sqrt{2}$ is excluded, we may use the rescaling given by the parameter $\varepsilon = \sqrt{2 - c^2}$ to prove that the travelling waves for the Gross-Pitaevskii equation converge towards the solitary waves for the Kadomtsev-Petviashvili equation when $\varepsilon$ tends to 0 (see the articles of A. de Bouard and J.C. Saut [5, 6] for more details on the solitary waves for the Kadomtsev-Petviashvili equation). In particular, in dimension $N \geq 3$, the energy of a nonconstant travelling wave for the Gross-Pitaevskii equation would tend to $+\infty$ when $\varepsilon$ tends to 0, which would presumably prevent the existence of nonconstant travelling waves of small energy.

In order to prove the nonexistence of nonconstant sonic travelling waves of finite energy in dimension $N \geq 3$, one fruitful argument seems to be to study their behaviour at infinity (see the conclusion for more details). In particular, we can already state their convergence at infinity towards a constant of modulus one.

**Theorem 2.** *Let $N \geq 3$ and $v$ be a travelling wave for the Gross-Pitaevskii equation of finite energy and speed $c = \sqrt{2}$. There exists a constant $\lambda_\infty$ of modulus one such that*

$$v(x) \underset{|x| \to +\infty}{\to} \lambda_\infty.$$

**Remarks.** 1. In dimension two, F. Béthuel and J.C. Saut [1] gave mathematical evidence of the limit at infinity of subsonic travelling waves of finite energy. We complemented their work in dimension $N \geq 3$ [8].

2. C.A. Jones, S.J. Putterman, and P.H. Roberts [12, 13] derived a formal asymptotic expansion of subsonic travelling waves which are axisymmetric around axis $x_1$. In dimension two, they computed

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})x_2^2}, \tag{0.6}$$

[1] In particular, if this is true, a scattering theory for a small-energy solution to equation (0.1) is possible, although presumably difficult.

while in dimension three, they obtained

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}}. \tag{0.7}$$

Here, the constant $\alpha$ is the stretched dipole coefficient linked to the energy $E(v)$ and to the scalar momentum $p(v) = P_1(v)$ by the formulae

$$2\pi\alpha\sqrt{1 - \frac{c^2}{2}} = cE(v) + 2\Big(1 - \frac{c^2}{4}\Big)p(v) \tag{0.8}$$

in dimension two and

$$4\pi\alpha = \frac{c}{2}E(v) + 2p(v) \tag{0.9}$$

in dimension three. In [9, 11], we derived rigorously conjectures (0.6), (0.7), (0.8), and (0.9). However, the study of the asymptotic behaviour of sonic travelling waves is much more involved than in the subsonic case. Indeed, in the subsonic case, it relies on a lemma (Lemma 10 of [9]) which is not valid anymore for $c = \sqrt{2}$.

3. In dimension $N \geq 3$, F. Béthuel, G. Orlandi, and D. Smets [4] showed the existence of travelling waves of finite energy when $c$ is small. A. Farina [7] proved a universal bound for their modulus.

Our paper is organized around the proofs of Theorems 1 and 2. In the first part, we recall some preliminary results mentioned in [8, 9, 10]. In particular, we derived some convolution equations from equation (0.2). They are the basic ingredient of the proofs.

The second part is devoted to the proof of Theorem 1. It relies on the same argument as in [10]: the singularity at the origin of the Fourier transforms of the kernels which appear in the convolution equations of the first part.

Finally, the last part deals with the proof of Theorem 2. It follows from the use of the convolution kernels as Fourier multipliers and from Proposition 2 of [8].

## 1. Some convolution equations

In this part, we write some convolution equations which are the key ingredient of all the proofs of this article. In order to state them, we first recall two useful propositions mentioned in [8, 9, 10] and based on arguments taken from F. Béthuel and J.C. Saut [1, 2].

**Proposition 1** ([9]). *Let $c > 0$ and $N \geq 2$. Consider a solution $v$ of equation* (0.2) *in $L^1_{loc}(\mathbb{R}^N)$ of finite energy. Then, $v$ is of class $C^\infty$ and* *bounded on $\mathbb{R}^N$. Moreover, its gradient $\nabla v$ and the function $\eta := 1 - \rho^2$ belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $p \in [2, +\infty]$.*

**Remark.** By Proposition 1, any weak solution of finite energy of (0.2) is a classical solution.

We deduce from Proposition 1 a first lemma which gives the convergence of the modulus of a travelling wave at infinity.

**Lemma 1** ([8, 9, 10])**.** *Let $c > 0$ and $N \geq 2$. Consider a solution $v$ of equation (0.2) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy. The modulus $\rho$ of $v$ uniformly converges to 1 at infinity.*

In particular, there is some real number $R_0$ such that $\rho \geq \frac{1}{2}$ on $B(0, R_0)^c$. Thus, using a standard degree argument in dimension two, we can construct a lifting $\theta$ of $v$ on $B(0, R_0)^c$; that is, a function in $C^\infty(B(0, R_0)^c, \mathbb{R})$ such that $v = \rho e^{i\theta}$.

We next compute new equations for the new functions $\eta$ and $\theta$. However, since $\theta$ is not well-defined on $\mathbb{R}^N$, we must introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^N, [0, 1])$ such that

$$\begin{cases} \psi = 0 & \text{on } \mathrm{B}(0, 2\mathrm{R}_0), \\ \psi = 1 & \text{on } \mathrm{B}(0, 3\mathrm{R}_0)^{\mathrm{c}}. \end{cases}$$

All the results in the following will be independent of the choice of $R_0$ and $\psi$. Finally, we deduce

**Proposition 2** ([9])**.** *Let $c > 0$ and $N \geq 2$. Consider a solution $v$ of equation (0.2) in $L^1_{loc}(\mathbb{R}^N)$ of finite energy. Then, the functions $\eta$ and $\psi\theta$ satisfy*

$$\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}) \tag{1.1}$$

*and*

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\mathrm{G}), \tag{1.2}$$

*where*

$$F = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v \cdot v - 2c\partial_1(\psi\theta) \tag{1.3}$$

*and*

$$G = i\nabla v \cdot v + \nabla(\psi\theta). \tag{1.4}$$

**Remark.** The functions $F$ and $G$ are related to the density of energy and of vectorial momentum. In order to clarify this claim, we must remove a difficulty in the definition of $\vec{P}(v)$. Indeed, the integral which appears in definition (0.4) is not always convergent for a travelling wave of finite energy.

In order to give a rigorous definition of the vectorial momentum $\vec{P}(v)$, we state

$$\vec{P}(v) = \frac{1}{2}\int_{\mathbb{R}^N} (i\nabla v \cdot v + \nabla(\psi\theta)). \tag{1.5}$$

This new definition is rather suitable (see for instance [9, 10, 11]). In particular, it is now straightforward to link the functions $F$ and $G$ to the density of energy and of vectorial momentum.

Finally, equations (1.1) and (1.2) lead to the desired convolution equations

$$\eta = K_0 * F + 2\sqrt{2}\sum_{j=1}^{N} K_j * G_j \tag{1.6}$$

and for every $j \in \{1, \dots, N\}$,

$$\partial_j(\psi\theta) = \frac{1}{\sqrt{2}} K_j * F + 2\sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k, \tag{1.7}$$

where $K_0$, $K_j$, $L_{j,k}$, and $R_{j,k}$ are the kernels of the Fourier transform

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi_\perp|^2}, \tag{1.8}$$

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi_\perp|^2}, \tag{1.9}$$

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{|\xi|^2(|\xi|^4 + 2|\xi_\perp|^2)}, \tag{1.10}$$

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j\xi_k}{|\xi|^2}. \tag{1.11}$$

**Remarks.** 1. Here, we denoted by $\xi_\perp$ the variable given by

$$\forall \xi \in \mathbb{R}^N,\ \xi_\perp = (\xi_2, \dots, \xi_N).$$

In particular, the value of $|\xi_\perp|^2$ is

$$|\xi_\perp|^2 = \sum_{j=2}^{N} \xi_j^2.$$

2. We only wrote equations (1.6) and (1.7) in the sonic case $c = \sqrt{2}$. However, we can compute similar equations for other values of $c$.

Now, thanks to equations (1.6) and (1.7), we turn to the proofs of Theorems 1 and 2.

## 2. Nonexistence of nonconstant travelling waves of finite energy in dimension two

The proof of Theorem 1 relies on the form of the Fourier transforms of the kernels $K_0$ and $K_j$. They are singular at the origin, in particular in direction $\xi_1$. In dimension two, we deduce from this singularity a new integral relation (formula (2.1) just below), which provides the nonexistence of nonconstant sonic travelling waves of finite energy.

**Proposition 3.** *Let $N = 2$. Any sonic travelling wave $v$ of finite energy satisfies the integral equation*

$$\int_{\mathbb{R}^2} (|\nabla v|^2 + \eta^2) = 0. \tag{2.1}$$

**Remark.** Actually, we recover formula (6) of [10],

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + \eta^2) = 2c\Big(1 - \frac{2}{c^2}\Big)p(v),$$

in the specific case $c = \sqrt{2}$ and $N = 2$. As in the present paper, it was the key ingredient of the nonexistence of nonconstant supersonic travelling waves of finite energy.

Theorem 1 is a direct consequence of Proposition 3.

**Proof of Theorem 1.** By equation (2.1), the gradient of $v$ vanishes on $\mathbb{R}^2$. Therefore, $v$ is constant on $\mathbb{R}^2$. Moreover, it is a constant of modulus one since the function $\eta$ also vanishes on $\mathbb{R}^2$ by equation (2.1). □

Now, it remains to prove Proposition 3. In order to explain the difficulty which appears in dimension $N \geq 3$, we keep in our analysis the dimension $N \geq 2$ arbitrary and only specify the case of dimension two at the very end.

**Proof of Proposition 3.** By Proposition 1, the functions $\eta$, $F$, and $G$ respectively belong to $H^4(\mathbb{R}^N)$, $W^{2,1}(\mathbb{R}^N)$, and $W^{2,1}(\mathbb{R}^N)$. Therefore, we can write for almost every $\xi \in \mathbb{R}^N$, by taking the Fourier transform of equation (1.6),

$$\widehat{\eta}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi_\perp|^2}\widehat{F}(\xi) + 2\sqrt{2}\sum_{j=1}^{N} \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi_\perp|^2}\widehat{G_j}(\xi). \tag{2.2}$$

The strategy of the proof now relies on the finiteness of the energy. Indeed, since the energy is finite, the function $\eta$ belongs to $L^2(\mathbb{R}^N)$. By Plancherel's theorem, the function $\widehat{\eta}$ is also in $L^2(\mathbb{R}^N)$. On the other hand, equation (2.2) gives an expression of the function $\widehat{\eta}$. We are going to integrate its square modulus on a suitable subset of $\mathbb{R}^N$ and prove that this integral cannot be finite unless equality (2.1) holds. The choice of the set of integration is motivated by the singularity at the origin of the Fourier transforms of the kernels $K_0$ and $K_j$. Indeed, by formulae (1.8) and (1.9), they are both more singular in case $\xi_\perp$ vanishes. That is the reason why we are going to integrate the function $|\widehat{\eta}|^2$ on the set $\Omega = \{\xi \in \mathbb{R}^N,\ 0 \leq \xi_1 \leq 1,\ |\xi_\perp| \leq \xi_1^2\}$. Indeed, it follows from equation (2.2) that

$$\int_\Omega |\widehat{\eta}(\xi)|^2 d\xi =$$
$$\int_0^1 \int_{|\xi_\perp| \leq \xi_1^2} \frac{\big||\xi|^2 \widehat{F}(\xi_1, \xi_\perp) + 2\sqrt{2}(\xi_1^2 \widehat{G_1}(\xi_1, \xi_\perp) + \xi_1 \xi_\perp \cdot \widehat{G_\perp}(\xi_1, \xi_\perp))\big|^2}{(|\xi|^4 + 2|\xi_\perp|^2)^2} d\xi_\perp \, d\xi_1.$$

Consider then the function $H$ defined by

$$H(\xi_1) = \int_{|y| \leq \xi_1^2} \frac{\big|(|y|^2 + \xi_1^2)\widehat{F}(\xi_1, y) + 2\sqrt{2}(\xi_1^2 \widehat{G_1}(\xi_1, y) + \xi_1 y \cdot \widehat{G_\perp}(\xi_1, y))\big|^2}{((|y|^2 + \xi_1^2)^2 + 2|y|^2)^2} dy,$$

$\forall \xi_1 \in (0, 1]$, so that

$$\int_\Omega |\widehat{\eta}(\xi)|^2 d\xi = \int_0^1 H(\xi_1) d\xi_1. \tag{2.3}$$

We claim that

**Claim 1.** *If* $\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0) \neq 0$, *then*

$$H(\xi_1) \underset{\xi_1 \to 0}{\sim} \left( |\mathbb{S}^{N-2}| |\widehat{F}(0) + 2\sqrt{2}\widehat{G_1}(0)|^2 \int_{-\delta_{N,2}}^1 \frac{s^{N-2}}{(1+2s^2)^2} ds \right) \xi_1^{2(N-3)}.$$

Indeed, the function $H$ satisfies for every $\xi = (\xi_1, r\sigma)$,

$$H(\xi_1) = \int_{-\delta_{N,2}\xi_1^2}^{\xi_1^2} \int_{\mathbb{S}^{N-2}} \frac{\big|(r^2 + \xi_1^2)\widehat{F}(\xi) + 2\sqrt{2}(\xi_1^2 \widehat{G_1}(\xi) + \xi_1 r\sigma \cdot \widehat{G_\perp}(\xi))\big|^2}{((\xi_1^2 + r^2)^2 + 2r^2)^2} r^{N-2} \, d\sigma \, dr$$
$$:= \xi_1^{2(N-3)} I(\xi_1),$$

where, denoting $\xi' = (\xi_1, \xi_1^2 s\sigma)$, we let

$$I(\xi_1) = \int_{-\delta_{N,2}}^1 \int_{\mathbb{S}^{N-2}} \frac{\big|(s^2\xi_1^2 + 1)\widehat{F}(\xi') + 2\sqrt{2}\widehat{G_1}(\xi') + 2\sqrt{2}\xi_1 s\sigma \cdot \widehat{G_\perp}(\xi')\big|^2}{((\xi_1^2 s^2 + 1)^2 + 2s^2)^2} s^{N-2} \, d\sigma \, ds.$$

Moreover, by Proposition 1, the functions $F$ and $G$ belong to $L^1(\mathbb{R}^N)$, so their Fourier transforms are continuous on $\mathbb{R}^N$. Therefore, the dominated convergence theorem yields

$$I(\xi_1) \underset{\xi_1\to 0}{\to} |\mathbb{S}^{N-2}||\widehat{F}(0)+2\sqrt{2}\widehat{G_1}(0)|^2 \int_{-\delta_{N,2}}^{1} \frac{s^{N-2}}{(1+2s^2)^2}ds.$$

In particular, if $\widehat{F}(0)+2\sqrt{2}\widehat{G_1}(0)\neq 0$, it gives

$$H(\xi_1) \underset{\xi_1\to 0}{\sim} \left(|\mathbb{S}^{N-2}||\widehat{F}(0)+2\sqrt{2}\widehat{G_1}(0)|^2 \int_{-\delta_{N,2}}^{1} \frac{s^{N-2}}{(1+2s^2)^2}ds\right)\xi_1^{2(N-3)},$$

which is the desired result.

We next argue by contradiction and assume that

$$\widehat{F}(0)+2\sqrt{2}\widehat{G_1}(0)\neq 0. \tag{2.4}$$

If assertion (2.4) were true, then, by Claim 1,

$$H(\xi_1) \underset{\xi_1\to 0}{\sim} A\xi_1^{2(N-3)}.$$

In particular, in dimension two, the function $\xi_1 \mapsto \frac{1}{\xi_1^2}$ is not integrable near 0. By formula (2.3), it yields

$$\int_\Omega |\widehat{\eta}(\xi)|^2 d\xi = +\infty.$$

Thus, it gives a contradiction with the fact that the function $\widehat{\eta}$ is in $L^2(\mathbb{R}^2)$. Therefore, assumption (2.4) does not hold and we find

$$\widehat{F}(0)+2\sqrt{2}\widehat{G_1}(0) = 0.$$

However, by formulae (1.3) and (1.4),

$$\begin{aligned}
&\widehat{F}(0)+2\sqrt{2}\widehat{G_1}(0)\\
&= 2\int_{\mathbb{R}^2}(|\nabla v|^2+\eta^2-\sqrt{2}i\partial_1 v\cdot v-\sqrt{2}\partial_1(\psi\theta))+\sqrt{8}\int_{\mathbb{R}^2}(i\partial_1 v\cdot v+\partial_1(\psi\theta))\\
&= 2\int_{\mathbb{R}^2}(|\nabla v|^2+\eta^2),
\end{aligned}$$

which gives

$$\int_{\mathbb{R}^2}(|\nabla v|^2+\eta^2) = 0.$$

□

**Remark.** The argument fails in dimension $N \geq 3$ since the function $\xi_1 \mapsto \xi_1^{2(N-3)}$ is then integrable near 0.

## 3. Limit at infinity in dimension $N \geq 3$

Theorem 2 follows from two arguments.

• The first one is to improve the $L^p$-integrability of the functions $\eta$ and $\nabla(\psi\theta)$, and of their derivatives.

**Proposition 4.** *Consider $\alpha \in \mathbb{N}^N$ such that $|\alpha| \geq 2$. Then, we claim*

(i) $(\eta, \nabla(\psi\theta)) \in L^p(\mathbb{R}^N)$ *for every* $p > \frac{2N-1}{2N-3}$,

(ii) $(\nabla\eta, d^2(\psi\theta)) \in L^p(\mathbb{R}^N)$ *for every* $p > \frac{2N-1}{2N-2}$,

(iii) $(\partial^\alpha\eta, \partial^\alpha\nabla(\psi\theta)) \in L^p(\mathbb{R}^N)$ *for every* $p > 1$.

Proposition 4 follows from Lizorkin's theorem [14].

**Lizorkin's theorem** ([14])**.** *Let $0 \leq \beta < 1$ and $\widehat{K}$ a bounded function in $C^N(\mathbb{R}^N \setminus \{0\})$. Assume*

$$\prod_{j=1}^N (\xi_j^{k_j+\beta})\partial_1^{k_1} \cdots \partial_N^{k_N} \widehat{K}(\xi) \in L^\infty(\mathbb{R}^N)$$

*as soon as $(k_1, \ldots, k_N) \in \{0,1\}^N$ satisfies*

$$0 \leq \sum_{j=1}^N k_j \leq N.$$

*Then, $\widehat{K}$ is a multiplier from $L^p(\mathbb{R}^N)$ to $L^{\frac{p}{1-\beta p}}(\mathbb{R}^N)$ for every $1 < p < \frac{1}{\beta}$.*

By Lizorkin's theorem, the kernels $K_0$, $K_j$, and $L_{j,k}$ are multipliers from some spaces $L^p(\mathbb{R}^N)$ to some other spaces $L^q(\mathbb{R}^N)$. For instance, the kernel $K_0$ satisfies the assumptions of Lizorkin's theorem for $\beta = \frac{2}{2N-1}$. Therefore, the function $\widehat{K_0}$ is a Fourier multiplier from $L^p(\mathbb{R}^N)$ to $L^{\frac{(2N-1)p}{2(N-p)-1}}(\mathbb{R}^N)$. By convolution equations (1.6) and (1.7), this enables us to improve the $L^p$-integrability of the functions $\eta$ and $\nabla(\psi\theta)$, and of their derivatives.

• The second argument follows from Proposition 4. Since the function $\nabla v$ belongs to some spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$ for $1 < p_0 < N-1 < p_1 < +\infty$, we can use the following proposition to prove the convergence of the function $v$ at infinity.

**Proposition 5** ([8])**.** *Consider a smooth function $v$ on $\mathbb{R}^N$ and assume that $N \geq 3$ and that the gradient of $v$ belongs to the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$ where $1 < p_0 < N-1 < p_1 < +\infty$. Then, there is a constant*

$v_\infty \in \mathbb{C}$ *which satisfies*

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

The proof of Theorem 2 is then a consequence of Propositions 4 and 5. That is the reason why we first show Proposition 4.

**Proof of Proposition 4.** We split the proof into three steps. In the first one, we specify the form of some derivatives of the Fourier transform of the kernel $K_0$. Our goal is to prove that the kernel $K_0$ satisfies the assumptions of Lizorkin's theorem in order to show that $\widehat{K_0}$ is a Fourier multiplier from some space $L^p(\mathbb{R}^N)$ to another space $L^q(\mathbb{R}^N)$.

**Step 1.** *Consider $\alpha \in \{0,1\}^N$. Then, the function $\partial^\alpha \widehat{K_0}$ may be written*

$$\forall \xi \in \mathbb{R}^N \setminus \{0\},\ \partial^\alpha \widehat{K_0}(\xi) = \frac{\xi^\alpha P_\alpha(\xi)}{(|\xi|^4 + 2|\xi_\perp|^2)^{1+|\alpha|}}, \tag{3.1}$$

*where $P_\alpha$ is a polynomial function of degree $d_\alpha \leq 2|\alpha| + 2$ which satisfies the following:*

- *(i) For every $j \in \{1,\dots,N\}$, $P_\alpha$ is even in the variable $\xi_j$.*
- *(ii) The term of lowest degree of $P_\alpha$ is equal to $(-1)^{|\alpha|-1}(|\alpha|-1)!4^{|\alpha|}|\xi_\perp|^2$ if $\alpha_1 = 1$, and to $(-1)^{|\alpha|-1}|\alpha|!4^{|\alpha|}\xi_1^2$, if $\alpha_1 = 0$ and $|\alpha| \neq 0$.*

Step 1 follows from an inductive argument on $|\alpha|$. Indeed, if $|\alpha| = 0$ or $|\alpha| = 1$, we compute by formula (1.8) for every $j \in \{2,\dots,N\}$,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi_\perp|^2}$$

$$\partial_1 \widehat{K_0}(\xi) = \frac{\xi_1(-2|\xi|^4 + 4|\xi_\perp|^2)}{(|\xi|^4 + 2|\xi_\perp|^2)^2}, \qquad \partial_j \widehat{K_0}(\xi) = \frac{\xi_j(-2|\xi|^4 + 4\xi_1^2)}{(|\xi|^4 + 2|\xi_\perp|^2)^2}.$$

Thus, Step 1 holds in this case.

Now, assume that Step 1 is valid for $|\alpha| = p \geq 1$ and fix some $\alpha \in \{0,1\}^N$ such that $|\alpha| = p + 1$. There are two cases to consider. If $\alpha_1 = 0$, there is some integer $j \in \{2,\dots,N\}$ such that $\alpha_j = 1$, so we can state

$$\partial^\alpha \widehat{K_0} = \partial_j \partial^\beta \widehat{K_0}$$

with $|\beta| = p$. Applying the inductive assumption, it yields for every $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$\partial^\alpha \widehat{K_0}(\xi) = \frac{\xi^\beta}{(|\xi|^4 + 2|\xi_\perp|^2)^{|\alpha|+1}} \Big(\partial_j P_\beta(\xi)(|\xi|^4+2|\xi_\perp|^2) - (1+|\beta|)P_\beta(\xi)(4\xi_j|\xi|^2+4\xi_j)\Big). \tag{3.2}$$

However, by assumption $(i)$, $P_\beta$ is even in every variable $\xi_k$, so there is some polynomial function $R_\beta$, even in every variable $\xi_k$, such that

$$\partial_j P_\beta(\xi) = \xi_j R_\beta(\xi).$$

Moreover, by assumption $(ii)$, either $R_\beta$ is equal to 0 or the term of lowest degree of $R_\beta$ is of degree at least equal to one.

Then, let us denote

$$P_\alpha(\xi) = R_\beta(\xi)(|\xi|^4 + 2|\xi_\perp|^2) - 4(1 + |\beta|)P_\beta(\xi)(|\xi|^2 + 1). \tag{3.3}$$

By the inductive assumption, the functions $P_\beta$ and $R_\beta$ are even in every variable $\xi_k$, so by equation (3.3), $P_\alpha$ is also even in every variable $\xi_k$. Likewise, the term of lowest degree of $P_\beta$ is equal to $(-1)^{p-1}p!4^p\xi_1^2$ and, if $R_\beta$ is not equal to 0, the term of lowest degree of $R_\beta$ is of degree at least equal to one. Therefore, by equation (3.3), the term of lowest degree of $P_\alpha$ is $(-1)^p(p+1)!4^{p+1}\xi_1^2$. On the other hand, by the inductive assumption and formula (3.3), the degree $d_\alpha$ of $P_\alpha$ is less than $2|\alpha|+2$. Finally, equation (3.1) is a straightforward consequence of equations (3.2) and (3.3). Therefore, the proof of the inductive step is valid in the case $\alpha_1 = 0$.

In the case $\alpha_1 = 1$, we can always assume that we first differentiated $\widehat{K_0}$ by the partial operator $\partial_1$. Therefore, there is some integer $j \in \{2, \dots, N\}$ such that $\alpha_j = 1$, and we can state

$$\partial^\alpha \widehat{K_0} = \partial_j \partial^\beta \widehat{K_0}$$

with $|\beta| = p$. Applying the inductive assumption, it yields for every $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$\partial^\alpha \widehat{K_0}(\xi) = \frac{\xi^\beta}{(|\xi|^4 + 2|\xi_\perp|^2)^{|\alpha|+1}} \Big(\partial_j P_\beta(\xi)(|\xi|^4 + 2|\xi_\perp|^2) - 4(1+|\beta|)\xi_j P_\beta(\xi)(1+|\xi|^2)\Big).$$

Likewise, by assumption $(i)$, $P_\beta$ is even in every variable $\xi_k$, so there is some polynomial function $R_\beta$, even in every variable $\xi_k$, such that

$$\partial_j P_\beta(\xi) = \xi_j R_\beta(\xi).$$

Moreover, by assumption $(ii)$, the term of lowest degree of $R_\beta$ is equal to $2(-1)^{p-1}(p-1)!4^p$.

Denoting

$$P_\alpha(\xi) = R_\beta(\xi)(|\xi|^4 + 2|\xi_\perp|^2) - 4(1 + |\beta|)P_\beta(\xi)(|\xi|^2 + 1),$$

we can prove equation (3.1) and assumptions $(i)$ and $(ii)$, and compute the suitable bound of the degree of $P_\alpha$ by the same argument as in the case $\alpha_1 = 0$. By induction, this completes the proof of Step 1.

In the second step, we use Step 1 and Lizorkin's theorem to state some properties of the Fourier multipliers $\widehat{K_0}$, $\widehat{K_j}$, and $\widehat{L_{j,k}}$.

**Step 2.** *Let $1 < p < +\infty$. The functions $\widehat{K_0}$, $\widehat{K_j}$, and $\widehat{L_{j,k}}$ are Fourier multipliers from $L^p(\mathbb{R}^N)$ to $L^{\frac{(2N-1)p}{2(N-p)-1}}(\mathbb{R}^N)$ if $1 < p < N - \frac{1}{2}$, while the functions $\widehat{d^2K_0}$, $\widehat{d^2K_j}$, and $\widehat{d^2L_{j,k}}$ are $L^p$ multipliers.*

Indeed, consider $\alpha \in \{0,1\}^N$ and set $\beta = \frac{2}{2N-1}$. By equation (3.1), we compute

$$\prod_{j=1}^{N}(\xi_j^{\alpha_j+\beta})\partial^\alpha \widehat{K_0}(\xi) = \prod_{j=1}^{N} \xi_j^{\beta} \frac{\xi^{2\alpha} P_\alpha(\xi)}{(|\xi|^4 + 2|\xi_\perp|^2)^{1+|\alpha|}}.$$

Therefore, by Step 1, if $|\xi| \geq 1$,

$$\Big|\prod_{j=1}^{N}(\xi_j^{\alpha_j+\beta})\partial^\alpha \widehat{K_0}(\xi)\Big| \leq A\frac{|\xi|^{N\beta+4|\alpha|+2}}{|\xi|^{4+4|\alpha|}} \leq A|\xi|^{N\beta-2} \leq A.$$

On the other hand, if $|\xi| \leq 1$, denoting $\xi = \rho\sigma$ where $\rho \geq 0$ and $\sigma \in \mathbb{S}^{N-1}$, we compute by Step 1

$$\Big|\prod_{j=1}^{N}(\xi_j^{\alpha_j+\beta})\partial^\alpha \widehat{K_0}(\xi)\Big| \leq A\frac{\rho^{2|\alpha|+N\beta+2}|\sigma_\perp|^{(N-1)\beta+2|\alpha|-2}\max\{\rho^2, |\sigma_\perp|^2\}}{\rho^{2|\alpha|+2}(\rho^2+2|\sigma_\perp|^2)^{1+|\alpha|}}$$
$$\leq A\max\{\rho^2, |\sigma_\perp|^2\}^{(2N-1)\beta-2} \leq A.$$

Thus, it follows that

$$\forall \alpha \in \{0,1\}^N, \xi \mapsto \prod_{j=1}^{N}(\xi_j^{\alpha_j+\beta})\partial^\alpha \widehat{K_0}(\xi) \in L^\infty(\mathbb{R}^N).$$

By Lizorkin's theorem, $\widehat{K_0}$ is a Fourier multiplier from $L^p(\mathbb{R}^N)$ to $L^{\frac{(2N-1)p}{2(N-p)-1}}(\mathbb{R}^N)$ for every $1 < p < N - \frac{1}{2}$.

Moreover, by equations (1.9) and (1.10), the Fourier transforms of the functions $K_j$ and $L_{j,k}$ may be written

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^2}\widehat{K_0}(\xi), \qquad \widehat{L_{j,k}}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{|\xi|^4}\widehat{K_0}(\xi).$$

By standard Riesz operator theory (see for instance the book of E.M. Stein and G. Weiss [17] for more details), the functions $\xi \mapsto \frac{\xi_1\xi_j}{|\xi|^2}$ and $\xi \mapsto \frac{\xi_1^2\xi_j\xi_k}{|\xi|^4}$

are $L^p$ multipliers for every $p > 1$. Therefore, $\widehat{K_j}$ and $\widehat{L_{j,k}}$ are also Fourier multipliers from $L^p(\mathbb{R}^N)$ to $L^{\frac{(2N-1)p}{2(N-p)-1}}(\mathbb{R}^N)$ for every $1 < p < N - \frac{1}{2}$.

Now, consider the Fourier transform of the kernel $\Delta K_0$. Leibniz's formula yields for every $\alpha \in \{0,1\}^N$

$$\partial^\alpha(|\xi|^2\widehat{K_0}(\xi)) = 2\sum_{j=1}^N \delta_{\alpha_j,1}\xi_j\partial^{\beta^j}\widehat{K_0}(\xi) + |\xi|^2\partial^\alpha\widehat{K_0}(\xi),$$

where $\beta^j$ is defined by

$$\forall k \in \{1,\dots,N\}, \beta_k^j = \begin{cases} \alpha_k, \text{ if } k \neq j, \\ 0, \text{ otherwise.} \end{cases}$$

Therefore, we compute

$$\Big|\prod_{j=1}^N \xi_j^{\alpha_j}\partial^\alpha(|\xi|^2\widehat{K_0}(\xi))\Big| \leq A\Big(\sum_{j=1}^N \Big(\frac{|\xi^{2\alpha}||P_{\beta^j}(\xi)|}{(|\xi|^4+2|\xi_\perp|^2)^{|\alpha|}}\Big) + \frac{|\xi|^2|\xi^{2\alpha}||P_\alpha(\xi)|}{(|\xi|^4+2|\xi_\perp|^2)^{1+|\alpha|}}\Big).$$

By Step 1, if $|\xi| \geq 1$,

$$\Big|\prod_{j=1}^N \xi_j^{\alpha_j}\partial^\alpha(|\xi|^2\widehat{K_0}(\xi))\Big| \leq A\Big(\frac{|\xi|^{4|\alpha|}}{|\xi|^{4|\alpha|}} + \frac{|\xi|^{4|\alpha|+4}}{|\xi|^{4|\alpha|+4}}\Big) \leq A.$$

Likewise, by Step 1, if $|\xi| < 1$, denoting $\xi = \rho\sigma$ where $\rho \geq 0$ and $\sigma \in \mathbb{S}^{N-1}$,

$$\Big|\prod_{j=1}^N \xi_j^{\alpha_j}\partial^\alpha(|\xi|^2\widehat{K_0}(\xi))\Big|$$
$$\leq A\Big(\frac{\rho^{2|\alpha|+2}|\sigma_\perp|^{2|\alpha|-2}}{\rho^{2|\alpha|}(\rho^2+2|\sigma_\perp|^2)^{|\alpha|}} + \frac{\rho^{4+2|\alpha|}|\sigma_\perp|^{2|\alpha|-2}\max\{|\sigma_\perp|^2,\rho^2\}}{\rho^{2|\alpha|+2}(\rho^2+2|\sigma_\perp|^2)^{1+|\alpha|}}\Big) \leq A,$$

which yields

$$\forall \alpha \in \{0,1\}^N,\ \xi \mapsto \prod_{j=1}^N \xi_j^{\alpha_j}\partial^\alpha\widehat{\Delta K_0}(\xi) \in L^\infty(\mathbb{R}^N).$$

By Lizorkin's theorem, we conclude that $\widehat{\Delta K_0}$ is a $L^p$ multiplier for every $p > 1$. By standard Riesz operator theory, it follows that $\widehat{d^2K_0}$, $\widehat{d^2K_j}$, and $\widehat{d^2L_{j,k}}$ are $L^p$ multipliers for every $p > 1$.

**Remark.** By standard Riesz operator theory, the functions $\widehat{R_{j,k}}$ are also $L^p$ multipliers for every $p > 1$.

At this stage, by Proposition 1 and formulae (1.3) and (1.4), the functions $F$ and $G$ are in all the spaces $L^p(\mathbb{R}^N)$ for every $p \geq 1$. Therefore, by Proposition 1, Step 2, and equations (1.6) and (1.7), the functions $\eta$ and $\nabla(\psi\theta)$ are in $L^p(\mathbb{R}^N)$ for every $p > \frac{2N-1}{2N-3}$, while their second-order derivatives are in $L^p(\mathbb{R}^N)$ for every $p > 1$. Thus, it only remains to prove

**Step 3.** *The functions $\nabla\eta$ and $d^2(\psi\theta)$ belong to $L^p(\mathbb{R}^N)$ for every $p > \frac{2N-1}{2N-2}$.*

Indeed, consider $p > \frac{2N-1}{2N-2}$. There are some real numbers $q > \frac{2N-1}{2N-3}$ and $r > 1$ such that

$$\frac{1}{p} = \frac{1}{2}\Big(\frac{1}{q} + \frac{1}{r}\Big).$$

In particular, by the Gagliardo-Nirenberg inequality, we derive

$$\|\nabla\eta\|_{L^p(\mathbb{R}^N)} \leq A\|\eta\|_{L^q(\mathbb{R}^N)}^{\frac{1}{2}}\|d^2\eta\|_{L^r(\mathbb{R}^N)}^{\frac{1}{2}} < +\infty.$$

Thus, the function $\nabla\eta$ is in $L^p(\mathbb{R}^N)$ for every $p > \frac{2N-1}{2N-2}$. The proof being identical for the function $d^2(\psi\theta)$, we omit it. □

Now, we end the proof of Theorem 2.

**Proof of Theorem 2.** By Proposition 1, the function $\nabla v$ is $C^\infty$ on $\mathbb{R}^N$ and is equal to

$$\nabla v = \Big(-\frac{\nabla\eta}{2\sqrt{1-\eta}} + i\sqrt{1-\eta}\nabla(\psi\theta)\Big)e^{i(\psi\theta)}$$

on a neighbourhood of infinity. However, by Lemma 1, the function $1-\eta$ converges to 1 at infinity, so by Proposition 4, there are some real numbers $1 < p_0 < N-1 < p_1 < +\infty$ such that $\nabla v$ belongs to $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$. Therefore, by Proposition 5, there is some constant $\lambda_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x|\to+\infty}{\to} \lambda_\infty.$$

Finally, by Lemma 1, the modulus of $\lambda_\infty$ is necessarily equal to one. □

## 4. CONCLUSION

To our knowledge, the question of the nonexistence of nonconstant sonic travelling waves of finite energy remains open in dimension $N \geq 3$. However, we can expect to prove such a conjecture by studying the asymptotic behaviour of the sonic travelling waves. Here, the key idea is to prove integral equation (2.1) by some integrations by parts. Indeed, let $B_R$ be the ball of center 0 and of radius $R > 0$ and $S_R$ the related sphere. By multiplying equation (0.2) by the function $v$ and integrating by parts on $B_R$, we find

$$\int_{B_R} (|\nabla v|^2 + \eta^2) = \int_{B_R} (\eta + \sqrt{2} i \partial_1 v.v) + \int_{S_R} \partial_\nu v \cdot v. \tag{4.1}$$

However, the multiplication of (0.2) by the function $iv$ gives

$$\partial_1 \eta + \sqrt{2} \mathrm{div}(i \nabla v \cdot v) = 0,$$

so, by multiplying by the function $x_1$ and integrating by parts on $B_R$,

$$\int_{B_R} (\eta + \sqrt{2} i \partial_1 v \cdot v) = \int_{S_R} x_1 (\nu_1 \eta + \sqrt{2} i \partial_\nu v \cdot v). \tag{4.2}$$

The sum of equations (4.1) and (4.2) is then

$$\int_{B_R} (|\nabla v|^2 + \eta^2) = \int_{S_R} (\partial_\nu v \cdot v + x_1 (\nu_1 \eta + \sqrt{2} i \partial_\nu v \cdot v)). \tag{4.3}$$

The question is now to prove that the integral of the second member of equation (4.3) tends to 0 when $R$ tends to $+\infty$. One possible argument in this direction is to derive some algebraic decay for the functions $\eta$ and $\nabla(\psi\theta)$. Actually, it seems rather difficult because Lemma 10 of [9], which gives a crucial decay estimate in the subsonic case, is not yet available for sonic travelling waves.

## Appendix. Travelling waves for the Gross-Pitaevskii equation in dimension one

In this appendix, we classify the travelling waves for the Gross-Pitaevskii equation of finite energy and of speed $c > 0$ in dimension one (see also the article of M. Maris [16] for more details).

**Theorem 3.** *Assume $N = 1$ and $c > 0$. Let $v$ be a solution of finite energy of equation* (0.2)*. Then,*

- *if $c \geq \sqrt{2}$, $v$ is a constant of modulus one.*
- *if $0 < c < \sqrt{2}$, up to a multiplication by a constant of modulus one and a translation, $v$ is either identically equal to* 1*, or to the function*

$$v(x) = \sqrt{1 - \frac{2 - c^2}{2 \mathrm{ch}^2 (\frac{\sqrt{2 - c^2}}{2} x)}}$$
$$\times \exp\Big( i \arctan \big( \frac{e^{\sqrt{2 - c^2} x} + c^2 - 1}{c \sqrt{2 - c^2}} \big) - i \arctan \big( \frac{c}{\sqrt{2 - c^2}} \big) \Big).$$

**Proof.** Indeed, let us denote $v = v_1 + iv_2$. Equation (0.2) then may be written

$$v_1'' - cv_2' + v_1(1 - v_1^2 - v_2^2) = 0, \tag{4.4}$$

$$v_2'' + cv_1' + v_2(1 - v_1^2 - v_2^2) = 0. \tag{4.5}$$

The multiplication of equation (4.4) by $v_2$ and of equation (4.5) by $v_1$ gives

$$(v_1 v_2' - v_2 v_1')' = \frac{c}{2}\eta'. \tag{4.6}$$

However, Proposition 1 also holds in the case $N = 1$. In particular, it follows that the functions $\eta$ and $v'$ uniformly converge to 0 at infinity. Thus, by integrating equation (4.6), we get

$$v_1 v_2' - v_2 v_1' = \frac{c}{2}\eta. \tag{4.7}$$

Likewise, we multiply equation (4.4) by $v_1'$ and equation (4.5) by $v_2'$ to deduce

$$\Big(\frac{|v'|^2}{2}\Big)' = \Big(\frac{\eta^2}{4}\Big)',$$

which yields

$$|v'|^2 = \frac{\eta^2}{2}. \tag{4.8}$$

Finally, we compute

$$\eta'' = -2|v'|^2 - 2(v_1 v_1'' + v_2 v_2'') = -2|v'|^2 - 2c(v_1 v_2' - v_2 v_1') + 2\eta - 2\eta^2.$$

Therefore, by equations (4.7) and (4.8),

$$\eta'' + (c^2 - 2)\eta + 3\eta^2 = 0. \tag{4.9}$$

Finally, we multiply equation (4.9) by the function $\eta'$ and integrate to obtain

$${\eta'}^2 + (c^2 - 2)\eta^2 + 2\eta^3 = 0. \tag{4.10}$$

Now, we consider different cases according to the value of $c$.

• If $c > \sqrt{2}$, then, by equation (4.10),

$$(c^2 - 2 + 2\eta)\eta^2 = -{\eta'}^2 \leq 0.$$

Therefore, for every $x \in \mathbb{R}$, $\eta(x)$ is either equal to 0, or less than $1 - \frac{c^2}{2}$. Since the function $\eta$ is continuous and in $L^2(\mathbb{R})$, we deduce that $\eta$ is identically equal to 0. By equation (4.8), $v'$ also vanishes, which means that $v$ is a constant of modulus one.

• If $c = \sqrt{2}$, then, by equation (4.10),

$$\eta^3 = -\frac{\eta'^2}{2} \leq 0,$$

so $\eta$ is a nonpositive function on $\mathbb{R}$. Now, assume for the sake of contradiction that there is some real number $x_0$ such that $\eta(x_0) < 0$. Since $\eta$ is smooth on $\mathbb{R}$ by Proposition 1, we deduce that there are some positive real number $\delta$ and some integer $\varepsilon \in \{-1, 1\}$ such that

$$\forall x_0 - \delta \leq x \leq x_0 + \delta,\ \eta'(x) = \varepsilon\sqrt{-2\eta^3(x)}.$$

Denoting $x_1 = x_0 - \varepsilon\sqrt{-\frac{2}{\eta(x_0)}}$, it follows that

$$\forall x_0 - \delta \leq x \leq x_0 + \delta,\ \eta(x) = -\frac{2}{(x - x_1)^2}.$$

In particular, such a solution cannot be extended to a function in $L^2(\mathbb{R})$, which yields a contradiction and proves that $\eta = 0$. As in the case $c > \sqrt{2}$, it follows that $v$ is a constant of modulus one.

• Assume finally that $0 < c < \sqrt{2}$ and $\eta \neq 0$ (indeed, if $\eta = 0$, it follows from equation (4.8) that $\eta$ is a constant of modulus one). By equation (4.10),

$$(c^2 - 2 + 2\eta)\eta^2 = -\eta'^2 \leq 0,$$

so

$$\eta \leq 1 - \frac{c^2}{2}. \tag{4.11}$$

Now, suppose for the sake of contradiction that there is some real number $x_0$ such that $\eta(x_0) < 0$. Since $\eta$ is smooth on $\mathbb{R}$ by Proposition 1, there are some positive real number $\delta$ and some integer $\varepsilon \in \{-1, 1\}$ such that

$$\forall x_0 - \delta \leq x \leq x_0 + \delta, \eta'(x) = \varepsilon\eta(x)\sqrt{2 - c^2 - 2\eta(x)}.$$

Denoting $x_1 = x_0 + \frac{2\varepsilon}{\sqrt{2-c^2}}\coth^{-1}\big(\sqrt{\frac{2-c^2-2\eta(0)}{2-c^2}}\big)$, it yields

$$\forall x_0 - \delta \leq x \leq x_0 + \delta,\ \eta(x) = -\frac{1 - \frac{c^2}{2}}{\operatorname{sh}^2(\frac{\sqrt{2-c^2}}{2}(x - x_1))}.$$

Since such a solution cannot be extended to a function in $L^2(\mathbb{R})$, it yields a contradiction and proves that $\eta \geq 0$. Moreover, by equation (4.11), since the constant function $1 - \frac{c^2}{2}$ is not in $L^2(\mathbb{R})$ and since we made the additional assumption that $\eta \neq 0$, we can assume up to a translation that

$$0 < \eta(0) < 1 - \frac{c^2}{2}.$$

Therefore, there are some positive real number $\delta$ and some integer $\varepsilon \in \{-1, 1\}$ such that

$$\forall -\delta \leq x \leq \delta,\ \eta'(x) = \varepsilon \eta(x) \sqrt{2 - c^2 - 2\eta(x)},$$

which gives

$$\forall -\delta \leq x \leq \delta,\ \eta(x) = \frac{1 - \frac{c^2}{2}}{\mathrm{ch}^2(\frac{\sqrt{2-c^2}}{2}(x - x_1))},$$

where $x_1 = \frac{2\varepsilon}{\sqrt{2-c^2}} \mathrm{ch}^{-1}\big(\sqrt{\frac{2-c^2}{2\eta(0)}}\big)$. Naturally, this solution can be extended to a smooth function in $L^2(\mathbb{R})$. Therefore, up to another translation, we conclude that

$$\forall x \in \mathbb{R},\ \eta(x) = \frac{1 - \frac{c^2}{2}}{\mathrm{ch}^2(\frac{\sqrt{2-c^2}}{2} x)}. \tag{4.12}$$

In particular, we find

$$\forall x \in \mathbb{R},\ |v(x)| = \sqrt{1 - \eta(x)} \geq \frac{c}{\sqrt{2}} > 0.$$

Therefore, we can construct a smooth lifting $\theta$ of $v$ which satisfies

$$\forall x \in \mathbb{R},\ v(x) = \rho(x) e^{i\theta(x)}.$$

By equation (4.7), the function $\theta$ satisfies the differential equation

$$\theta' = \frac{c\eta}{2 - 2\eta}.$$

Thus, there is some real number $\theta_0$ such that

$$\forall x \in \mathbb{R},\ \theta(x) = \theta_0 + \arctan\Big(\frac{e^{\sqrt{2-c^2}x} + c^2 - 1}{c\sqrt{2 - c^2}}\Big).$$

By equation (4.12), up to a multiplication by a constant of modulus one, we finally obtain

$$v(x) = \sqrt{1 - \frac{2 - c^2}{2\mathrm{ch}^2(\frac{\sqrt{2-c^2}}{2} x)}} \times \exp\Big(i \arctan\big(\frac{e^{\sqrt{2-c^2}x} + c^2 - 1}{c\sqrt{2 - c^2}}\big) - i \arctan\big(\frac{c}{\sqrt{2 - c^2}}\big)\Big),$$

which concluses the proof of Theorem 3. □

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