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EXISTENCE OF A PERIODIC SOLUTION IN A CHUA'S CIRCUIT WITH SMOOTH NONLINEARITY

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Abstract. In this paper, we consider Chua's circuit:

$$\varepsilon u' = z + f(u), \quad z' = u + w - z, \quad w' = -\beta z - \gamma w,$$

where $f(u)$ is chosen as a cubic function, $\beta > 0$ and $\gamma \geqslant 0$ are constants, and $\varepsilon > 0$ is a small parameter. We prove that the flow defines a Poincaré map from a compact set which is homeomorphic to the unit disk to itself and then apply Brouwer's fixed-point theorem to conclude that the system has a "big" periodic solution. This global analysis is viewed as a step towards understanding chaos in this model analytically.

1. Introduction

Chua's circuit is a physical electric circuit made up of four linear circuit elements (one resistor, one inductor, and two capacitors) and a two-terminal nonlinear resistor characterized by a piecewise-linear $i - v$ curve. The state

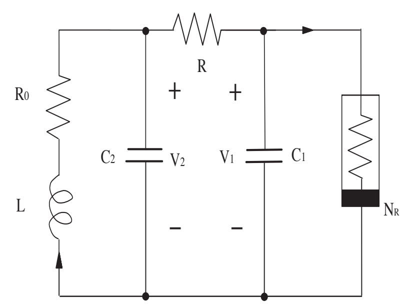

Figure 1. Chua's circuit

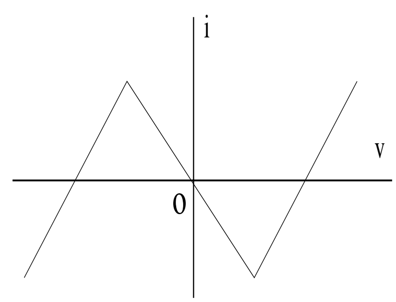

Figure 2. Piecewise linear $i - v$ curve

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equations of the circuit are

$$\left\{\begin{aligned} \frac{dv_1}{dt} &= \tfrac{1}{C_1}(G(v_2 - v_1) - h(v_1)) \\ \frac{dv_2}{dt} &= \tfrac{1}{C_2}(G(v_1 - v_2) + i_3) \\ \frac{di_3}{dt} &= -\tfrac{1}{L}(v_2 + R_0 i_3), \end{aligned}\right. \tag{1.1}$$

where $h(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)(|v_1 + E| - |v_1 - E|)$, $G = \frac{1}{R}$, and G_a, G_b and E are constants which characterize the nonlinearity of the nonlinear element. We introduce the dimensionless variables suggested by Kocarev and Roska ([16], 1993):

$$x \equiv \frac{v_1}{E}, \quad y \equiv \frac{v_2}{E}, \quad z \equiv \frac{i_3}{GE}, \quad \tau \equiv \frac{G}{C_2}t, \quad m_1 \equiv \frac{G_a}{G} + 1,$$
$$m_0 \equiv \frac{G_b}{G} + 1, \quad \alpha \equiv \frac{C_2}{C_1}, \quad \beta \equiv \frac{C_2}{LG^2}, \quad \gamma \equiv \frac{C_2 R_0}{G^2}.$$

System (1.1) is transformed to the dimensionless form,

$$\left\{\begin{aligned} \frac{dx}{d\tau} &= \alpha(y - g(x)) \\ \frac{dy}{d\tau} &= x - y + z \\ \frac{dz}{d\tau} &= -\beta y - \gamma z, \end{aligned}\right. \tag{1.2}$$

where $g(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| - |x-1|)$ is the canonical piecewise-linear function given by Chua and Ying ([9]). It is an odd-symmetric, three-segment, piecewise-linear curve having breaking points at $x = 1$ and $x = -1$ and a slope equal to $m_0 < 0$ at the inner segment and $m_1 > 0$ at the outer segments respectively. Note that the $i - v$ curve of a physical nonlinear element does not have to be symmetric. The nonlinearity can be shifted up and down by adding a constant current source.

The circuit was originally conceived and designed by L.O. Chua in 1983. Then in 1984 T. Matsumoto wrote a computer program, verified its chaotic nature, and named it Chua's circuit. Experimental confirmation of chaos was done in [20] (Zhong and Ayrom, 1985). Hundreds of research papers have been published since then. Here are some of them we think related to our work: Kocarev and Roska ([16], 1993) gave a comparison of the Lorénz equation and Chua's equation on the numerical results of the existence of homoclinic orbits, period-doubling phenomena, and chaotic attractors. Lozi and Ushiki ([17], 1991) gave a symbolic dynamic analysis on Chua's circuit. A. Khibnik, Roose, and Chua ([15], 1993) used a cubic nonlinearity and ignored the resistance caused by the inductor ($\gamma = 0$) in (1.2) and studied the bifurcations of periodic orbits numerically by fixing β and changing α in (1.2). They also analyzed two-parameter bifurcations on how periodic orbits can be generated from homoclinic orbits. A. Algaba et al. ([1], 2000) studied periodic and quasiperiodic behavior exhibited by Chua's equation with a cubic nonlinearity near a Hopf-Pitchfork bifurcation. Existence of homoclinic orbits is proved in [8] (Leon O. Chua, Komuro Motomasa, and Matsumoto Takashi, 1986) and then "chaos" in the Shi'lnikov sense. The existence of "small" periodic solutions is implied by their proof. But no rigorous results on the smooth nonlinearity has been found so far. In this paper we are interested in the model (1.2) with a smooth nonlinearity.

We consider the following system:

$$\begin{cases} \varepsilon u' = z + f(u) \\ z' = u + w - z \\ w' = -\beta z - \gamma w, \end{cases} \tag{1.3}$$

where $\beta > 0$, $\gamma \geqslant 0$, $\varepsilon > 0$, and $f(u)$ is a cubic function. Here we take $f(u) = u(u - u_2)(u_3 - u)$, where $u_3 > u_2 > 0$. The parameters β, ε, u_2, and u_3 are to be chosen in some open set so that there is a periodic solution $P = (u_P(t), z_P(t), w_P(t))$ such that

$$P(t) = P(t+T) \quad \text{for some} \quad T > 0, \quad \text{and} \quad t \in [0, \infty)$$

with T the smallest period of $P(t)$.

We will first prove the result for $\gamma = 0$, then by a regular perturbation to γ, we obtain the existence of a periodic orbit for small $\gamma > 0$.

Numerical bifurcation studies show that unlike the "small" periodic solutions, which begin to exist after Hopf bifurcations, the "big" periodic solutions show up after a saddle-node bifurcation. After the saddle-node bifurcation, a pair of "big" periodic solutions appears. One of these solutions is stable, the other is unstable. Figure 3 shows the saddle-node bifurcation for fixed $\beta = 14$, $u_2 = 0.375$, and $u_3 = 0.75$. Figure 4 contains the corresponding solutions projected onto the (u, z) plane. These two pictures are generated by numerical bifurcation software AUTO97 ([11], 1998). Khibnik et al. have done a thorough study of the bifurcation phenomenon in [15] (1993). They show the extension of the diagram in Figure 3 from solution 9 (see also Figure 4) to close to a homoclinic orbit that tends to the equilibrium point $(u_3, 0, -u_3)$ as $t \to \infty$. Figure 5 to Figure 9 are obtained by G.B. Er-

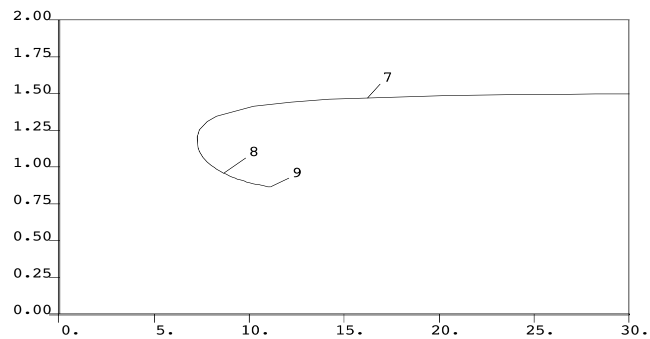

FIGURE 3. maximum u versus $1/\varepsilon$

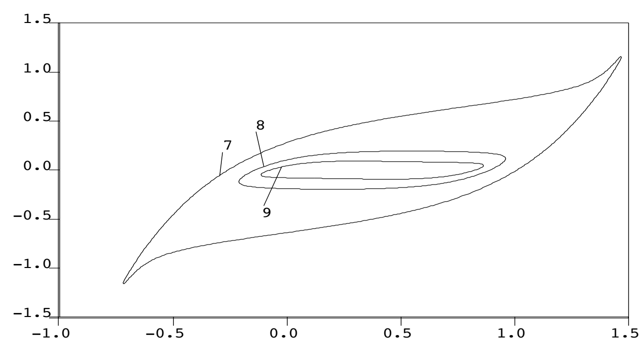

FIGURE 4. Corresponding trajectories in figure 3

mentrout's numerical differential equation solver XPP. They are solutions of (1.3) for fixed $\beta = 14$, $u_2 = 0.375$, $u_3 = 0.75$, $\gamma = 0$, and various εs projected onto the (u, z) plane. Figure 5 shows the two stable, small periodic solutions around the two equilibrium points $(0, 0, 0)$ and $(u_3, 0, -u_3)$ generated through Hopf bifurcation. While ε decreases, there appears a "big" stable periodic solution along with the two "small" stable periodic solutions; see Figure 6. Continuing to decrease ε, we have the coexistence of the butterfly chaotic attractor with the stable "big" periodic solution shown in Figure 7. After a crucial small enough ε, all other attractors disappear; the only stable attractor left is the "big" periodic solution shown in Figure 8 and Figure 9. As one can see, once the stable "big" periodic appears, the smaller the

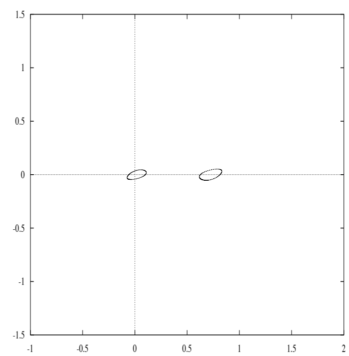

FIGURE 5. $\varepsilon = 1/7$

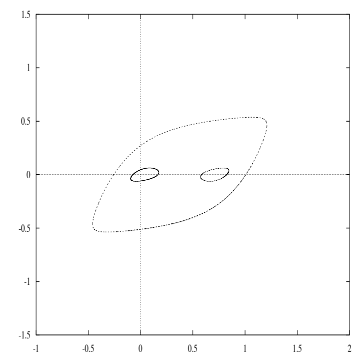

FIGURE 6. $\varepsilon = 1/7.25$

ε is, the narrower the projections of the periodic solutions onto the (u, z) plane look. Numerical simulations show that the existence of the strange attractors of either the butterfly type or the Rössler type around the equilibrium points has to be accompanied by a stable "big" periodic solution. In this paper, we will prove the existence of a "big" periodic solution as shown in Figure 8 and Figure 9. For the exact conditions on the parameters, see Theorem 2.2 and the definitions of B and Λ in the next section.

2. EXISTENCE PROOF

We first consider system (1.3) for $\gamma = 0$, namely

$$\begin{cases} \varepsilon u' = z + f(u) \\ z' = u + w - z \\ w' = -\beta z, \end{cases} \tag{2.1}$$

and the result can be extended to the case when $\gamma > 0$ but small by a regular perturbation.

2.1. **Global boundedness and definitions.** Here we define a cut-off function, $\zeta(s) \in C^1(-\infty, \infty)$, as follows:

$$\zeta(s) = \begin{cases} 1, \ s > 1 \\ -1, \ s < -1 \\ s, \ |s| < \frac{1}{2} \\ 0 \le \zeta'(s) \le 2, \ s \in (-\infty, \infty). \end{cases} \tag{2.2}$$

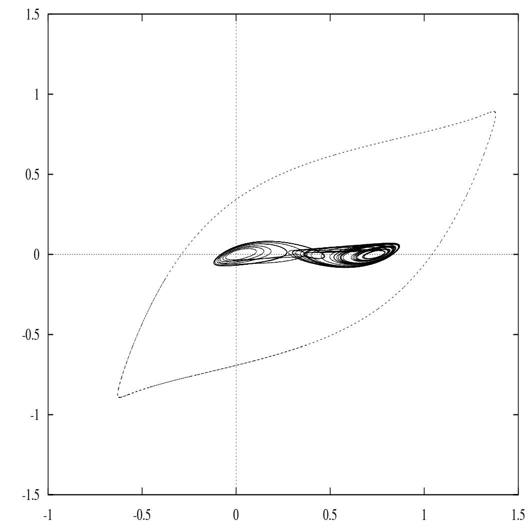

FIGURE 7. $\varepsilon = 1/9$

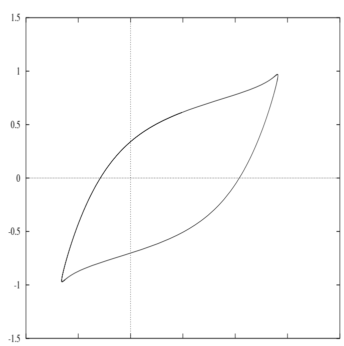

FIGURE 8. $\varepsilon = 1/10$

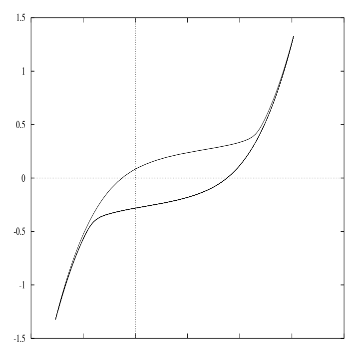

FIGURE 9. $\varepsilon = 1/100$

We will use this function([3], Chen, 1999) to construct a Lyapunov function for system (2.1).

Theorem 2.1. *For system* (2.1)*, there exists an* $M > 0$*, independent of* ε*, such that all the solutions will enter the box* R:

$$R = \{(u, z, w) : |u| \leqslant M,\ |z| \leqslant M,\ |w| \leqslant M\}$$

in finite time if they start outside of R and R is positively invariant.

To prove Theorem 2.1, we define the following Lyapunov function for system (2.1):

$$H(u,z,w)=\frac{z^2}{2}+\frac{w^2}{2\beta}+\varepsilon(\frac{1}{4}-\sigma\sqrt{\beta})u^2-\frac{1}{2}\sigma z\zeta(\frac{w}{\sqrt{\beta}}), \tag{2.3}$$

where $\sigma=\min(\frac{1}{8\sqrt{\beta}},\ \frac{1}{2})$.

Proposition 2.1. *$H(u,z,w)$ is positive definite.*

Proof. Since $\frac{1}{4}-\sigma\sqrt{\beta}\geqslant 0$, $\varepsilon>0$, and $\beta>0$, we only have to consider

$$\frac{z^2}{2}-\frac{1}{2}\sigma z\zeta(\frac{w}{\sqrt{\beta}})+\frac{w^2}{2\beta}=\frac{1}{2}(z-\frac{1}{2}\sigma\zeta(\frac{w}{\sqrt{\beta}}))^2-\frac{1}{8}\sigma^2\zeta^2(\frac{w}{\sqrt{\beta}})+\frac{w^2}{2\beta}.$$

We want to show that $\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2\zeta^2(\frac{w}{\sqrt{\beta}})>0$ for all $w\in\mathbb{R}$.

Case 1. $|\frac{w}{\sqrt{\beta}}|>1$. Then $w^2>\beta$, and therefore

$$\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2\zeta^2(\frac{w}{\sqrt{\beta}})=\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2>\frac{1}{2}-\frac{1}{8}\sigma^2;$$

if $\frac{1}{8\sqrt{\beta}}<\frac{1}{2}$, then $\frac{1}{2}-\frac{1}{8}\sigma^2=\frac{1}{2}-\frac{1}{8}(\frac{1}{8\sqrt{\beta}})^2>\frac{1}{2}-\frac{1}{32}>0$; if $\frac{1}{8\sqrt{\beta}}>\frac{1}{2}$, then it is straightforward.

Case 2. $|\frac{w}{\sqrt{\beta}}|<\frac{1}{2}$. Then

$$\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2\zeta^2(\frac{w}{\sqrt{\beta}})=\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2(\frac{w^2}{\beta})>\frac{w^2}{2\beta}-\frac{1}{32}\frac{w^2}{\beta}>0.$$

Case 3. $\frac{1}{2}<\frac{w}{\sqrt{\beta}}<1$. Then $0\leqslant\zeta'(\frac{w}{\sqrt{\beta}})\leqslant 2$ and $\int_0^t\zeta'(s)ds=\zeta(t)\leqslant 2t$ for $\frac{1}{2}\leqslant t\leqslant 1$. Therefore,

$$-\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2\zeta^2(\frac{w}{\sqrt{\beta}})\geqslant\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2 4(-\frac{w^2}{\beta})=\frac{w^2}{2\beta}(1-\sigma^2)\geqslant\frac{w^2}{2\beta}(1-\frac{1}{4})>0.$$

Case 4. $-1<\frac{w}{\sqrt{\beta}}<-\frac{1}{2}$. Then $0\leqslant\zeta'(\frac{w}{\sqrt{\beta}})\leqslant 2$ and $0\leqslant\int_0^t\zeta'(s)\,ds=-\zeta(t)\leqslant -2t$ for $-1\leqslant t\leqslant -\frac{1}{2}$ or $\zeta^2(t)\leqslant 4t^2$. Therefore,

$$\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2\zeta^2(\frac{w}{\sqrt{\beta}})\geqslant\frac{w^2}{2\beta}-\frac{1}{8}\sigma^2 4(\frac{w^2}{\beta})=\frac{w^2}{2\beta}(1-\sigma^2)\geqslant\frac{w^2}{2\beta}(1-\frac{1}{4})>0.$$

This completes the proof of the proposition. □

Proof of Theorem 2.1. We calculate $H'(u,z,w)$ along the trajectories of system (2.1):

$$
\begin{aligned}
H' &= zz' + \frac{ww'}{\beta} + 2\varepsilon(\frac{1}{4} - \sigma\sqrt{\beta})uu' - \frac{1}{2}\sigma z'\zeta(\frac{w}{\sqrt{\beta}}) - \frac{\sigma z}{2\sqrt{\beta}}\zeta'(\frac{w}{\sqrt{\beta}})w' \\
&= (\frac{1}{2} - 2\sigma\sqrt{\beta})uf(u) + (\frac{3}{2} - 2\sigma\sqrt{\beta})zu - z^2 - \frac{1}{2}\sigma u\zeta(\frac{w}{\sqrt{\beta}}) \\
&- \frac{1}{2}\sigma w\zeta(\frac{w}{\sqrt{\beta}}) + \frac{1}{2}\sigma z\zeta(\frac{w}{\sqrt{\beta}}) + \frac{\sigma z^2\sqrt{\beta}}{2}\zeta'(\frac{w}{\sqrt{\beta}}).
\end{aligned}
$$

We claim that $k_1 = \frac{1}{2} - 2\sigma\sqrt{\beta} > 0$. This is because if $\frac{1}{8\sqrt{\beta}} < \frac{1}{2}$, then $k_1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0$; if $\frac{1}{8\sqrt{\beta}} > \frac{1}{2}$, $k_1 = \frac{1}{2} - \frac{2}{2}\sqrt{\beta} > \frac{1}{2} - \frac{1}{4} = \frac{1}{4} > 0$. Then $2k_2 = \frac{3}{2} - 2\sigma\sqrt{\beta} = 1 + k_1 > 0$. Since $zu \leqslant \frac{u^2+z^2}{2}$, we have

$$
\begin{aligned}
H' &< k_1uf(u) + (\frac{3}{4} - \sigma\sqrt{\beta})u^2 + (\frac{3}{4} - \sigma\sqrt{\beta})z^2 - z^2 \\
&- \frac{1}{2}u\sigma\zeta(\frac{w}{\sqrt{\beta}}) - \frac{1}{2}w\sigma\zeta(\frac{w}{\sqrt{\beta}}) + \frac{1}{2}z\sigma\zeta(\frac{w}{\sqrt{\beta}}) + \sigma\sqrt{\beta}z^2 \\
&= -k_1u^4 + k_1(u_2 + u_3)u^3 - k_1u_2u_3u^2 + k_2u^2 \\
&- \frac{1}{2}u\sigma\zeta(\frac{w}{\sqrt{\beta}}) - \frac{1}{4}z^2 + \frac{1}{2}z\sigma\zeta(\frac{w}{\sqrt{\beta}}) - \frac{1}{2}w\sigma\zeta(\frac{w}{\sqrt{\beta}}).
\end{aligned}
$$

Since $u^3 \leqslant |u|^3$, $u \leqslant |u|$, and $|\zeta(\frac{w}{\sqrt{\beta}})| \leqslant 1$, we have

$$
\begin{aligned}
H' &< -k_1u^4 + k_1(u_2 + u3)|u|^3 + (k_2 - k_1u_2u3)u^2 \\
&+ \frac{1}{2}\sigma|u| - \frac{1}{4}z^2 + \frac{1}{2}\sigma|z| - \frac{1}{2}w\sigma\zeta(\frac{w}{\sqrt{\beta}}).
\end{aligned}
$$

Let

$$
\begin{aligned}
&\phi_1(u) = -k_1u^4 + k_1(u_2 + u3)|u|^3 + (k_2 - k_1u_2u_3)u^2 + \frac{1}{2}\sigma|u|, \\
&\phi_2(z) = -\frac{1}{4}z^2 + \frac{1}{2}\sigma|z|, \qquad \phi_3(w) = -\frac{1}{2}w\sigma\zeta(\frac{w}{\sqrt{\beta}}).
\end{aligned}
$$

We know that ϕ_1, ϕ_2, and ϕ_3 are all bounded above, and they all go to negative infinity as $|u|$, $|z|$, and $|w|$ go to infinity respectively. Let K_1, K_2, and K_3 be such that

$$
\begin{aligned}
\phi_1 &\leqslant K_1, \text{ for all } u \in \mathbb{R} \\
\phi_2 &\leqslant K_2, \text{ for all } z \in \mathbb{R} \\
\phi_3 &\leqslant K_3, \text{ for all } w \in \mathbb{R}.
\end{aligned}
$$

Let $K = K_1 + K_2 + K_3$; then for this K there exists an $M > 0$ such that

$$\begin{aligned} \phi_1 &\leqslant -K, \text{ as } |u| > M \\ \phi_2 &\leqslant -K, \text{ as } |z| > M \\ \phi_3 &\leqslant -K, \text{ as } |w| > M. \end{aligned}$$

Actually, inside the closed box R, there is a largest level surface $H(u, w, z) =$ const such that $H' < 0$ outside this level surface. Thus, as long as the solutions are outside of the box R so that it's outside of the level surface, it will get into the level surface so that it gets into R. This completes the proof of Theorem 2.1. □

Here we define some sets Σ_1, Σ_2, Ω_1, Ω_2, Ω_3, Ω_4, Σ_0, and Σ_{01} for the proof of the main theorem.

Let $-m_1 < 0$ be the local minimum of $f(u)$ and u_{m_1} be such that $f(u_{m_1}) = -m_1$, and let m_2 be the local maximum of $f(u)$ and u_{m_2} be such that $f(u_{m_2}) = m_2$. Define $\Lambda \in \mathbb{R}^2$ as

$$\begin{aligned} \Lambda = \Big\{(u_2, u_3) &: u_{m_2} - u_3 + m_2 < 0, u_{m_1} - m_1 > 0, \\ & f\Big(\frac{1}{2}\Big(u_2 + u_3 - \sqrt{(u_2 - u_3)^2 + 4}\Big) + \sigma\Big) > m_1 + \delta, \text{ and} \\ & f\Big(\frac{1}{2}\Big(u_2 + \sqrt{u_2^2 + 4}\Big) - \gamma\Big) < -m_2 - \delta\Big\}, \end{aligned} \tag{2.4}$$

where $\gamma > 0$, $\sigma > 0$, and $\delta > 0$ are small and will be chosen in the proof of the main theorem. To show that Λ is not empty, we take $u_2 = 0.25$ and $u_3 = 1$; then $u_{m_1} = 0.116204$, $u_{m_2} = 0.717129$, $m_1 = 0.0137409$, and $m_2 = 0.0947595$, and so $u_{m_2} - u3 + m_2 = -0.188112 < 0$, $u_{m_1} - m_1 = 0.102463 > 0$, $f(\frac{1}{2}(u_2 + u_3 - \sqrt{(u_2 - u_3)^2 + 4})) = 0.443 > m_1$, and $-f(\frac{1}{2}(u_2 + \sqrt{u_2^2 + 4})) = 0.132782 > m_2$. Therefore, for small enough $\gamma > 0$, $\sigma > 0$, and $\delta > 0$, Λ is not empty. Let

$$B = \begin{cases} \{\beta : \beta < (u_3 - u_2 + m_1)^2/(m_2^2 - m_1^2)\}, & \text{if } m_2 > m_1; \\ \{\beta : \beta < (m_2 + u_{m_2})^2/(m_1^2 - m_2^2)\}, & \text{if } m_1 > m_2; \\ \text{All real numbers}, & \text{if } m_1 = m_2. \end{cases} \tag{2.5}$$

For example, if we take the same u_2 and u_3 as above, $\{\beta : 0 < \beta < 80\} \subset B$. We will see in the following proof how we obtain the set B.

Now let $\Sigma_1 = \{(u, z, w) : u = u_2,\ 0 < m_1 \leqslant z \leqslant M,\ z - u_2 \leqslant w \leqslant M\}$. We can see that Σ_1 is a compact set; see figure 10.

$$\Omega_1 = \{(u, z, w) : u > u_2,\ m_1 < z < M,\ z - u < w < M\}$$

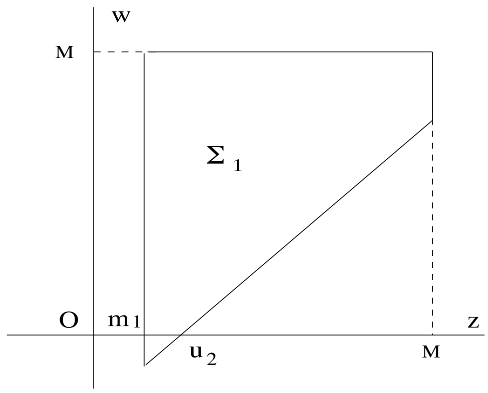

FIGURE 10. The set Σ_1

so that

$$p(t) \in \Omega_1 \Leftrightarrow z'(t) > 0, u(t) > u_2,\ z(t) > m_1, \text{ and } w(t) < M,$$
$$\Sigma_0 = \{(u,z,w) : u > u_2,\ m_1 < z < M,\ w = z - u\}$$

so that

$$p(t) \in \Sigma_0 \Leftrightarrow z'(t) = 0,\ u(t) > u_2,\ m_1 < z(t) < M.$$

Also let

$$\Omega_2 = \{(u,z,w) : u > u_2,\ z < M,\ -M < w < z - u\}$$

so that

$$p(t) \in \Omega_2 \Leftrightarrow z'(t) < 0,\ u(t) > u_2,\ z(t) < M, \text{ and } -M < w(t).$$

Similarly, set

$$\Sigma_2 = \{(u,z,w) : u = u_2,\ z \leqslant -m_2,\ -M \leqslant w \leqslant z - u_2\}$$

so that

$$p(t) \in \Sigma_2 \Leftrightarrow z'(t) < 0,\ u(t) = u_2,\ z(t) < -m_2,$$
$$\Omega_3 = \{(u,z,w) : u < u_2,\ z < -m_2,\ -M < w < z - u\}$$

so that

$$p(t) \in \Omega_3 \Leftrightarrow z'(t) < 0,\ u(t) < u_2,\ z \leqslant -m_2,$$
$$\Sigma_{01} = \{(u,z,w) : u < u_2,\ -M < z < -m_2,\ w = z - u\}$$

so that

$$p(t) \in \Sigma_{01} \Leftrightarrow z'(t) = 0,\ u(t) < u_2,\ z < -m_2.$$

Finally, set

$$\Omega_4 = \{(u,z,w) : u < u_2,\ -M < z,\ z - u < w < M\}$$

so that

$$p(t) \in \Omega_4 \Leftrightarrow z'(t) > 0,\ u(t) < u_2, \text{ and } w(t) < M.$$

2.2. **Properties of the system when $\varepsilon = 0$.** Let $\varepsilon = 0$ in (2.1). We obtain the following differential-algebraic equations (DAE):

$$\begin{cases} 0 = z + f(u) \\ z' = u + w - z \\ w' = -\beta z. \end{cases} \tag{2.6}$$

Since $f(u)$ is a cubic function, solving for u from (2.6a), we obtain three branches:

$$\begin{aligned} u &= u^1(z) \text{ if } u \in (-\infty, u_{m_1}), \\ u &= u^2(z) \text{ if } u \in (u_{m_1}, u_{m_2}), \\ u &= u^3(z) \text{ if } u \in (u_{m_2}, \infty). \end{aligned}$$

We define the following three manifolds corresponding to the three branches:

$$\begin{aligned} M_1 &= \{(u, w, z) : u = u^1(z), z \in (-\infty, m_1)\} \\ M_2 &= \{(u, w, z) : u = u^2(z), z \in (-m_2, m_1)\} \\ M_3 &= \{(u, w, z) : u = u^3(z), z \in (-m_2, \infty)\}. \end{aligned}$$

There are three equilibrium points in the DAE (2.6): $(w, z) = (0, 0)$, $(-u_2, 0)$, and $(-u_3, 0)$. Their stability depends on the choice of the parameters in the system.

1. For $(0, 0) \in M_1$. If $u_2u_3 < 1$, then both eigenvalues have positive real parts, and thus (0,0) is an unstable node. If $u_2u_3 > 1$, then both eigenvalues have negative real parts, and thus (0,0) is a stable node.

2. For $(-u_2, 0) \in M_2$. Since $\frac{1}{u_2^2 - u_2u_3} - 1 < 0$ for all $0 < u_2 < u_3$, both eigenvalues have negative real parts, and thus $(-u_2, 0)$ is a stable node.

3. For $(-u_3, 0) \in M_3$. If $u_3^2 - u_2u_3 < 1$, then both eigenvalues have positive real parts, and thus $(-u_3, 0)$ is an unstable node. If $u_3^2 - u_2u_3 > 1$, then both eigenvalues have negative real parts, and thus (0,0) is a stable node.

We shall prove two propositions:

Proposition 2.2. *If $f(\frac{1}{2}(u_2 + u_3 - \sqrt{(u_2 - u_3)^2 + 4})) > m_1$, then any solution of system* (2.6) *starting from M_1 will reach $z = m_1$ (or $u = u_{m_1}$).*

Proof. We define $E_1(w,z)$ as the energy function

$$E_1 = \frac{(w')^2}{2} + \frac{\beta}{2}w^2, \text{ with } \beta > 0.$$

We can see that E_1 is positive definite.

$$E_1' = z\beta^2(z'-w) = z\beta^2(u-z) = \beta^2(-f(u))(u+f(u)).$$

In order that $E_1' \geqslant 0$, either

$$\begin{cases} u(u-u_2)(u-u_3) \geqslant 0 \\ u(1-(u-u_2)(u-u_3)) \geqslant 0 \end{cases} \tag{2.7}$$

or

$$\begin{cases} u(u-u_2)(u-u_3) \leqslant 0 \\ u(1-(u-u_2)(u-u_3)) \leqslant 0. \end{cases} \tag{2.8}$$

If $1-(u-u_2)(u-u_3) = 0$, i.e., $-u^2+(u_2+u_3)u-u_2u_3+1 = 0$, then $u = \frac{1}{2}((u_2+u_3) \pm \sqrt{(u_2-u_3)^2+4})$. If $0 < u_2 < u_3$, then

$$\frac{1}{2}\left((u_2+u_3)+\sqrt{(u_2-u_3)^2+4}\right) > u_3$$

and

$$\frac{1}{2}\left((u_2+u_3)-\sqrt{(u_2-u_3)^2+4}\right) < u_2.$$

Also

$$\frac{1}{2}\left((u_2+u_3)-\sqrt{(u_2-u_3)^2+4}\right) < 0 \text{ if and only if } u_2u_3 < 1.$$

(2.7) implies

$$0 \leqslant u \leqslant u_2 \text{ or } u_3 \leqslant u \leqslant \frac{1}{2}\left((u_2+u_3)+\sqrt{(u_2-u_3)^2+4}\right),$$

and (2.8) implies

$$\frac{1}{2}\left((u_2+u_3)-\sqrt{(u_2-u_3)^2+4}\right) \leqslant u \leqslant 0.$$

Hence $E_1' > 0$ if and only if

$$\frac{1}{2}\left((u_2+u_3)-\sqrt{(u_2-u_3)^2+4}\right) \leqslant u \leqslant u_2$$

or

$$u_3 \leqslant u \leqslant \frac{1}{2}\left((u_2+u_3)+\sqrt{(u_2-u_3)^2+4}\right).$$

Since $f(\frac{1}{2}(u_2 + u_3 - \sqrt{(u_2 - u_3)^2 + 4})) > m_1$ and $f(u) + z = 0$,

$$u = \frac{1}{2}\left(u_2 + u_3 - \sqrt{(u_2 - u_3)^2 + 4}\right) \text{ implies } z < -m_1.$$

Therefore, $\frac{1}{2}((u_2 + u_3) - \sqrt{(u_2 - u_3)^2 + 4}) \leqslant u \leqslant u_{m_1}$, and so as $-m_1 \leqslant f(u) \leqslant m_1$ or as $|z| < m_1$, $E_1' > 0$.

Hence projections of the solutions on M_1 onto the (w, z) plane starting from outside of the ellipse $\{(w, z) : \frac{w'^2}{2} + \frac{\beta w^2}{2} \leqslant \frac{\beta^2}{2} m_1^2\}$ can not get into it; if they start inside of it, they will get out of it at some finite time. We can see from Figure 11 that the solutions starting from M_1 will get to $z = m_1$. This completes the proof of this proposition. □

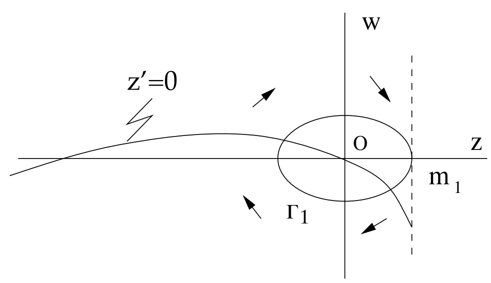

FIGURE 11. Vector field on M_1 projected onto (w, z) plane
$\Gamma_1 = \{(w, z) : \frac{w'^2}{2} + \frac{\beta w^2}{2} = \frac{\beta^2}{2} m_1^2\}$

Proposition 2.3. *If $f(\frac{1}{2}(u_2 + \sqrt{u_2^2 + 4})) < -m_2$, then any solution of system* (2.6) *starting from M_3 will reach $z = -m_2$ (or $u = u_{m_2}$).*

Proof. We define $E_2(w, z)$ as the energy function

$$E_2 = \frac{(w')^2}{2} + \frac{\beta}{2}(w + u_3)^2, \text{ with } \beta > 0.$$

We can see that E_2 is positive definite.

$$E_2' = z\beta^2(z' - w - u_3) = z\beta^2(u - u_3 - z) = \beta^2(-f(u))(u - u_3 + f(u)).$$

In order that $E_2' \geqslant 0$, either

$$\begin{cases} u(u - u_2)(u - u_3) \geqslant 0 \\ (u - u_3)(1 - u(u - u_2)) \geqslant 0 \end{cases} \tag{2.9}$$

or

$$\begin{cases} u(u - u_2)(u - u_3) \leqslant 0 \\ (u - u_3)(1 - u(u - u_2)) \leqslant 0. \end{cases} \tag{2.10}$$

If $1-(u-u_2)u=0$, then

$$u=\frac{1}{2}(u_2\pm\sqrt{u_2^2+4}).$$

Also

$$\frac{1}{2}\big(u_2+\sqrt{u_2^2+4}\big)>u_3 \text{ if and only if } u_3^2-u_2u_3<1.$$

(2.9) implies

$$u_3\leqslant u\leqslant\frac{1}{2}\big(u_2+\sqrt{u_2^2+4}\big),$$

and (2.10) implies

$$\frac{1}{2}\big(u_2-\sqrt{u_2^2+4}\big)\leqslant u\leqslant 0 \text{ or } u_2\leqslant u\leqslant u_3.$$

Hence, $E_2'>0$ if and only if

$$u_2\leqslant u\leqslant\frac{1}{2}\big(u_2+\sqrt{u_2^2+4}\big) \text{ or } \frac{1}{2}\big(u_2-\sqrt{u_2^2+4}\big)\leqslant u\leqslant 0.$$

Since $f(\frac{1}{2}(u_2+\sqrt{u_2^2+4}))<-m_2$ and $z+f(u)=0$, as $u=\frac{1}{2}(u_2+\sqrt{u_2^2+4})$, $z>m_2$. Therefore, $u_{m_2}\leqslant u\leqslant\frac{1}{2}(u_2+\sqrt{u_2^2+4})$, so as $-m_2\leqslant f(u)\leqslant m_2$ or as $|z|<m_2$, $E_2'>0$.

Hence, projections of the solutions on M_3 onto the (w,z) plane starting from outside of the ellipse $\big\{(w,z):\frac{w'^2}{2}+\frac{\beta(w+u_3)^2}{2}\leqslant\frac{\beta^2}{2}m_2^2\big\}$ can not get into it; if they start outside of it, they will get out of it at some finite time. We can see from Figure 12 that the solutions on M_3 will get to $z=-m_2$. This completes the proof of the proposition. □

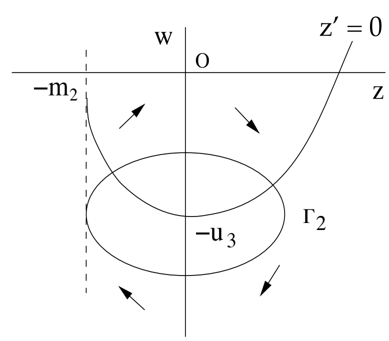

FIGURE 12. Vector field on M_3 projected onto (w,z) plane $\Gamma_2=\{(w,z):\frac{w'^2}{2}+\frac{\beta(w+u_3)^2}{2}=\frac{\beta^2}{2}m_2^2\}$

2.3. The main theorem and its proof.

Theorem 2.2. (Main Theorem) *If* $(u_2, u_3) \in \Lambda$ *and* $\beta \in B$*, where* Λ *and* B *are defined in* (2.4) *and* (2.5)*, then for small enough* $\varepsilon > 0$*, there exists a nonconstant periodic solution in system* (2.1)*.*

Proof. We will show that the flow of system (2.1) defines a Poincaré map from a compact set which is homeomorphic to the unit disk into itself, then apply Brouwer's fixed-point theorem to conclude that system (2.1) has a periodic orbit. The overall setup follows the ideas of [12] (S.P. Hastings, 1977). Here we choose the compact set as Σ_1 defined in Section 2.1. We decompose the proof into the proof of the following lemmas. In each of the following lemmas, we assume $(u_2, u_3) \in \Lambda$, $\beta \in B$, and $p(0) \in \Sigma_1$. Then by Theorem 2.1, $(u(t), w(t), z(t)) \in R$ for all $t \geqslant 0$.

Lemma 2.1. *Let* (u, z, w) *be a solution of system* (2.1)*. If there exists a* $t_0 \geqslant 0$ *such that* $u'(t_0) > 0$ *and* $z'(t_0) > 0$*, then* $u'(t)$ *does not change sign before* z' *does, as* t *increases.*

Proof. Suppose that there is a $\tau > t_0$ such that

$$u'(\tau) = 0,\, u'(t) > 0 \text{ in } [t_0, \tau), z'(t) > 0 \text{ in } [t_0, \tau].$$

Then $u''(\tau) \leqslant 0$. However $u'' = \frac{1}{\varepsilon} z'(\tau) > 0$, a contradiction. This proves Lemma 2.1. □

Remark 2.1. Similar to the above lemma, if there exists a $t_0 \geqslant 0$ such that $u'(t_0) < 0$ and $z'(t_0) < 0$, then $u'(t)$ does not change sign before z' does as t increases.

In Lemmas 2.2–2.21 except Lemma 2.7 and Lemma 2.17, we fix an arbitrary $p(0) \in \Sigma_1$ as our initial value and let $p(t) = (u(t), z(t), w(t))$ be the corresponding solution of system (2.1). Then we have

$$\begin{aligned}
u'(0) &= \frac{1}{\varepsilon}(z(0) + f(u(0)) > 0 \\
z'(0) &= u_2 + w(0) - z(0) = \frac{1}{\varepsilon} z(0) \geqslant 0 \\
w'(0) &= -\beta z(0) \\
u''(0) &= \frac{1}{\varepsilon}(z^{'}(0) + f^{'}(u_2) u^{'}(0)) > 0 \\
z''(0) &= w^{'}(0) + u^{'}(0) - z'(0) = (\frac{1}{\varepsilon} - \beta) z(0) - z^{'}(0) \\
w''(0) &= -\beta z'(0) \leqslant 0.
\end{aligned}$$

Note that the existence of $\tau_1, \tau_2, \ldots$ in the following lemmas depend on the parameters $\varepsilon > 0$, $\beta > 0$, and $(u_2, u_3) \in \Lambda$. We will specify the dependency whenever we need to.

Lemma 2.2. *For small enough $\varepsilon > 0$, there exists an $\eta > 0$ such that $p(t) \in \Omega_1$ in $(0, \eta)$.*

Proof. Since $u'(0) > 0$ and $u(0) = u_2$, we have $u(t) > u_2$ in $(0, \eta)$ for some $\eta > 0$.

Case 1. $z'(0) > 0$. Since $z(0) \geqslant m_1$, $m_1 < z(t) < u(t) + w(t)$ in $(0, \eta)$ for some $\eta > 0$.

Case 2. $z'(0) = 0$. Since $z''(0) = u'(0) + w'(0) = z(0)(\frac{1}{\varepsilon} - \beta)$, for a $\beta > 0$, we can choose ε small enough such that $(\frac{1}{\varepsilon} - \beta) > 0$ and so that $z''(0) > 0$. Then there exists an $\eta > 0$ such that $z'(t) > 0$, and thus $z(t) > m_1$ and $w(t) > z(t) - u(t)$ in $(0, \eta)$. Since $p(0) \in R$, by Theorem 2.1, $|w(t)| \leqslant M$ and $|z(t)| \leqslant M$ for all $t \geqslant 0$.

Hence, we proved that there exists an $\eta > 0$ such that $u(t) > u_2$, $m_1 < z(t) < M$, and $z(t) - u(t) < w(t) < M$ in $(0, \eta)$; i.e., $p(t) \in \Omega_1$ in $(0, \eta)$. This completes the proof of Lemma 2.2. □

Lemma 2.3. *For small enough $\varepsilon > 0$, there exists a $\tau_1 > 0$ such that $p(\tau_1) \in \Sigma_0$, $p(t) \in \Omega_1$ in $(0, \tau_1)$, and $p(t) \in \Omega_2$ in $(\tau_1, \tau_1 + \eta)$ for some $\eta > 0$.*

Proof. By Lemma 2.2, $z'(t) > 0$ and $u(t) > u_2$ in $(0, \eta)$ for some $\eta > 0$, and by Lemma 2.1, $u'(t) > 0$ in $(0, \eta)$; thus, $u(t) > u_2$ as long as $z'(t)$ keeps positive.

Now we show that there exists a $\tau_1 > 0$ such that $z'(\tau_1) = 0$ and $p(t) \in \Omega_1$ in $(0, \tau_1)$. Suppose that $z^{'}(t) > 0$ for all $t > 0$. Then either $z(t) \to \infty$ or $z(t) \to L$, as $t \to \infty$, where $m_1 < L < \infty$.

The first possibility contradicts Theorem 2.1. If the second possibility occurs, then there exists a $T > 0$ such that $z(t) > \frac{L}{2}$ for $t > T$. Then $w' < -\frac{1}{2}\beta L$ for $t > T$, and thus $w \to -\infty$. Again this contradicts Theorem 2.1.

In the following, we show that $z''(\tau_1) < 0$, so that $z(t) \in \Omega_2$ in $(\tau_1, \tau_1 + \eta)$ for some $\eta > 0$.We know that $z''(\tau_1) \leqslant 0$. Now suppose that $z''(\tau_1) = 0$.

We consider the following cases:

Case 1. $z'''(\tau_1) \geqslant 0$, $z'(\tau_1) = z''(\tau_1) = 0$. Since $z(\tau_1) > 0$ and

$$z''(\tau_1) = u'(\tau_1) + w'(\tau_1) = (\frac{1}{\varepsilon} - \beta)z(\tau_1) + \frac{1}{\varepsilon}f(u(\tau_1))$$

we have that for small enough $\varepsilon > 0$, $f(u(\tau_1)) < 0$; i.e., $u(\tau_1) > u_3$. However,

$$z'''(\tau_1) = u'' + w'' + z'' = u''(\tau_1) = \frac{1}{\varepsilon}f'(u(\tau_1))u'(\tau_1) \geqslant 0.$$

By Lemma 2.1, $u'(\tau_1) > 0$, and therefore $f'(u(\tau_1)) \geqslant 0$; this implies that $u_2 \leqslant u(\tau_1) \leqslant u_{m_2}$, a contradiction.

Case 2. $z'''(\tau_1) < 0$, $z'(\tau_1) = z''(\tau_1) = 0$. This implies that $z'(\tau_1) = 0$ is a local maximum, a contradiction. This completes the proof of Lemma 2.3. □

Lemma 2.4. *For small enough $\varepsilon > 0$, $u(\tau_1) > u_3$.*

Proof. By Lemma 2.3, $z'(\tau_1) = 0$, and by the proof of Lemma 2.3,

$$z''(\tau_1) = u'(\tau_1) + w'(\tau_1) = (\frac{1}{\varepsilon} - \beta)z(\tau_1) + \frac{1}{\varepsilon}f(u(\tau_1)) < 0.$$

For small enough ε, $\frac{1}{\varepsilon} - \beta > 0$, and since $z(\tau_1) > m_1 > 0$ and $\frac{1}{\varepsilon} > 0$, it follows that $f(u(\tau_1)) < 0$. Since $u(t) > u_2$ in $(0, \tau_1]$, we have $u(\tau_1) > u_3$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *For small enough $\varepsilon > 0$, there exists a $\tau_2 > \tau_1$ such that $u'(\tau_2) = 0$ and $u''(\tau_2) < 0$, and $p(t) \in \Omega_2$, $z(t) > 0$, and $u'(t) \geqslant 0$ on $(\tau_1, \tau_2]$.*

Before proving the lemma, we give an outline of the proof. First we show that if there exists a τ_2 such that $u'(\tau_2) = 0$ and $u'(t) \geqslant 0$ in (τ_1, τ_2), then $z(t) > 0$ on $[\tau_1, \tau_2]$. Then we show that $z'(t) < 0$ on $(\tau_1, \tau_2]$ so that $p(t) \in \Omega_2$ on $(\tau_1, \tau_2]$. Finally we prove the existence of τ_2.

Proof. If, for the solutions in Lemmas 2.2–2.4, there exists a $\tau_2 > \tau_1$ such that $u'(\tau_2) = 0$ and $u'(t) \geqslant 0$ in (τ_1, τ_2), then by Lemma 2.4 $u(t) > u_3$, and so $f(u(t)) < 0$ in $[\tau_1, \tau_2]$. Also, the fact that $u'(t) = -\frac{b}{a\varepsilon}(z(t) + f(u(t))) \geqslant 0$ on $[\tau_1, \tau_2]$ implies that $z(t) \geqslant -f(u(t)) > 0$ on $[\tau_1, \tau_2]$.

Next we show that $z'(t) < 0$ on $(\tau_1, \tau_2]$. Suppose that there exists a $\tilde{\tau} \in (\tau_1, \tau_2]$ such that $z'(\tilde{\tau}) = 0$ and $z'(t) < 0$ in $(\tau_1, \tilde{\tau})$. Then $z''(\tilde{\tau}) \geqslant 0$ and $p(t) \in \Omega_2$ in $(\tau_1, \tilde{\tau})$.

$$z''(\tilde{\tau}) = u'(\tilde{\tau}) + w'(\tilde{\tau}) = \frac{1}{\varepsilon}(z(\tilde{\tau}) + f(u(\tilde{\tau}))) - \beta z(\tilde{\tau}) \geqslant 0;$$

i.e.,

$$(\frac{1}{\varepsilon} - \beta)z(\tilde{\tau}) \geqslant -\frac{1}{\varepsilon}f(u(\tilde{\tau})) > 0. \tag{2.11}$$

Then $z' < 0$ on $(\tau_1, \tilde{\tau})$ implies that $z(\tau_1) > z(\tilde{\tau}) > 0$. For small enough $\varepsilon > 0$, $\frac{1}{\varepsilon} - \beta > 0$; thus,

$$(\frac{1}{\varepsilon} - \beta)z(\tau_1) > (\frac{1}{\varepsilon} - \beta)z(\tilde{\tau}). \tag{2.12}$$

By Lemma 2.4 and the fact that $u'(t) \geqslant 0$ on $(\tau_1, \tilde{\tau})$, $u(\tilde{\tau}) > u(\tau_1) > u_3$, and thus $f(u(\tilde{\tau})) < f(u(\tau_1)) < 0$; we have

$$-\frac{1}{\varepsilon}f(u(\tilde{\tau})) > -\frac{1}{\varepsilon}f(u(\tau_1)). \tag{2.13}$$

Inequalities (2.11), (2.12), and (2.13) imply that

$$(\frac{1}{\varepsilon} - \beta)z(\tau_1) > -\frac{1}{\varepsilon}f(u(\tau_1));$$

i.e.,

$$z''(\tau_1) = \frac{1}{\varepsilon}(z(\tau_1) + f(u(\tau_1))) - \beta z(\tau_1) > 0,$$

a contradiction, since by the proof of Lemma 2.4, $z''(\tau_1) < 0$. So far we have proved that if there exists a $\tau_2 > \tau_1$ such that $u'(\tau_2) = 0$ and $u'(t) \geqslant 0$ in (τ_1, τ_2), then $p(t) \in \Omega_2$ and $z(t) > 0$ in $(\tau_1, \tau_2]$.

Finally, we prove that there exists a $\tau_2 > \tau_1$ such that $u'(\tau_2) = 0$, $u''(\tau_2) < 0$, and $u'(t) \geqslant 0$ in (τ_1, τ_2). Suppose that $u'(t) \geqslant 0$ for $t \in (\tau_1, \infty)$. Since $u(\tau_1) > u_3$, there are two possibilities: (i) $u \to \infty$ or (ii) $u \to L$ as $t \to \infty$ with $u_3 < L < \infty$.

By Theorem 2.1, $p(t)$ is bounded, and (i) is not possible. For (ii), since $u \to L > u_3$, there exists a $\delta_0 > 0$ and a $T > 0$ such that $u(t) > u_3 + \delta_0$, for $t > T$, and so $-f(u(t)) > -f(u_3 + \delta_0)$ for $t > T$. Since $u'(t) = \frac{1}{\varepsilon}(z(t) + f(u(t))) \geqslant 0$, $z(t) \geqslant -f(u(t)) > -f(u_3 + \delta_0)$, and thus $w'(t) < \beta f(u_3 + \delta_0)$, for $t > T$. This gives $w(t) \to -\infty$ as $t \to \infty$ and contradicts Theorem 2.1. By the above proof, $u''(\tau_2) = \frac{1}{\varepsilon}z'(\tau_2) < 0$. This completes the proof of Lemma 2.5. □

Lemma 2.6. *For small enough $\varepsilon > 0$, $p(t) \in \Omega_2$ and $u'(t) < 0$ for $t > \tau_2$ as long as $z(t) > 0$.*

Proof. By the proof of Lemma 2.5, there exist $\eta > 0$ such that $u^{'}(t) < 0$ and $z'(t) < 0$ in $(\tau_2, \tau_2 + \eta)$. By Lemma 2.1, $u'(t)$ cannot reach zero before $z'(t)$ reaches zero.

Now we show that $z'(t) < 0$ for $t > \tau_2$ as long as $z(t) > 0$. Suppose that there exists a $\tau^* > \tau_2$ such that $z'(\tau^*) = 0$, $z(t) > 0$ in $[\tau_2, \tau^*]$, and $z'(t) < 0$ in $[\tau_2, \tau^*)$, and so $z''(\tau^*) = u'(\tau^*) + w^{'}(\tau^*) \geqslant 0$; i.e., $u'(\tau^*) \geqslant \beta z(\tau^*) > 0$. Then there exists $\tau^{**} \in [\tau_2, \tau^*]$ such that $u'(\tau^{**}) = 0$, a contradiction, since by Lemma 2.1, $u'(t)$ can't reach zero before $z'(t)$ does.

Since $u' < 0$ for $t > \tau_2$ as long as $z(t) > 0$, $u'(t) = \frac{1}{\varepsilon}(z + f(u)) < 0$ implies that $f(u) < -z < 0$, and thus $u > u_3 > u_2$ for $t > \tau_2$ before $z(t)$ reaches zero.

Hence, $p(t) \in \Omega_2$ and $u'(t) < 0$ for $t < \tau_2$ before $z(t)$ reaches zero. This completes the proof of the Lemma 2.6. □

Lemma 2.7 is a general result for the solutions of system (2.1) starting from M_3.

Lemma 2.7. *For any* $\delta > 0$ *and* $(u(t_0), w(t_0), z(t_0)) \in M_3$ *for some* $t_0 \geqslant 0$, *there exists* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$, *then the solution satisfies* $|f(u(t)) + z(t)| < \delta$ *as long as* $t > t_0$ *and* $u(t) \geqslant u_{m_2}$.

Proof. Since $(u(t_0), w(t_0), z(t_0)) \in M_3$,

$$u'(t_0) = \frac{1}{\varepsilon}(f(u(t_0)) + z(t_0)) = 0.$$

Suppose that there exists $\delta_0 > 0$ such that for any $\varepsilon > 0$, there exists $T_\varepsilon > t_0$ such that $|f(u(T_\varepsilon)) + z(T_\varepsilon)| = \delta_0$ with $u(T_\varepsilon) \geqslant u_{m_2}$ and $|f(u(t)) + z(t)| < \delta_0$ for $t_0 < t < T_\varepsilon$. By continuity of the solutions in t, there is a small interval containing T_ε such that $|f(u(t)) + z(t)| > \frac{\delta_0}{2}$. Let $(T_\varepsilon - \eta_\varepsilon^-, T_\varepsilon + \eta_\varepsilon^+)$ be the maximal interval such that $u \geqslant u_{m_2}$ and $|f(u(t)) + z(t)| > \frac{\delta_0}{2}$. Then the change of u in this interval satisfies

$$\begin{aligned} 2M > |\Delta u| &= |u(T_\varepsilon + \eta_\varepsilon^+) - u(T_\varepsilon - \eta_\varepsilon^-)| \\ &= \int_{T_\varepsilon - \eta_\varepsilon^-}^{T_\varepsilon + \eta_\varepsilon^+} \frac{1}{\varepsilon}(f(u(s)) + z(s))\, ds \geqslant \frac{\delta_0}{2\varepsilon}(\eta_\varepsilon^+ + \eta_\varepsilon^-). \end{aligned}$$

Since u is bounded, by Theorem 2.1, Δu is bounded.

Hence, $\eta_\varepsilon^+ + \eta_\varepsilon^- \to 0$; i.e., $\eta_\varepsilon^+ \to 0$ and $\eta_\varepsilon^- \to 0$ as $\varepsilon \to 0$. Since $z'(t) = u(t) + w(t) - z(t)$ is bounded, by Theorem 2.1,

$$|z(T_\varepsilon) - z(T_\varepsilon - \eta_\varepsilon^-)| \longrightarrow 0 \text{ as } \varepsilon \to 0; \text{ i.e., } \eta_\varepsilon^- \to 0.$$

We will show that this is impossible.

Case 1. $u'(T_\varepsilon) < 0$; i.e., $f(u(T_\varepsilon)) + z(T_\varepsilon) = -\delta_0$. From the definition of η_ε^-, we have

$$f(u(T_\varepsilon - \eta_\varepsilon^-)) + z(T_\varepsilon - \eta_\varepsilon^-) = -\frac{\delta_0}{2}.$$

Since

$$u'(t) < 0 \text{ in } (T_\varepsilon - \eta_\varepsilon^-, T_\varepsilon) \text{ and } u(T_\varepsilon) \geqslant u_{m_2},$$

we have

$$u(T_\varepsilon - \eta_\varepsilon^-) > u(T_\varepsilon) \geqslant u_{m_2},$$

and since $f'(u) < 0$ in (u_{m_2}, ∞),

$$f(u(T_\varepsilon - \eta_\varepsilon^-)) < f(u(T_\varepsilon)).$$

Hence,

$$f(u(T_\varepsilon - \eta_\varepsilon^-)) + z(T_\varepsilon - \eta_\varepsilon^-) - (f(u(T_\varepsilon)) + z(T_\varepsilon)) = -\frac{\delta_0}{2} - (-\delta_0) = \frac{\delta_0}{2}$$

$$z(T_\varepsilon - \eta_\varepsilon^-) - z(T_\varepsilon) = \frac{\delta_0}{2} + f(u(T_\varepsilon)) - f(u(T_\varepsilon - \eta_\varepsilon^-) > \frac{\delta_0}{2}.$$

This contradicts $z(T_\varepsilon - \eta_\varepsilon^-) - z(T_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Case 2. $u'(T_\varepsilon) > 0$. The proof is similar to that of Case 1. This completes the proof of Lemma 2.7. □

Before we get into Lemma 2.8, we define an energy function:

$$H_2(z, w) = \frac{\beta^2}{2} z^2 + \frac{\beta}{2}(w + u_3)^2.$$

Now we want to choose $\beta > 0$ such that the intersection of the projection of Σ_1 onto the (w, z) plane with D_2 is the empty set, where

$$D_2 = \{(w, z) : \frac{\beta^2}{2} z^2 + \frac{\beta}{2}(w + u_3)^2 \leqslant \frac{\beta^2}{2} m_2^2\}.$$

On the (w, z) plane, let Q_1 be the point where $z = m_1$ and $z = w + u_2$ intersect and Q_2 be the point where $z = m_1$ and $\frac{\beta^2}{2} z^2 + \frac{\beta}{2}(w + u_3)^2 = \frac{\beta^2}{2} m_2^2$ intersect. We want to choose $\beta > 0$ such that Q_1 is above Q_2. See Figure 13.

If $m_1 > m_2$, then Q_2 does not exist. We can also say that Q_1 is above Q_2 for any β. Therefore there is no restriction on β in this case.

If $m_1 \leqslant m_2$, then choose β so small that $m_1 - u_2 \geqslant \sqrt{\beta(m_2^2 - m_1^2)} - u_3$ or $\beta \leqslant (u_3 + m_1 - u_2)^2/(m_2^2 - m_1^2)$.

For example, when $u_2 = 0.25$ and $u_3 = 1$, $(u_3 + m_1 - u_2)^2/(m_2^2 - m_1^2) > 80$.

Let $B_2 = \{\beta : \beta < (u_3 + m_1 - u_2)^2/(m_2^2 - m_1^2)\}$. We can see from Figure 13 that if $m_2 > m_1$, then as long as $\beta \in B_2$, the projections of the solutions starting from Σ_1 onto the (w, z) plane start outside of region D_2.

Lemma 2.8. *Let $(u_2, u_3) \in \Lambda$ and $\beta \in B$. Then for small enough $\varepsilon > 0$, $H_2' > 0$ as long as $|z(t)| < m_2$, and therefore the projection of the solution onto the (w, z) plane does not enter the region D_2.*

Proof. $H_2' = \beta^2 z z' + \beta w'(w + u_3) = \beta^2 z(u - u_3 - z)$. Let $f(u(t)) + z(t) = \eta(t)$, so that $z(t) = \eta(t) - f(u(t))$, and so

$$H_2' = \beta^2(\eta(t) - f(u(t)))(u - u_3 - \eta(t) + f(u(t))).$$

By the proof of Proposition 2.3,

$$E_2'(u) = \beta^2(-f(u))(u - u_3 + f(u)) > 0$$

as long as $u_2 < u < \frac{1}{2}(u_2 + \sqrt{u_2^2 + 4})$.

By Lemma 2.7 for any $\delta > 0$, there exists an ε_0 such that if $\varepsilon < \varepsilon_0$, then $|\eta(t)| < \delta$ for $t < \tau_2$ as long as $u(t) \geqslant u_{m_2}$.

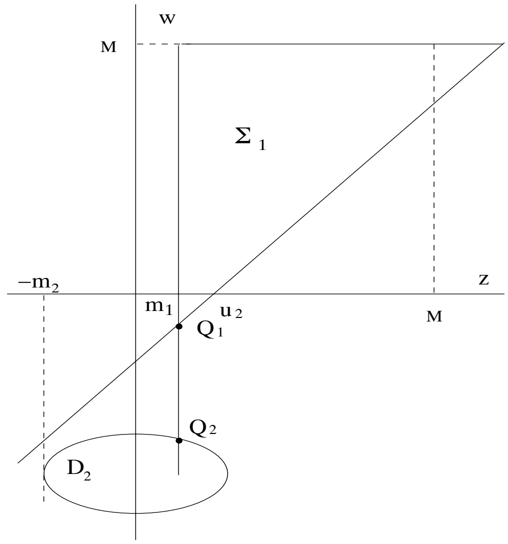

FIGURE 13. Positions of Q_1 and Q_2

Comparing H_2' and $E_2'(u)$, there exist small $\gamma(\delta)$ and $\gamma'(\delta)$ such that $H_2' > 0$ as long as

$$u_2 + \gamma' < u < \frac{1}{2}\big(u_2 + \sqrt{u_2^2 + 4}\big) - \gamma.$$

By Lemma 2.7 again, when $u(t) \geqslant u_{m_2}$, we have $|\eta(t)| < \delta$. Therefore, $H_2' > 0$ as long as

$$u_{m_2} \leqslant u \leqslant \frac{1}{2}\big(u_2 + \sqrt{u_2^2 + 4}\big) - \gamma.$$

By continuity of the solutions in t, $\gamma \to 0$ as $\delta \to 0$. Hence, $H_2' > 0$ as long as

$$u \geqslant u_{m_2} \text{ and } f(u) \geqslant f\Big(\frac{1}{2}\big(u_2 + \sqrt{u_2^2 + 4}\big) - \gamma\Big),$$

or equivalently $H_2' > 0$ as long as

$$u \geqslant u_{m_2} \text{ and } z(t) = \eta(t) - f(u(t)) \leqslant \eta(t) - f\Big(\frac{1}{2}\big(u_2 + \sqrt{u_2^2 + 4}\big) - \gamma\Big),$$

or equivalently $H_2' > 0$ as long as

$$u \geqslant u_{m_2} \text{ and } -f\Big(\frac{1}{2}\big(u_2 + \sqrt{u_2^2 + 4}\big) - \gamma\Big) > z(t) + \delta.$$

We showed earlier in an example after the definition of Λ that there exist u_2 and u_3 such that

$$-f\big(\frac{1}{2}(u_2+\sqrt{u_2^2+4})\big) > m_2.$$

Therefore, we can choose δ^* small enough and γ^* small enough so that there exist u_2 and u_3 such that

$$-f\big(\frac{1}{2}(u_2+\sqrt{u_2^2+4}) - \gamma^*\big) > m_2 + \delta^*.$$

For the δ^*, γ^*, u_2, and u_3 chosen, we consider two cases:

Case 1. $\gamma(\delta^*) \leqslant \gamma^*$. Then $H_2' > 0$ as long as $u \geqslant u_{m_2}$ and $z(t) < m_2$ because

$$m_2 < -\delta^* - f\big(\frac{1}{2}(u_2+\sqrt{u_2^2+4}) - \gamma^*\big) \leqslant -\delta^* - f\big(\frac{1}{2}(u_2+\sqrt{u_2^2+4}) - \gamma(\delta^*)\big).$$

Case 2. $\gamma(\delta^*) > \gamma^*$. Then for the fixed u_2 and u_3, we choose smaller $\delta^{**} < \delta^*$ such that

$$m_2 < -\delta^{**} - f\big(\frac{1}{2}(u_2+\sqrt{u_2^2+4}) - \gamma(\delta^{**})\big).$$

This is always possible, because $\gamma(\delta^{**}) \to 0$ as $\delta^{**} \to 0$.

Therefore, for small enough $\varepsilon > 0$, $|\eta(t)| < \delta^{**}$, and so when $u \geqslant u_{m_2}$ and $z < m_2$, $H_2' > 0$.

Next we show that if $z(t) > -m_2$, then $u(t) \geqslant u_{m_2}$ so that $H_2' > 0$ as long as $-m_2 < z(t) < m_2$. Suppose that there exists a $\tau > \tau_2$ such that $z(t) > -m_2$ in $[\tau_2, \tau]$ and $u(\tau) \leqslant u_{m_2}$. Then there exists a $\tau^* \in (\tau_2, \tau]$ such that $u(\tau^*) = u_{m_2}$ and $u'(\tau^*) \leqslant 0$; i.e., $z(\tau^*) + f(u(\tau^*)) = z(\tau^*) + m_2 \leqslant 0$ or $z(\tau^*) \leqslant -m_2$, a contradiction.

Since the projection of the solution onto the (w, z) plane starts outside of region D_2, where $-m_2 < z(t) < m_2$, it will not enter the region D_2. This completes the proof of Lemma 2.8. □

Lemma 2.9. *If we choose parameters as in Lemma* 2.8*, then for small enough* $\varepsilon > 0$*, there exists a* $\tau_3 > \tau_2$ *such that* $z(\tau_3) = 0$ *and* $p(t) \in \Omega_2$ *in* $[\tau_2, \tau_3]$.

Proof. We first show that there exists a $\tau_3 > \tau_2$ such that $z(\tau_3) = 0$, and then we show that $p(t) \in \Omega_2$ in $[\tau_2, \tau_3]$; i.e., $z'(t) < 0$ and $u(t) > u_2$ in $[\tau_2, \tau_3]$.

By Lemma 2.6, either there exists a $\tau_3 > \tau_2$ such that $z(\tau_3) = 0$ or $z(t) \to 0$ as $t \to \infty$. In the latter case, since $z''(t)$ is bounded for any fixed $\varepsilon > 0$, $z'(t) = u + w - z \to 0$; i.e., $u + w \to 0$ as $t \to \infty$. But by Lemma 2.1 and Lemma 2.6, $u' < 0$ for $t > \tau_2$; i.e., $f(u(t)) < -z(t) < 0$, for $t > \tau_2$, and so $u(t) \geqslant u_3$, for $t > \tau_2$. Thus, $u(t)$ is monotonically decreasing for $t > \tau_2$ and

$\lim_{t\to\infty} u(t) \geqslant u_3$, and thus $\lim_{t\to\infty} w(t) \leqslant -u_3$, because $P_3 = (u_3, 0, -u_3)$ is the only equilibrium with $u \geqslant u_3$. We then have $p(t) \to (u_3, 0, -u_3)$ as $t \to \infty$, and thus the projection onto the (w, z) plane of the solution has to enter the region D_2, contradicting Lemma 2.8. Hence, only the first case happens.

Next we show that $z'(\tau_3) < 0$ and $u'(\tau_3) < 0$. By Lemma 2.6, $u'(\tau_3) \leqslant 0$ and $z'(\tau_3) \leqslant 0$. We consider three cases:

Case 1. $u'(\tau_3) < 0$ and $z'(\tau_3) = 0$. So $z''(\tau_3) \geqslant 0$. However, $z''(\tau_3) = u'(\tau_3) < 0$, a contradiction.

Case 2. $u'(\tau_3) = 0$ and $z'(\tau_3) < 0$. So $u''(\tau_3) \geqslant 0$. However, $u''(\tau_3) = \frac{1}{\varepsilon} z'(\tau_3) < 0$, a contradiction.

Case 3. $u'(\tau_3) = 0$ and $z'(\tau_3) = 0$. $u'(\tau_3) = \frac{1}{\varepsilon}(z(\tau_3) + f(u(\tau_3)) = 0$ and $u(\tau_3) \geqslant u_3$, which gives $u(\tau_3) = u_3$, and thus $w(\tau_3) = -u_3$. However, by uniqueness of initial-value problems, this is not possible. Lemma 2.6 also implies that $u(t) \geqslant u_3 > u_2$ in $[\tau_2, \tau_3]$. Hence, $p(t) \in \Omega_2$ in $[\tau_2, \tau_3]$. This completes the proof of Lemma 2.9. □

Lemma 2.10. *If we choose the parameters as in Lemma* 2.8*, then for* $\varepsilon > 0$ *small enough, there exists a* $\tau_4 > \tau_3$ *such that* $z(\tau_4) = -m_2$ *and* $p(t) \in \Omega_2$ *in* $[\tau_3, \tau_4]$.

Proof. Suppose that there exists a $\tau > \tau_3$ such that $z'(\tau) = 0$ and $-m_2 \leqslant z(\tau) < 0$. Then Lemma 2.8 implies that

$$H_2'(\tau) = \beta^2 z z' - \beta^2 (w + u_3) z = -\beta^2 (w(\tau) + u_3) z(\tau) \geqslant 0,$$

and thus $w(\tau) \geqslant -u_3$.

Since the projection onto the (w, z) plane of the solution which starts from Σ_1 is outside of the region D_2, by Lemma 2.8, it can't get into the region D_2. Then Lemma 9 implies that $w(\tau) < -u_3$, a contradiction.

Hence, $z'(t) < 0$ for $t \geqslant \tau_3$ with $z(t) \geqslant -m_2$, and therefore there exists $\tau_4 > \tau_3$ such that $z(\tau_4) = -m_2$ and $z'(t) < 0$ in $[\tau_3, \tau_4]$. By the proof of Lemma 2.8, $u(t) \geqslant u_{m_2} > u_2$ in $[\tau_3, \tau_4]$. Thus $p(t) \in \Omega_2$ in $[\tau_3, \tau_4]$, and this completes the proof of Lemma 2.10. □

Lemma 2.11. *If we choose the parameters as in Lemma* 2.8*, then for small enough* $\varepsilon > 0$*, there exists a* $\tau_5 > \tau_4$ *such that* $p(\tau_5) \in \Sigma_2$*,* $p(t) \in \Omega_2$ *in* $[\tau_4, \tau_5]$.

Proof. First we show that if we choose u_2 and u_3 properly, $z'(\tau_4) < -\Delta$ for some $\Delta > 0$ independent of ε, $z'(\tau_4) = u(\tau_4) + w(\tau_4) - z(\tau_4)$. By Lemma 2.10 $z(\tau_4) = -m_2$, by Lemma 2.8 $w(\tau_4) \leqslant -u_3$, and by Lemma 2.7, for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, $|f(u(t)) + z(t)| < \delta$ as $t > \tau_2$ with $u(t) \geqslant u_{m_2}$, and

$$|f(u(\tau_4)) - m_2| < \delta$$

$$\Longleftrightarrow m_2 - \delta < f(u(\tau_4)) \leqslant m_2 \Longleftrightarrow u_{m_2} \leqslant u(\tau_4) < f^{-1}(m_2 - \delta).$$

We can choose u_2 and u_3 such that $u_{m_2} - u_3 + m_2 < 0$ and δ small for small enough $\varepsilon > 0$, we have that

$$z'(\tau_4) < f^{-1}(m_2 - \delta) - u_3 + m_2 < \frac{1}{2}(u_{m_2} - u_3 + m_2) = -\Delta < 0.$$

Next we show that $z'(t) < 0$ for $t \geqslant \tau_4$ with $u \geqslant u_{m_2}$. Suppose that there exists $\tau \geqslant \tau_4$ such that $z'(\tau) = 0$, and $u(t) \geqslant u_{m_2}$ in $[\tau_4, \tau]$; then by Lemma 2.1, $u'(t) < 0$ in $[\tau_4, \tau]$. Thus by Theorem 2.1, there exists a $K > 0$ such that

$$z''(t) = u'(t) + w'(t) - z'(t) < w' - z' < K \text{ in } [\tau_4, \tau]$$

$$|z(\tau_4) - z(\tau)| = |\int_{\tau_4}^{\tau} z'(t)\,dt| > \frac{1}{2K}(z'(\tau_4))^2.$$

On the other hand, if $\varepsilon > 0$ is small enough, because $u(t) \geqslant u_{m_2}$ in $[\tau_4, \tau]$,

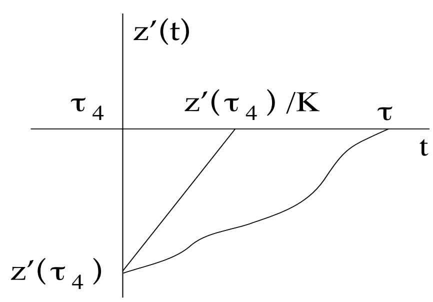

FIGURE 14. $z'(t)$ versus t around τ_4

then

$$|f(u(\tau)) + z(\tau)| < \delta \Longleftrightarrow -f(u(\tau)) - \delta < z(\tau) < -f(u(\tau)) + \delta$$

adding $-m_2$ to each of the three parts of the above inequality, we obtain

$$-m_2 + f(u(\tau)) - \delta < -m_2 - z(\tau) < -m_2 + f(u(\tau)) + \delta.$$

Since

$$|f(u(\tau_4)) - m_2| < \delta \Longleftrightarrow -\delta < f(u(\tau_4)) - m_2 < \delta$$

and $u'(t) < 0$ in $[\tau_4, \tau]$, these imply that $f(u(\tau)) > f(u(\tau_4))$ and $f(u(\tau)) < m_2$. Thus we have $-2\delta < -m_2 - z(\tau) < 2\delta$, i.e., $-2\delta < z(\tau_4) - z(\tau) < 2\delta$, and so $|z(\tau_4) - z(\tau)| < 2\delta$, a contradiction.

Now suppose that there exists $\tau > \tau_4$ such that $z'(\tau) = 0$ with $u_2 < u(\tau) < u_{m_2}$ and $z'(t) < 0$ in $[\tau_4, \tau)$, and thus $z''(\tau) \geqslant 0$ and $u_2 \leqslant u(\tau) < u_{m_2}$.

By the above proof,

$$z(\tau) < -m_2 - \frac{1}{2K}(z'(\tau_4))^2 \text{ and } f(u) < m_2;$$

thus,

$$\begin{aligned} u'(\tau) = \frac{1}{\varepsilon}(z + f(u)) &< \frac{1}{\varepsilon}(-m_2 - \frac{1}{2K}(z'(\tau_4))^2 + m_2) \\ &= -\frac{1}{2\varepsilon K}(z'(\tau_4))^2 \to -\infty, \text{ as } \varepsilon \to 0. \end{aligned}$$

By Theorem 2.1, z' and w' are bounded in $[0, \infty)$, and for ε small enough

$$z''(\tau) = u'(\tau) + w'(\tau) - z'(\tau) < 0,$$

a contradiction. Thus there exists $\tau_5 > \tau_4$ such that $p(t) \in \Sigma_2$ and $p(t) \in \Omega_2$ in $[\tau_4, \tau_5]$. This completes the proof of Lemma 2.11. □

Lemma 2.12. *If we choose the parameters as in Lemma* 2.11*, then for small enough $\varepsilon > 0$, there exists an $\eta > 0$ such that $p(t) \in \Omega_3$ on $(\tau_5, \tau_5 + \eta)$.*

Proof. By Lemma 2.11, $z'(\tau_5) < 0$, $z(\tau_5) \leqslant -m_2$, $u'(\tau_5) < 0$, and $u(\tau_5) = u_2$, we have $u(t) < u_2$ in $(\tau_5, \tau_5 + \eta)$ and $-m_2 > z(t) > u(t) + w(t)$ in $(\tau_5, \tau_5 + \eta)$ for some $\eta > 0$. Since $p(\tau_5) \in R$, $|w(\tau_5)| \leqslant M$; by Theorem 2.1 $w(t) > -M$ for all $t \geqslant \tau_5$. Hence we proved that there exists an $\eta > 0$ such that $u(t) < u_2$, $-M < z(t) < -m_2$, and $-M < w(t) < z(t) - u(t)$ in $(\tau_5, \tau_5 + \eta)$; i.e., $p(t) \in \Omega_3$ in $(\tau_5, \tau_5 + \eta)$ for some $\eta > 0$. This completes the proof of Lemma 2.12. □

Lemma 2.13. *Let the parameters be as in Lemma* 2.11*. Then for small enough $\varepsilon > 0$, there exists a $\tau_6 > \tau_5$ such that $p(\tau_6) \in \Sigma_{01}$, and $p(t) \in \Omega_4$ in $(\tau_6, \tau_6 + \eta)$ for some $\eta > 0$.*

Proof. By Lemma 2.12, $z'(t) < 0$, and $u(t) < u_2$ in $(\tau_5, \tau_5 + \eta)$ for some $\eta > 0$, and by Remark 2.1, $u'(t) < 0$ in $(\tau_5, \tau_5 + \eta)$; thus, $u(t) < u_2$ as long as $z'(t)$ keeps negative.

Now we show that there exists a $\tau_6 > 0$ such that $z'(\tau_6) = 0$ and $p(t) \in \Omega_3$ in (τ_5, τ_6). Suppose that $z'(t) < 0$ for all $t > \tau_5$. Then either $z(t) \to -\infty$ or $z(t) \to L$, as $t \to \infty$, where $L < -m_2$.

The first possibility contradicts Theorem 2.1. If the second possibility occurs, then there exists $T > 0$ such that $z(t) < \frac{L}{2}$ for $t > T$. Then $w' > -\frac{1}{2}\beta L$ for $t > T$, and thus $w \to \infty$. Again this contradicts Theorem 2.1.

In the following, we show that $z''(\tau_6) > 0$, so that $z(t) \in \Omega_4$ in $(\tau_6, \tau_6 + \eta)$ for some $\eta > 0$.We know that $z''(\tau_6) \geqslant 0$. Now suppose that $z''(\tau_6) = 0$.

Case 1. $z'''(\tau_6) \leqslant 0$ and $z'(\tau_6) = z''(\tau_6) = 0$. Then

$$z''(\tau_6) = u'(\tau_6) + w'(\tau_6) = (\frac{1}{\varepsilon} - \beta)z(\tau_6) + \frac{1}{\varepsilon}f(u(\tau_6))$$

Thus for small enough $\varepsilon > 0$, $f(u(\tau_6)) > 0$; i.e., $u(\tau_6) < 0$. However,

$$z'''(\tau_6) = u'' + w'' + z'' = u''(\tau_6) = \frac{1}{\varepsilon}f'(u(\tau_6))u'(\tau_6) \leqslant 0.$$

By Remark 2.1, $u'(\tau_6) < 0$, and therefore $f'(u(\tau_6)) \geqslant 0$. This implies that $u_2 \leqslant u \leqslant u_{m_2}$, a contradiction.

Case 2. $z'''(\tau_6) > 0$ and $z'(\tau_6) = z''(\tau_6) = 0$. This implies that $z'(\tau_6) = 0$ is a local minimum, a contradiction. This completes the proof of Lemma 2.13. □

Lemma 2.14. *Let the parameters be as in Lemma* 2.11*. Then for small enough* $\varepsilon > 0$ $u(\tau_6) < 0$.

Proof. By Lemma 2.13 $z'(\tau_6) = 0$, and by the proof of Lemma 2.13

$$z''(\tau_6) = u'(\tau_6) + w'(\tau_6) = (\frac{1}{\varepsilon} - \beta)z(\tau_6) + \frac{1}{\varepsilon}f(u(\tau_6)) > 0.$$

For small enough ε, $\frac{1}{\varepsilon} - \beta > 0$, and since $z(\tau_6) < -m_2 < 0$, it follows that $f(u(\tau_6)) > 0$. Since $u(t) < u_2$ in $(\tau_5, \tau_6]$, we have $u(\tau_6) < 0$. This completes the proof of Lemma 2.14. □

Lemma 2.15. *Let the parameters be as in Lemma* 2.11*. Then for small enough* $\varepsilon > 0$*, there exists a* $\tau_7 > \tau_6$ *such that* $u'(\tau_7) = 0$ *and* $u''(\tau_7) > 0$*, and* $p(t) \in \Omega_4$*,* $z(t) < 0$*, and* $u'(t) \leqslant 0$ *on* $(\tau_6, \tau_7]$.

Before proving the lemma, we give an outline of the proof. First we show that if there exists a τ_7 such that $u'(\tau_7) = 0$ and $u'(t) \leqslant 0$ in (τ_6, τ_7), then $z(t) < 0$ on $[\tau_6, \tau_7]$. Then we show that $z'(t) > 0$ on $(\tau_6, \tau_7]$ so that $p(t) \in \Omega_4$ on $(\tau_6, \tau_7]$. Finally we prove the existence of τ_7.

Proof. If, for the solutions in Lemmas 2.2–2.14, there exists $\tau_7 > \tau_6$ such that $u'(\tau_7) = 0$ and $u'(t) \leqslant 0$ in (τ_6, τ_7), then by Lemma 2.14 $u(t) < 0$, and so $f(u(t)) > 0$ in $[\tau_6, \tau_7]$. Also $u^{'}(t) = \frac{1}{\varepsilon}(z(t) + f(u(t))) \leqslant 0$ on $[\tau_6, \tau_7]$ implies that $-z(t) \geqslant f(u(t)) > 0$ on $[\tau_6, \tau_7]$.

Next we show that $z'(t) > 0$ on $(\tau_6, \tau_7]$. Suppose that there exist $\tilde{\tau} \in (\tau_6, \tau_7]$ such that $z'(\tilde{\tau}) = 0$ and $z'(t) > 0$ in $(\tau_6, \tilde{\tau})$. Then $z''(\tilde{\tau}) \leqslant 0$ and $p(t) \in \Omega_4$ in $(\tau_6, \tilde{\tau})$.

$$z''(\tilde{\tau}) = u'(\tilde{\tau}) + w'(\tilde{\tau}) = \frac{1}{\varepsilon}(z(\tilde{\tau}) + f(u(\tilde{\tau}))) - \beta z(\tilde{\tau}) \leqslant 0;$$

i.e.,

$$(\frac{1}{\varepsilon} - \beta)z(\tilde{\tau}) \leqslant -\frac{1}{\varepsilon}f(u(\tilde{\tau})) < 0. \tag{2.14}$$

Then $z' > 0$ on $(\tau_6, \tilde{\tau})$ implies that $z(\tau_6) < z(\tilde{\tau}) < 0$. For small enough $\varepsilon > 0$, $\frac{1}{\varepsilon} - \beta > 0$; thus,

$$(\frac{1}{\varepsilon} - \beta)z(\tau_6) < (\frac{1}{\varepsilon} - \beta)z(\tilde{\tau}). \tag{2.15}$$

By Lemma 2.14 and the fact that $u^{'}(t) \leqslant 0$ on $(\tau_6, \tilde{\tau})$, $u(\tilde{\tau}) < u(\tau_6) < 0$ and thus $f(u(\tilde{\tau})) > f(u(\tau_6)) > 0$; we have

$$-\frac{1}{\varepsilon}f(u(\tilde{\tau})) < -\frac{1}{\varepsilon}f(u(\tau_6)). \tag{2.16}$$

Inequalities (2.14), (2.15), and (2.16) imply that

$$(\frac{1}{\varepsilon} - \beta)z(\tau_6) < -\frac{1}{\varepsilon}f(u(\tau_6));$$

i.e.,

$$z^{''}(\tau_6) = \frac{1}{\varepsilon}(z(\tau_6) + f(u(\tau_6))) - \beta z(\tau_6) < 0,$$

a contradiction, since by the proof of Lemma 2.14, $z''(\tau_6) > 0$. So far we have proved that if there exists a $\tau_7 > \tau_6$ such that $u'(\tau_7) = 0$ and $u'(t) \geqslant 0$ in (τ_6, τ_7), then $p(t) \in \Omega_4$ and $z(t) < 0$ in $(\tau_6, \tau_7]$.

Finally, we prove that there exists a $\tau_7 > \tau_6$ such that $u'(\tau_7) = 0$, $u''(\tau_7) > 0$, and $u'(t) \leqslant 0$ in (τ_6, τ_7). Suppose that $u'(t) \leqslant 0$ for $t \in (\tau_6, \infty)$. Since $u(\tau_6) < 0$, there are two possibilities:

(i) $u \to -\infty$ or (ii) $u \to L$ as $t \to \infty$ with $-\infty < L < 0$.

By Theorem 2.1, $p(t)$ is bounded, and (i) is not possible. For (ii), since $u \to L < 0$, there exist $\delta_0 > 0$ and $T > 0$ such that $u(t) < -\delta_0$ for $t > T$, and so $-f(u(t)) < -f(-\delta_0)$ for $t > T$. Since $u'(t) = -\frac{1}{\varepsilon}(z(t) + f(u(t))) \leqslant 0$, $z(t) \leqslant -f(u(t)) < -f(-\delta_0)$, and thus $w'(t) > \beta f(-\delta_0)$, for $t > T$. This gives $w(t) \to \infty$ as $t \to \infty$ and contradicts Theorem 2.1. By the above proof, $u''(\tau_7) = \frac{1}{\varepsilon}z'(\tau_7) > 0$. This completes the proof of Lemma 2.15. □

Lemma 2.16. *Let the parameters be as in Lemma 2.11. Then for small enough $\varepsilon > 0$, $p(t) \in \Omega_4$ and $u'(t) > 0$ for $t > \tau_7$ as long as $z(t) < 0$.*

Proof. By the proof of Lemma 2.15, there exists $\eta > 0$ such that $u'(t) > 0$ and $z'(t) > 0$ in $(\tau_7, \tau_7 + \eta)$. By Remark 2.1, $u'(t)$ cannot reach zero before $z'(t)$ reaches zero.

Now we show that $z'(t) > 0$ for $t > \tau_7$ as long as $z(t) < 0$. Suppose that there exists a $\tau^* > \tau_7$ such that $z'(\tau^*) = 0$, $z(t) < 0$ in $[\tau_7, \tau^*]$, and $z'(t) > 0$ in $[\tau_7, \tau^*)$, and so $z''(\tau^*) = u'(\tau^*) + w'(\tau^*) \leqslant 0$; i.e., $u'(\tau^*) \leqslant \beta z(\tau^*) < 0$. Then there exists $\tau^{**} \in (\tau_7, \tau^*)$ such that $u'(\tau^{**}) = 0$, a contradiction, because $u'(t)$ cannot reach zero before $z'(t)$ does.

Since $u' > 0$ for $t > \tau_7$ as long as $z(t) < 0$, $u'(t) = \frac{1}{\varepsilon}(z + f(u)) > 0$ implies that $f(u) > -z > 0$, and thus $u < 0 < u_2$ for $t > \tau_7$ before $z(t)$ reaches zero.

Hence $p(t) \in \Omega_4$ and $u'(t) > 0$ for $t < \tau_7$ before $z(t)$ reaches zero. This completes the proof of Lemma 2.16. □

Lemma 2.17 is a general result for all the solutions starting from M_1.

Lemma 2.17. *For any $\delta > 0$ and $(u(t_0), w(t_0), z(t_0)) \in M_1$ for some $t_0 \geqslant 0$, there exists an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $|f(u(t)) + z(t)| < \delta$ as long as $t > t_0$ and $u(t) \leqslant u_{m_1}$.*

Proof. Since $(u(t_0), w(t_0), z(t_0)) \in M_1$,

$$u'(t_0) = \frac{1}{\varepsilon}(f(u(t_0)) + z(t_0)) = 0.$$

Suppose that there exists $\delta_0 > 0$ such that for any $\varepsilon > 0$ there exists $T_\varepsilon > t_0$, such that $|f(u(T_\varepsilon)) + z(T_\varepsilon)| = \delta_0$ with $u(T_\varepsilon) \leqslant u_{m_1}$ and $|f(u(t)) + z(t)| < \delta_0$ for $t_0 < t < T_\varepsilon$. By continuity of the solutions in t, there is a small interval containing T_ε such that $|f(u(t)) + z(t)| > \frac{\delta_0}{2}$.

Let $(T_\varepsilon - \eta_\varepsilon^-, T_\varepsilon + \eta_\varepsilon^+)$ be the maximal interval such that $u \leqslant u_{m_1}$ and $|f(u(t)) + z(t)| > \frac{\delta_0}{2}$. Then the change of u in the interval satisfies

$$|\Delta u| = |u(T_\varepsilon + \eta_\varepsilon^+) - u(T_\varepsilon - \eta_\varepsilon^-)| \geqslant \frac{\delta_0}{2\varepsilon}(\eta_\varepsilon^+ + \eta_\varepsilon^-).$$

Since u is bounded, by Theorem 2.1, Δu is bounded. Hence $\eta_\varepsilon^+ + \eta_\varepsilon^- \to 0$; i.e., $\eta_\varepsilon^+ \to 0$ and $\eta_\varepsilon^- \to 0$ as $\varepsilon \to 0$.

Since $z'(t) = u(t) + w(t) - z(t)$ is bounded, by Theorem 2.1,

$$|z(T_\varepsilon) - z(T_\varepsilon - \eta_\varepsilon^-)| \longrightarrow 0 \text{ as } \varepsilon \to 0; \text{ i.e., } \eta_\varepsilon^- \to 0.$$

We will show that this is impossible.

Case 1. $u'(T_\varepsilon) < 0$; i.e., $f(u(T_\varepsilon)) + z(T_\varepsilon) = -\delta_0$. From the definition of η_ε^- we have

$$f(u(T_\varepsilon - \eta_\varepsilon^-)) + z(T_\varepsilon - \eta_\varepsilon^-) = -\frac{\delta_0}{2}.$$

Since $u'(t) > 0$ in $(T_\varepsilon - \eta_\varepsilon^-, T_\varepsilon)$ and $u(T_\varepsilon) \leqslant u_{m_1}$, we have

$$u(T_\varepsilon - \eta_\varepsilon^-) < u(T_\varepsilon) \leqslant u_{m_1},$$

and since $f'(u) < 0$ $(-\infty, u_{m_1})$, then $f(u(T_\varepsilon - \eta_\varepsilon^-)) > f(u(T_\varepsilon))$. Hence,

$$f(u(T_\varepsilon - \eta_\varepsilon^-)) + z(T_\varepsilon - \eta_\varepsilon^-) - (f(u(T_\varepsilon)) + z(T_\varepsilon)) = -\frac{\delta_0}{2} - (-\delta_0) = \frac{\delta_0}{2}$$

$$z(T_\varepsilon - \eta_\varepsilon^-) - z(T_\varepsilon) = \frac{\delta_0}{2} + f(u(T_\varepsilon)) - f(u(T_\varepsilon - \eta_\varepsilon^-)) > \frac{\delta_0}{2}.$$

This contradicts $z(T_\varepsilon - \eta_\varepsilon^-) - z(T_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Case 2. $u'(T_\varepsilon) > 0$. The proof is similar to that of Case 1. This proves Lemma 2.17. □

Before we state Lemma 2.18 we define an energy function:

$$H_1(w, z) = \frac{\beta^2}{2} z^2 + \frac{\beta}{2} w^2.$$

Now we want to choose $\beta > 0$ such that the intersection of the projection of Σ_2 onto (w, z) plane and D_1 is empty set, where

$$D_1 = \{(w, z) : \frac{\beta^2}{2} z^2 + \frac{\beta}{2} w^2 \leqslant \frac{\beta^2}{2} m_1^2\}.$$

Let Q_3 be the point where $z = -m_2$ and $z = w + u_2$ intersect and Q_4 be the point where $z = -m_2$ and $\frac{\beta^2}{2} z^2 + \frac{\beta}{2} w^2 = \frac{\beta^2}{2} m_1^2$ intersect. On the (w, z) plane we want to choose $\beta > 0$ such that Q_4 is above Q_3. See Figure 15.

If $m_2 \geqslant m_1$, Q_3 does not exist, and so there is no restriction on β in this case.

If $m_1 > m_2$, then choose β so small that $m_2 + u_2 > \sqrt{\beta(m_1^2 - m_2^2)}$ or $\beta < (m_2 + u_2)^2/(m_1^2 - m_2^2)$. Let $B_1 = \{\beta | \beta < (m_2 + u_2)^2/(m_1^2 - m_2^2)\}$. We can see from Figure 15 that if $m_1 > m_2$, as long as $\beta \in B_1$, the projections onto the (w, z) plane of the solutions starting from Σ_2 start outside of the region D_1.

Lemma 2.18. *Let the parameters be as in Lemma* 2.11. *Then for small enough* $\varepsilon > 0$, $H_1' > 0$ *as long as* $|z(t)| < m_1$; *i.e., the projections of the solutions onto the* (w, z) *plane do not enter the region* D_1.

Proof. Since $H_1' = \beta^2 z z' + \beta w' w = \beta^2(z z' - z w) = \beta^2 z(u - z)$. Let $f(u(t)) + z(t) = \eta(t)$, so that $z(t) = \eta(t) - f(u(t))$ and so

$$H_1' = \beta^2(\eta(t) - f(u(t)))(u - \eta(t) + f(u(t))).$$

By the proof of Proposition 2.2,

$$E_1'(u) = \beta^2(-f(u))(u + f(u)) > 0$$

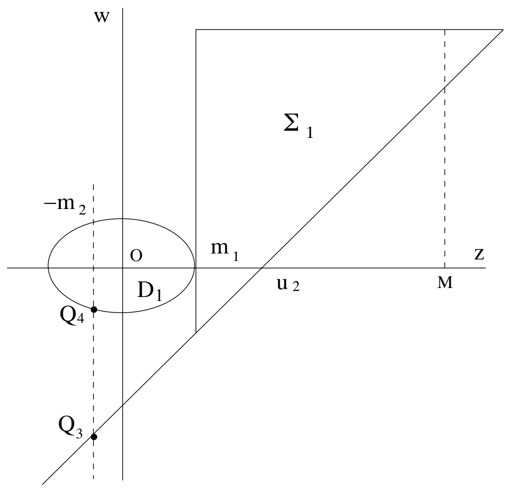

FIGURE 15. Positions of Q_3 and Q_4

as long as

$$\frac{1}{2}\left(u_2 + u_3 - \sqrt{(u_3 - u_2)^2 + 4}\right) < u < u_2.$$

By Lemma 2.17 for any $\delta > 0$, there exists an ε_0 such that if $\varepsilon < \varepsilon_0$, then $|\eta(t)| < \delta$ as $t < \tau_7$ with $u(t) \leqslant u_{m_1}$.

Comparing H_1' and $E_1(u)$, there exist $\sigma(\delta)$ and $\sigma'(\delta)$ such that $H_1' > 0$ as long as

$$\frac{1}{2}\left(u_2 + u_3 - \sqrt{(-u_3 - u_2)^2 + 4}\right) + \sigma < u < u_2 - \sigma'.$$

Because by Lemma 2.17 again, when $u(t) \leqslant u_{m_1}$, we have $|\eta(t)| < \delta$. Therefore, $H_1' > 0$ as long as

$$\frac{1}{2}\left(u_2 + u_3 - \sqrt{(-u_3 - u_2)^2 + 4}\right) + \sigma < u < u_{m_1},$$

and by continuity of the solutions in t, $\sigma \to 0$ as $\delta \to 0$. Hence, $H_1' > 0$ as long as

$$u \leqslant u_{m_1} \text{ and } f(u) \leqslant f\Big(\frac{1}{2}(u_2 + u_3 - \sqrt{(u_3 - u_2)^2 + 4}) + \sigma\Big),$$

or equivalently $H_1' > 0$ as long as $u \leqslant u_{m_1}$ and

$$z(t) = \eta(t) - f(u(t)) \geqslant \eta(t) - f\Big(\frac{1}{2}(u_2 + u_3 - \sqrt{(u_3 - u_2)^2 + 4}) + \sigma\Big),$$

or equivalently $H_1' > 0$ as long as $u \leqslant u_{m_1}$ and

$$z(t) - \delta > -f\Big(\frac{1}{2}(u_2 + u_3 - \sqrt{(u_3 - u_2)^2 + 4}) + \sigma\Big).$$

We showed in an example in Section 2 after the definition of Λ that there exist u_2 and u_3 such that

$$f\Big(\frac{1}{2}\big(u_2 + u_3 - \sqrt{(u_3 - u_2)^2 + 4}\big)\Big) > m_1.$$

We can choose δ^* small enough and σ^* small enough so that there exist u_2 and u_3 such that

$$-f\Big(\frac{1}{2}\big(u_2 + u_3 - \sqrt{(u_3 - u_2)^2 + 4}\big) + \sigma^*\Big) < -m_1 - \delta^*.$$

For the δ^*, u_2, and u_3 chosen, we consider two cases:

Case 1. $\sigma(\delta^*) \leqslant \sigma^*$. Then $H_1' > 0$ as long as $u \leqslant u_{m_1}$ and $z(t) > -m_1$, because

$$\begin{aligned} -m_1 &> \delta^* - f\Big(\frac{1}{2}\big(u_2 + u_3 - \sqrt{(-u_3 - u_2)^2 + 4}\big) + \sigma^*\Big) \\ &\geqslant \delta^* - f\Big(\frac{1}{2}\big(u_2 + u_3 - \sqrt{(u_3 - u_2)^2 + 4}\big) + \delta(\sigma^*)\Big). \end{aligned}$$

Case 2. $\sigma(\delta^*) > \sigma^*$. Then for the fixed u_2 and u_3, we choose a smaller $\delta^{**} < \delta^*$ such that

$$-f\left(\frac{1}{2}\left(u_2 + u_3 - \sqrt{(u_3 - u_2)^2 + 4}\right) + \sigma^{**}\right) < -m_1 - \delta^{**}.$$

This is always possible, because $\sigma(\delta^{**}) \to 0$ as $\delta^{**} \to 0$.

Therefore, for small enough $\varepsilon > 0$, $H_1^{'} > 0$, and so when $u \leqslant u_{m_1}$ and $z > -m_1$, $H_1^{'} > 0$.

Next we show that if $z(t) < m_1$, then $u(t) \leqslant u_{m_1}$, so that $H_1' > 0$ as long as $-m_1 < z(t) < m_1$. Suppose that there exists $\tau > \tau_7$ such that $z(t) < m_1$ in $[\tau_7, \tau]$ and $u(\tau) \geqslant u_{m_1}$. Then there exists $\tau^* \in (\tau_7, \tau]$ such that $u(\tau^*) = u_{m_1}$ and $u^{'}(\tau^*) \geqslant 0$; i.e., $z(\tau^*) + f(u(\tau^*)) = z(\tau^*) + m_1 \geqslant 0$ or $z(\tau^*) \geqslant m_1$, a contradiction.

Since the projection of the solution onto the (w, z) plane starts outside of the region D_1, where $-m_1 < z(t) < m_1$, it does not enter the region D_1 for any $t > 0$. This completes the proof of Lemma 2.18. □

Lemma 2.19. *Let the parameters be as in Lemma* 2.11*. Then for small enough* $\varepsilon > 0$*, there exists a* $\tau_8 > \tau_7$ *such that* $z(\tau_8) = 0$ *and* $p(t) \in \Omega_4$ *in* $[\tau_7, \tau_8]$.

Proof. We first show that there exists a $\tau_8 > \tau_7$ such that $z(\tau_8) = 0$, and then we show that $p(t) \in \Omega_4$ in $[\tau_7, \tau_8]$; i.e., $z'(t) < 0$ and $u(t) < 0$ in $[\tau_7, \tau_8]$. By Lemma 2.16, either there exists a $\tau_8 > \tau_7$ such that $z(\tau_8) = 0$ or $z(t) \to 0$ as $t \to \infty$. In the latter case, since $z''(t)$ is bounded for any fixed $\varepsilon > 0$, $z'(t) = u + w - z \to 0$; i.e., $u + w \to 0$ as $t \to \infty$. By Remark 2.1 and Lemma 2.16, $u' > 0$ for $t > \tau_7$; i.e.,

$$f(u(t)) > -z(t) > 0, \text{ for } t > \tau_7,$$

and so $u(t) \leqslant 0$, for $t > \tau_7$. Thus, $u(t)$ is monotonically increasing for $t > \tau_7$ and $\lim_{t\to\infty} u(t) \leqslant 0$, and thus $\lim_{t\to\infty} w(t) \geqslant 0$, because $P_3 = (0,0,0)$ is the only equilibrium with $u \leqslant 0$. We then have $p(t) \to (0,0,0)$ as $t \to \infty$, and thus the projection onto the (w, z) plane of the solution has to enter the region D_1 as $u(t) \leqslant 0 < u_{m_1}$, contradicting Lemma 2.18. Hence only the first case happens.

Next we show that $z'(\tau_8) < 0$ and $u'(\tau_8) < 0$. By Lemma 2.16, $u'(\tau_8) \geqslant 0$ and $z'(\tau_8) \geqslant 0$. We consider three cases:

Case 1. $u'(\tau_8) > 0$ and $z'(\tau_8) = 0$. So $z''(\tau_8) \leqslant 0$. However, $z''(\tau_8) = u'(\tau_8) > 0$, a contradiction.

Case 2. $u'(\tau_8) = 0$ and $z'(\tau_8) > 0$. So $u''(\tau_8) \leqslant 0$. However, $u''(\tau_8) = \frac{1}{\varepsilon} z'(\tau_8) > 0$, a contradiction.

Case 3. $u'(\tau_8) = 0$ and $z'(\tau_8) = 0$. Then $u'(\tau_8) = \frac{1}{\varepsilon}(z(\tau_8) + f(u(\tau_8)) = 0$ and $u(\tau_8) \leqslant 0$ gives that $u(\tau_8) = 0$, and thus $w(\tau_8) = 0$. However, by uniqueness of initial-value problems, this is not possible. Lemma 2.16 also implies that $u(t) \leqslant 0 < u_2$ in $[\tau_7, \tau_8]$. Hence $p(t) \in \Omega_4$ in $[\tau_7, \tau_8]$. This completes the proof of Lemma 2.19. □

Lemma 2.20. *Let the parameters be as in Lemma* 2.11. *Then for small enough $\varepsilon > 0$, there exists a $\tau_9 > \tau_8$ such that $z(\tau_9) = m_1$ and $p(t) \in \Omega_4$ in $[\tau_8, \tau_9]$.*

Proof. Suppose that there exists $\tau > \tau_8$ such that $z'(\tau) = 0$ and $0 \leqslant z(\tau) < m_1$. Then Lemma 2.18 implies that

$$H_1'(\tau) = \beta^2 zz' - \beta^2 wz = -\beta^2 w(\tau)z(\tau) \geqslant 0,$$

and thus $w(\tau) \leqslant 0$. Since the projection onto the (w, z) plane of the solution which starts from Σ_2 is outside of the region D_1, by Lemma 2.18, it cannot get into the region D_1. Then Lemma 2.19 implies that $w(\tau) > 0$, a contradiction.

Hence $z'(t) > 0$ for $t \geqslant \tau_8$ with $z(t) \leqslant m_1$, and therefore there exists a $\tau_9 > \tau_8$ such that $z(\tau_9) = m_1$ and $z'(t) > 0$ in $[\tau_8, \tau_9]$. By the proof

Lemma 2.18, $u(t) \leqslant u_{m_1} < u_2$ in $[\tau_8, \tau_9]$. Thus $p(t) \in \Omega_4$ in $[\tau_8, \tau_9]$, and this completes the proof of Lemma 2.20. □

Lemma 2.21. *Let the parameters be as in Lemma* 2.11*. Then for small enough* $\varepsilon > 0$*, there exists a* $\tau_{10} > \tau_9$ *such that* $p(\tau_{10}) \in \Sigma_1$*, and* $p(t) \in \Omega_4$ *in* $[\tau_9, \tau_{10})$.

Proof. First we show that if we choose u_2 and u_3 properly, $z'(\tau_9) > \Delta'$ for some $\Delta' > 0$ independent of ε,

$$z'(\tau_9) = u(\tau_9) + w(\tau_9) - z(\tau_9).$$

By Lemma 2.20, $z(\tau_9) = m_1$ and $w(\tau_9) \geqslant 0$, and by Lemma 2.17, for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, $|f(u(t)) + z(t)| < \delta$ as $t > \tau_7$, with $u(t) \leqslant u_{m_1}$ and

$$|f(u(\tau_9)) + m_1| < \delta$$
$$\Longleftrightarrow -m_1 < f(u(\tau_9)) \leqslant -m_1 + \delta \Longleftrightarrow f^{-1}(-m_1 + \delta) \leqslant u(\tau_9) < u_{m_1}.$$

We can choose u_2 and u_3 such that $u_{m_1} - m_1 > 0$, and for δ small and small enough $\varepsilon > 0$, we have that

$$z'(\tau_9) > f^{-1}(-m_1 + \delta) - m_1 > \frac{1}{2}(u_{m_1} - m_1) = \Delta' > 0.$$

Next we show that $z'(t) > 0$ for $t \geqslant \tau_9$ with $u \leqslant u_{m_1}$. Suppose that there exists $\tau \geqslant \tau_9$ such that $z'(\tau) = 0$, and $u(t) \leqslant u_{m_1}$ in $[\tau_9, \tau]$; then by Remark 2.1 $u'(t) > 0$ in $[\tau_9, \tau]$. Thus by Theorem 2.1, there exists a $K' > 0$ such that

$$z''(t) = u'(t) + w'(t) - z'(t) > -K' \text{ in } [\tau_9, \tau]$$
$$|z(\tau_9) - z(t)| = \Big| \int_{\tau_9}^{\tau} z'(t)\, dt \Big| > \frac{1}{2K'}(z'(\tau_9))^2.$$

On the other hand, if $\varepsilon > 0$ is small, because $u(t) \leqslant u_{m_1}$ in $[\tau_9, \tau]$, then

$$|f(u(\tau)) + z(\tau)| < \delta \Longleftrightarrow -f(u(\tau)) - \delta < z(\tau) < -f(u(\tau)) + \delta;$$

adding m_1 to each of the three parts of the above inequality, we obtain

$$m_1 + f(u(\tau)) - \delta < m_1 - z(\tau) < m_1 + f(u(\tau)) + \delta.$$

Since

$$|f(u(\tau_9)) + m_1| < \delta \Longleftrightarrow -\delta < f(u(\tau_9)) + m_1 < \delta$$

and $u'(t) > 0$ in $[\tau_9, \tau]$. These imply that $f(u(\tau)) > f(u(\tau_9))$ and $f(u(\tau)) > -m_1$. Thus we have

$$-2\delta > m_1 - z(\tau) < 2\delta; \quad \text{i.e.,} \quad -2\delta < z(\tau_9) - z(\tau) < 2\delta,$$

and so $|z(\tau_9) - z(\tau)| < 2\delta$, a contradiction.

Now suppose that there exists $\tau > \tau_9$ such that $z'(\tau) = 0$ and $z'(t) > 0$ in $[\tau_9, \tau)$, and thus $z''(\tau) \leqslant 0$ and $u_{m_1} \leqslant u(\tau) < u_2$.

By the above proof, $z(\tau) > m_1 - \frac{1}{2K'}(z'(\tau_9))^2$; thus,

$$u'(\tau) = \frac{1}{\varepsilon}(z + f(u)) > \frac{1}{\varepsilon}(-\frac{1}{2K'}(z'(\tau_9))^2) = \frac{1}{2\varepsilon K'}(z'(\tau_9))^2 \to \infty, \text{ as } \varepsilon \to 0.$$

By Theorem 2.1, z' and w' are bounded in $[0, \infty)$; for ε small enough

$$z''(\tau) = u'(\tau) + w'(\tau) - z'(\tau) > 0,$$

a contradiction. This completes the proof of Lemma 2.21. □

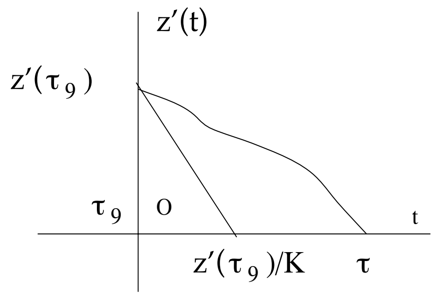

FIGURE 16. $z'(t)$ versus t around τ_9

So far we've proved that for the parameters chosen as in Lemma 2.11 and for each $p(0) \in \Sigma_1$, there exists an $\varepsilon(p(0)) > 0$ such that if $\varepsilon < \varepsilon(p(0))$, then $p(\tau_{10})$ is in Σ_1. From the proof of the above lemmas, we know that we require ε is small enough such that $-\frac{b}{a\varepsilon} - \beta > 0$ and Lemmas 2.8, 2.11, 2.18, and 2.21 are true. For the uniformity of ε in Σ_1, it suffices to prove the following two lemmas. Now for fixed u_2, u_3, and β, let $F(t, \varepsilon, u(0), w(0), z(0)) = |f(u)+z|$. By Theorem 2.1 and the continuous dependence of the solutions on parameters and initial conditions, F is continuous for $(u(0), w(0), z(0)) \in \mathbb{R}^3$, $\varepsilon > 0$, and $t \geqslant 0$. Then we define an extension of F to $\varepsilon = 0$ and prove the following two lemmas:

$$F^{ex}(t, \varepsilon, u(0), w(0), z(0)) = \begin{cases} F(t, \varepsilon, u(0), w(0), z(0)) & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon = 0. \end{cases}$$

Lemma 2.22. *For any $\delta > 0$, there exists an $\varepsilon_0' > 0$ such that for all $p(0) \in \Sigma_1$, if $\varepsilon < \varepsilon_0'$, $|f(u) + z| < \delta$ as long as $t \geqslant \tau_2$ and $u(t) \geqslant u_{m_2}$.*

Proof. From the above proof we know that for any $p(0) \in \Sigma_1$, and any $\delta > 0$, there exists an $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, then

$$|f(u) + z| < \delta \text{ as long as } t \geqslant \tau_2 \text{ and } u(t) \geqslant u_{m_2}, \tag{2.17}$$

where τ_2 is such that $u'(\tau_2) = 0$ (see Lemma 2.5).

Let $\varepsilon_{00}(p(0)) = \sup\{\varepsilon_0(p(0))\}$ be such that if $\varepsilon < \varepsilon_0(p(0))$, then (2.17) holds. For the sake of contradiction, suppose that $\inf_{p(0)\in\Sigma_1}\{\varepsilon_{00}(p(0))\} = 0$. In particular, suppose that there exists a sequence $p_n(0) \in \Sigma_1$ such that for some $\delta_0 > 0$, $\varepsilon_{00}(p_n(0)) \to 0$ as $n \to \infty$. Since Σ_1 is compact, the sequence has a cluster point, say $p^*(0)$, in Σ_1. Without loss of generality, let $p_n(0) \to p^*(0) \in \Sigma_1$ as $n \to \infty$; then by Lemma 2.7, there exists an $\varepsilon_{00}(p^*(0)) > 0$ such that if $0 < \varepsilon < \varepsilon_{00}(p^*(0))$, then

$$|f(u) + z| = F(t, \varepsilon, u^*(0), w^*(0), z^*(0)) < \delta_0,$$

for $\tau_2(\varepsilon, p^*(0)) \leqslant t \leqslant \tau^*(\varepsilon, p^*(0))$, where $u(\tau^*) = u_{m_2}$.

Since $u' = \frac{1}{\varepsilon}(f(u) + z)$, integrating from 0 to τ_2, we have

$$\varepsilon(u(\tau_2) - u(0)) = \int_0^{\tau_2} (f(u) + z).$$

Therefore, by the mean-value theorem and the boundedness of the solutions, $\tau_2 \to 0$ as $\varepsilon \to 0$ for any fixed $p(0) \in \Sigma_1$. We define $\tau_2(0, p(0)) = 0$. Then by the definition of F^{ex}, $F^{ex} < \delta_0$ in the following closed set:

$$\Big\{(u(0), w(0), z(0), \varepsilon, t) : (u(0), w(0), z(0)) = p^*(0),$$
$$0 \leqslant \varepsilon \leqslant \varepsilon_{00}(p^*(0)), \tau_2(\varepsilon, p^*(0)) \leqslant t \leqslant \tau^*(\varepsilon, p^*(0))\Big\}.$$

We also see that F^{ex} is continuous for $t \geqslant 0$, $\varepsilon \geqslant 0$, and $p(0) \in \Sigma_1$. Then there exists some open five-dimensional set containing the above closed set such that $F^{ex} < \delta_0$ in the open set

$$\begin{aligned} &t \in (\tau_2(\varepsilon, p^*(0)) - \eta_1, \tau^*(\varepsilon, p^*(0)) + \eta_1) \\ &p(0) \in B_r(p^*(0)), \qquad \varepsilon \in [0, \varepsilon_{00}(p^*(0) + \eta_2] \end{aligned} \tag{2.18}$$

for some small $\eta_1 > 0$, $\eta_2 > 0$, and $r > 0$. Since for fixed $\varepsilon > 0$ and every $p(0) \in \Sigma_1$ there is a τ_2 and a τ^* such that $u'(\tau_2) = 0$, $u(\tau^*) = u_{m_2}$, $z'(\tau_2) < 0$, and $u'(\tau^*) < 0$, by the implicit-function theorem and the continuity of solutions in initial conditions (see [10]), τ_2 and τ^* are continuous functions of $p(0)$. Then for the $\eta_1 > 0$ in (2.18), there exists an r_0 such that if $p(0) \in B_{r_0}(p^*(0))$,

$$\tau_2 \in (\varepsilon, p(0)) \in (\tau_2(\varepsilon, p^*(0)) - \eta_1, \tau^*(\varepsilon, p^*(0)) + \eta_1),$$
$$\tau^* \in (\varepsilon, p(0)) \in (\tau_2(\varepsilon, p^*(0)) - \eta_1, \tau^*(\varepsilon, p^*(0)) + \eta_1).$$

Hence $\varepsilon_{00}(p(0)) \geqslant \frac{1}{2}\varepsilon_{00}(p^*(0))$ for all $p(0)$ in $B_{r_0}(p^*(0))$, a contradiction. This completes the proof of Lemma 2.22. □

Lemma 2.23. *For any $\delta > 0$, there exists $\varepsilon_0'' > 0$ such that for all $p(\tau_7) \in \Sigma_2$, if $\varepsilon < \varepsilon_0''$, $|f(u) + z| < \delta$ as long as $t \geqslant \tau_7$ and $u(t) \leqslant u_{m_1}$.*

Proof. The proof is similar to that of Lemma 2.22. It is left for readers. □

The times $t = 0, \tau_1, \ldots, \tau_{10}$ on the trajectories projected onto the (u, z) plane given in the above lemmas is shown in Figure 17. So far we have proved

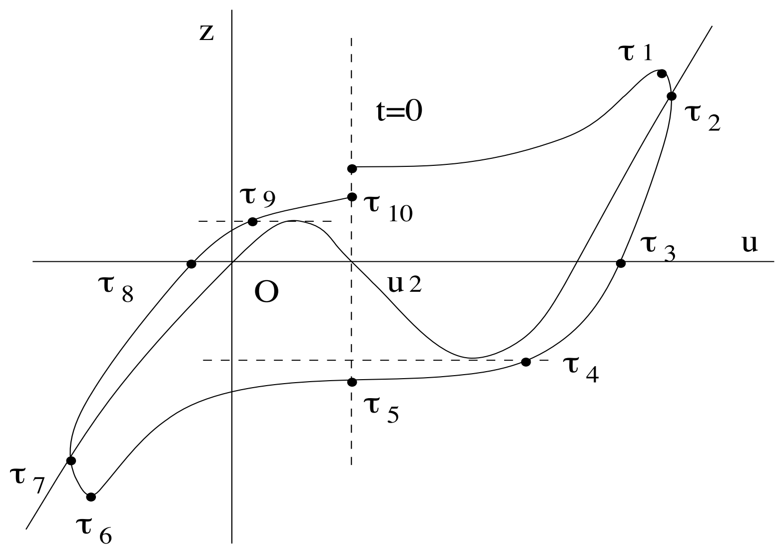

FIGURE 17. The 10 τ's used in the proof

that there exists an $\varepsilon^* > 0$ such that for every $p(0) \in \Sigma_1$, if $\varepsilon < \varepsilon^*$, then there exists a τ_{10} such that $p(\tau_{10}) \in \Sigma_1$. We define a map $\pi : \Sigma_1 \to \Sigma_1$ as $p(\tau_{10}) = \pi(p(0))$. Since Σ_1 is homeomorphic to the unit disk, by Brouwer's fixed-point theorem we conclude that the map π has a fixed point, and therefore system (2.1) has a nonconstant periodic solution. This completes the proof of our main theorem. □

The result of Theorem 2.2 can be extended to $\gamma > 0$ and small. We have the following theorem.

Theorem 2.3. (*Existence theorem for $\gamma > 0$*) *For the parameters (u_2, u_3), β and $\varepsilon > 0$ chosen fixed as in Theorem 2.2, for small enough $\gamma > 0$, there exists a nonconstant periodic solution in system* (1.3).

Proof. Let the parameters (u_2, u_3), β, and $\varepsilon > 0$ be chosen fixed as in Theorem 2.2. In the proof of Theorem 2.2, we've shown that for small enough $\varepsilon > 0$, $p(\tau_{10}) \in \Sigma_1$. Since Σ_1 is a compact set, the set $\Sigma_3 = \{p(\tau_{10}) : p(0) \in \Sigma_1\}$ is compact by continuity of solutions in the initial conditions (see [10]). Here it suffices to show that $p(\tau_{10})$ is in the interior of the two-dimensional set Σ_1; i.e., $m_1 < z(\tau_{10}) < M$ and $z(\tau_{10}) - u_2 < w(\tau_{10}) < M$. Theorem 2.1 implies $|z(t)| < M$ and $|w(t)| < M$ for all $t > 0$. Lemmas 2.20 and 2.21 imply that $z(\tau_{10}) > m_1$ and $z'(\tau_{10}) > 0$; i.e., $w(\tau_{10}) > z(\tau_{10}) - u_2$. Therefore we have that Σ_3 is in the interior of Σ_1; i.e., $dist(\Sigma_1, \Sigma_3) > 0$. Hence by the continuity of solutions in parameters (see [10]), for $\gamma > 0$ but small, $p(\tau_{10}) \in \Sigma_1$. By Brouwer's fixed-point theorem again as in the proof of Theorem 2.2, the map π has a fixed point, and therefore system (1.3) has a nonconstant periodic solution if $\gamma > 0$ is small. This completes the proof of Theorem 2.3. □

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