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ON THE UNIQUENESS OF SOLUTIONS FOR A SEMILINEAR ELLIPTIC PROBLEM IN CONVEX DOMAINS

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Abstract. We exhibit a class of convex and nonsymmetric domains Ω in $\mathbb{R}^N$, $N \geq 4$, such that the slightly subcritical problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

does not have any solutions blowing up at more than one point in Ω as ε goes to zero. Moreover if Ω is a small perturbation of a convex and symmetric domain, we prove that such a problem has a unique solution provided ε is small enough.

0. Introduction

Let us consider the problem

$$\begin{cases} -\Delta u = u^{p-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is an open, bounded domain in $\mathbb{R}^N$, $N \geq 4$, $p = \frac{N+2}{N-2}$, and $\varepsilon > 0$ is a small parameter.

It is well known that the geometry of the domain Ω plays a crucial role in finding solutions to (0.1). In fact, a good condition for existence and multiplicity of solutions to (0.1) involves the Green's function of the Laplacian with Dirichlet boundary conditions on Ω (see, for example, [1], [2], and [13]).

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In this paper we are interested in the following question:

$$\text{\textit{if } } \Omega \text{ \textit{is a convex domain, does} (0.1) \textit{have a unique solution}} \\ \text{\textit{if} } \varepsilon \text{ \textit{is small enoguh?}} \tag{0.2}$$

A positive answer was given by Grossi in [8] when Ω is symmetric about $x_i = 0$ and convex in the x_i direction for $i = 1, \ldots, N$. We would like to point out that the symmetry assumption on the domain Ω is a crucial key to get his result. Here we remove "partially" the symmetry assumption on the domain Ω.

First of all we consider solutions u_ε to (0.1) which blow up as ε goes to zero in the sense of the following definition.

Definition 0.1. *Let u_ε be a family of solutions to* (0.1). *We say that u_ε blows up at k points $\xi_1, \ldots, \xi_k$ of $\overline{\Omega}$ as ε goes to zero if*

$$|\nabla u_\varepsilon|^2 \rightharpoonup S^{N/2} \sum_{i=1}^{k} \delta_{\xi_i} \quad \text{and} \quad u_\varepsilon^{p+1} \rightharpoonup S^{N/2} \sum_{i=1}^{k} \delta_{\xi_i} \quad \text{as } \varepsilon \to 0,$$

in the sense of measures, where δ_{ξ_i} denotes the Dirac measure at ξ_i and

$$S = \inf \left\{ \int_\Omega |\nabla u|^2 dx \mid \int_\Omega |u|^{\frac{2N}{N-2}} dx = 1 \right\}.$$

In Section 1 we prove that (0.1) does not have any solutions blowing up at more than one point for a particular class of domains Ω, which are not necessarily symmetric. We recall that if Ω is a symmetric domain, such a result simply follows by [5].

Theorem 0.2. *Assume Ω is an admissible domain* (*see Definition* 1.3). *For any integer $k \geq 2$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ problem* (0.1) *does not have any solutions blowing up at k points as ε goes to zero.*

Examples of admissible domains are given in Example 1.4, Lemma 1.5, and Example 1.6.

We would like to point out that Dancer and Yan in [6] (see also [20]) proved a similar result for the subcritical problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when Ω is a strictly convex domain. Here $q \in (1, p)$ if $N \geq 3$ and $q \in (1, +\infty)$ if $N = 2$.

In Section 2, we prove that (0.1) has a unique solution blowing up at one point, for a particular class of nonsymmetric domains, which are obtained by perturbing a symmetric domain.

Theorem 0.3. *Let Ω_0 be an open, bounded domain with C^2 boundary, symmetric about $x_i = 0$ for $i = 1, \dots, N$. Assume that the Gauss-Kronecker curvature at $\partial\Omega_0$ is nowhere zero. There exists $\delta > 0$ such that for any C^2 function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\|\phi\|_{C^2} \leq \delta$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the problem*

$$\begin{cases} -\Delta u = u^{p-\varepsilon} & \text{in } \Omega_\phi, \\ u > 0 & \text{in } \Omega_\phi, \\ u = 0 & \text{on } \partial\Omega_\phi, \end{cases} \tag{0.3}$$

where $\Omega_\phi := \{x + \phi(x) : x \in \Omega\}$, has a unique solution which blows up at one point as ε goes to zero.

Finally, using some of Schoen's ideas (see [17] and [11]), Theorem 0.2, and Theorem 0.3 we exhibit a class of nonsymmetric domains for which question (0.2) has a positive answer.

Theorem 0.4. *Let Ω_0 be an open, bounded domain with C^2 boundary, symmetric about $x_i = 0$ for $i = 1, \dots, N$. Assume that the Gauss-Kronecker curvature at $\partial\Omega_0$ is nowhere zero. There exists $\delta > 0$ such that for any C^2 function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\|\phi\|_{C^2} \leq \delta$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ problem* (0.3), *namely*

$$\begin{cases} -\Delta u = u^{p-\varepsilon} & \text{in } \Omega_\phi, \\ u > 0 & \text{in } \Omega_\phi, \\ u = 0 & \text{on } \partial\Omega_\phi, \end{cases}$$

has a unique solution.

1. A nonexistence result

We recall the procedure of moving up planes perpendicular to a fixed direction as in [18]. Let Ω be an open, bounded domain in $\mathbb{R}^N$ with smooth boundary. Let γ be a unit vector in $\mathbb{R}^N$, let λ a real number, and let us introduce the hyperplane

$$T_\lambda^\gamma := \{x \in \mathbb{R}^N : \gamma \cdot x = \lambda\},$$

where λ is large enough and T_λ^γ is disjoint from $\overline{\Omega}$. Let the plane move continuously toward Ω, preserving the same normal; i.e., decrease λ until it begins to intersect $\overline{\Omega}$. From that moment on, at every stage the plane T_λ^γ will cut off from Ω an open cap $\Sigma_\gamma(\lambda)$, the part of Ω on the same side of T_λ^γ :

$$\Sigma_\gamma(\lambda) := \{x \in \Omega : \gamma \cdot x > \lambda\}. \tag{1.1}$$

Let $\Sigma'_\gamma(\lambda)$ denote the reflection of $\Sigma_\gamma(\lambda)$ in the plane T_λ^γ. At the beginning $\Sigma'_\gamma(\lambda)$ will be in Ω, and as λ decreases the reflected cap $\Sigma'_\gamma(\lambda)$ will remain in Ω and at least one of the following occurs:

(i) $\Sigma'_\gamma(\lambda)$ becomes internally tangent to $\partial\Omega$ at some point which is not in T_λ^γ,

(ii) T_λ^γ reaches a position where it is orthogonal to the boundary of Ω.

We denote by $T_{\lambda_\gamma}^\gamma$ the plane T_λ^γ when it first reaches one of these positions and we call

$$\Sigma_\gamma := \Sigma_\gamma(\lambda_\gamma) \text{ the maximal cap associated with } \gamma. \tag{1.2}$$

Now let us consider the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here f is a C^1 function.

The following theorem was proved in [5]:

Theorem 1.1. *Let $u \in \mathrm{C}^2(\overline{\Omega})$ be solution to* (1.3). *For any unit vector $\gamma \in \mathbb{R}^N$ it holds that*

$$\gamma \cdot \nabla u(x) < 0 \quad \forall\, x \in \Sigma_\gamma.$$

For any $y \in \partial\Omega$ let $\nu(y)$ be the inward normal at $\partial\Omega$ in y. Let

$$C_\Omega := \overline{\Omega} \setminus \bigcup_{y \in \partial\Omega} \Sigma_{\nu(y)}. \tag{1.4}$$

The following result holds.

Lemma 1.2. *Let $u \in \mathrm{C}^2(\overline{\Omega})$ be solution to* (1.3). *If $\nabla u(x) = 0$, then $x \in C_\Omega$.*

Proof. By Theorem 1.1 we deduce that if $x \in \Sigma_{\nu(y)}$ for some $y \in \partial\Omega$, then $\nabla u(x) \neq 0$. The claim follows. $\square$

Now let us introduce the following definition.

Definition 1.3. *We say that Ω is an admissible domain if*

$$2^{\frac{1}{N-2}} \operatorname{diam} C_\Omega < \operatorname{dist}\, (C_\Omega, \partial\Omega)\,, \tag{1.5}$$

where

$$\text{diam } C_\Omega = \max\{|x-y| : x, y \in C_\Omega\}$$

and

$$\text{dist } (C_\Omega, \partial\Omega) = \min\{\text{dist } (x, \partial\Omega) : x \in C_\Omega\}.$$

A first example of admissible domain is the following.

Example 1.4. Let Ω be symmetric about $x_i = 0$ and convex in the x_i direction for $i = 1, \dots, N$. Then Ω is an admissible domain.

Proof. It is enough to point out that $\cup_{y\in\partial\Omega} \Sigma_{\nu(y)} = \Omega \backslash \{0\}$ and so $C_\Omega = \{0\}$. Therefore diam $C_\Omega = 0$ and dist $(C_\Omega, \partial\Omega) > 0$. □

The next result shows that condition (1.5) is stable with respect to small perturbation of the domain.

Lemma 1.5. *Assume Ω is an admissible domain. There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and for any C^1 function $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with $\|\phi\|_{C^1} \leq \delta$, if $\Omega_\phi := \{x + \phi(x) : x \in \Omega\}$, then Ω_ϕ is an admissible domain.*

Proof. We argue by contradiction. Assume there exists a sequence of C^1 functions $\phi_n : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with $\|\phi_n\|_{C^1} \to 0$ such that

$$\Omega_n := \{x + \phi_n(x) : x \in \Omega\}$$

does not satisfy (1.5); namely,

$$2^{\frac{1}{N-2}} \text{diam } C_{\Omega_n} \geq \text{dist } (C_{\Omega_n}, \partial\Omega_n)\,. \tag{1.6}$$

First of all let $\hat{x}_n \in C_{\Omega_n}$. Then $\hat{x}_n = x_n + \phi_n(x_n)$ for $x_n \in \Omega$. Up to a subsequence, we can assume that $x := \lim_n x_n = \lim_n \hat{x}_n$. We claim that $x \in C_\Omega$. In fact by (1.4), (1.1), and (1.2) we deduce that

$$\hat{x}_n \cdot \nu_{\partial\Omega_n}(\hat{y}_n) \leq \lambda_{\nu_{\partial\Omega_n}(\hat{y}_n)} \quad \forall\ \hat{y}_n \in \partial\Omega_n. \tag{1.7}$$

Since $\hat{y}_n = y + \phi_n(y)$ for $y \in \partial\Omega$, we have that $\hat{y}_n \to y$, $\nu_{\partial\Omega_n}(\hat{y}_n) \to \nu_{\partial\Omega}(y)$, and $\lambda_{\nu_{\partial\Omega_n}(\hat{y}_n)} \to \lambda_{\nu_{\partial\Omega}(y)}$. Therefore by (1.7) it follows that

$$x \cdot \nu_{\partial\Omega}(y) \leq \lambda_{\nu_{\partial\Omega}(y)} \quad \forall\ y \in \partial\Omega.$$

That proves our claim.

Now by (1.6) we deduce that there exist sequences $\hat{x}_n, \hat{y}_n, \hat{z}_n \in C_{\Omega_n}$ and $\hat{w}_n \in \partial\Omega_n$ such that

$$2^{\frac{1}{N-2}} |\hat{x}_n - \hat{y}_n| \geq |\hat{z}_n - \hat{w}_n|. \tag{1.8}$$

Since $\lim_n \hat{w}_n \in \partial\Omega$ and, because of the previous result, $\lim_n \hat{x}_n, \lim_n \hat{y}_n, \lim_n \hat{z}_n \in C_\Omega$, by passing to the limit in (1.8) we deduce that Ω does not satisfy (1.5) and a contradiction arises. □

In the next example we construct an admissible convex domain, which is not a perturbation of a symmetric domain.

Example 1.6. For any $a \in (0,1)$ and $h > 0$ let $g : [0,2] \to \mathbb{R}^+$ be a C^1 function with $g(0) = g(2) = 0$, such that $h = g(a)$, $g'(a) = 0$, $g'(t) > 0$ if $t \in (0,a)$, $g'(t) < 0$ if $t \in (a,2)$, $\lim_{t\to 0^+} g'(t) = +\infty$, and $\lim_{t\to 2^-} g'(t) = -\infty$. Moreover, assume

$$g(2-t) < g(t) \quad \forall\, t \in (0,1) \quad \text{and} \quad g(2a-t) < g(t) \quad \forall\, t \in (a,2a).$$

Then there exists $a_0 \in (0,1)$ and $h_0 > 0$ such that for any $h \geq h_0$ and $a \in (a_0, 1)$ the domain

$$\Omega = \{(x_1, x_2, \dots, x_N) : x_1 \in (0,2),\ |(x_2, \dots, x_N)| < g(x_1)\}$$

is admissible.

Proof. It is easy to see that $C_{\Omega_h} = \{(x_1, 0, \dots, 0) : x_1 \in [a,1]\}$ and so $\operatorname{diam} C_{\Omega_h} = 1 - a$. Moreover

$$\lim_{h\to+\infty} \operatorname{dist}\ (C_{\Omega_h}, \partial\Omega_h) = a.$$

We can choose a such that $2^{\frac{1}{N-2}}(1-a) < a$ and the claim follows. □

Let us denote by G_Ω the Green's function of the Laplacian on Ω with Dirichlet boundary condition and by H_Ω its regular part, chosen in such a way that

$$H_\Omega(x,y) = \frac{\alpha_N}{|x-y|^{N-2}} - G_\Omega(x,y), \quad \forall\ (x,y) \in \Omega^2,$$

where $\alpha_N = \left[(N-2)\text{meas}\left(S^{N-1}\right)\right]^{-1}$ and S^{N-1} is the $(N-1)$-dimensional unit sphere.

The following lemma plays a crucial role in proving nonexistence result Theorem 0.2.

Lemma 1.7. *Assume* Ω *is an admissible domain* (*see Definition* 1.3). *Then* (*see* (1.4))

$$\det \begin{pmatrix} H_\Omega(x,x) & -G_\Omega(x,y) \\ -G_\Omega(x,y) & H_\Omega(y,y) \end{pmatrix} < 0 \quad \forall\ x,y \in C_\Omega.$$

Proof. Set

$$\varphi_\Omega(x,y) = H_\Omega^{1/2}(x,x) H_\Omega^{1/2}(y,y) - G_\Omega(x,y).$$

We prove that

$$\max_{x,y\in C_\Omega} \varphi_\Omega(x,y) < 0. \tag{1.9}$$

The claim is an immediate consequence. If $x, y \in C_\Omega$ it holds that

$$\begin{aligned}
&\varphi_\Omega(x,y) = H^{1/2}(x,x)H^{1/2}(y,y) - G(x,y)\\
&= H^{1/2}(x,x)H^{1/2}(y,y) + H(x,y) - \frac{\alpha_N}{|x-y|^{N-2}}\\
&\le H^{1/2}(x,x)H^{1/2}(y,y) + \frac{1}{2}[H(x,x)+H(y,y)] - \frac{\alpha_N}{|x-y|^{N-2}}\\
&= \frac{1}{2}\left[H^{1/2}(x,x)+H^{1/2}(y,y)\right]^2 - \frac{\alpha_N}{|x-y|^{N-2}}\\
&\le \frac{1}{2}\Big[\frac{\alpha_N^{\frac{1}{2}}}{\big(\text{dist }(x,\partial\Omega)\big)^{\frac{N-2}{2}}} + \frac{\alpha_N^{\frac{1}{2}}}{\big(\text{dist }(y,\partial\Omega)\big)^{\frac{N-2}{2}}}\Big]^2 - \frac{\alpha_N}{|x-y|^{N-2}}\\
&\le \frac{2\alpha_N}{\text{dist }(C_\Omega,\partial\Omega)^{N-2}} - \frac{\alpha_N}{(\text{diam } C_\Omega)^{N-2}}.
\end{aligned}$$

Therefore, (1.9) is proved. Here we used the following well-known estimates (see, for example, [3])

$$2H(x,y) \le H(x,x) + H(y,y) \quad \forall\ x,y\in\Omega$$

and

$$H(x,x) \le \frac{\alpha_N}{\big(\text{dist }(x,\partial\Omega)\big)^{N-2}} \quad \forall\ x\in\Omega. \qquad \square$$

Proof of Theorem 0.2. We argue by contradiction. Let u_ε be a solution to (0.1) which blows up at k points $\xi_1,\dots,\xi_k$ in $\overline{\Omega}$ as ε goes to zero. By Theorem 1 of [2] we deduce that $\xi_1,\dots,\xi_k \in \Omega$ and the matrix

$$M_\Omega(\xi_1,\dots,\xi_k) := \begin{pmatrix} H_\Omega(\xi_1,\xi_1) & -G_\Omega(\xi_1,\xi_2) & \dots & -G_\Omega(\xi_1,\xi_k)\\ -G_\Omega(\xi_2,\xi_1) & H_\Omega(\xi_2,\xi_2) & \dots & -G_\Omega(\xi_2,\xi_k)\\ \vdots & \vdots & \ddots & \vdots\\ -G_\Omega(\xi_k\xi_1) & -G_\Omega(\xi_k\xi_2) & \dots & H_\Omega(\xi_k,\xi_k)\end{pmatrix}$$

is positively or null definite. Therefore any submatrix

$$M_\Omega^{(2)}(\xi_i,\xi_j) := \begin{pmatrix} H_\Omega(\xi_i,\xi_i) & -G_\Omega(\xi_i,\xi_j)\\ -G_\Omega(\xi_i,\xi_j) & H_\Omega(\xi_j,\xi_j)\end{pmatrix}$$

is positively or null definite. In particular

$$\det M_{\Omega}^{(2)}(\xi_i, \xi_j) \geq 0. \tag{1.10}$$

On the other hand it is easy to see that for ε small enough there exist k local maximum points $x_{1\varepsilon}, \ldots, x_{k\varepsilon}$ of the function u_ε such that

$$\lim_{\varepsilon \to 0} x_{i\varepsilon} = \xi_i \quad \text{for } i = 1, \ldots, k. \tag{1.11}$$

Since $\nabla u_\varepsilon(x_{i\varepsilon}) = 0$, by Lemma 1.2 we deduce that $x_{i\varepsilon} \in C_\Omega$, and therefore by (1.11) also $\xi_i \in C_\Omega$. Finally by Lemma 1.7 and (1.10) a contradiction arises. □

2. A uniqueness result

Let us recall the Robin's function $\tau_\Omega : \Omega \to \mathbb{R}$ which is defined as the leading term of the regular part of the Green's function; i.e., $\tau_\Omega(x) = H_\Omega(x, x)$.

First of all we recall the following result concerning convex domains proved in [4].

Theorem 2.1. *Let Ω be a convex, bounded domain in $\mathbb{R}^N$. Then the Robin's function τ_Ω is strictly convex. In particular τ_Ω has a unique critical point.*

Moreover, if the domain is also symmetric, the following more precise information about the unique critical point of the Robin's function was given in [8].

Theorem 2.2. *Let Ω be symmetric about $x_i = 0$ and convex in the x_i direction for $i = 1, \ldots, N$. Then the Robin's function τ_Ω has a unique critical point, which is nondegenerate.*

Finally, let us recall the following uniqueness result proved in [2].

Theorem 2.3. *If ξ_0 is a nondegenerate critical point of the Robin's function τ_Ω, then problem* (0.1) *has a unique family of solutions blowing up at ξ_0 according to Definition* 0.1.

Let us prove a perturbation result which will play a crucial role in proving Theorem 0.3.

Let Ω_0 be an open, bounded domain in $\mathbb{R}^N$. Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function and set $\Omega_\phi := \{x + \phi(x) : x \in \Omega\}$.

Lemma 2.4. *It holds that $\tau_{\Omega_\phi} \to \tau_\Omega$ C^2-uniformly on compact sets of Ω, as $\|\phi\|_{C^0} \to 0$.*

Proof. Let $\phi_n : \mathbb{R}^N \to \mathbb{R}^N$ be a sequence of C^2 functions such that $\|\phi_n\|_{C^0} \to 0$. Set $\Omega_n := \Omega_{\phi_n}$, $H_n := H_{\Omega_n}$, and $H_0 := H_{\Omega_0}$. For any $y \in \Omega$, $H_n(\cdot, y)$ is a solution to

$$\begin{cases} -\Delta_x H_n(x,y) = 0 & x \in \Omega_n, \\ H_n(x,y) = \frac{1}{|x-y|^{N-2}} & x \in \partial\Omega_n \end{cases}$$

and $H_0(\cdot, y)$ is a solution to

$$\begin{cases} -\Delta_x H_0(x,y) = 0 & x \in \Omega_0, \\ H_0(x,y) = \frac{1}{|x-y|^{N-2}} & x \in \partial\Omega_0. \end{cases}$$

Since $\|\phi_n\|_{C^0} \to 0$, it is easy to check that

$$\lim_n \Big(\sup_{x \in \Omega_n \Delta \Omega_0} \operatorname{dist}(x, \partial\Omega) \Big) = 0,$$

where $\Omega_n \Delta \Omega_0 = (\Omega_0 \setminus \Omega_n) \cup (\Omega_n \setminus \Omega_0)$. Therefore, by using Mosco's convergence in [12] (see also, for example, Theorem 1 and Corollary 4 of [16]), we deduce that

$$H_n(\cdot, y) \longrightarrow H_0(\cdot, y) \quad \text{strongly in } H^1(B), \tag{2.1}$$

where $B \subset \mathbb{R}^N$ is a fixed ball which contains Ω_n and Ω_0. Since $H_n(\cdot, y)$ and $H_0(\cdot, y)$ are harmonic functions, by (2.1) we deduce that $H_n(\cdot, y)$ converges to $H_0(\cdot, y)$ pointwise in Ω_0. Then by Theorem 2.9 in [7] it follows that

$$H_n(\cdot, y) \longrightarrow H_0(\cdot, y) \quad \text{uniformly on compact sets in } \Omega_0,$$

and finally by Theorem 2.10 in [7] we get that

$$H_n(\cdot, y) \longrightarrow H_0(\cdot, y) \quad C^2\text{-uniformly on compact sets in } \Omega_0. \tag{2.2}$$

By using (2.2) and the fact that H_n and H_0 are harmonic with respect to x and y, it is easy to prove that

$$H_n(x, x) \longrightarrow H_0(x, x) \quad C^2\text{-uniformly on compact sets in } \Omega_0.$$

That proves our claim. □

Let us recall the notion of Gauss-Kronecker curvature (see [19]). Let S be an $(N-1)$-manifold in $\mathbb{R}^N$. The Gauss-Kronecker curvature of S at a point x is the product of the principle curvatures of S at x which are stationary values of the normal curvature on the tangent space T_x. The following result holds (see Theorem 5, [19]).

Theorem 2.5. *Let Ω be an open, bounded domain with C^2 boundary. If the Gauss-Kronecker curvature at $\partial\Omega$ is nowhere zero, then Ω is strictly convex.*

Proposition 2.6. *Let Ω_0 be an open, bounded domain with C^2-boundary, symmetric about $x_i = 0$ for $i = 1, \dots, N$. Assume that the Gauss-Kronecker curvature at $\partial\Omega_0$ is nowhere zero. There exists $\delta > 0$ such that for any C^2 function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\|\phi\|_{C^2} \leq \delta$, the domain $\Omega_\phi := \{x + \phi(x) : x \in \Omega\}$ is such that τ_{Ω_ϕ} has a unique critical point, which is nondegenerate.*

Proof. First of all by Theorem 2.5 we deduce that Ω_0 is strictly convex. Then by using both the symmetry and the convexity, by Theorem 2.1 and Theorem 2.2 we deduce that the Robin's function τ_{Ω_0} has a unique critical point x_0 which is nondegenerate.

Secondly, if $\|\phi\|_{C^2}$ is small enough, it is easy to check that the Gauss-Kronecker curvature at $\partial\Omega_\phi$ is nowhere zero. Therefore by Theorem 2.5 we deduce again that Ω_ϕ is strictly convex. Then by Theorem 2.1 it follows that the Robin's function τ_{Ω_ϕ} has a unique critical point x_ϕ. By Lemma 2.4 we deduce that x_ϕ must converge to x_0 as $\|\phi\|_{C^2}$ goes to 0. Finally, by using again Lemma 2.4 and the fact that x_0 is a nondegenerate critical point of τ_{Ω_0}, we can claim that x_ϕ is nondegenerate critical point of τ_{Ω_ϕ} provided $\|\phi\|_{C^2}$ is small enough. □

Proof of Theorem 0.3. First of all by Example 1.4, Lemma 1.5, and Proposition 2.6 we deduce that there exists $\delta > 0$ such that for any C^2 function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\|\phi\|_{C^2} \leq \delta$, the domain $\Omega_\phi := \{x + \phi(x) : x \in \Omega\}$ is a convex domain and moreover the unique critical point of the Robin's function τ_{Ω_ϕ} is nondegenerate. Now let Ω_ϕ be fixed as above.

Let u_ε be a family of solutions to (0.3) which blow up at one point of Ω_ϕ as ε goes to 0. By [15] we get that such a point has to be a critical point of the Robin's function τ_{Ω_ϕ}. Since τ_{Ω_ϕ} has a unique critical point which is nondegenerate, by Theorem 2.3 we deduce the uniqueness of the solutions blowing up at such a point. That proves our claim. □

Proof of Theorem 0.4. First of all by Example 1.4, Lemma 1.5, and Proposition 2.6 we deduce that there exists $\delta > 0$ such that for any C^2 function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ with $\|\phi\|_{C^2} \leq \delta$, the domain $\Omega_\phi := \{x + \phi(x) : x \in \Omega\}$ is admissible according to Definition 1.3, is a convex domain, and moreover the unique critical point of the Robin's function τ_{Ω_ϕ} is nondegenerate. Now let Ω_ϕ be fixed as above, so that Theorem 0.2 and Theorem 0.3 hold.

Let u_ε be a family of solutions to (0.3) as ε goes to 0. First we claim that u_ε is uniformly bounded in $\mathrm{H}^1_0(\Omega)$. In fact u_ϵ is uniformly bounded near a fixed neighborhood of $\partial\Omega_\phi$ (see [10]). After this step we argue exactly as in the proof of Theorem 0.3 in [11] to get the claim.

Then u_ε, as ε goes to zero, may either converge to a solution to (0.3) with $\varepsilon = 0$ or blow up at a finite number of points of Ω_ϕ. On the other hand, problem (0.3) with $\varepsilon = 0$ has only the trivial solution, because of the convexity of the domain Ω_ϕ (see [14]). Therefore, using results in [2], we can claim that u_ε blows up at k points of Ω_ϕ according to Definition 0.1. Finally by Theorem 0.3 we deduce that k has to be 1, and then by Theorem 0.2 the claim follows. □

References

[1] A. Bahri, "Critical Points at Infinity in some Variational Problems," Pitman Research Notes in Mathematics Series, 182, Longman, 1982.

[2] A. Bahri, Y. Li, and O. Rey, *On a variational problem with lack of compactness: the topological effect of the critical points at infinity,* Calc. Var. P.D.E., 3 (1995), 67–93.

[3] C. Bandle and M. Flucher, *Harmonic radius and concentration of energy; hyperbolic radius and Liouville's equations* $\Delta U = e^U$ *and* $\Delta U = U^{(n+2)/(n-2)}$, SIAM Rev., 38 (1996), 191–238.

[4] P. Cardaliaguet and R. Tahraoui, *On the strict concavity of the harmonic radius in dimension* $N \geq 3$, J. Math. Pures Appl., 81 (2002), 223–240.

[5] B. Gidas, W.M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle,* Comm. Math. Phys., 68 (1979), 209–243.

[6] E.N. Dancer and S. Yan, *Effect of the domain geometry on the existence of multipeak solutions for an elliptic problem,* Topol. Methods Nonlinear Anal., 14 (1999), 1–38.

[7] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, Berlin, 1977.

[8] M. Grossi, *On the nondegeneracy of the critical points of the Robin function in symmetric domains,* C. R. Math. Acad. Sci. Paris, 335 (2002), 157–160.

[9] M. Grossi, *A uniqueness result for a semilinear elliptic equation in symmetric domains,* Adv. Differential Equations, 5 (2000), 193–212.

[10] Z.C. Han, *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent,* Ann. Inst. Poincaré Anal. non linéaire, 8 (1991), 159–174.

[11] Y.Y. Li, *Prescribing scalar curvature on* S^n *and related problems, Part I,* J. Diff. Equations, 120 (1995), 319–410.

[12] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities,* Adv. in Math., 3 (1969), 510–585.

[13] M. Musso and A. Pistoia, *Multispike solutions for a nonlinear elliptic problem involving critical Sobolev exponent,* Indiana Univ. Math. J., 51 (2002), 541–579.

[14] S.I. Pohožaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$ (Russian), Dokl. Akad. Nauk SSSR, 165 (1965), 36–39.

[15] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent,* J. Funct. Anal., 89 (1990), 1–52.

[16] G. Savaré and G. Schimperna, *Domain perturbations and estimates for the solutions of second order elliptic equations,* J. Math. Pures Appl., 81 (2002), 1071–1112.

[17] R. Schoen, *On the number of constant scalar curvature metrics in a conformal class,* "Differential Geometry," 311–320, Pitman Monogr. Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991.

[18] J. Serrin, *A symmetry problem in potential theory,* Arch. Rational Mech. Anal., 43 (1971), 304–318.

[19] J.A. Thorpe, "Elementary Topics in Differential Geometry," Springer-Verlag, Berlin, 1978.

[20] J. Wei, *Conditions for two-peaked solutions of singularly perturbed elliptic equations,* Manuscripta Math., 96 (1998), 113–131.