

## ON BASIS PROPERTY OF A HYPERBOLIC SYSTEM WITH DYNAMIC BOUNDARY CONDITION

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(Submitted by: Viorel Barbu)

**Abstract.** This paper addresses the basis property of a linear hyperbolic system with dynamic boundary condition in one space variable whose general form was first studied in [15]. It is shown that under a regularity assumption, the spectrum of the system displays a distribution on the complex plane similar to zeros of a sine-type function and the generalized eigenfunctions of the system constitute a Riesz basis for its root subspace. The state space thereby decomposes into a topological direct sum of the root subspace with another invariant subspace in which the associated semigroup is superstable: that is to say, the semigroup is identical to zero after a finite time. As a consequence, the spectrum-determined growth condition is established.

### 1. INTRODUCTION

This paper studies the basis property of a linear hyperbolic system with dynamic boundary condition in one space variable whose general form was first proposed and studied in [15]. The case of a static boundary condition has been investigated in [8]. This system is often encountered in the counter-flow heat exchange process, gas absorber process, tubular reactor process, and many other applications. In particular, it is connected with wave equations describing the dynamics of serially connected vibrating strings with

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Accepted for publication: March 2004.

AMS Subject Classifications: 93C20, 93D15, 35B35, 35P10.

This work is supported by the National Nature Science of China and supported partially by Lui Hui center for Applied Mathematics, Nan Kai University and Tianjin University.

joint stabilizers, as well as the vibrating strings with tip masses (see e.g. [13]). The system is described by the following linear homogeneous hyperbolic system in one space variable in normal form:

$$\begin{cases} \frac{\partial}{\partial t} \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} + K(x) \frac{\partial}{\partial x} \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} + C(x) \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = 0, \\ \frac{\partial}{\partial t} [v(1, t) - Du(1, t)] = Fu(1, t) + Gv(1, t), \\ u(0, t) = Ev(0, t), \end{cases} \quad 0 < x < 1, t > 0, \quad (1.1)$$

where (i)  $K(x) = \text{diag}\{\lambda_1(x), \lambda_2(x), \dots, \lambda_m(x), \mu_1(x), \mu_2(x), \dots, \mu_k(x)\}$  is a diagonal  $n \times n$  ( $n = m + k$ ) matrix with real entries  $\lambda_i(x), \mu_j(x) \in C^1[0, 1]$ ,  $\lambda_i(x) > 0$ ,  $\mu_j(x) < 0$ ,  $\forall x \in [0, 1]$ ,  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, k$ . In this article, we adopt the assumption that  $m \geq k$ .

(ii)  $C(x) = \text{diag}\{c_1(x), c_2(x), \dots, c_n(x)\}$  is an  $n \times n$  diagonal matrix with continuous entries in  $x \in [0, 1]$ ;

(iii)  $u(x) = [u_1(x), u_2(x), \dots, u_m(x)]^\top$  is a column vector in  $\mathbb{C}^m$  and  $v(x) = [v_1(x), v_2(x), \dots, v_k(x)]^\top$  is a column vector in  $\mathbb{C}^k$ ;

(iv)  $D, E, F$ , and  $G$  are real (or complex) constant matrices of appropriate size:  $D = \{d_{ij}\}_{k \times m}$ ,  $E = \{e_{ij}\}_{m \times k}$ ,  $F = \{f_{ij}\}_{k \times m}$ ,  $G = \{g_{ij}\}_{k \times k}$ .

In this paper, we show, under a regularity assumption, that the spectrum of the system displays a distribution as good as the zeros of a sine-type function in the complex plane, and the generalized eigenfunctions of the system constitute a Riesz basis for the root subspace. The state space thereby decomposes into a topological direct sum of the root subspace with another invariant subspace in which the associated semigroup is superstable: in other words, the semigroup is identical to zero after a finite time. As a consequence, the spectrum-determined growth condition is justified.

The organization of the paper is as follows. In the next section, some preliminary results are presented. Section 3 is devoted to the expansion of the solution of equation (1.1) in terms of the series of eigenprojections.

## 2. PRELIMINARY RESULTS

Consider the system (1.1) in the underlying Hilbert space  $\mathcal{H} = (L^2(0, 1))^n \times \mathbb{C}^k$  endowed with the usual inner product. Define the operator  $\mathcal{A} : D(\mathcal{A}) \subset$

$\mathcal{H}) \rightarrow \mathcal{H}$  by

$$\left\{ \begin{array}{l} \mathcal{A} \begin{bmatrix} u(x) \\ v(x) \\ d \end{bmatrix} = \begin{bmatrix} -K(x) \frac{\partial}{\partial x} \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} - C(x) \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} \\ Fu(1) + Gv(1) \end{bmatrix}, \\ D(\mathcal{A}) = \left\{ [u, v, d]^\top \in (H^1(0, 1))^m \times (H^1(0, 1))^k \times \mathbb{C}^k, \right. \\ \left. u(0) = Ev(0), d = v(1) - Du(1) \right\}. \end{array} \right. \quad (2.1)$$

Then the system (1.1) can be written as an evolutionary equation in  $\mathcal{H}$ :

$$\left\{ \begin{array}{l} \frac{dW(t)}{dt} = \mathcal{A}W(t), \quad t > 0 \\ W(0) = W_0 \end{array} \right. \quad (2.2)$$

with  $W(t) = [u(\cdot, t), v(\cdot, t), d(t)]^\top$  and initial data  $W_0 \in \mathcal{H}$ .

The following Theorem 2.1 was proved in [15] (details can also be found in [13]).

**Theorem 2.1.** (i) *The operator  $\mathcal{A}$  defined by (2.1) generates a  $C_0$  semigroup  $T(t)$  on  $\mathcal{H}$ .*

(ii) *The resolvent operator  $R(\lambda, \mathcal{A})$  can be represented as*

$$\begin{aligned} R(\lambda, \mathcal{A}) \begin{bmatrix} f \\ g \\ b \end{bmatrix} (x) &= \begin{bmatrix} u(x) \\ v(x) \\ d \end{bmatrix} \\ &= \begin{bmatrix} T(x, 0, \lambda) \begin{bmatrix} Ev(0) \\ v(0) \end{bmatrix} + \int_0^x T(x, s, \lambda) K^{-1}(s) \begin{bmatrix} f(s) \\ g(s) \end{bmatrix} \\ v(1) - Du(1) \end{bmatrix} \end{aligned}$$

where

$$T(x, s, \lambda) = \begin{bmatrix} Y_1(x, s, \lambda) & 0 \\ 0 & Y_2(x, s, \lambda) \end{bmatrix}$$

with

$$\left\{ \begin{array}{l} Y_1(x, s, \lambda) = \text{diag}\{e_{\lambda_1}(x, s, \lambda), e_{\lambda_2}(x, s, \lambda), \dots, e_{\lambda_m}(x, s, \lambda)\}, \\ e_{\lambda_i}(x, s, \lambda) = e^{-\lambda \int_s^x \frac{ds}{\lambda_i(s)} - \int_s^x \frac{c_i(s)}{\lambda_i(s)} ds}, \quad i = 1, 2, \dots, m, \\ Y_2(x, s, \lambda) = \text{diag}\{e_{\mu_1}(x, s, \lambda), e_{\mu_2}(x, s, \lambda), \dots, e_{\mu_k}(x, s, \lambda)\}, \\ e_{\mu_j}(x, s, \lambda) = e^{-\lambda \int_s^x \frac{ds}{\mu_j(s)} - \int_s^x \frac{c_j(s)}{\mu_j(s)} ds}, \quad j = 1, 2, \dots, k \end{array} \right.$$

and  $v(0)$  satisfying

$$H(\lambda)v(0) = -[\lambda D + F, G - \lambda I_k] \int_0^1 T(1, s, \lambda) K^{-1}(s) \begin{bmatrix} f(s) \\ g(s) \end{bmatrix} ds - b, \quad (2.3)$$

where  $I_k$  stands for the  $k \times k$  identity matrix and  $H(\lambda) = [\lambda D + F, G - \lambda I_k] T(1, 0, \lambda) \begin{bmatrix} E \\ I_k \end{bmatrix}$ . Therefore,  $\mathcal{A}$  is a discrete operator; that is to say, for any  $\lambda \in \sigma(\mathcal{A})$ ,  $R(\lambda, \mathcal{A})$  is compact on  $\mathcal{H}$ .

(iii)

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda : h(\lambda) = \det(H(\lambda)) = 0\}$$

and for each  $\lambda \in \sigma(\mathcal{A})$ , all eigenfunctions associated with  $\lambda$  can be represented as

$$Y_\lambda = \begin{bmatrix} Y_1(x, 0, \lambda) E v(0) \\ Y_2(x, 0, \lambda) v(0) \end{bmatrix} \text{ for some nonzero } v(0) \text{ satisfying } H(\lambda)v(0) = 0.$$

Hence the geometric multiplicity of each eigenvalue is less than or equal to  $k$ .

Recall that an entire function  $f(z)$  is said to be of *exponential type* if the inequality

$$|f(z)| \leq A e^{B|z|} \quad (2.4)$$

holds for some positive constants  $A$  and  $B$  and all complex values of  $z$ . It is seen obviously from (iii) of Theorem 2.1 that  $\sigma(\mathcal{A})$  consist of the zeros of a function of exponential type.

The function of exponential type  $f$  is called *sine-type* if the following two conditions are fulfilled (Definition II.1.27, [1]):

- (a) the zeros of  $f$  lie in a strip  $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq h\}$  for some  $h > 0$ ;
- (b) there are  $c_1, c_2 > 0$  and  $x_0 \in \mathbb{R}$  such that for all  $y \in \mathbb{R}$ ,  $c_1 \leq |f(x_0 + iy)| \leq c_2$ .

The class of sine-type functions was first introduced in [11] to cope with the problems of interpolation by entire functions and Riesz basis generation of exponential polynomials in  $L^2$  space. The distribution of the zeros of sine-type function is given by the next important proposition (Proposition II.1.28, [1]).

**Proposition 2.1.** *Let  $f$  be a sine-type function. Then its set of zeros (a multiple zero is repeated the number of times of its multiplicity) is a finite unification of separable sets; that is,*

$$\text{zeros of } f = \bigcup_{i=1}^M \Lambda_i, \quad \inf_{p \neq q, \lambda_p^i, \lambda_q^i \in \Lambda_i} |\lambda_p^i - \lambda_q^i| > 0$$

for some integer  $M < \infty$ . Consequently, the multiplicities of zeros of a sine-type function must be uniformly upper bounded.

**Lemma 2.1.** *The resolvent  $R(\lambda, \mathcal{A})$  can be expressed as*

$$R(\lambda, \mathcal{A})[f, g, b]^\top = \frac{\mathcal{I}G(f, g, b, \lambda)}{h(\lambda)} \quad (2.5)$$

where  $\mathcal{I}G(f, g, b, \lambda)$  is an  $H$ -valued entire function of exponential type.

**Proof.** Let  $R(\lambda, \mathcal{A})(f, g, b)^\top = (u, v, d)^\top$ , and let  $H(\lambda)$  be defined by (2.3). Then when  $h(\lambda) = \det(H(\lambda)) \neq 0$ , one has

$$H^{-1}(\lambda) = \frac{L(\lambda)}{h(\lambda)},$$

where  $L(\lambda)$  is a  $k \times k$  matrix whose entries consist of algebraic cofactors of corresponding entries of  $H(\lambda)$ . From (2.3), it follows that

$$v(0) = -\frac{L(\lambda)}{h(\lambda)}[\lambda D + F, G - \lambda I_k] \int_0^1 T(1, s, \lambda) K^{-1} \begin{bmatrix} f(s) \\ g(s) \end{bmatrix} ds - \frac{L(\lambda)}{h(\lambda)} b. \quad (2.6)$$

Noting that  $h(\lambda)I_k = H(\lambda)L(\lambda)$  and the transition property of  $T : T(1, 0, \lambda) = T(1, x, \lambda)T(x, 0, \lambda)$ ,  $T(1, s, \lambda) = T(1, x, \lambda)T(x, s, \lambda)$ , and substituting (2.6) into (ii) of Theorem 2.1, we obtain

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \frac{1}{h(\lambda)} \int_0^1 \mathcal{G}(x, s, \lambda) \begin{bmatrix} f(s) \\ g(s) \end{bmatrix} ds - \frac{1}{h(\lambda)} V(x, 0, \lambda) b$$

and

$$d = -\frac{1}{h(\lambda)} \left\{ \int_0^1 [D, -I_k] \mathcal{G}(1, s, \lambda) \begin{bmatrix} f(s) \\ g(s) \end{bmatrix} ds - [D, -I_k] V(1, 0, \lambda) b \right\},$$

where

$$\mathcal{G}(x, s, \lambda) = \begin{cases} -V(x, 0, \lambda)U(1, s, \lambda)K^{-1}(s), & x \leq s \leq 1, \\ [U(1, x, \lambda)V(x, 0, \lambda) - V(x, 0, \lambda)U(1, x, \lambda)]T(x, s, \lambda)K^{-1}(s), & 0 \leq s \leq x \end{cases} \quad (2.7)$$

with

$$U(1, x, \lambda) = [\lambda D + F, G - \lambda I_k]T(1, x, \lambda), \quad V(x, 0, \lambda) = T(x, 0, \lambda) \begin{bmatrix} E \\ I_k \end{bmatrix} L(\lambda). \quad (2.8)$$

Therefore, as  $\lambda \in \rho(\mathcal{A})$ , the resolvent operator is given by

$$R(\lambda, \mathcal{A})[f, g, b]^\top = \frac{\mathcal{I}G(f, g, b, \lambda)}{h(\lambda)}.$$

Because all  $U(1, x, \lambda)$ ,  $V(x, 0, \lambda)$ , and  $T(x, s, \lambda)$  are entire functions of exponential type, so is  $\mathcal{IG}(f, g, b, \lambda)$  with values in  $\mathcal{H}$ . The result follows.  $\square$

Since  $m \geq k$ ,  $E = \{e_{ij}\}_{m \times k}$ , and  $D = \{d_{ij}\}_{k \times m}$ , a straightforward calculation shows that

$$\begin{aligned} & \det(DY_1(1, 0, \lambda)E) \tag{2.9} \\ &= \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq m} \left[ \sum_{[i_1, i_2, \dots, i_k] \in [j_1, j_2, \dots, j_k]} [e_{i_1 1} e_{i_2 2} \dots e_{i_k k}] \cdot \det(\{d_{p i_p}\}_{k \times k}) \right] \\ & \times e_{\lambda_{j_1}}(1, 0, \lambda) e_{\lambda_{j_2}}(1, 0, \lambda) \dots e_{\lambda_{j_k}}(1, 0, \lambda), \end{aligned}$$

where  $[i_1, i_2, \dots, i_k]$  stands for an arrangement of  $[j_1, j_2, \dots, j_k]$  with  $j_1 < j_2 < \dots < j_k$  and both summations are summed up over all possible arrangements.

**Definition 2.1.** *System (1.1) is called regular if*

$$\det(DY_1(1, 0, \lambda)E) \neq 0 \text{ for } \lambda \in \mathbb{C}. \tag{2.10}$$

*This is equivalent to saying that there is at least one arrangement  $(j_1, j_2, \dots, j_k)$ ,  $1 \leq j_1 < j_2 < \dots < j_k \leq m$  satisfying*

$$\left[ \sum_{[i_1, i_2, \dots, i_k] \in [j_1, j_2, \dots, j_k]} [e_{i_1 1} e_{i_2 2} \dots e_{i_k k}] \cdot \det(\{d_{p i_p}\}_{k \times k}) \right] \neq 0.$$

The following lemma is from Lemma 6.10 of [13].

**Lemma 2.2.** *Let  $f$  be an exponential polynomial of the form*

$$f(\lambda) = \sum_{n=1}^N a_n e^{b_n \lambda} \quad (b_n \text{ are real}) \tag{2.11}$$

*where  $a_n$  and  $b_n$  are constants. Let  $\Omega = \{\lambda : f(\lambda) = 0\}$ . Then for all  $\lambda$  satisfying  $\text{dist}(\lambda, \Omega) \geq \delta > 0$ ,  $\alpha \leq \text{Re}\lambda \leq \beta$ , and  $|\text{Im}\lambda| \geq M$ , there exists a constant  $m(\delta, \alpha, \beta, M) > 0$  such that  $|f(\lambda)| > m(\delta, \alpha, \beta, M)$ , where  $M$ ,  $\delta$ ,  $\alpha$ , and  $\beta$  are arbitrary constants.*

It is clear from Lemma 2.2 that the exponential polynomial of the form (2.11) must be a sine-type function.

**Theorem 2.2.** *Suppose the system (1.1) is regular. Then*

(i) *The zeros of  $h$  are located on a strip paralleling the imaginary axis and decompose into a finite union of separable sets.*

(ii) *Suppose  $\sigma(\mathcal{A}) = \{\lambda_n\}_{n \in \mathcal{J}}$ , where henceforth  $\mathcal{J}$  stands for some subset of integers. Let  $m_a(\lambda_n)$  denote the algebraic multiplicity of  $\lambda_n$ . Then*

$$\sup_{n \in \mathcal{J}} m_a(\lambda_n) < \infty.$$

**Proof.** (ii) follows directly from (i), and Lemma 2.1. Indeed, in light of a general formula on page 148 of [13]

$$m_a(\lambda_n) \leq p_{\lambda_n} m_g(\lambda_n),$$

where  $m_g(\lambda_n)$  stands for the geometric multiplicity of  $\lambda_n$ , which is less than or equal to  $k$  by (iii) of Theorem 2.1, and  $p_{\lambda_n}$  stands for the order of the pole of  $R(\lambda, \mathcal{A})$  at  $\lambda_n$ , which is less than or equal to the multiplicity of  $\lambda_n$  as a zero of  $h(\lambda)$  from (2.5) and hence is uniformly bounded by (i). Therefore, we need only show (i). Note that

$$H(\lambda) = (\lambda D + F)Y_1(1, 0, \lambda)E + (G - \lambda I_k)Y_2(1, 0, \lambda). \quad (2.12)$$

It follows that

$$\lambda^{-1}H(\lambda) = (D + \lambda^{-1}F)Y_1(1, 0, \lambda)E + (\lambda^{-1}G - I_k)Y_2(1, 0, \lambda).$$

When  $\operatorname{Re}\lambda \rightarrow +\infty$ ,  $(D + \lambda^{-1}F)Y_1(1, 0, \lambda)E \rightarrow 0$ . Hence

$$\lim_{\operatorname{Re}\lambda \rightarrow +\infty} \frac{h(\lambda)}{\lambda^k \det(-Y_2(1, 0, \lambda))} = 1. \quad (2.13)$$

While  $\operatorname{Re}\lambda \rightarrow -\infty$ ,  $(\lambda^{-1}G - I_k)Y_2(1, 0, \lambda) \rightarrow 0$ . Hence

$$\lim_{\operatorname{Re}\lambda \rightarrow -\infty} \frac{h(\lambda)}{\lambda^k \det((D + \lambda^{-1}F)Y_1(1, 0, \lambda)E)} = 1. \quad (2.14)$$

Next, similar to (2.9), we have

$$\begin{aligned} & \det((D + \lambda^{-1}F)Y_1(1, \lambda)E) \\ &= \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq m} \left[ \sum_{[i_1, i_2, \dots, i_k] \in [j_1, j_2, \dots, j_k]} [e_{i_1 1} e_{i_2 2} \dots e_{i_k k}] \right. \\ & \quad \left. \cdot \det(\{d_{p i_p} + \lambda^{-1} f_{p i_p}\}_{k \times k}) \right] e_{\lambda_{j_1}}(1) e_{\lambda_{j_2}}(1) \dots e_{\lambda_{j_k}}(1). \end{aligned}$$

Since the system (1.1) is regular, it follows that

$$\begin{aligned} & \lim_{\operatorname{Re}\lambda \rightarrow +\infty} |\lambda^k \det(-Y_2(1, 0, \lambda))| = \infty, \\ & \lim_{\operatorname{Re}\lambda \rightarrow -\infty} |\lambda^k \det((D + \lambda^{-1}F)Y_1(1, 0, \lambda)E)| = \infty. \end{aligned}$$

By virtue of (2.13) and (2.14), for any  $\epsilon > 0$ , there exists a positive constant  $h_\epsilon > 0$  such that

$$|h(\lambda)| > \epsilon \text{ as } |\operatorname{Re}\lambda| > h_\epsilon. \quad (2.15)$$

Therefore,  $\sigma(\mathcal{A}) \subset \{\lambda : |\operatorname{Re}\lambda| < h_\epsilon\}$ . Finally, observe that  $h$  can be expanded as

$$h(\lambda) = \sum_{n=1}^N c_n(\lambda) e^{d_n \lambda} \quad (d_n \text{ are real})$$

for some positive integer  $N$ , constants  $d_n$ , and polynomials  $c_n(\lambda)$  of degree less than or equal to  $k$ . Let  $m$  be the maximal degree of all polynomials  $c_k(\lambda)$ . Then

$$\lambda^{-m} h(\lambda) = \sum_{n=1}^N a_n e^{b_n \lambda} + \mathcal{O}(\lambda^{-1}) \quad \text{as } |\operatorname{Re}\lambda| < h_\epsilon \text{ and } |\lambda| \rightarrow \infty \quad (2.16)$$

for some constants  $a_n$  and  $b_n$ , where  $\mathcal{O}(\lambda^{-1})$  means that

$$\sup_{|\operatorname{Re}\lambda| < h_\epsilon, |\lambda| \rightarrow \infty} |\lambda \mathcal{O}(\lambda^{-1})| < \infty$$

and the  $b_n$  are real. Since the exponential polynomial

$$f(\lambda) = \sum_{n=1}^N a_n e^{b_n \lambda}$$

is a sine-type function, it follows from Proposition 2.1 that the zeros of  $f$  consist of a union of a finite number of separable sets. Suppose (counted according to their multiplicities)

$$\text{zeros of } f = \Xi = \bigcup_{n=1}^M \Xi_n, \quad \inf_{i \neq j, \lambda_i, \lambda_j \in \Xi_n} |\lambda_i - \lambda_j| > 0, \quad \forall 1 \leq n \leq M, \quad 1 < M < \infty.$$

Let  $\gamma = \min_{1 \leq n \leq M} \inf_{i \neq j, \lambda_i, \lambda_j \in \Xi_n} |\lambda_i - \lambda_j| > 0$ . Then for any  $r = \delta/(2M)$ , by Lemma 1 of [3] (see also [8]), there exist  $\Xi^p = \{\lambda_{j,p}\}_{j=1}^{M^p}$ ,  $M^p \leq M$ ,  $p \in \mathcal{J}$ , the  $p$ -th connected component of intersection of  $\Xi$  with  $\bigcup_{n \in \mathcal{J}} D_{\lambda_n}(r)$ , where  $D_{\lambda_n}(r)$  is the circle centered at  $\lambda_n$  with radius  $r$ , such that

$$\text{zeros of } f = \bigcup_{p \in \mathcal{J}} \Xi^p.$$

We may assume without loss of generality that there exists a  $\delta_0 > 0$  such that  $\inf_{n, m \in \mathcal{J}, n \neq m} \operatorname{disp}(\Xi_n, \Xi_m) > 0$ . In light of this fact and applying Rouché's theorem to (2.16), we conclude immediately the assertion (i).  $\square$

**Theorem 2.3.** *Suppose (2.15). Then*

$$\sup_{\operatorname{Re}\lambda = -h_\epsilon - \epsilon} \|R(\lambda, \mathcal{A})\| < \infty. \quad (2.17)$$



**Proof.** By assumption, the straight line  $\operatorname{Re}\lambda = -h_\epsilon - \epsilon$  lies in the resolvent set of  $\mathcal{A}$ , so for any  $W = [f, g, b]^T \in \mathcal{H}$ ,  $[u, v, d]^T = R(\lambda, \mathcal{A})W \in D(\mathcal{A})$  is given by (ii) of Theorem 2.1; that is,

$$\left\{ \begin{array}{l} u(x) = Y_1(x, 0, \lambda)Ev(0) + \int_0^x Y_1(x, s, \lambda)K_1^{-1}(s)f(s)ds, \\ v(x) = Y_2(x, 0, \lambda)v(0) + \int_0^x Y_2(x, s, \lambda)K_2^{-1}(s)g(s)ds, \\ d = v(1) - Du(1), \\ H(\lambda)v(0) = -(\lambda D + F) \int_0^1 Y_1(1, s, \lambda)K_1^{-1}(s)f(s)ds \\ \quad - (G - \lambda I_k) \int_0^1 Y_2(1, s, \lambda)K_2^{-1}(s)g(s)ds - b, \end{array} \right. \quad (2.18)$$

where  $H(\lambda)$  is given by (2.12). Obviously

$$\begin{aligned} \sup_{\operatorname{Re}\lambda = -h_\epsilon - \epsilon, x \in [0, 1]} [\|Y_1(x, 0, \lambda)\|_{\mathbb{C}^m} + \|Y_2(x, 0, \lambda)\|_{\mathbb{C}^k}] &< \infty, \\ \sup_{\operatorname{Re}\lambda = -h_\epsilon - \epsilon, x \in [0, 1]} \left\| \int_0^x Y_1(x, s, \lambda)K_1^{-1}f(s)ds \right\|_{\mathbb{C}^m} &< \infty, \\ \sup_{\operatorname{Re}\lambda = -h_\epsilon - \epsilon, x \in [0, 1]} \left\| \int_0^x Y_2(x, s, \lambda)K_2^{-1}g(s)ds \right\|_{\mathbb{C}^k} &< \infty. \end{aligned}$$

It follows that there exists a constant  $K_1 > 0$  such that

$$\begin{aligned} &\left\| \int_0^1 (\lambda D + F)Y_1(1, s, \lambda)K_1^{-1}(s)f(s)ds \right. \\ &+ \left. \int_0^1 (G - \lambda I_k)Y_2(1, s, \lambda)K_2^{-1}(s)g(s)ds + b \right\|_{\mathbb{C}^k} \leq K_1|\lambda|, \quad \forall \operatorname{Re}\lambda = -h_\epsilon - \epsilon. \end{aligned} \quad (2.19)$$

Note that  $H^{-1}(\lambda) = \frac{L(\lambda)}{h(\lambda)}$  and the entries  $L_{ij}(\lambda)$  of  $L(\lambda)$  are of the form

$$L_{ij}(\lambda) = \sum_p a_p(\lambda)e^{\omega_p \lambda} \quad (\omega_p \text{ are real}),$$

where  $a_p(\lambda)$  are polynomials of  $\lambda$  of degree less than or equal to  $k - 1$  and  $\omega_p$  are constants. Hence there exists a constant  $K_2$  such that

$$\|L(\lambda)\| \leq K_2|\lambda|^{k-1}, \quad \forall \operatorname{Re}\lambda = -h_\epsilon - \epsilon.$$

By (2.13) and (2.14), we may assume without loss of generality that

$$\sup_{\operatorname{Re}\lambda=-h_\epsilon-\epsilon} |\lambda^k h(\lambda)| > 0.$$

Therefore,

$$\|H^{-1}(\lambda)\|_{\mathbb{C}^k} \leq \frac{K_3}{|\lambda|}, \quad \forall \operatorname{Re}\lambda = -h_\epsilon - \epsilon$$

with some constant  $K_3$  independent of  $\lambda$ . This together with (2.18)–(2.19) gives

$$\sup_{\operatorname{Re}\lambda=-h_\epsilon-\epsilon} \|v(0)\| < \infty.$$

By virtue of (2.18), we have thus obtained that

$$\sup_{\operatorname{Re}\lambda=-h_\epsilon-\epsilon, x \in [0,1]} (\|u(x)\|_{\mathbb{C}^m} + \|v(x)\|_{\mathbb{C}^k}) < \infty, \quad \sup_{\operatorname{Re}\lambda=-h_\epsilon-\epsilon} \|d\|_{\mathbb{C}^k} < \infty.$$

Therefore,

$$\sup_{\operatorname{Re}\lambda=-h_\epsilon-\epsilon} \|R(\lambda, \mathcal{A})W\|_{\mathbb{C}^k} < \infty, \quad \forall W \in \mathcal{H}.$$

The result then follows from the uniform bounded principle.  $\square$

Let us recall that  $x \in D(A)$  is called a generalized eigenvector of the linear operator  $A$  in a Hilbert space  $\mathbf{H}$  associated with an eigenvalue  $\lambda$  if there is an integer  $n \geq 1$  such that  $(\lambda - A)^n x = 0$ . The root subspace  $Sp(A)$  of  $A$  is the closed subspace of  $\mathbf{H}$  which is spanned by all generalized eigenfunctions of  $A$ . The semigroup  $S(t)$  generated by  $A$  is called hyperbolic when the space  $\mathbf{H}$  decomposes into  $\mathbf{H} = \mathbf{H}_- \oplus \mathbf{H}_+$  such that  $S(t)\mathbf{H}_\pm \in \mathbf{H}_\pm$ ,

$$S_-(t) : \mathbf{H}_- \rightarrow \mathbf{H}_-, \quad S_-(t)x = S(t)x$$

extends to a  $C_0$  group on  $\mathbf{H}_-$  over  $-\infty < t < \infty$ , and there are positive constants  $K, \alpha$ , and  $\beta$  such that

$$\begin{aligned} \|S(t)x\| &\leq Ke^{\beta t} \|\mathbb{P}x\|, \quad t \leq 0; \\ \|S(t)(I - \mathbb{P})x\| &\leq Ke^{-\alpha t} \|(I - \mathbb{P})x\|, \quad t \geq 0. \end{aligned}$$

Here  $\mathbb{P} : \mathbf{H} \rightarrow \mathbf{H}$  denotes the projection of  $\mathbf{H}$  along  $\mathbf{H}_+$  onto  $\mathbf{H}_-$ . The following theorem characterizes the hyperbolicity of the  $C_0$  semigroup (see e.g. [10]).

**Theorem 2.4** [Gearhart and Herbst].  *$S(t)$  is hyperbolic if and only if there exists an open strip containing the imaginary axis on which the resolvent of  $A$  is uniformly bounded.*

The following lemma is available in [19].

**Lemma 2.3.** *Let  $A$  be the generator of a  $C_0$  semigroup in a Hilbert space  $\mathbf{H}$ . Assume that  $A$  is discrete (so is  $A^*$ ) and for  $\lambda \in \rho(A^*)$ ,  $R(\lambda, A^*)$  is of the form*

$$R(\lambda, A^*)x = \frac{G(\lambda)x}{F(\lambda)}, \quad \forall x \in \mathbf{H}$$

where for each  $x \in \mathbf{H}$ ,  $G(\lambda)x$  is an  $H$ -valued entire function with order less than or equal to  $\rho_1$  and  $F(\lambda)$  is a scalar entire function of order  $\rho_2$ . Denote by  $\rho = \max\{\rho_1, \rho_2\} < \infty$  and an integer  $n$  so that  $n - 1 \leq \rho < n$ . If there are  $n + 1$  rays  $\gamma_j$ ,  $j = 0, 1, 2, \dots, n$  on the complex plane,

$$\arg \gamma_0 = \frac{\pi}{2} < \arg \gamma_1 < \arg \gamma_2 < \dots < \arg \gamma_n = \frac{3\pi}{2}$$

with

$$\arg \gamma_{j+1} - \arg \gamma_j \leq \frac{\pi}{n}, \quad 0 \leq j \leq n - 1$$

so that  $R(\lambda, A^*)x$  is bounded on each ray  $\gamma_j$ ,  $0 < j < n$  as  $|\lambda| \rightarrow \infty$  for any  $x \in \mathbf{H}$ , then  $Sp(A) = Sp(A^*) = \mathbf{H}$ .

Now we are in a position to show the main result of this section.

**Theorem 2.5.** *Suppose the system (1.1) is regular. Let  $\sigma(\mathcal{A}) = \{\lambda_n\}_{n \in \mathcal{J}}$ . Then the decomposition*

$$\mathcal{H} = Sp(\mathcal{A}) \oplus M_\infty(\mathcal{A}) \quad (\text{topological direct sum}) \quad (2.20)$$

holds with

$$M_\infty(\mathcal{A}) = \{W \in \mathbf{H} : \mathbb{P}_{\lambda_n} W = 0, \quad \forall n \in \mathcal{J}\}$$

and  $T(t)$  extends to a  $C_0$  group on  $Sp(\mathcal{A})$ . Moreover, there exists a  $t_0 > 0$  such that

$$T(t)W = 0, \quad \forall t > t_0 \text{ and } W \in M_\infty(\mathcal{A}).$$

**Proof.** By virtue of Theorem 2.3 and the resolvent identity, there exists a constant  $\delta > 0$  such that  $\{\lambda : |\lambda + h_\epsilon + \epsilon| \leq \delta\} \subset \rho(\mathcal{A})$  and

$$\sup_{\lambda \in \{\lambda : |\lambda + h_\epsilon + \epsilon| \leq \delta\}} \|R(\lambda, \mathcal{A})\| < \infty.$$

Let  $\mathcal{B} = \mathcal{A} + (h_\epsilon + \epsilon)I$ , where  $I$  is the identity operator of  $\mathcal{H}$ . Then  $\sigma(\mathcal{B}) \subset \{\mu | \epsilon \leq \operatorname{Re} \mu \leq 2h_\epsilon + \epsilon\}$  and

$$\sup_{|\mu| \leq \delta} \|R(\mu, \mathcal{B})\| < \infty.$$

Apply Theorem 2.4 to  $\mathcal{B}$  to obtain that  $\mathcal{B}$  is hyperbolic and hence  $\mathcal{H}$  decomposes into two closed subspaces  $\mathcal{H}_u$  and  $\mathcal{H}_s$ :  $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s = (I - \mathbb{P})\mathcal{H} \oplus \mathbb{P}\mathcal{H}$  (topological direct sum),  $e^{\mathcal{B}t}\mathcal{H}_u \subset \mathcal{H}_u$ , and  $e^{\mathcal{B}t}\mathcal{H}_s \subset \mathcal{H}_s$ , where  $\mathbb{P}$  denotes

the spectral projection of  $e^{\mathcal{B}}$  corresponding to the spectral set  $\{\mu \in \sigma(e^{\mathcal{B}}) : |\mu| < 1\}$ .  $e^{\mathcal{B}t}$  extends to a  $C_0$  group on  $\mathcal{H}_u$ , and the restriction of  $e^{\mathcal{B}t}$  on  $\mathcal{H}_s$  is an exponentially stable  $C_0$  semigroup. On the other hand, by the spectral mapping theorem (note that  $\sigma(\mathcal{B}) = \sigma_p(\mathcal{B})$ ),  $e^{\sigma(\mathcal{B})} \subset \sigma_p(e^{\mathcal{B}}) \subset e^{\sigma(\mathcal{B})} \cup \{0\} \subset \{\mu \mid |\mu| \geq e^\epsilon\} \cup \{0\}$  and the generalized eigenfunctions of  $\mathcal{B}$  corresponding to  $\mu$  are those of  $e^{\mathcal{B}}$  corresponding to  $e^\mu$  ([16]), whence  $Sp(\mathcal{B}) \subset (I - \mathbb{P})\mathcal{H} = \mathcal{H}_u$ . However, since  $\mathcal{B}$  generates a  $C_0$  group on  $\mathcal{H}_u$ , it follows from the Hille-Yosida theorem that

$$\|R(\lambda, \mathcal{B})\|_{\mathcal{H}_u} \rightarrow 0 \text{ as } |\lambda| \rightarrow +\infty.$$

Taking  $\rho_2 = \rho = 1$ ,  $n = 2$ , and  $\gamma_1 = \{\lambda : \arg \lambda = \pi\}$  and noticing the representation of  $R(\lambda, \mathcal{A})$  in Lemma 2.1,  $R(\lambda, \mathcal{A}^*)[f, g, b]^\top = \mathcal{I}G^*(f, g, b, \lambda)/h(\lambda)$ , we see that all conditions of Lemma 2.3 are satisfied for  $\mathcal{B}$  in  $\mathcal{H}_u$ . Therefore, the root subspace of  $\mathcal{B}$  is complete in  $\mathcal{H}_u$ :  $Sp(\mathcal{B}) = \mathcal{H}_u$  and hence  $Sp(\mathcal{A}) = \mathcal{H}_u$ . Furthermore, since  $\sigma(\mathcal{A}) = \sigma(\mathcal{A}|_{Sp(\mathcal{A})}) \cup \sigma(\mathcal{A}|_{\mathcal{H}_s})$ , we can assert  $\sigma(\mathcal{A}|_{\mathcal{H}_s}) = \emptyset$ , and so for any  $W \in \mathcal{H}_s$ ,  $R(\lambda, \mathcal{A})W$  is an  $H$ -valued entire function of  $\lambda$ . By Lemma 6 on page 2296 of [4],  $\mathcal{H}_s \subset M_\infty(\mathcal{A})$ . Therefore,  $\mathcal{H}_s = M_\infty(\mathcal{A})$ . Finally, by Lemma 2.1, for any  $W \in M_\infty(\mathcal{A})$ ,  $R(\lambda, \mathcal{A})W$  is an  $H$ -valued entire function of exponential type; in other words,

$$\|R(\lambda, \mathcal{A})W\| \leq M e^{t_0|\lambda|} \|W\|, \forall W \in M_\infty(\mathcal{A})$$

holds for some positive constants  $M$  and  $t_0$  and all complex  $\lambda$ . Since there exists an  $\omega \in \mathbb{R}$  so that  $\sup_{\sigma > \omega} \int_{\mathbb{R}} \|R(\sigma + i\tau, \mathcal{A})W\|^2 d\tau < \infty$ , it follows from the Paley-Wiener theorem (Theorem 18, [22], page 101) that there exists an  $H$ -valued function  $\phi_W$  in  $L^2(-t_0, t_0)$  satisfying

$$R(\lambda, \mathcal{A})W = \int_{-t_0}^{t_0} e^{-\lambda t} \phi_W(t) dt.$$

On the other hand, it always holds that

$$R(\lambda, \mathcal{A})W = \int_0^\infty e^{-\lambda t} T(t)W dt, \forall \operatorname{Re} \lambda > \omega, W \in M_\infty(\mathcal{A}).$$

By the uniqueness of the Laplace transform, we have  $T(t)W = 0$  for all  $t > t_0$ . The proof is complete.  $\square$

### 3. EXPANSION OF THE SOLUTION

In this section, we discuss the expansion of the solution of (1.1) in terms of the generalized eigenfunctions or Riesz basis property of the root subspace of the operator  $\mathcal{A}$ . We always assume that the system (1.1) is regular.

As was mentioned in Proposition 2.1, a scalar sequence of complex numbers  $\{\mu_n : n \in \mathcal{J}\}$  is called *separated* if

$$\inf_{n \neq m, n, m \in \mathcal{J}} |\mu_n - \mu_m| > 0. \quad (3.1)$$

The sequence  $\{W_n\}_{n \in \mathcal{J}}$  is called a *basis* for  $\mathcal{H}$  if any element  $W \in \mathcal{H}$  has a unique representation

$$W = \sum_{n \in \mathcal{J}} c_n W_n, \quad (3.2)$$

the series being convergent with respect to the norm of  $\mathcal{H}$ .  $\{W_n\}_{n \in \mathcal{J}}$  is called a *Riesz basis* for  $\mathcal{H}$  if

- a)  $\overline{\text{span}}\{W_n\} = \mathcal{H}$  and
- b) there exist positive constants  $m$  and  $M$  such that for any numbers  $c_n$ ,  $n \in \mathcal{I}$ , where  $\mathcal{I}$  is a finite subset of  $\mathcal{J}$ , we have

$$m \sum_{n \in \mathcal{I}} |c_n|^2 \leq \left\| \sum_{n \in \mathcal{I}} c_n W_n \right\|^2 \leq M \sum_{n \in \mathcal{I}} |c_n|^2.$$

A basis  $\{W_n, n \in \mathcal{J}\}$  of  $\mathcal{H}$  is called a *Riesz basis with parentheses* ([17]) if (3.2) converges in  $\mathcal{H}$  after putting some of its terms in parentheses, the arrangement of which does not depend on  $W$ .

In a Hilbert space, the most important bases are orthonormal. Second in importance are Riesz bases that are bases equivalent to some orthonormal basis. We refer to [22] for more details on Riesz bases.

Denote again  $\sigma(\mathcal{A}) = \{\lambda_n\}_{n \in \mathcal{J}}$ . Assume that each  $\lambda_n$  is of algebraic multiplicity  $m_n$ . Thus we have a set of complex exponentials associated with  $\lambda_n$

$$E_n(t) = \{e^{\lambda_n t}, t e^{\lambda_n t}, \dots, t^{m_n-1} e^{\lambda_n t}\}, \quad n \in \mathcal{J}.$$

The Riesz basis property of  $\{E_n(t), n \in \mathcal{J}\}$  in  $L^2(0, T)$  for some  $T > 0$  has been studied extensively by former Soviet mathematicians (Levin, Pavlov, Nikolskii, and many others) and necessary and sufficient conditions are already available in literature ([1], [9], and [22]) for the case that  $\{\lambda_n\}$  are separable. It was shown in [19] that the Riesz basis property of  $\{E_n(t), n \in \mathcal{J}\}$  in  $L^2(0, T)$  is closely related to the Riesz basis generation of the root subspace of  $\mathcal{A}$  if each eigenvalue is algebraically simple ( $m_n = 1$ ). Since it is hard to check the algebraic multiplicity and the separability of the eigenvalues, we shall make use of the concept of generalized divided difference (GDD) ([3] and [2]).

Now, since the zeros of  $h(\lambda)$  consist of eigenvalues of  $\mathcal{A}$  (however, the multiplicity as a zero of  $h$  may be different from algebraic multiplicity as an eigenvalue of  $\mathcal{A}$ ), and by Theorem 2.2, the zeros of  $h(\lambda)$  decompose into a finite union of separable sets (counted according to their multiplicities as

zeros of  $h$ ), and for all  $\lambda_n \in \sigma(A)$ , their algebraic multiplicity  $m_n$  is uniformly bounded,

$$\sup_{\lambda_n \in \sigma(\mathcal{A})} m_n < \infty, \quad (3.3)$$

hence the eigenvalues of  $\mathcal{A}$  consist of a finite union of separable sets (counted according to their algebraic multiplicities):

$$\text{eigenvalues of } \mathcal{A} = \Lambda = \bigcup_{n=1}^N \Lambda_n, \quad \inf_{i \neq j, \lambda_i, \lambda_j \in \Lambda_n} |\lambda_i - \lambda_j| > 0, \quad \forall 1 \leq n \leq N. \quad (3.4)$$

Let  $\delta = \min_{1 \leq n \leq N} \inf_{i \neq j, \lambda_i, \lambda_j \in \Lambda_n} |\lambda_i - \lambda_j| > 0$ . Then for any  $r < r_0 = \delta/(2N)$ , by Lemma 1 of [3], there exists  $\Lambda^p = \{\lambda_{j,p}\}_{j=1}^{M^p}$ ,  $M^p \leq N$ ,  $p \in \mathcal{J}$ , the  $p$ -th connected component of intersection of  $\Lambda$  with  $\bigcup_{n \in \mathcal{J}} D_{\lambda_n}(r)$ , where  $D_{\lambda_n}(r)$  is the circle centered at  $\lambda_n$  with radius  $r$ , such that

$$\sigma(A) = \bigcup_{p \in \mathcal{J}} \Lambda^p. \quad (3.5)$$

We may assume without loss of generality that the  $\{\lambda_n\}$  are arranged so that  $\text{Im}\lambda_n$  are nondecreasing and  $\text{Re}\lambda_{1,p} \geq \text{Re}\lambda_{2,p} \geq \dots \geq \text{Re}\lambda_{M^p,p}$ . Form a family of GDD as follows:

$$E^p(\Lambda, r) = \{[\lambda_{1,p}](t), [\lambda_{1,p}, \lambda_{2,p}](t), \dots, [\lambda_{1,p}, \lambda_{2,p}, \dots, \lambda_{M^p,p}](t)\}, \quad p \in \mathcal{J}.$$

It is known from [7] that  $D^+(\Lambda) < \infty$ . From Theorem 3 of [3], for any  $T > 2\pi D^+(\Lambda)$ , the family of GDD  $\{E^p(\Lambda, r)\}_{p \in \mathcal{J}}$  constitutes a Riesz basis in the closed subspace spanned by itself in  $L^2(0, T)$ . Specifically, suppose each  $\Lambda^p = \{\lambda_j^p\}_{j=1}^{N^p}$  has  $N^p$ , a different number of elements, and each appears  $m_{pj}$  times,  $\sum_{j=1}^{N^p} m_{pj} = M^p$ . Since  $M^p \leq N$ , all conditions of Theorem A1 of [8] are satisfied. We thus have proved the following result on the Riesz basis generation of the root subspace of  $\mathcal{A}$ .

**Theorem 3.1.** *Suppose the system (1.1) is regular. Then the operator  $\mathcal{A}$  defined by (2.1) has the following properties:*

(a) *There exists an  $\epsilon > 0$  such that  $\sigma(\mathcal{A}) = \bigcup_{p \in \mathcal{J}} \{\lambda_i^p\}_{i=1}^{N^p}$  (the multiplicity is not counted) where  $\sup_p N^p < \infty$ ,  $|\lambda_i^p - \lambda_j^q| \geq \epsilon$ ,  $\forall p, q \in \mathcal{J}$ ,  $p \neq q$ ,  $1 \leq i \leq N^p$ ,  $1 \leq j \leq N^q$ .*

(b) *The algebraic multiplicity  $m_{pi}$  of  $\lambda_i^p \in \sigma(\mathcal{A})$  is uniformly bounded:*

$$\sup_{p \in \mathcal{J}, 1 \leq i \leq N^p} m_{pi} < \infty.$$

(c)

$$Sp(\mathcal{A}) = \{W : W = \sum_{p \in \mathcal{J}} \sum_{i=1}^{N^p} \mathbb{P}_{\lambda_i^p} W\}, \quad (3.6)$$

where  $\mathbb{P}_{\lambda_i^p}$  is the eigenprojection of  $\mathcal{A}$  corresponding to  $\lambda_i^p$ .

(d) There are constants  $M_1, M_2 > 0$  such that

$$M_1 \sum_{p \in \mathcal{J}} \left\| \sum_{i=1}^{N^p} \mathbb{P}_{\lambda_i^p} W \right\|^2 \leq \|W\|^2 \leq M_2 \sum_{p \in \mathcal{J}} \left\| \sum_{i=1}^{N^p} \mathbb{P}_{\lambda_i^p} W \right\|^2, \quad \forall W \in Sp(\mathcal{A}). \quad (3.7)$$

The properties expressed by (3.6) and (3.7) are nothing but the *Riesz basis with parentheses* in  $Sp(\mathcal{A})$ . Furthermore, if we know that the eigenvalues of  $\mathcal{A}$  are separable, the Riesz basis with parentheses reduces the usual Riesz basis. Unfortunately, it is not clear whether or not the separability holds.

**Corollary 3.1.** *Suppose the system (1.1) is regular. Then the spectrum-determined growth condition holds:  $S(\mathcal{A}) = \omega(\mathcal{A})$ , where  $S(\mathcal{A})$  denotes the spectral bound of  $\mathcal{A}$ ,  $S(\mathcal{A}) = \sup_{\lambda \in \sigma(\mathcal{A})} \operatorname{Re} \lambda$ , and  $\omega(\mathcal{A})$  stands for the growth bound of the semigroup  $T(t)$ . Moreover, for any  $[u_0, v_0, d_0]^\top \in \mathcal{H}$ , decompose  $[u_0, v_0, d_0]^\top$  into  $[u_0, v_0, d_0]^\top = [u_{0u}, v_{0u}, d_{0u}]^\top + [u_{0s}, v_{0s}, d_{0s}]^\top \in Sp(\mathcal{A}) \oplus M_\infty(\mathcal{A})$  claimed by Theorem 2.5. Then the solution of (1.1) can be represented as*

$$\begin{aligned} & \begin{bmatrix} u(x, t) \\ v(x, t) \\ d_0 \end{bmatrix} \\ &= \sum_{p \in \mathcal{J}} \sum_{i=1}^{N^p} e^{\lambda_i^p t} \sum_{j=1}^{m_{pi}} \frac{(\mathcal{A} - \lambda_i^p)^{j-1}}{(j-1)!} t^{j-1} \mathbb{P}_{\lambda_i^p} \begin{bmatrix} u_{0u}(x) \\ v_{0u}(x) \\ d_{0u} \end{bmatrix} + T(t) \begin{bmatrix} u_{0s}(x) \\ v_{0s}(x) \\ d_{0s} \end{bmatrix}. \end{aligned} \quad (3.8)$$

The first term on the right-hand side of (3.8) converges unconditionally with parentheses, i.e.,

$$\sum_{p \in \mathcal{J}} \left\| \sum_{i=1}^{N^p} e^{\lambda_i^p t} \sum_{j=1}^{m_{pi}} \frac{(\mathcal{A} - \lambda_i^p)^{j-1}}{(j-1)!} t^{j-1} \mathbb{P}_{\lambda_i^p} \begin{bmatrix} u_{0u}(x) \\ v_{0u}(x) \\ d_{0u} \end{bmatrix} \right\|^2 < \infty,$$

and the second term is zero whenever  $t > t_0$ .

**Proof.** The spectrum-determined growth condition follows from Theorem 2.5 and (c) of Theorem A1 of [8]. Others follow from Theorem 2.5 and Theorem 3.1.  $\square$

To end this section, we state a result for the special case of  $m = k$  for which the generalized eigenfunctions of  $\mathcal{A}$  does form a Riesz basis with parentheses in the whole space  $\mathcal{H}$ .

**Theorem 3.2.** *Suppose the system (1.1) is regular and  $m = k$ . Then  $\mathcal{A}$  generates a  $C_0$  group on  $\mathcal{H}$ , and hence the root subspace of  $\mathcal{A}$  is complete in  $\mathcal{H}$ :  $Sp(\mathcal{A}) = \mathcal{H}$ . In this case, the generalized eigenfunctions of  $\mathcal{A}$  form a Riesz basis with parentheses in  $\mathcal{H}$ .*

**Proof.** Since (1.1) is regular and  $m = k$ , it follows from (2.10) that both  $D^{-1}$  and  $E^{-1}$  exist. Let

$$\tilde{u}(x, t) = u(x, -t), \quad \tilde{v}(x, t) = v(x, -t).$$

Then  $(\tilde{u}, \tilde{v})$  still satisfies equation (1.1), for  $0 < x < 1$  and  $t > 0$ :

$$\begin{cases} \frac{\partial}{\partial t} \begin{bmatrix} \tilde{u}(x, t) \\ \tilde{v}(x, t) \end{bmatrix} - K(x) \frac{\partial}{\partial x} \begin{bmatrix} \tilde{u}(x, t) \\ \tilde{v}(x, t) \end{bmatrix} - C(x) \begin{bmatrix} \tilde{u}(x, t) \\ \tilde{v}(x, t) \end{bmatrix} = 0, \\ \frac{\partial}{\partial t} (\tilde{u}(1, t) - D^{-1} \tilde{v}(1, t)) = -D^{-1} F \tilde{u}(1, t) - D^{-1} G \tilde{v}(1, t), \\ \tilde{v}(0, t) = E^{-1} \tilde{u}(0, t). \end{cases} \quad (3.9)$$

Exchanging the positions of  $\tilde{u}$  and  $\tilde{v}$ , we see, from Theorem 2.1, that the system (3.9) is associated with a  $C_0$  semigroup on  $\mathcal{H}$ . Hence,  $\mathcal{A}$  generates a  $C_0$  group on  $\mathcal{H}$ . Thus by the Hille-Yosida theorem ([16]), we have

$$\|R(\lambda, \mathcal{A})\| \rightarrow 0 \quad \text{as } |\lambda| \rightarrow +\infty.$$

Taking  $\rho_2 = \rho = 1$ ,  $n = 2$ , and  $\gamma_1 = \{\lambda : \arg \lambda = \pi\}$  in Lemma 2.3 proves the desired result.  $\square$

Finally, we examine the following example, which was considered in [14] and [6].

**Example 1.** Consider a pinched vibration cable with a tip mass whose motion is described by the following partial differential equation in  $H = H_E^1(0, 1) \times L^2(0, 1) \times \mathbb{R}$ ,  $H_E^1(0, 1) = \{f \in H^1(0, 1), f(0) = 0\}$ :

$$\begin{cases} y_{tt}(x, t) - y_{xx}(x, t) = 0, & 0 < x < 1, t > 0, \\ y(0, t) = 0, \\ y_x(1, t) + my_{tt} + ay_{xt}(1, t) + \alpha y_t(1, t) = 0, & m, a, \alpha > 0. \end{cases} \quad (3.10)$$



It was pointed out in [6] (see also [13]) that the system (3.9) is equivalent to the system following in the state Hilbert space  $\mathcal{H} = (L^2(0, 1))^2 \times \mathbb{C}$ :

$$\begin{cases} \frac{\partial}{\partial t} \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} + K \frac{\partial}{\partial x} \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = 0, & 0 < x < 1, t > 0, \\ \frac{\partial}{\partial t} \left( v(1, t) + \frac{m-a}{m+a} u(1, t) \right) = \frac{1-\alpha}{m+a} u(1, t) - \frac{1+\alpha}{m+a} v(1, t), \\ u(0, t) = -v(0, t), \end{cases} \quad (3.11)$$

where

$$(u(x, t), v(x, t)) = \frac{1}{2}(y_t(x, t) - y_x(x, t), y_t(x, t) + y_x(x, t)), \quad K = \text{diag}\{1, -1\}.$$

It is seen that the system (3.10) is of the form (1.1) with  $m = k = 1$ ,  $C = 0$ ,  $D = -(m-a)/(m+a)$ ,  $F = (1-\alpha)/(m+a)$ ,  $G = -(1+\alpha)/(m+a)$ , and  $E = -1$ . The system (3.10) is regular if and only if  $m \neq a$ .

By Theorem 3.2, we obtained the following result.

**Theorem 3.3.** *Suppose  $m \neq a$ . Then the generalized eigenfunctions of the system (3.10) form a Riesz basis with parentheses in the state space  $H$ , and the spectrum-determined growth condition holds true for the semigroup associated with system (3.10).*

A Riesz basis property was obtained in [14] under the assumption that  $\alpha = m/a \neq 1$  with additional conditions. Here we get rid of these assumptions but with a little difference: we know only that our basis is a Riesz basis with parentheses. This example also tells us something about the necessity of the regularity assumption for the completeness of the root subspace. For instance, when  $m \neq a$  and  $\alpha = 1$ , the system (3.10) ([13]) has only one eigenvalue  $\lambda = -1/m$ . So, the root subspace is never complete.

#### 4. AN ABSTRACT RESULT ON RIESZ BASIS WITH PARENTHESES IN [8]

Part of this section was proved in [5] for  $C_0$ -groups in a separable Hilbert space. Here we want to conclude the same results for the  $C_0$ -semigroups, which were announced in [21]. For the sake of completeness, we give here a detailed discussion of the problem. As in [5], the generalized divided difference introduced in [3] and [2] seems necessary in studying Riesz basis generation of the family of complex exponentials:

$$E_n(t) = \{e^{\lambda_n t}, te^{\lambda_n t}, \dots, t^{m_n-1} e^{\lambda_n t}\}$$

in some  $L^2$  space when  $\{\lambda_n\}$  are not necessarily simple and separable.

**Definition A1.** *Let  $\mu_k, k = 1, 2, \dots, m$ , be arbitrary complex numbers (not necessarily different). The generalized divided difference (GDD) of order*

zero of the function  $e^{\mu t}$  corresponding to the point  $\mu_1$  is defined as  $[\mu_1](t) = e^{\mu_1 t}$ . GDD of the order  $n-1, n \leq m$  of  $e^{\mu t}$  corresponding to  $\{\mu_k, k = 1, 2, \dots, n\}$  is defined by

$$[\mu_1, \mu_2, \dots, \mu_n](t) = \begin{cases} \frac{[\mu_1, \mu_2, \dots, \mu_{n-1}](t) - [\mu_2, \mu_3, \dots, \mu_n](t)}{\mu_1 - \mu_n}, & \mu_1 \neq \mu_n, \\ \frac{\partial}{\partial \mu} [\mu, \mu_2, \dots, \mu_{n-1}](t)|_{\mu=\mu_1}, & \mu_1 = \mu_n. \end{cases}$$

The following formula is valid for any  $\{\mu_k\}$

$$\begin{aligned} & [\mu_1, \mu_2, \dots, \mu_n](t) & (4.1) \\ & = \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{n-2}} d\tau_{n-1} t^{n-1} e^{t[\mu_1 + \tau_1(\mu_2 - \mu_1) + \dots + \tau_{n-1}(\mu_n - \mu_{n-1})]} \end{aligned}$$

and hence if  $\operatorname{Re}\mu_n \leq \operatorname{Re}\mu_{n-1} \leq \dots \leq \operatorname{Re}\mu_1$ , then

$$|[\mu_1, \mu_2, \dots, \mu_n](t)| \leq t^{n-1} e^{\operatorname{Re}\mu_1 t}, \quad \forall t \geq 0. \quad (4.2)$$

The following arguments may help us to understand the function GDD. If  $\mu_i = \mu, i = 1, 2, \dots, n$ , then

$$[\mu_1, \mu_2, \dots, \mu_i](t) = t^{i-1} e^{\mu t}, \quad 1 \leq i \leq n.$$

Generally, suppose there are  $m$  different elements in  $\{\mu_1, \mu_2, \dots, \mu_n\}$ , that is,  $\{\mu_1, \mu_2, \dots, \mu_n\} = \{\nu_1, \nu_2, \dots, \nu_m\}, \nu_i \neq \nu_j$  whenever  $i \neq j$ . Suppose that each  $\nu_k$  repeats  $n_k$  times:  $\sum_{k=1}^m n_k = n$ . Then Lemma 3.1 of [2] claims that the GDD  $[\mu_1, \mu_2, \dots, \mu_n](t)$  is a linear combination of functions  $t^{j-1} e^{\nu_k t}, 1 \leq j \leq n_k, 1 \leq k \leq m$  and the coefficients of the leading terms  $t^{n_k-1} e^{\nu_k t}$  are not equal to zero. The latter in return implies that for any  $1 \leq k \leq n_m, t^{k-1} e^{\nu_m t}$  is also a linear combination of  $[\mu_1](t), [\mu_1, \mu_2](t), \dots, [\mu_1, \mu_2, \dots, \mu_n](t)$ . Summarizing, we have the following proposition.

**Proposition A1.** *Let  $\{\mu_1, \mu_2, \dots, \mu_n\} = \{\nu_1, \nu_2, \dots, \nu_m\}, \nu_i \neq \nu_j$  when  $i \neq j, 1 \leq i, j \leq m$  and each  $\nu_j$  repeats  $n_j$  times:  $\sum_{j=1}^m n_j = n$ . Then any*

$$\phi(t) = \sum_{j=1}^m e^{\nu_j t} \sum_{i=1}^{n_j} a_{ij} t^{i-1} \text{ can be represented as}$$

$$\phi(t) = \sum_{i=1}^n G_i [\mu_1, \mu_2, \dots, \mu_i](t)$$

where  $G_1 = \sum_{j=1}^m a_{1j}$ .

Let  $\Omega = \{\mu_k\}_{k \in \mathcal{J}}$  be a sequence in  $\mathbb{C}$  ordered in such a way that  $\{\operatorname{Im}\mu_k\}$  forms a nondecreasing sequence. This is always possible because, in what follows, we always assume that  $\sup_{k \in \mathcal{J}} \{\operatorname{Re}\mu_k\} < \infty$ . Suppose  $\Omega$  is a union

of  $N$  separable sets  $\Omega_j$  (each  $\mu_k$  is repeated the number of times of its multiplicity):  $\Omega = \bigcup_{j=1}^N \Omega_j$ . Define

$$D^+(\Omega) = \lim_{r \rightarrow \infty} \frac{n^+(r)}{r}$$

where  $n^+(r) = \sup_{x \in \mathbb{R}} \{ \text{the cardinality of } \text{Im}(\Omega) \cap [x, x+r] \}$ . By a simple argument similar to proposition 1 of [7], we have

$$D^+(\Omega) < \infty. \quad (4.3)$$

For any  $\mu \in \mathbb{C}$ , denote by  $D_\mu(r)$  a disk with center  $\mu$  and radius  $r$ . Let  $G^p(r)$ ,  $p \in \mathcal{J}$ , be the connected components of the union  $\bigcup_{\mu \in \Omega} D_\mu(r)$  and write  $\Omega^p(r) = \{\mu_{j,p}\}$  to be the subsequence of  $\Omega$  in  $G^p(r)$ :  $\Omega^p(r) = \Omega \cap G^p(r)$ . Let  $\delta = \min_{1 \leq j \leq m} [\inf_{a,b \in \Omega_j, a \neq b} |a-b|]$ . Then Lemma 1 of [3] says that for any  $r < r_0 = \delta/(2N)$ , the cardinality of  $\Omega^p(r)$  is less than or equal to  $N$ . Set

$$\Omega^p(r) = \{\mu_{j,p}\}, j = 1, 2, \dots, M^p(r), M^p(r) \leq N. \quad (4.4)$$

Denote by

$$\varepsilon^p(\Omega, r) = \{[\mu_{1,p}], [\mu_{1,p}, \mu_{2,p}], \dots, [\mu_{1,p}, \mu_{2,p}, \dots, \mu_{M^p,p}]\}$$

the family of GDD corresponding to  $\Omega^p(r)$ . Then we have the following proposition A2 which is part of the result of Theorem 3 of [3].

**Proposition A2.** *Assume  $\Omega = \{\mu_k\}_1^\infty$  is defined as above. Then for any  $2\pi D^+(\Omega) < T < \infty$ , the family  $\varepsilon^p(\Omega, r)$  constitutes a Riesz basis in the closed subspace of  $L^2(0, T)$  spanned by itself.*

With these preliminaries, we come to the following abstract result which links the Riesz basis property of the corresponding GDD family of  $L^2$  spaces with that of the root subspace of associated operators in the case that the eigenvalues are not necessarily simple and separable but are some finite union of separable sets.

**Lemma A1.** *Let  $\mathbf{H}$  be a separable Hilbert space. Suppose  $\{e_n(t)\}_{n \in \mathcal{J}}$  forms a Riesz basis for the closed subspace spanned by itself in  $L^2(0, T)$ ,  $T > 0$ . Then for any  $\phi \in L^2(0, T; \mathbf{H})$ ,  $\phi(t) = \sum_{n \in \mathcal{J}} e_n(t) \phi_n$ , there exist constants  $C_1, C_2 > 0$  such that*

$$C_1 \sum_{n \in \mathcal{J}} \|\phi_n\|_{\mathbf{H}}^2 \leq \|\phi\|_{L^2(0, T; \mathbf{H})}^2 \leq C_2 \sum_{n \in \mathcal{J}} \|\phi_n\|_{\mathbf{H}}^2. \quad (4.5)$$

**Proof.** Take some orthonormal basis  $\{\psi_n\}$  of  $\mathbf{H}$  and for almost all  $t \in [0, T]$ , expand  $\phi$  as

$$\phi(t) = \sum_{n \in \mathcal{J}} \langle \phi(t), \psi_n \rangle_{\mathbf{H}} \psi_n, \quad t \in [0, T] \text{ a.e.}$$

Then

$$\|\phi(t)\|_{\mathbf{H}}^2 = \sum_{n \in \mathcal{J}} |\langle \phi(t), \psi_n \rangle_{\mathbf{H}}|^2, \quad \forall t \in [0, T] \text{ a.e.} \quad (4.6)$$

Since for any  $m \in \mathcal{J}$ ,  $\langle \phi(t), \psi_m \rangle_{\mathbf{H}} \in \text{span}\{e_n(t)\}_{n \in \mathcal{J}}$ , and

$$\langle \phi(t), \psi_m \rangle_{\mathbf{H}} = \sum_{n \in \mathcal{J}} \langle \phi_n, \psi_m \rangle_{\mathbf{H}} e_n(t), \quad \forall m \in \mathcal{J} \text{ in } L^2(0, T). \quad (4.7)$$

It can be seen from the assumption that

$$\begin{aligned} C_1 \sum_{n \in \mathcal{J}} |\langle \phi_n, \psi_m \rangle_{\mathbf{H}}|^2 &\leq \int_0^T |\langle \phi(t), \psi_m \rangle_{\mathbf{H}}|^2 dt \\ &\leq C_2 \sum_{n \in \mathcal{J}} |\langle \phi_n, \psi_m \rangle_{\mathbf{H}}|^2 \end{aligned} \quad (4.8)$$

for some constants  $C_1, C_2 > 0$  that are depending on  $\{e_n(t)\}$ . Hence it follows from (4.6) that

$$\begin{aligned} C_1 \sum_{m \in \mathcal{J}} \sum_{n \in \mathcal{J}} |\langle \phi_n, \psi_m \rangle_{\mathbf{H}}|^2 &\leq \sum_{m \in \mathcal{J}} \int_0^T |\langle \phi(t), \psi_m \rangle_{\mathbf{H}}|^2 dt \\ &= \int_0^T \|\phi(t)\|_{\mathbf{H}}^2 dt \leq C_2 \sum_{m \in \mathcal{J}} \sum_{n \in \mathcal{J}} |\langle \phi_n, \psi_m \rangle_{\mathbf{H}}|^2. \end{aligned} \quad (4.9)$$

Note that

$$\phi_n = \sum_{m \in \mathcal{J}} \langle \phi_n, \psi_m \rangle_{\mathbf{H}} \psi_m, \quad \|\phi_n\|^2 = \sum_{m \in \mathcal{J}} |\langle \phi_n, \psi_m \rangle_{\mathbf{H}}|^2. \quad (4.10)$$

and then (4.5) follows from (4.9).  $\square$

Lemma A1, which generalizes the result of Lemma 1 of [19], is going to be used to establish the following abstract result on Riesz basis generation that can be considered as a generalization of the main result in [19] without the assumption of the separability of eigenvalues.

Assume that  $B$  is a discrete operator on a separable Hilbert space  $\mathbf{H}$ , and  $B$  generates a  $C_0$ -semigroup on  $\mathbf{H}$  (as opposed to the  $C_0$ -group assumption in [5]). Suppose  $\sigma(B) = \bigcup_{p \in \mathcal{J}} \Omega(p)$ ,  $\Omega(p) = \{\nu_j^p\}_{j=1}^{N^p}$ ,  $\text{Re} \nu_1^p \geq \text{Re} \nu_2^p \geq \dots \geq \text{Re} \nu_{N^p}^p$ ,  $\nu_j^p \neq \nu_i^p$  when  $i \neq j$  and  $\nu_i^p \neq \nu_j^m$  unless  $m = n$  and  $i = j$ . Assume that each  $\nu_j^p$  has algebraic multiplicity  $m_j^p$  and both  $N^p$

and  $m_j^p$  have uniform upper bounds, i.e., there exists an  $N > 0$  such that  $\sup_p [N^p \max_{1 \leq j \leq N^p} m_j^p] \leq N$ . Set  $\tilde{m}_0^p = 0$ ,  $\tilde{m}_l^p = \sum_{q=1}^l m_q^p$ ,  $l = 1, \dots, N^p$ . Arranging  $\Omega(p)$  again by taking the multiplicity into account, we obtain a new set  $\Lambda^p = \{ \{ \mu_{i+\tilde{m}_{j-1}^p}^p \}_{i=1}^{m_j^p} \}_{j=1}^{N^p}$

$$\mu_{i+\tilde{m}_{j-1}^p}^p = \nu_j^p, \quad 1 \leq i \leq m_j^p, \quad 1 \leq j \leq N^p \quad (4.11)$$

and we hence write ( $\nu_j^p$  is repeated the number of times of its multiplicity)  $\sigma(B) = \bigcup_{p \in \mathcal{J}} \Lambda^p$ . Construct the family of GDD as follows:

$$E_p(t) = \{ [\mu_1^p](t), [\mu_1^p, \mu_2^p](t), \dots, [\mu_1^p, \mu_2^p, \dots, \mu_{\tilde{m}_{N^p}^p}^p](t) \}. \quad (4.12)$$

**Theorem A1.** *If there exists  $T > 0$  such that the family of GDD  $\{E_p(t)\}_{p \in \mathcal{J}}$  defined by (4.12) forms a Riesz basis for the closed subspace spanned by itself in  $L^2(0, T)$ , then*

(a)

$$S_\infty(B) = \{ x \in \mathbf{H} | x = \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x \} = \overline{S_\infty(B)} \quad (4.13)$$

where  $\mathbb{P}_{\nu_j^p}$  denotes the eigen-projection of  $B$  corresponding to the eigenvalue  $\nu_j^p$ .

(b) *There exists a constant  $M_1 > 0$  such that*

$$M_1 \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x \right\|^2 \leq \|x\|^2 \leq M_1^{-1} \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x \right\|^2, \quad \forall x \in S_\infty(B). \quad (4.14)$$

(c) *The spectrum-determined growth condition holds in the sense of*

$\omega(B) = \inf\{\omega \mid \text{there exists } M > 1 \text{ such that}$

$$\sup_{x \in S_\infty(B), \|x\|=1} \|e^{Bt}x\| \leq M e^{\omega t}\} = S(B) = \sup_{\nu \in \sigma_p(B)} \operatorname{Re} \nu.$$

**Proof.** We first prove (4.14). Since  $B$  generates a  $C_0$ -semigroup, there are constants  $M, \omega > 0$  such that

$$\|e^{Bt}\| \leq M e^{\omega t}, \quad \forall t \geq 0.$$

Take  $x_0 \in S_\infty(B)$ ,  $x_0 = \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0$  to find

$$e^{Bt}x_0 = \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} e^{\nu_j^p t} \sum_{i=1}^{m_j^p} \frac{(B - \nu_j^p)^{i-1}}{(i-1)!} t^{i-1} \mathbb{P}_{\nu_j^p} x_0 = \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} e^{\nu_j^p t} \sum_{i=1}^{m_j^p} a_{ij}^p t^{i-1} \quad (4.15)$$

where we set  $a_{ij}^p = \frac{(B-\nu_j^p)^{i-1}}{(i-1)!} \mathbb{P}_{\nu_j^p} x_0$ . By Proposition A1, we can write, in terms of GDD  $[\mu_1^p, \mu_2^p, \dots, \mu_{i+\tilde{m}_{j-1}^p}^p](t)$ , that

$$e^{Bt} x_0 = \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} \sum_{i=1}^{m_j^p} G_{i+\tilde{m}_{j-1}^p}^p(x_0) [\mu_1^p, \mu_2^p, \dots, \mu_{i+\tilde{m}_{j-1}^p}^p](t) \quad (4.16)$$

By assumption and Lemma A1, there are constants  $C_1, C_2 > 0$  such that

$$\begin{aligned} C_1 \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} \sum_{i=1}^{m_j^p} \|G_{i+\tilde{m}_{j-1}^p}^p(x_0)\|^2 &\leq \int_0^T \|e^{Bt} x_0\|^2 dt \\ &\leq C_2 \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} \sum_{i=1}^{m_j^p} \|G_{i+\tilde{m}_{j-1}^p}^p(x_0)\|^2. \end{aligned} \quad (4.17)$$

In particular,

$$\begin{aligned} C_1 \sum_{p \in \mathcal{J}} \|G_1^p(x_0)\|^2 &= C_1 \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0 \right\|^2 \leq \int_0^T \|e^{Bt} x_0\|^2 dt \\ &\leq \frac{M^2}{2\omega} (e^{2\omega T} - 1) \|x_0\|^2. \end{aligned} \quad (4.18)$$

Since  $S_\infty(B) \subset \overline{S_\infty(B)}$  is dense in  $\overline{S_\infty(B)}$ , we see that (4.18) holds for all  $x_0 \in \overline{S_\infty(B)}$ . Setting  $M_1 = C_1 \frac{2\omega}{M^2} (e^{2\omega T} - 1)^{-1}$  we obtain the left-hand side of the inequality of (4.14). Finally, to establish the inequality of the right-hand side of (4.14), we need a trick played in [20]. Note that  $Sp(B) \subset S_\infty(B)$  is dense in  $\overline{S_\infty(B)}$  and  $Sp(B)$  is an invariant subspace of  $e^{Bt}$  in  $\mathbf{H}$ , and so is  $\overline{S_\infty(B)}$  for  $e^{Bt}$  in  $\mathbf{H}$ . Consider the adjoint operator  $B^+$  of  $B|_{\overline{S_\infty(B)}}$  in  $\overline{S_\infty(B)}$  where  $B|_{\overline{S_\infty(B)}}$  denotes the restriction of  $B$  to the closed subspace  $\overline{S_\infty(B)}$  of  $\mathbf{H}$ . Along the same lines as the proof for  $B$  above, we have

$$M_1 \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p}^* x \right\|^2 \leq \|x\|^2, \quad \forall x \in \overline{S_\infty(B^+)} \quad (4.19)$$

where  $\mathbb{P}_{\nu_j^p}^*$  denotes the adjoint of  $\mathbb{P}_{\nu_j^p}$ . Now, for any  $x_0 \in S_\infty(B)$ ,  $x_0 = \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0$ , it follows from (4.19) that

$$\|x_0\|^2 = \left\langle \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0, x_0 \right\rangle = \sum_{p \in \mathcal{J}} \left\langle \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0, \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p}^* x_0 \right\rangle$$

$$\begin{aligned}
&\leq \left( \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0 \right\|^2 \right)^{1/2} \left( \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p}^* x_0 \right\|^2 \right)^{1/2} \\
&\leq \left( \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0 \right\|^2 \right)^{1/2} M_1^{-1/2} \|x_0\|,
\end{aligned}$$

and hence

$$\|x_0\|^2 \leq M_1^{-1} \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0 \right\|^2.$$

This is the inequality in the right-hand side of (4.14) in light of the density argument.

Now we turn to the proof of (4.13). It should be shown that  $S_\infty(B)$  is a closed subspace of  $\mathbf{H}$ . By virtue of Theorem 3.5 on page 63 of [12], it suffices to show that there exists  $M_0 > 0$ , such that

$$\left\| \sum_{p \in \mathcal{I}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x \right\|^2 \leq M_0 \|x\|^2, \quad \forall x \in \overline{S_\infty(B)}$$

where  $\mathcal{I}$  is any finite set of  $\mathcal{J}$ . Let  $x = \sum_{p \in \mathcal{I}} \sum_{j=1}^{N^p} P_{\nu_j^p} z$ ,  $z \in \overline{S_\infty(B)}$ . Then  $\sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x = \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} z$ . From (4.14)

$$M_1^2 \|x\|^2 \leq M_1 \sum_{p \in \mathcal{I}} \left\| \sum_{j=1}^{N^p} P_{\nu_j^p} x \right\|^2 = M_1 \sum_{p \in \mathcal{I}} \left\| \sum_{j=1}^{N^p} P_{\nu_j^p} z \right\|^2 \leq \|z\|^2, \quad \forall z \in \overline{S_\infty(B)},$$

and hence

$$\left\| \sum_{p \in \mathcal{I}} \sum_{j=1}^{N^p} P_{\nu_j^p} z \right\|^2 \leq M_1^{-2} \|z\|^2, \quad \forall z \in \overline{S_\infty(B)}.$$

(4.13) is thus proved. Finally, since  $\operatorname{Re} \mu_1^p \geq \operatorname{Re} \mu_2^p \geq \dots \geq \operatorname{Re} \mu_{i+\tilde{m}_{j-1}^p}^p$ , it follows from (4.2) that

$$|[\mu_1^p, \mu_2^p, \dots, \mu_{i+\tilde{m}_{j-1}^p}^p](t)| \leq t^N e^{S(B)t}, \quad \forall t \geq 1. \quad (4.20)$$

Combining (4.14), (4.15), (4.16) and (4.20) yields

$$\|e^{Bt} x_0\|^2 \leq M_1^{-1} \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} e^{Bt} x_0 \right\|^2 \quad (4.21)$$

$$\begin{aligned}
&= M_1^{-1} \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} e^{\nu_j^p t} \sum_{i=1}^{m_j^p} \frac{(B - \nu_j^p)^{i-1}}{(i-1)!} t^{i-1} \mathbb{P}_{\nu_j^p} x_0 \right\|^2 \\
&= M_1^{-1} \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \sum_{i=1}^{m_j^p} G_{i+\tilde{m}_{j-1}^p}^p(x_0) [\mu_1^p, \mu_2^p, \dots, \mu_{i+\tilde{m}_{j-1}^p}^p](t) \right\|^2 \\
&\leq M_1^{-1} \sum_{p \in \mathcal{J}} \sum_{j=1}^{N^p} \sum_{i=1}^{m_j^p} \|G_{i+\tilde{m}_{j-1}^p}^p(x_0)\|^2 N t^{2N} e^{2S(B)t}, \quad \forall x_0 \in \overline{S_\infty(B)}.
\end{aligned}$$

It follows from (4.17) that

$$C_1 \sum_{j=1}^{N^p} \sum_{i=1}^{m_j^p} \|G_{i+\tilde{m}_{j-1}^p}^p(x_0)\|^2 \leq \int_0^T \|e^{Bt} x_0\|^2 dt \leq \frac{M^2}{2\omega} (e^{2\omega T} - 1) \|x_0\|^2$$

as  $x_0 \in S_\infty(B)$ . In particular, by letting  $x_0 = \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x \in S_\infty(B)$ ,  $p \in \mathcal{J}$ ,  $x \in \overline{S_\infty(B)}$ , we obtain

$$\begin{aligned}
C_1 \sum_{j=1}^{N^p} \sum_{i=1}^{m_j^p} \|G_{i+\tilde{m}_{j-1}^p}^p(x)\|^2 &= C_1 \sum_{j=1}^{N^p} \sum_{i=1}^{m_j^p} \|G_{i+\tilde{m}_{j-1}^p}^p(x_0)\|^2 \\
&\leq \int_0^T \|e^{Bt} x_0\|^2 dt \leq \frac{M^2}{2\omega} (e^{2\omega T} - 1) \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x \right\|^2, \quad \forall x \in \overline{S_\infty(B)}.
\end{aligned} \tag{4.22}$$

The assertion is then concluded from (4.21), (4.22) and (4.14) that

$$\|e^{Bt} x_0\|^2 \leq C t^{2N} e^{2S(B)t} \sum_{p \in \mathcal{J}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x_0 \right\|^2 \leq C M_1^{-1} t^{2N} e^{2S(B)t} \|x_0\|^2,$$

$\forall x_0 \in \overline{S_\infty(B)}$  for some constant  $C$ . □

**Theorem A2.** *Suppose all conditions of Theorem A1 are satisfied. Then*

(i) *The space  $\mathbf{H}$  has the decomposition:*

$$\mathbf{H} = \overline{Sp(B) \oplus M_\infty(B)} \quad (\text{topological direct sum}) \tag{4.23}$$

where

$$M_\infty(B) = \{x \in \mathbf{H} \mid \mathbb{P}_{\nu_j^p} x = 0, \forall 1 \leq j \leq N^p, p \in \mathcal{J}\}. \tag{4.24}$$

(ii) *The space  $\mathbf{H}$  has the decomposition:*

$$\mathbf{H} = Sp(B) \oplus M_\infty(B) \quad (\text{topological direct sum})$$



if and only if

$$\sup_{\mathcal{I}} \left\| \sum_{p \in \mathcal{I}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} \right\| < \infty$$

where again  $\mathcal{I}$  is any finite set of  $\mathcal{J}$ .

**Proof.** Obviously, both  $Sp(B)$  and  $M_\infty(B)$  are  $e^{Bt}$ -invariant closed subspaces of  $\mathbf{H}$  and  $Sp(B) \cap M_\infty(B) = \{0\}$ . Let  $B^*$  be the adjoint operator of  $B$  in  $\mathbf{H}$ . Then all conclusions of Theorem A1 are still valid for  $B^*$ . It then follows from Lemma 5 on page 2355 of [4] that

$$\mathbf{H} = Sp(B^*) \oplus M_\infty(B) = Sp(B) \oplus M_\infty(B^*) \quad (\text{orthogonal sum})$$

where  $\mathbb{P}_{\nu_j^p}^*$  is the adjoint of  $\mathbb{P}_{\nu_j^p}$ . For any  $x \in \mathbf{H}$ ,  $x \perp \overline{Sp(B) \oplus M_\infty(B)}$ , we see that  $x \perp Sp(B)$ ,  $x \perp M_\infty(B)$  and  $x \in M_\infty(B^*) \cap Sp(B^*) = \{0\}$ . This is part (i). As for part (ii), we write  $\mathbb{P}_p = \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p}$ . Then  $\mathbb{P}_p$  is also a projection and  $\mathbb{P}_p x = 0$  if and only if  $\mathbb{P}_{\nu_j^p} x = 0$  for all  $1 \leq j \leq N^p$ . The sufficiency of part (ii) then follows from Theorem 3.5 on page 63 of [12].

On the other hand, since the projection  $\mathbb{P}$  from  $\mathbf{H}$  to  $Sp(B)$  along  $M_\infty(B)$  is a bounded operator, by Theorem 12.2 on page 247 of [18], for any  $x \in \mathbf{H}$ ,  $\mathbb{P}x \in Sp(B)$  we have from (4.13) and (4.14) that

$$\begin{aligned} \left\| \sum_{p \in \mathcal{I}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} x \right\|^2 &= \left\| \sum_{p \in \mathcal{I}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} \mathbb{P}x \right\|^2 \leq M_1^{-1} \sum_{p \in \mathcal{I}} \left\| \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} \mathbb{P}x \right\|^2 \\ &\leq M_1^{-2} \|\mathbb{P}x\|^2 \leq M_1^{-2} \|\mathbb{P}\|^2 \|x\|^2 \end{aligned}$$

and hence

$$\left\| \sum_{p \in \mathcal{I}} \sum_{j=1}^{N^p} \mathbb{P}_{\nu_j^p} \right\| \leq M_1^{-1} \|\mathbb{P}\|$$

and the proof is complete.  $\square$

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