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ON A NONLINEAR VARIANT OF THE BEAM EQUATION WITH WENTZELL BOUNDARY CONDITIONS

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1. Introduction

A mathematical model for the transverse deflection of an extensible beam whose ends are held at fixed distance apart is

$$\frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial^4 u}{\partial x^4}+\Big(\rho+k\int_0^1\Big(\frac{\partial u}{\partial \xi}(\xi,t)\Big)^2 d\xi\Big)\Big(-\frac{\partial^2 u}{\partial x^2}\Big)=0 \tag{1.1}$$

$$u\,(0,t)=u\,(1,t)=\frac{\partial^2 u}{\partial x^2}\,(0,t)=\frac{\partial^2 u}{\partial x^2}\,(1,t)=0, \tag{1.2}$$

which has been proposed by Woinowsky and Krieger [20]. Here $k,\alpha>0$ are constants, $\rho\in\mathbf{R}$, and the nonlinear term represents the change in tension of the beam due to its extensibility. The model has also been discussed by Eisley [8], Dickey [7], and Ball [1]–[2], while related experimental results have been given by Burgreen [6].

Nonlinear beams have been the subject of much recent activity. Ball uses a Galerkin method to obtain weak solutions to (1.1) and obtains classical solutions by placing further restrictions on the regularity of the data. The abstract formulation of (1.1)–(1.2) is the equation

$$u_{tt}+\alpha A^2u+M\big(\big\|A^{\frac{1}{2}}u\big\|^2\big)Au=0, \tag{1.3}$$

where A is a positive self-adjoint operator in a Hilbert space H and M is a real function. This model has been studied by Medeiros [16]. He supposed that $M\in C^1[0,\infty)$ be such that $M(\lambda)\geq m_0+m_1\lambda$, for any $\lambda; m_0,m_1>0$, and with A having compact resolvent. The same equation (1.3), but with a dissipative term, was studied by Brito [4]–[5], Pereira [17], and Holmes and Marsden [15].

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In this paper we are concerned with the nonlinear beam-like equation,

$$u_{tt} + \alpha\Delta^2 u + \Big(\rho + k\int_\Omega |\nabla u|^2 dx + k\int_{\partial\Omega} |\nabla u|^2 \frac{dS}{\beta}\Big)(-\Delta u) = 0, \tag{1.4}$$

subject to the general Wentzell boundary conditions (or GWBC) of the form

$$\Delta u + \beta\frac{\partial u}{\partial n} + \gamma u = 0 \text{ on } \partial\Omega, \tag{1.5}$$

$$\Delta^2 u + \beta\frac{\partial}{\partial n}(\Delta u) + \gamma\Delta u = 0 \text{ on } \partial\Omega. \tag{1.6}$$

Here Ω is a bounded domain in $\mathbf{R}^n$ with smooth $\partial\Omega$; β, k, α, and γ are real positive constants and $u_t = \frac{\partial u}{\partial t}$. More generally, these coefficients can be positive functions on the boundary. The term $\int_\Omega \phi^2 dx$ is a term of generalized mass, where $\phi = |\nabla u|$ is the torsion function that also appears in the study of torsional vibrations of thin rods. It seems natural to consider a term like $\int_{\partial\Omega} \phi^2 \frac{dS}{\beta}$ in (1.4), which is the mass at the boundary with a certain weight β that also appears in the boundary conditions (1.5)–(1.6). For a discussion of the physical interpretation of (1.5)–(1.6), see [13].

2. General Wentzell boundary conditions

Based on previous work [11]–[12], we know that $A = -\Delta$ with GWBC (1.5) defines a positive self-adjoint operator on X_2 (see (2.1) below). Define X_2 as

$$X_2 = L^2(\Omega, dx) \oplus L^2\Big(\partial\Omega, \frac{dS}{\beta}\Big) \tag{2.1}$$

with norm for $u \in C\left(\overline{\Omega}\right) \subset X_2$

$$\|u\|_{X_2}^2 = \int_\Omega |u(x,t)|^2 dx + \int_{\partial\Omega} |u(x,t)|^2 \frac{dS}{\beta}.$$

Define A_0 to be the Laplacian with the boundary condition (1.5), with domain $D(A_0) = C^2(\overline{\Omega}) \subset X_2$. Thus A, the closure of A_0, is self-adjoint on X_2 (by [11]). Let $G = A^2 = \Delta^2$ be the operator with GWBC (1.5)–(1.6). Then the corresponding beam equation with the same GWBC (1.5)–(1.6) (with domain $D(A^2) \oplus D(A)$) is governed by a skew-adjoint operator that generates a unitary group on $H = D(A) \oplus X_2$. Let G_0 be G restricted to $C^4(\overline{\Omega}) \subset X_2$. Then G_0 is essentially self-adjoint on X_2, by [11]. The symmetry of G_0 can be seen from the following calculation:

$$\langle G_0 u, v\rangle_{X_2} = \int_\Omega \left(\Delta^2 u\right) \bar{v}\, dx + \int_{\partial\Omega} \left(\Delta^2 u\right) \bar{v}\, \frac{dS}{\beta}$$

$$= -\int_\Omega \nabla(\Delta u)\cdot(\nabla\,\bar v)dx + \int_{\partial\Omega}\Big(\frac{\partial}{\partial n}(\Delta u)\Big)\,\bar v\;dS + \int_{\partial\Omega}(\Delta^2 u)\;\bar v\;\frac{dS}{\beta}$$
$$= \int_\Omega \Delta u\cdot\,\bar{\Delta} v\;dx - \int_{\partial\Omega}\Delta u\cdot\frac{\partial\,\bar v}{\partial n}dS + \int_{\partial\Omega}\beta\Big(\frac{\partial}{\partial n}(\Delta u)\Big)\;\bar v\;\frac{dS}{\beta} + \int_{\partial\Omega}(\Delta^2 u)\;\bar v\;\frac{dS}{\beta}$$
$$= \int_\Omega \Delta u\cdot\,\bar{\Delta} v\;dx - \int_{\partial\Omega}\Delta u\cdot\frac{\partial\,\bar v}{\partial n}dS - \int_{\partial\Omega}\gamma\Delta u\cdot\overline{v}\frac{dS}{\beta}$$

since $\Delta^2 u + \beta\frac{\partial}{\partial n}(\Delta u) + \gamma\Delta u = 0$ on $\partial\Omega$, so

$$\langle G_0 u, v\rangle_{X_2} = \int_\Omega \Delta u\cdot\,\bar{\Delta} v\;dx + \int_{\partial\Omega}\Delta u\cdot\overline{\Delta v}\frac{dS}{\beta} = \langle u, G_0 v\rangle_{X_2}$$

since the GWBC (1.5)–(1.6) holds for v as well.

As has already been noted in our previous paper [12], we know that the one-dimensional Laplacian with GWBC (1.5) is resolvent compact when the dimension of the underlying bounded set is $n = 1$. This has been proved by Binding, Brown, and Watson [3], who established an orthonormal basis of eigenvectors in the space $\mathcal{H} = L^2(0,1)\oplus\mathbf{C}^2$ with norm

$$\|u\|^2_{\mathcal{H}} = \int_0^1 |u(x)|^2\,dx + \sum_{j=0}^{1}\frac{|u(j)|^2}{\beta}$$

and the real eigenvalues tend to $-\infty$. Thus G, the one-dimensional bi-Laplacian $\frac{d^4}{dx^4}$ with GWBC (1.5)–(1.6), is positive self-adjoint and resolvent compact.

Fractional powers of G can be defined in the usual way that appears in books (see, for example, [14]). For $0 < \mu < 1$, let

$$G^{-\mu} = \frac{1}{2\pi i}\int_\Gamma s^{-\mu}(s - G)^{-1}\,ds,$$

where Γ is a piecewise-smooth path in $\sum\backslash\mathbf{R}_+$ going from $\infty e^{-i\delta}$ to $\infty e^{i\delta}$ for some $\delta 0$, and $\sum$ is an open sector in $\mathbf{C}$ such that $\mathbf{R}_+\subset\sum\subset\rho(G)$, the resolvent set of G. To show that $G = A^2$ (with (1.5)–(1.6)) is one-to-one, it is enough to show that A (equipped with (1.5)) is one-to-one. We use Maz$'$ja's inequality (see e.g. [11]), that is,

$$\|u\|^2_{L^{\frac{2n}{n-1}}(\Omega)} \le C_1(n, vol(\Omega))\big(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\big).$$

It follows easily that there exists $C_2 > 0$ independent of u such that

$$\|u\|^2_{X_2} \le C_2\big(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)}\big). \tag{2.2}$$

We compute

$$\begin{aligned}\langle Au,u\rangle_{X_2} &= -\int_\Omega \Delta u u\,dx - \int_{\partial\Omega} \Delta u u \frac{dS}{\beta}\\ &= -\int_{\partial\Omega}\Big(\beta\frac{\partial u}{\partial n}u\Big)\frac{dS}{\beta} + \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega}\Big(\beta\frac{\partial u}{\partial n}+\gamma u\Big)u\frac{dS}{\beta},\end{aligned}$$

since $\Delta u = -\beta\frac{\partial u}{\partial n} - \gamma u$ on $\partial\Omega$. In conclusion, we have

$$\langle Au,u\rangle_{X_2} = \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega}\frac{\gamma}{\beta}|u|^2\,dS \geq \frac{\min\{1,\frac{\gamma}{\beta}\}}{C_2}\|u\|^2_{X_2},\ \text{by}\ (2.2)\,.$$

Hence A is one-to-one since $\gamma>0$ and $\beta>0$ on $\partial\Omega$. More generally, we may assume that $\beta,\gamma\in C(\partial\Omega)$ with $\beta>0$ and $\gamma\geq 0$, but γ is not identically zero on $\partial\Omega$.

The operators $G^{-\mu}$ are injective for every $\mu\geq 0$, and they form a strongly continuous semigroup, since the operator G is densely defined. The operators $G^{\mu}=(G^{-\mu})^{-1}$ $(\mu>0)$ are also positive self-adjoint operators, and in the one-dimensional case, $G^{-\mu}$ are compact. In particular, $G^{\frac12}=A=-\Delta$, equipped with GWBC. The quantity $\int_\Omega|\nabla u|^2+\int_{\partial\Omega}|\nabla u|^2\frac{dS}{\beta}$ can be expressed as

$$\int_\Omega |\nabla u|^2 + \int_{\partial\Omega}|\nabla u|^2\frac{dS}{\beta} = \big\|G^{\frac14}u\big\|^2_{X_2}. \tag{2.3}$$

3. Existence and uniqueness

Let us define the operator $F: D(A)\subset X_2\to X_2$ by

$$Fu = \big(\rho + k\big\|G^{\frac14}u\big\|^2_{X_2}\big)G^{\frac12}u.$$

In this setting, the initial-value problem for our original problem (1.4)–(1.6) becomes

$$u_{tt}+\alpha G u + Fu(t) = 0,\quad u(0,x)=\varphi(x)\,,\quad u_t(0,x)=\psi(x)\,, \tag{3.7}$$

where the GWBC (1.5)–(1.6) are incorporated in the domain of G.

Now we rewrite (3.7) as a first-order abstract Cauchy problem:

$$v_t = \mathcal{A}v + \mathcal{F}v,\quad v(0)=v_0=(\varphi,\psi)\,, \tag{3.8}$$

where $v=(u,u_t)=(u_1,u_2)$ and the operators $\mathcal{A}:\mathcal{X}\to\mathcal{X}$ and $\mathcal{F}:\mathcal{X}\to\mathcal{X}$ are defined as follows:

$$\mathcal{A} = \begin{pmatrix} 0 & I\\ -\alpha G & 0\end{pmatrix} \tag{3.9}$$

and

$$\mathcal{F} = \begin{pmatrix} 0 \\ -F \end{pmatrix} \tag{3.10}$$

on the energy Hilbert space $\mathcal{X} = \mathcal{D}(G^{\frac{1}{2}}) \times X_2$ and the norm on $\mathcal{X}$ is given by

$$\|v\|_{\mathcal{X}}^2 = \left\|G^{\frac{1}{2}} u_1\right\|_{X_2}^2 + \|u_2\|_{X_2}^2 .$$

As subsequent analysis shall reveal, we lose no generality assuming $\alpha = 1$. By results from semigroup theory [14], $\mathcal{A}$ generates a (C_0) unitary group $\{\mathcal{T}(t) : t \in \mathbf{R}\}$ on $\mathcal{X}$. Let $v_1, v_2 \in \mathcal{X}$ with $v_j = (u_{j1}, u_{j2})$. We compute

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\mathcal{X}}^2 = \left\| \begin{pmatrix} 0 \\ Fu_{12} - Fu_{22} \end{pmatrix} \right\|_{\mathcal{X}}^2 = \|Fu_{12} - Fu_{22}\|_{X_2}^2 .$$

It is not hard to see that there exists a positive nondecreasing function $L(\cdot)$, so that for any $v_1, v_2 \in \mathcal{X}$ with $\max \|v_i\|_{\mathcal{X}} \leq R \ (i = 1, 2)$, we have

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\mathcal{X}} \leq L(R) \|v_1 - v_2\|_{\mathcal{X}} .$$

Therefore, $\mathcal{F}$ is locally Lipschitz continuous on $\mathcal{X}$. We can obtain global existence for our problem (3.8).

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{F}$ be defined via* (3.9) *and* (3.10). *If $v_0 = (\varphi, \psi) \in D(\mathcal{A})$, then there exists a unique continuous function $v : [0, \infty) \to \mathcal{X}$ which satisfies*

$$v(t) = \mathcal{T}(t) v_0 + \int_0^t \mathcal{T}(t-s) \mathcal{F}v(s)\, ds, \quad v(0) = v_0 \tag{3.11}$$

for all $t \in [0, \infty)$. Moreover, there exists a constant C which depends on $\|\psi\|$ and $\|G^{\frac{1}{2}}\varphi\|$ so that

$$\sup_{t\in[0,\infty)} \|v(t)\|_{\mathcal{X}} \leq C, \qquad \sup_{t\in[0,\infty)} \|G^{\frac{1}{2}} u\| \leq C. \tag{3.12}$$

Proof. The proof is based on methods due to Segal [18]. We use the version employed by Fitzgibbon in his papers [9, 10]. The local Lipschitz continuity of $\mathcal{F}$ allows us to use standard applications of the Banach fixed-point theorem to obtain local existence and uniqueness of the solution to (3.11). The usual arguments permit us to continue these solutions to a maximal interval of existence $[0, T_{\max})$. To show that $T_{\max} = \infty$, we prove that the solution of (3.11) can be continued beyond any finite interval of $[0, T)$. The bounds which allows us this continuation are global bounds given by (3.12).

Recall that u satisfies (3.7) (in the integral equation sense):

$$u_{tt} + Gu + \left(\rho + k\|G^{\frac{1}{4}}u\|^2_{X_2}\right)G^{\frac{1}{2}}u\,(t) = 0. \tag{3.13}$$

We take the inner product of $u_t\,(t)$ with (3.13) in X_2:

$$\langle u_{tt}\,(t)\,, u_t\,(t)\rangle + \langle Gu\,(t)\,, u_t\,(t)\rangle + \left(\rho + k\|G^{\frac{1}{4}}u\|^2_{X_2}\right)\cdot\langle G^{\frac{1}{2}}u\,(t)\,, u_t\,(t)\rangle = 0,$$

$$\frac{1}{2}\frac{d}{dt}\,\|u_t\,(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|G^{\frac{1}{2}}u\,(t)\,\|^2 + \frac{\rho}{2}\frac{d}{dt}\|G^{\frac{1}{4}}u\,(t)\,\|^2 + \frac{k}{4}\frac{d}{dt}\|G^{\frac{1}{4}}u\,(t)\,\|^4 = 0.$$

Integrate it from 0 to t, so that

$$\|u_t\,(t)\|^2 + \left\|G^{\frac{1}{2}}u\,(t)\,\right\|^2 + \rho\left\|G^{\frac{1}{4}}u\,(t)\,\right\|^2 + \frac{k}{2}\left\|G^{\frac{1}{4}}u\,(t)\,\right\|^4 = C_1,$$

where

$$C_1 = \|u_t(0)\|^2 + \left\|G^{\frac{1}{2}}u(0)\right\|^2 + \rho\left\|G^{\frac{1}{4}}u(0)\right\|^2 + \frac{k}{2}\left\|G^{\frac{1}{4}}u(0)\right\|^4.$$

If $\rho < 0$, then the expression $\rho\|G^{\frac{1}{4}}u(t)\|^2 + \frac{k}{2}\|G^{\frac{1}{4}}u(t)\|^4$ may be negative, but the function $y(x) = \rho x^2 + \frac{k}{2}x^4$ is bounded from below, and these bounds depend on ρ and k. But since $u\,(0) = \varphi\,(x)$ and $u_t\,(0) = \psi\,(x)$, we have produced the desired bounds for $\|u_t\,(t)\|$ and $\|G^{\frac{1}{2}}u(t)\|$ in terms of C_1, ρ and k, and these bounds are independent of t.

These bounds show that $\|Fu\,(t)\|_{X_2}$ is bounded, so we can use a variation-of-parameters formula to extend the solution $u\,(t)$ beyond any preassigned T. Therefore, $T_{\max} = \infty$, and this proves global existence and uniqueness of the mild solution. □

Remark 1. If $v \in AC_{loc}\,(\mathbf{R}^+, \mathcal{X})$ [respectively $C^1\,(\mathbf{R}^+, \mathcal{X})$] and satisfies (3.11), then (3.7) and (3.8) hold almost everywhere [respectively, for all $t \geq 0$]. This will follow from Theorem 3 below.

In order to insure more regularity of the solution of (3.7) we make use of a theorem of Travis and Webb [19], which we state it here as a lemma. They used the theory of strongly continuous cosine families to obtain existence results to semilinear second-order Volterra integrodifferential equations in Banach spaces.

Lemma 1. *Consider the initial-value problem*

$$u_{tt}\,(t) = Au + f\,(t)\,, \quad u\,(0) = \varphi\,(x)\,, \quad u_t\,(0) = \psi\,(x)\,, \tag{3.14}$$

where $(\varphi, \psi) \in D(A) \times D((-A)^{\frac{1}{2}})$. *If* A *is the generator of a strongly continuous cosine family on a Banach space* $\mathbf{X}$ *and* $f : \mathbf{R} \to \mathbf{X}$ *is continuously* *differentiable, then there exists a unique solution* $u : \mathbf{R} \to \mathbf{X}$ *such that* u *is twice continuously differentiable on* $\mathbf{X}$, *and* u *satisfies* (3.14).

With the use of this lemma, we can state the theorem:

Theorem 2. *Let* $G = \Delta^2$ *with GWBC* (1.5)–(1.6) *so that* $-G$ *is the generator of a strongly continuous cosine family* $\{\mathcal{C}(t) : t \in \mathbf{R}\}$ *on* X_2. *Let* $\mathcal{S}(t) = \int_0^t \mathcal{C}(s)\,ds$ *be the associated sine family. If* $(\varphi, \psi) \in D(G) \times D(G^{\frac{1}{2}})$ *and* $f(t) = Fu(t) : [0,\infty) \to X_2$, *then the solution of* (3.7) *has the variation-of-parameter representation*

$$u(t) = \mathcal{C}(t)\varphi + \mathcal{S}(t)\psi - \int_0^t \mathcal{S}(t-s) f(s)\,ds, \ \ t \geq 0,$$

and $u : [0,\infty) \to X_2$ *is twice continuously differentiable. Thus the mild solution* u *of Theorem* 1 *is a global solution of* (3.7), (1.5)–(1.6) *and* $u \in C^2(\mathbf{R}^+, X_2) \cap C^1(\mathbf{R}^+, D(G^{\frac{1}{2}}))$, *where* $D(G^{\frac{1}{2}})$ *is equipped with its graph norm.*

Proof. Notice that

$$\begin{aligned} \frac{d}{dt} f(t) &= \rho G^{\frac{1}{2}} u_t(t) + 2k\langle G^{\frac{1}{4}} u_t(t), G^{\frac{1}{4}} u(t)\rangle G^{\frac{1}{2}} u(t) \\ &\quad + k\langle G^{\frac{1}{4}} u(t), G^{\frac{1}{4}} u(t)\rangle G^{\frac{1}{2}} u_t(t). \end{aligned}$$

We know that if $v = (u, u_t) \in \mathcal{X}$ is continuous in t, then $u_t(t) \in X_2$ is also continuous in t. If we can show that for any $T > 0$, $G^{\frac{1}{2}} u_t(\cdot) \in L^\infty([0,T), X_2)$, since $G^{\frac{1}{2}}$ is a closed operator and $u_t(t)$ is continuous, that would insure the continuity of $G^{\frac{1}{2}} u_t(t)$ so that f is continuously differentiable. We show this estimate now.

Recall the second-order equation

$$u_{tt} + Gu + \left(\rho + k\left\|G^{\frac{1}{4}} u\right\|_{X_2}^2\right) G^{\frac{1}{2}} u(t) = 0. \tag{3.15}$$

We take the inner product of $G^{\frac{1}{2}} u_t(t)$ with (3.15) in X_2 and obtain

$$\langle u_{tt}(t), G^{\frac{1}{2}} u_t(t)\rangle + \langle Gu(t), G^{\frac{1}{2}} u_t(t)\rangle + \left(\rho + k\|G^{\frac{1}{4}} u\|_{X_2}^2\right) \cdot \langle G^{\frac{1}{2}} u(t), G^{\frac{1}{2}} u_t(t)\rangle = 0,$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|G^{\frac{1}{4}} u_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|G^{\frac{3}{4}} u(t)\|^2 + \frac{\rho}{2}\frac{d}{dt}\|G^{\frac{1}{2}} u(t)\|^2 \\ &\quad + \frac{k}{2}\left(\frac{d}{dt}\|G^{\frac{1}{2}} u(t)\|^2\right) \cdot \|G^{\frac{1}{4}} u(t)\|^2 = 0. \end{aligned} \tag{3.16}$$

Integrate on $[0,t]$, and use integration by parts and the fact that $u_t(0)=\psi$ and $u(0)=\varphi$.

$$\|G^{\frac{1}{4}}u_t(t)\|^2+\|G^{\frac{3}{4}}u(t)\|^2+\rho\|G^{\frac{1}{2}}u(t)\|^2+k\|G^{\frac{1}{2}}u(t)\|^2\|G^{\frac{1}{4}}u(t)\|^2$$
$$-k\int_0^t \|G^{\frac{1}{2}}u\left(s\right)\|^2\cdot\frac{d}{ds}\|G^{\frac{1}{4}}u\left(s\right)\|^2=C_2. \tag{3.17}$$

We observe that $\|G^{\frac{1}{2}}u(t)\|$ is bounded from a previous argument, and we can use the Schwarz inequality to deduce

$$\left|\frac{d}{dt}\|G^{\frac{1}{4}}u(t)\|^2\right|\le\|G^{\frac{1}{2}}u(t)\|^2+\|u_t(t)\|^2, \tag{3.18}$$

so that we could produce a bound for $\|G^{\frac{3}{4}}u(t)\|$ from (3.17):

$$\|G^{\frac{3}{4}}u(t)\|\le C_3\left(1+t\right)\ \text{ for all } t\ge 0 \text{ and some constant } C_3>0.$$

Now we take the inner product of (3.15) with $Gu_t(t)$ in X_2:

$$\langle u_{tt}(t),Gu_t(t)\rangle+\langle Gu(t),Gu_t(t)\rangle+\left(\rho+k\|G^{\frac{1}{4}}u\|_{X_2}^2\right)\cdot\langle G^{\frac{1}{2}}u(t),Gu_t(t)\rangle=0,$$

$$\frac{1}{2}\frac{d}{dt}\|G^{\frac{1}{2}}u_t(t)\|^2+\frac{1}{2}\frac{d}{dt}\|Gu(t)\|^2+\frac{\rho}{2}\frac{d}{dt}\|G^{\frac{3}{4}}u(t)\|^2$$
$$+\frac{k}{2}\|G^{\frac{1}{4}}u(t)\|^2\cdot\frac{d}{dt}\|G^{\frac{3}{4}}u(t)\|^2=0.$$

Integrating on $[0,t]$, we obtain

$$\|G^{\frac{1}{2}}u_t(t)\|^2+\|Gu(t)\|^2+\rho\|G^{\frac{3}{4}}u(t)\|^2+k\int_0^t\|G^{\frac{1}{4}}u\left(s\right)\|^2\cdot\frac{d}{ds}\|G^{\frac{3}{4}}u\left(s\right)\|^2\le C_4,$$

so that the same argument as in (3.18) and the at most linear growth of $\|G^{\frac{3}{4}}u(t)\|^2$ permit us to obtain at most quadratic growth of $\|G^{\frac{1}{2}}u_t(t)\|^2$, and that completes the proof. □

Let us define the domain $D^{\infty}\left(A\right)=\bigcap_{n=1}^{\infty}D\left(A^n\right)$. If the initial condition (φ,ψ) of (3.7) has more regularity, then we can state the following theorem.

Theorem 3. *Let ρ be a real constant and $G=A^2=\Delta^2$, equipped with GWBC* (1.5)–(1.6). *If $u_t(0)=\psi\in D^{\infty}(A)$, $u(0)=\varphi\in D^{\infty}(A)$. Then the solution of*

$$u_{tt}+\alpha Gu+(\rho+k\|G^{\frac{1}{4}}u\|_{X_2}^2)G^{\frac{1}{2}}u=0,\quad u(0,x)=\varphi(x),\quad u_t(0,x)=\psi(x) \tag{3.19}$$

satisfies the following conditions:

$$u(t),u_t(t)\in D^{\infty}(A),\ u \text{ and } u_t\in C([0,\infty);D(A^n)), \text{ for all } n\in\mathbf{N}.$$

Proof. First we notice that $G^{\frac{\lambda}{2}} = A^\lambda$, for all $\lambda \in \mathbf{N}$.

We take the inner product of $2A^{2\lambda}u_t(t)$ with (3.19) in X_2. We obtain

$$\begin{aligned} &\frac{d}{dt}\|A^\lambda u_t(t)\|^2 + \frac{d}{dt}\|A^{\lambda+1}u(t)\|^2 + \rho\frac{d}{dt}\|A^{\lambda+\frac{1}{2}}u(t)\|^2 \\ &+ k\|A^{\frac{1}{2}}u(t)\|^2 \cdot \frac{d}{dt}\|A^{\lambda+\frac{1}{2}}u(t)\|^2 = 0. \end{aligned} \tag{3.20}$$

Let us set $a_\lambda(t) = \|A^\lambda u_t(t)\|^2 + \|A^{\lambda+1}u(t)\|^2$, $b_\lambda(t) = \|A^{\lambda+\frac{1}{2}}u(t)\|^2$, and $c(t) = k\|A^{\frac{1}{2}}u(t)\|^2 + \varepsilon$, with $\varepsilon > 0$ arbitrary. It follows that equation (3.20) can be written as

$$\frac{da_\lambda(t)}{dt} + (\rho - \varepsilon)\,\frac{db_\lambda(t)}{dt} + c(t)\frac{db_\lambda(t)}{dt} = 0. \tag{3.21}$$

Let us define the function

$$d_\lambda(t) = a_\lambda(t) + (\rho - \varepsilon)\,b_\lambda(t) + c(t)b_\lambda(t). \tag{3.22}$$

Thus we notice that $d_\lambda(t) - (\rho - \varepsilon)b_\lambda(t)$ is always a positive function for all $t \geq 0$. Take the derivative with respect to t of both sides of (3.22) and we obtain

$$\frac{d}{dt}\left(d_\lambda(t) - (\rho - \varepsilon)\,b_\lambda(t)\right) = (\varepsilon - \rho)\,\frac{db_\lambda(t)}{dt} + \frac{dc(t)}{dt}b_\lambda(t). \tag{3.23}$$

We can use the Schwarz inequality to deduce that

$$\left|\frac{db_\lambda(t)}{dt}\right| \leq a_\lambda(t),\ \lambda \in \mathbf{N}. \tag{3.24}$$

From (3.23) and (3.24) we find

$$\left|\frac{d}{dt}(d_\lambda(t) - (\rho - \varepsilon)b_\lambda(t))\right| \leq |\varepsilon - \rho|a_\lambda(t) + \left|\frac{dc(t)}{dt}\right|b_\lambda(t),$$

and from (3.22) this last inequality changes into the following:

$$\left|\frac{d}{dt}\left(d_\lambda(t) - (\rho - \varepsilon)\,b_\lambda(t)\right)\right| \leq \left(|\varepsilon - \rho| + \frac{1}{c(t)}\left|\frac{dc(t)}{dt}\right|\right)\left(d_\lambda(t) - (\rho - \varepsilon)\,b_\lambda(t)\right). \tag{3.25}$$

We need to find an estimate for $|\frac{dc(t)}{dt}|$ as follows:

$$\left|\frac{dc(t)}{dt}\right| = k\left|\frac{d}{dt}\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}u\rangle\right| = 2k\,|\langle Au, u_t\rangle| \leq 2k\left(\|Au\|^2 + \|u_t\|^2\right) \leq C_5,$$

where C_5 is a constant independent of t. Since $c(t) \geq \varepsilon > 0$, the inequality (3.25) can be written as

$$\left| \frac{d}{dt} \left(d_\lambda(t) - (\rho - \varepsilon)\, b_\lambda(t) \right) \right| \leq C_6 \left(d_\lambda(t) - (\rho - \varepsilon)\, b_\lambda(t) \right) \tag{3.26}$$

with $C_6 = |\varepsilon - \rho| + \frac{C_5}{\varepsilon}$. From (3.26) and Gronwall's inequality, we obtain

$$d_\lambda(t) - (\rho - \varepsilon)\, b_\lambda(t) \leq \left(d_\lambda(0) - (\rho - \varepsilon)\, b_\lambda(0) \right) e^{C_6 t}, \tag{3.27}$$

for all $t \geq 0$.

Finally, we let $\varepsilon \nearrow 1^+$. It follows from (3.27) and the assumption on the initial data that there is a constant $C_7 = C_7(t)$ such that

$$\|A^\lambda u_t(t)\|^2 + \|A^{\lambda+1} u(t)\|^2 + \|A^{\lambda+\frac{1}{2}} u(t)\|^2 < C_7,$$

for all $\lambda \in \mathbf{N}$ and all $T > 0$. This completes the proof. □

Also to be noted is that another interesting problem is the case when the nonlinear evolution equation (1.4)–(1.6) has the corresponding coefficients α, ρ, k, β, and γ as functions of the space variable x, as well the time variable t. For that we need to use the theory developed by Kato and Goldstein for the time-dependent hyperbolic evolution equations. The details will appear elsewhere.

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