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A WAVE EQUATION IN THE CURL-FREE SPACE RELATED TO A SMECTICS LIQUID CRYSTAL

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Abstract. A wave equation in the curl-free space is treated, and its weak solutions are constructed by use of a minimizing method. Numerical calculations are also done. Physical backgrounds are on equilibrium states of the smectics liquid crystal. Motion of zero points for the director vector is shown under control of an equation of wave type.

1. Introduction

In this paper, we treat a equation of wave type related to the smectics liquid crystal model. In the smectic model, the director vectors are considered to have a "small curl" and constant length almost everywhere. Equilibrium states can be written as an elliptic minimizing problem of the functional J, which will be introduced in Section 2 (see [3]). The focus of this problem is on the degenerate leading term of the equation and curl-free request. Elliptic or parabolic cases were already studied numerically by the author in [10] for treating eikonal equations (see [1] and [2]). These results are applicable to a magnetic thin-film problem introduced in [4].

In a usual liquid crystal other than smectics, ordering of director vectors is mainly caused by the free energy and behaves like a liquid. In the case of smectics, they have layer structures and behave similar to an elastic body against deformation of vertical direction to layers. We should consider kinetic energy and use the second time derivative for describing motion caused by deformation. Thus we adopt a new equation (1.1) for our starting analysis. The mathematical technique depends on the decomposition of H^1 and recovery of degeneracy using a special domain such as a ball or cube.

The basic tool is the discrete Morse flow, which is constructed by minimizers of a time-semidiscretized functional. By use of minimality, an energy

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estimate can be obtained, and consequently we get weak solutions from approximate solutions. It works well to use numerical calculations also. Our scheme has been developed to solve a Ginzburg–Landau-type problem and has been shown to be useful (see [8], [12], and [5]).

We introduce our problem. Let Ω be $\mathbf{B}_N = \{x \in \boldsymbol{R}^N; |x| < 1\}$ or $\mathbf{Q}_N = \{x \in \boldsymbol{R}^N; |x_i| < 1\}$ in $\boldsymbol{R}^N (N = 2, 3)$. We look for a vector-valued function $u : \Omega \times [0, \infty) \to \boldsymbol{R}^N$ satisfying a degenerate hyperbolic system with curl-free equation:

$$\begin{cases} u_{tt} - \nabla(\operatorname{div} u) &= \dfrac{1}{\delta}(1 - |u|^2)u \quad + \quad \operatorname{curl} \psi \\ \operatorname{curl} u &= 0, \\ u(0) &= u_0, \\ u_t(0) &= v_0, \end{cases} \tag{1.1}$$

for all $\psi \in W^{1,2}(\Omega; \boldsymbol{R}^N)$ and under suitable boundary conditions such as $\langle u, \nu \rangle = 0$ or $\langle u, \tau \rangle = 0$ where, ν is the unit outer normal of Ω and the τ are tangentials to $\partial\Omega$. Usually the first equation of (1.1) is a degenerate type. But by the appropriate choice of boundary conditions and the shape of Ω, we can recover coercivity. In a physical sense, u describes a director vector field of smectics liquid crystal. From an elliptic minimizing problem, one can see that zero points are an $N - 1$-dimensional set for small δ. So in the hyperbolic case, our numerical target is to know whether the set of zeros behaves like a membrane vibration (while in this case δ is positive). By our numerical experiment, it seems true.

Only the idea of how to construct a weak solution when $N = 2$ was introduced in [11], without giving proofs. Here we show proofs when $N = 3$ and numerical results ($N = 2, 3$).

2. Basic calculations

In this section, we introduce a discrete Morse flow of hyperbolic type and show coercivity of this variational treatment under some assumptions on the shape of Ω and boundary conditions.

We introduce an elliptic variational problem for (1.1): Find a minimizer of

$$J(u) = \int_\Omega \left(\frac{1}{2}(\operatorname{div} u)^2 + \frac{1}{\delta}\left(|u|^2 - 1\right)^2 \right) dx \tag{2.1}$$

$$\text{on } \mathcal{K}_\nu := \{u \in H^1(\Omega; \boldsymbol{R}^N) \cap L^4(\Omega; \boldsymbol{R}^N); \operatorname{curl} u = 0, \langle u, \nu \rangle = 0 \text{ on } \partial\Omega\}$$
$$\text{or } \mathcal{K}_\tau := \{u \in H^1(\Omega; \boldsymbol{R}^N) \cap L^4(\Omega; \boldsymbol{R}^N);\ \operatorname{curl} u = 0,$$
$$\langle u, \tau \rangle = 0 \ \text{ on } \partial\Omega \ \forall \tau \in T_x(\partial\Omega)\}.$$

We denote by $\mathcal{V}_\nu = \overline{\mathcal{K}}_\nu$ and $\mathcal{V}_\tau = \overline{\mathcal{K}}_\tau$ closures of $\mathcal{K}_\nu$ and $\mathcal{K}_\tau$ in the sense of $L^2(\Omega; \boldsymbol{R}^N)$, respectively. Sometimes we omit the subscripts ν and τ when no confusion may occur.

Here we introduce the following (hyperbolic) functionals:

$$J_n(u) := \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2} dx + J(u) \quad (n \geq 2). \tag{2.2}$$

We will determine a sequence $\{u_n\}$ in $\mathcal{K}$ by induction. For u_0 and $u_1 = u_0 + hv_0$, we define u_2 as a minimizer of J_2. A function $u_2 \in \mathcal{K}$ is determined as a minimizer of J_3. We can determine a sequence of minimizers $\{u_n\}$. By the use of $\{u_n\}$, we can construct an approximate weak solution which will be a weak solution to (1.1) when $h \to 0$. This method is called the discrete Morse flow in the hyperbolic type and was first introduced by Tachikawa [13], in the case when $J(u)$ is strictly convex. Related results are given in [7], [9], and [11].

A variational problem coming from (2.1) is not usually coercive for arbitrary Ω. But one can recover the coercivity by the appropriate choice of Ω and the boundary conditions. The following lemmas are essential:

Lemma 2.1. *If* $\Omega = \mathbf{B}_N$ $(N = 2, 3)$ *and* u *satisfies* $\langle u, \nu \rangle = 0$ *or* $\langle u, \tau \rangle = 0$, *then*

$$\int_\Omega \big((\text{div } u)^2 + |\text{curl } u|^2\big) dx = \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} |u|^2 ds.$$

Proof. We show the case $N = 3$. For $N = 2$, we can proceed in the same way here (see [10]). It is easy to see that

$$\int_\Omega [(\text{div } u)^2 + |\text{curl } u|^2] dx = \int_\Omega |\nabla u|^2 dx$$
$$+ 2 \int_\Omega (u^1_{x_1} u^2_{x_2} - u^2_{x_1} u^1_{x_2} + u^2_{x_2} u^3_{x_3} - u^3_{x_2} u^2_{x_3} + u^1_{x_1} u^3_{x_3} - u^3_{x_1} u^1_{x_3}) dx,$$

where $u = (u^1, u^2, u^3)$ and the subscript x_i $(i = 1, 2, 3)$ means the partial derivative in x_i direction. We here denote the last term by I. It is easy to see that

$$I = \int_{\partial\Omega} (u^1 u^2_{x_2} \nu_1 + u^1_{x_1} u^2 \nu_2 - u^1 u^2_{x_1} \nu_2 - u^1_{x_2} u^2 \nu_1) ds$$
$$+ \int_{\partial\Omega} (u^2 u^3_{x_3} \nu_2 + u^2_{x_2} u^3 \nu_3 - u^2_{x_3} u^3 \nu_2 - u^2 u^3_{x_2} \nu_3) ds$$
$$+ \int_{\partial\Omega} (u^1 u^3_{x_3} \nu_1 + u^1_{x_1} u^3 \nu_3 - u^1_{x_3} u^3 \nu_1 - u^1 u^3_{x_1} \nu_3) ds,$$

where $\nu = (\nu_1, \nu_2, \nu_3)$.

In the case of $\langle u, \nu\rangle = 0$ on $\partial\Omega$, we have

$$\begin{aligned} 0 = &(-u^2\nu_2 - u^3\nu_3)u^1_{x_1} + (u^1)^2 + u^1u^2_{x_1}\nu_2 + u^1u^3_{x_1}\nu_3 \\ &+u^2u^1_{x_2}\nu_1 + (-u^1\nu_1 - u^3\nu_3)u^2_{x_2} + (u^2)^2 + u^2u^3_{x_2}\nu_3 \\ &+u^3u^1_{x_3}\nu_1 + u^3u^2_{x_3}\nu_2 + (-u^1\nu_1 - u^2\nu_2)u^3_{x_3} + (u^3)^2; \end{aligned}$$

then $I = \int_{\partial\Omega} |u|^2 dx$.

In the same way, we can examine the case when $\langle u, \tau\rangle = 0$. □

The next lemma is well-known and can be proved by calculating I.

Lemma 2.2. *If* $\Omega = \mathbf{Q}_N$ $(N = 2, 3)$ *and* u *satisfies* $\langle u, \nu\rangle = 0$ *or* $\langle u, \tau\rangle = 0$ *on* $\partial\Omega$, *then*

$$\int_\Omega \big((\text{div } u)^2 + |\text{curl } u|^2\big)dx = \int_\Omega |\nabla u|^2 dx.$$

Using Lemmas 1.1 and 1.2, one can get the existence theorem.

Proposition 2.3. *If* Ω *is a ball or a rectangular domain, then the minimizer of the functional* (2.2) *exists in* $\mathcal{K}_\nu$ *and* $\mathcal{K}_\tau$.

3. Construction of weak solutions (friction term)

In this section, we construct weak solutions as in [8]. Here we put $f(u) = \frac{1}{\delta}(|u|^2 - 1)^2$, $\delta > 0$. We, at first, construct a weak solution of the equation adding a friction term,

$$\begin{aligned} &u_{tt} + \mu u_t &&= -\text{grad} J(u), \\ &\text{curl } u &&= 0, \\ &u(0) = u_0, && u_t(0) = v_0, \end{aligned} \tag{3.1}$$

with $u_0, v_0 \in \mathcal{K}$. We denote the weak solution of (3.1) by u^μ. Passing to the limit as $\mu \to +0$, we will get a weak solution of (1.1). Then we show the case $v_0 \in \mathcal{V}$.

For $h > 0$, let us define the sequence $\{u_n\} \subset \mathcal{K}_\nu$ (or $\mathcal{K}_\tau$) by

$$u_1 = u_0 + hv_0, \qquad J^\mu_n(u_n) = \inf_{u\in\mathcal{K}} J^\mu_n(u) \quad \text{for} \quad n \geq 2,$$

where

$$J^\mu_n(u) = \frac{\|u - 2u_{n-1} + u_{n-2}\|^2}{2h^2} + \mu\frac{\|u - u_{n-1}\|^2}{2h} + J(u),$$

denoting the $L^2(\Omega; \mathbf{R}^N)$ norm by $\|\cdot\|$.

Proposition 3.1. *For any $\epsilon \in (0,\mu)$ there exists $h_0 = h_0(C_1,\epsilon) > 0$, where C_1 is a positive constant with $\mathrm{Hess}_u f(u) \geq -C_1$, such that if $h \in (0,h_0]$, then for $n \geq 2$,*

$$\frac{\|u_n - u_{n-1}\|^2}{2h^2} + (\mu-\epsilon)\sum_{k=2}^{n}\frac{\|u_k - u_{k-1}\|^2}{h} + J(u_n) \leq \frac{1}{2}\|v_0\|^2 + J(u_1).$$

Proof. As in [8],

$$J_k^\mu(u_k) \leq J_k^\mu((1-\lambda)u_k + \lambda u_{k-1})\,\lambda \in (0,1)$$

implies

$$0 \leq \frac{1}{\lambda}\big\{J_k^\mu(u_k + \lambda(u_{k-1}-u_k)) - J_k^\mu(u_k)\big\}.$$

Letting $\lambda \to +0$, we have

$$\begin{aligned}
&\frac{d}{d\lambda}\big\|\{u_k + \lambda(u_{k-1}-u_k)\} - 2u_{k-1} + u_{k-2}\big\|^2\Big|_{\lambda=+0} \\
&\qquad \leq \|u_{k-1}-u_{k-2}\|^2 - \|u_k - u_{k-1}\|^2, \\
&\frac{d}{d\lambda}\big\|\{u_k + \lambda(u_{k-1}-u_k)\} - u_{k-1}\big\|^2\Big|_{\lambda=+0} = -2\|u_k - u_{k-1}\|^2, \\
&\frac{d}{d\lambda}J(u_k + \lambda(u_{k-1}-u_k))\Big|_{\lambda=+0} \leq J(u_{k-1}) - J(u_k) + \frac{C_1}{2}\|u_k - u_{k-1}\|^2.
\end{aligned}$$

If h satisfies that $\frac{\mu-\epsilon}{h} \leq \frac{\mu}{h} - \frac{C_1}{2}$, i.e., $h \leq \frac{2\epsilon}{C_1} \equiv h_0$, then

$$\frac{\|u_k - u_{k-1}\|^2}{2h^2} + (\mu-\epsilon)\frac{\|u_k - u_{k-1}\|^2}{h} + J(u_k) \leq \frac{\|u_{k-1} - u_{k-2}\|^2}{2h^2} + J(u_{k-1})$$

holds for $k \geq 2$. Summing them up with respect to k from 2 to n, we obtain the assertion. □

Definition 3.2 (Approximate solution)**.** *We define $\bar{u}^h$ and u^h on $\Omega \times (0,\infty)$ by*

$$\begin{aligned}
\bar{u}^h(x,t) &= u_n^h(x), \\
u^h(x,t) &= \frac{t-(n-1)h}{h}u_n^h(x) + \frac{nh-t}{h}u_{n-1}^h(x),
\end{aligned}$$

for $(x,t) \in \Omega \times ((n-1)h, nh]$, $n \in \boldsymbol{N}$.

Corollary 3.3. *If $h \in (0,h_0]$, then*

$$\frac{1}{2}\|u_t^h\|^2 + (\mu-\epsilon)\int_0^t \|u_t^h\|^2 dt + J(\bar{u}^h) \leq \Big\{\frac{1}{2} + (\mu-\epsilon)h\Big\}\|v_0\|^2 + J(u_0 + hv_0)$$

holds for almost every $t > 0$.

It follows from this corollary that for every $T > 0$,
$\{u_t^h\}_{h\in(0,h_0]}$ is bounded in $L^\infty(0,\infty;\mathcal{V}) \cap L^2(0,\infty;\mathcal{V})$,
$\{u^h\}_{h\in(0,h_0]}$ is bounded in $L^\infty(0,\infty;\mathcal{K}) \cap L^\infty(0,T;\, L^4(\Omega;\mathbf{R}^N))$, and
$\{\bar{u}^h\}_{h\in(0,h_0]}$ is bounded in $L^\infty(0,\infty;\mathcal{K}) \cap L^\infty(0,T;L^4(\Omega;\mathbf{R}^N))$. Thus, we can choose a subsequence such that u_t^h, u^h, and $\bar{u}^h$ converge to u_t and u in the weak (-star) topology of each space ($h \downarrow 0$). $\{u_n\}_{n\geq 2}$ satisfies $\mathrm{grad} J_n(u_n) = 0$; i.e.,

$$\int_h^\infty \int_\Omega \Big(\frac{u_t^h(t) - u_t^h(t-h)}{h}\phi(t) + \mu u_t^h \phi + \operatorname{div} \bar{u}^h \cdot \operatorname{div} \phi + \nabla_u f(\bar{u}^h)\phi\Big) dxdt = 0$$

for any $\phi \in C_0^1([0,\infty); C_0^1(\Omega))$ with $\mathrm{curl}\phi = 0$. Passing to the limit as $h \downarrow 0$ along the subsequence, we get

$$\int_0^\infty \int_\Omega (-u_t\phi_t + \mu u_t \phi + \operatorname{div} u \cdot \operatorname{div} \phi + \nabla_u f(u)\phi)\, dxdt - \int_\Omega v_0 \phi(0)\, dx = 0,$$

$u(0) = u_0$, in the trace sense, and $\langle u, \nu\rangle = 0$ or $\langle u, \tau\rangle = 0$; i.e., u is a *weak solution* to (1.1) with $v_0 \in \mathcal{K}$. By use of an argument similar to that in [12], it is deduced from Corollary 3.3 that

$$\frac{1}{2}\|u_t\|^2 + (\mu - \epsilon)\int_0^t \|u_t\|^2 dt + J(u) \leq \frac{1}{2}\|v_0\|^2 + J(u_0)$$

holds for almost every $t > 0$. Since $\epsilon \in (0, \mu)$ is arbitrary, we can replace $\mu - \epsilon$ by μ.

We denote the weak solution to (3.1) as above by u^μ. It satisfies

$$\frac{1}{2}\|u_t^\mu\|^2 + \mu \int_0^t \|u_t^\mu\|^2 dt + J(u^\mu) \leq \frac{1}{2}\|v_0\|^2 + J(u_0) \tag{3.2}$$

for almost every $t > 0$. This shows that $\{u_t^\mu\}_{\mu>0}$ is uniformly bounded in $L^\infty(0,\infty;\mathcal{V})$ and $\{u^\mu\}_{\mu>0}$ is uniformly bounded in $L^\infty(0,\infty;\mathcal{K}) \cap L^\infty(0,T;$ $L^4(\Omega;\mathbf{R}^N))$ for all $T > 0$. Hence, there exists a suitable subsequence of $\mu \downarrow 0$ such that u_t^μ and u^μ converge to u_t and u in the weak-star topology of the above spaces. Moreover, (3.2) gives the uniform bound of $\{\sqrt{\mu}u_t^\mu\}_{\mu>0}$ in $L^2(0,\infty;\mathcal{K})$. Thus, for any $\phi \in C_0^1([0,\infty); C^1(\Omega))$ with curl $\phi = 0$ for almost every t we have

$$\begin{aligned}\Big|\int_0^\infty \int_\Omega \mu u_t^\mu \phi \, dx\, dt\Big| &\leq \sqrt{\mu}\Big(\int_0^\infty \mu\|u_t^\mu\|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^\infty \|\phi\|^2 dt\Big)^{\frac{1}{2}} \\ &\leq \sqrt{\mu}\Big(\frac{1}{2}\|v_0\|^2 + J(u_0)\Big)^{\frac{1}{2}}\Big(\int_0^\infty \|\phi\|^2 dt\Big)^{\frac{1}{2}} \to 0\end{aligned}$$

as $\mu \downarrow 0$. Passing to the limit of (3.2) along the subsequence, we find that the limit function u solves (1.1) in a weak sense. The inequality (3.2) reduces the energy inequality

$$\frac{1}{2}\|u_t\|^2 + J(u) \leq \frac{1}{2}\|v_0\|^2 + J(u_0) \tag{3.3}$$

for almost every $t > 0$.

Now we construct a weak solution with the initial data $v_0 \in \mathcal{V}$. Since $\mathcal{K} \cap L^4(\Omega; \mathbf{R}^N)$ is dense in $\mathcal{V} \cap L^4(\Omega; \mathbf{R}^N)$, there exists $\{v_0^\epsilon\} \subset \mathcal{K} \cap L^4(\Omega; \mathbf{R}^N)$ such that $\|v_0^\epsilon - v_0\| \leq \epsilon$, $\|v_0^\epsilon\| \leq C_2$. Let u^ϵ be a weak solution to (1.1) with $v_0 = v_0^\epsilon$ constructed by the above procedure. It satisfies

$$\begin{aligned}
\int_0^\infty \int_\Omega \quad & (-u_t^\epsilon \phi_t + \operatorname{div} u^\epsilon \cdot \operatorname{div} \phi + \nabla_u f(u^\epsilon)\phi)\, dx\, dt - \int_\Omega v_0^\epsilon \phi(0)\, dx = 0, \\
& u^\varepsilon(0) = u_0, \ \text{ and} \langle u, \nu \rangle = 0 \ \text{ or} \langle u, \tau \rangle = 0 \quad \text{in the trace sense}, \\
& \frac{1}{2}\|u_t^\epsilon\|^2 + J(u^\epsilon) \leq \frac{1}{2}\|v_0^\epsilon\|^2 + J(u_0) \leq \frac{C_2^2}{2} + J(u_0).
\end{aligned}$$

The last inequality shows $\{u_t^\epsilon\}$ and $\{u^\epsilon\}$ are uniformly bounded respectively in $L^\infty(0, \infty; \mathcal{V})$ and $L^\infty(0, \infty; \mathcal{K}) \cap L^\infty(0, T; L^4(\Omega; \mathbf{R}^N))$ for any $T > 0$. Using an argument similar to that used before, we get a weak solution to (1.1) by the limit of u^ϵ as $\epsilon \downarrow 0$ along a suitable subsequence. It satisfies the energy inequality (3.3) for almost every $t > 0$.

4. Construction of weak solution (Gronwall's lemma)

In this section, we shall derive an inequality similar to that of the previous section, without the friction term. By this fact, we can say that minimizers of (2.2) also satisfy the energy inequality without extra terms. It assures us that by use of minimizers obtained directly from (2.2) we can construct approximate solutions for the hyperbolic problem.

We assume $v_0 \in \mathcal{K} \cap L^4(\Omega; \mathbf{R}^N)$, and redefine $\{u_n\}$ by

$$u_1 = u_0 + hv_0, \qquad J_n(u_n) = \inf_{u \in \kappa} J_n(u) \quad \text{for} \quad n \geq 2,$$

where $J_n(u) = \frac{\|u - 2u_{n-1} + u_{n-2}\|^2}{2h^2} + J(u)$.

Proposition 4.1. *If $h \in (0, C_1^{-\frac{1}{2}})$, then*

$$\frac{\|u_n - u_{n-1}\|^2}{2h^2} + J(u_n) \leq \frac{1}{1 - C_1 h^2} \exp\left\{ \frac{(n-2)_+ C_1 h^2}{1 - C_1 h^2} \right\} \left(\frac{1}{2}\|v_0\|^2 + J(u_1) \right),$$

where $(n-2)_+ = \max\{n-2, 0\}$.

Proof. We may assume $n \geq 2$. Similarly to the proof of Proposition 4.1, we get

$$\frac{\|u_k - u_{k-1}\|^2}{2h^2} + J(u_k) \leq \frac{\|u_{k-1} - u_{k-2}\|^2}{2h^2} + J(u_{k-1}) + \frac{C_1}{2}\|u_k - u_{k-1}\|^2,$$

and therefore

$$\frac{\|u_n - u_{n-1}\|^2}{2h^2} + J(u_n) \quad \leq \tfrac{1}{2}\|v_0\|^2 + J(u_1) + C_1 h \sum_{k=2}^{n} \frac{\|u_k - u_{k-1}\|^2}{2h^2} \cdot h.$$

Applying the discrete Gronwall's lemma, 1.3.19 ii) in [6], we get the assertion.

Corollary 4.2. *Assume* $h \in (0, C_1^{-\frac{1}{2}})$. *Then there exist a nondecreasing function* $C_3(\cdot)$ *of* h *and an absolute positive constant* C_4 *such that*

$$\lim_{h \downarrow 0} C_3(h) = 1,$$

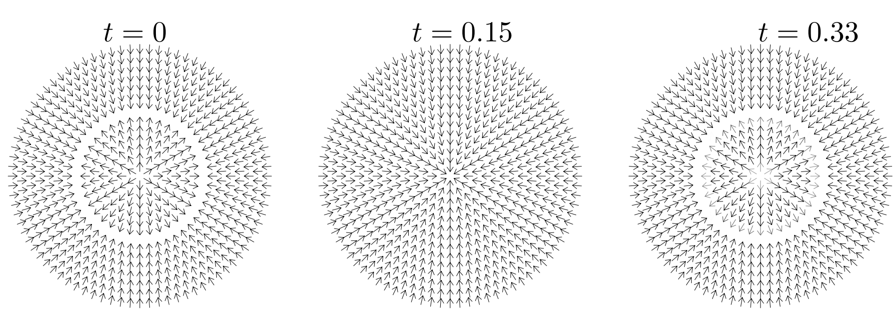
and

$$\frac{1}{2}\|u_t^h\|^2 + J(\bar{u}^h) \leq C_3(h) \exp(C_4 ht)\,(\frac{1}{2}\|v_0\|^2 + J(u_0 + hv_0))$$

hold for almost every $t > 0$.

Hence the argument in Section 3 is valid provided $L^p(0, \infty; *)$ is replaced by $L^p(0, T; *)$. Here $p = \infty$ or 2, and $* = \mathcal{V}$ or $\mathcal{K}$. That is, a subsequence of $\{u^h\}$ and $\{\bar{u}^h\}$ converges to a weak solution to (1.1) satisfying the energy inequality (3.3). The construction of weak solutions for $v_0 \in \mathcal{V}$ is exactly the same as in Section 3.

5. Numerical experiment



The numerical experiment was done by use of a minimizing method. In the hyperbolic case, vibration of the zero set can be seen.

Experiment 1. In the first case, we treat $\Omega = \mathbf{B}_2$. We put ring-shaped zero points on the initial data as is seen in the picture with zero initial velocity. The boundary condition is chosen as $\langle u, \tau \rangle = 0$.

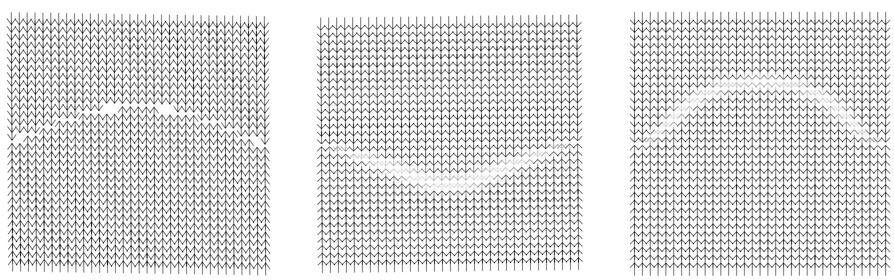
Zero points go to the center of $\mathbf{B}_2$ keeping the circle shape and disappear at $t = 0.15$. Zero points again appear from the center and spread to the same position as the initial data ($t = 0.33$).

From the stage $t = 0.33$, they move in the same way as $t = 0$ to $t = 0.33$.

Experiment 2. In the second case, we treat $\Omega = \mathbf{Q}_3$, and initial zero points are given on $0.375 \cos \pi x_1 \cos \pi x_2$ with zero initial velocity and satisfying $\langle u, \tau \rangle = 0$ on $|x_3| = 1$ and $\langle u, \nu \rangle = 0$ on $|x_1| = 1$ and $|x_2| = 1$.

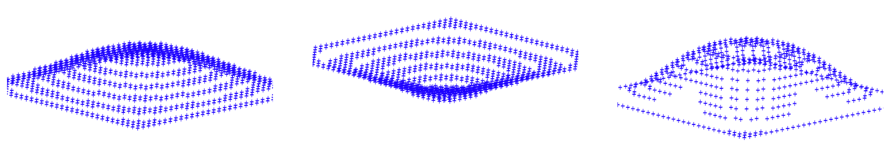
We show the vector field on the cross section $x_1 = 0$.

$t = 0$ $\qquad$ $t = 2.0$ $\qquad$ $t = 4.8$



Here we show the points in $\mathbf{Q}_3$ on which u satisfies $|u| \leq 0.15$.

$t = 0$ $\qquad$ $t = 2.0$ $\qquad$ $t = 4.8$



Conclusions. The dynamics of zero points of the director vectors are shown numerically. They behave like a membrane vibrating under control of a wave-type equation. Although local structures of zeros are also seen in some instances, they do not influence the global structure.

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