

## AN EQUIVALENT DEFINITION OF RENORMALIZED ENTROPY SOLUTIONS FOR SCALAR CONSERVATION LAWS

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**Abstract.** We introduce a new notion of renormalized dissipative solutions for a scalar conservation law  $u_t + \operatorname{div} \mathbf{F}(u) = f$  with locally Lipschitz  $\mathbf{F}$  and  $L^1$  data, and prove the equivalence of such solutions and renormalized entropy solutions in the sense of Benilan et al. The structure of renormalized dissipative solutions is more useful in dealing with relaxation systems than the renormalized entropy scheme. As an example, we apply our result to contractive relaxation systems in merely an  $L^1$  setting and construct a renormalized dissipative solution via relaxation.

### 1. INTRODUCTION

We consider the following Cauchy problem:

$$(CP) \quad \begin{cases} \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{F}(u) = f & \text{in } Q := (0, T) \times \mathbb{R}^N, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

where  $T > 0$  and  $N \geq 1$ . Here  $f \in L^1(Q)$  and  $u_0 \in L^1(\mathbb{R}^N)$  are given functions and the flux  $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^N$  is a locally Lipschitz-continuous function.

Kruřkov [8] proved that if  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then (CP) has a unique weak solution  $u \in C([0, T]; L^1(\mathbb{R}^N)) \cap L^\infty(Q)$  satisfying the entropy inequalities, which is the so-called entropy solution. In the case that the flux  $\mathbf{F}$  is globally Lipschitz, Portilheiro introduced a notion of dissipative solutions for (CP) and proved the equivalence of such solutions and entropy solutions.

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The relationship between the notions of various solutions for degenerate parabolic equations is also investigated in [7] and [9]. The dissipative solutions are more suitable to obtain relaxation limits for some hyperbolic systems than entropy solutions. Indeed, Portilheiro [11] used this notion to obtain certain relaxation limits for hyperbolic systems describing discrete velocity models and chemical-reaction models. His idea is also based on the perturbed test function method introduced by Evans [3] for conservation laws. These systems have already been studied by Katsoulakis and Tzavaras ([5], [6]), who obtained several important results including comparison results. On the other hand, it is known that if  $f \in L^1(Q)$  and  $u_0 \in L^1(\mathbb{R}^N)$ , then the mild solution  $u$  of (CP) constructed by nonlinear semigroup theory is a unique entropy solution which is unbounded in general. In the case that  $\mathbf{F}$  is only locally Lipschitz, the flux function  $\mathbf{F}(u)$  may fail to be locally integrable since no growth condition is assumed on the flux  $\mathbf{F}$ , and hence (CP) does not possess a solution even in the sense of distributions. To overcome this the notion of renormalized entropy solutions has been introduced in [1], where the existence and uniqueness of a renormalized entropy solution of (CP) has been established and the semigroup solutions of (CP) in  $L^1$  spaces are characterized. Renormalized solutions have been introduced first by DiPerna and Lions [2] for the Boltzmann equation and utilized for degenerate elliptic and parabolic problems in the  $L^1$  setting in the last decade. However, the argument in [10] does not work well in the case that  $\mathbf{F}$  is only locally Lipschitz and the solution  $u$  is unbounded.

Our purpose of this paper is to extend some results in [10] to the case of locally Lipschitz-continuous flux. In this paper, we introduce a new notion of renormalized dissipative solutions which is a generalization of dissipative solutions in [10], and show the equivalence of renormalized dissipative solutions and renormalized entropy solutions. In Section 4, as an application, we apply our result to contractive relaxation systems in merely an  $L^1$  setting and construct a renormalized dissipative solution via relaxation.

## 2. THE MAIN RESULT

We begin with some notation and definitions. For  $r, s \in \mathbb{R}$ , we set  $r \wedge s := \min(r, s)$ ,  $r \vee s := \max(r, s)$ ,  $r^+ := r \vee 0$ , and  $r^- := (-r) \vee 0$ . For  $r \in \mathbb{R}$  and  $j = 0, 1$ , we define a sign function  $S_j$  by  $S_j(r) = 1$  if  $r > 0$ ,  $S_j(r) = -1$  if  $r < 0$ ,  $S_j(0) = j$ . Then we denote  $S_j^+(r) := S_j(r) \vee 0$  and  $S_j^-(r) := S_j(r) \wedge 0$ .

Let  $u \in L^1(Q)$ . For  $(t, x) \in Q$  and  $r > 0$ , we set

$$B_r(t, x) := \{(s, y) \in Q; (s - t)^2 + |y - x|^2 \leq r^2\},$$

and define the upper- and lower-semicontinuous envelopes of  $u$  as

$$u^*(t, x) := \limsup_{r \downarrow 0} \{u(s, y); (s, y) \in B_r(t, x)\}$$

and

$$u_*(t, x) := \liminf_{r \downarrow 0} \{u(s, y); (s, y) \in B_r(t, x)\},$$

respectively. Then we see that  $u_* \leq u \leq u^*$ ,  $u^*$  is upper semicontinuous and  $u_*$  is lower semicontinuous.

We now recall from [1] the definition of renormalized entropy solutions.

**Definition 2.1.** (i) We say  $u \in L^1(Q)$  is a renormalized entropy subsolution of (CP) if for any  $k, \ell \in \mathbb{R}$ ,

$$\mu_{k,\ell} := (u \wedge \ell - k)_t^+ + \operatorname{div} \{S_0^+(u \wedge \ell - k) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k))\} - S_0^+(u \wedge \ell - k) f \quad (2.1)$$

is a Radon measure on  $Q$  such that for each  $k \in \mathbb{R}$ ,

$$\lim_{\ell \rightarrow \infty} \mu_{k,\ell}^+(Q) = 0,$$

and for each  $\ell \in \mathbb{R}$ ,

$$(u(t, \cdot) \wedge \ell - u_0 \wedge \ell)^+ \rightarrow 0 \text{ in } L_{loc}^1(\mathbb{R}^N) \text{ as } t \rightarrow 0 \text{ essentially.}$$

(ii) We say  $u \in L^1(Q)$  is a renormalized entropy supersolution of (CP) if for any  $k, \ell \in \mathbb{R}$ ,

$$\nu_{k,\ell} := (u \vee \ell - k)_t^- + \operatorname{div} \{S_0^-(u \vee \ell - k) (\mathbf{F}(u \vee \ell) - \mathbf{F}(k))\} - S_0^-(u \vee \ell - k) f \quad (2.2)$$

is a Radon measure on  $Q$  such that for each  $k \in \mathbb{R}$ ,

$$\lim_{\ell \rightarrow -\infty} \nu_{k,\ell}^+(Q) = 0,$$

and for each  $\ell \in \mathbb{R}$ ,

$$(u(t, \cdot) \vee \ell - u_0 \vee \ell)^- \rightarrow 0 \text{ in } L_{loc}^1(\mathbb{R}^N) \text{ as } t \rightarrow 0 \text{ essentially.}$$

(iii) We say  $u \in L^1(Q)$  is a renormalized entropy solution of (CP) if  $u$  is a renormalized entropy subsolution of (CP) and also a renormalized entropy supersolution of (CP).

Next, we introduce a new notion of renormalized dissipative solutions of (CP).

**Definition 2.2.** (i) We say  $u \in L^1(Q)$  is a renormalized dissipative subsolution of (CP) if there is a sequence  $\{\mu_\ell\} \subset \mathcal{M}_b(Q)^+$  with  $\mu_\ell(Q) \rightarrow 0$  as  $\ell \rightarrow \infty$  such that for each  $\ell \geq 1$  and  $\phi \in \mathcal{T}_\ell$ ,

$$\int_Q S_0^+(u \wedge \ell - \phi) (f - \phi_t - \operatorname{div} \mathbf{F}(\phi)) dx dt + \int_Q S_0^+(u^* \wedge \ell - \phi) d\mu_\ell \geq 0 \quad (2.3)$$

and

$$(u(t, \cdot) \wedge \ell - u_0 \wedge \ell)^+ \rightarrow 0 \text{ in } L^1_{loc}(\mathbb{R}^N) \text{ as } t \rightarrow 0 \text{ essentially,}$$

where  $\mathcal{T}_\ell := C^1_0(Q) \cap \{\phi; \phi(t, x) \equiv k \text{ for } (t, x) \in Q \text{ if } |x| > R \text{ for some } k \in (-\ell, \ell) \text{ and } R > 0\}$  and  $\mathcal{M}_b(Q)^+$  denotes the space of all nonnegative bounded measures on  $Q$ .

(ii) We say  $u \in L^1(Q)$  is a renormalized dissipative supersolution of (CP) if there is a sequence  $\{\nu_\ell\} \subset \mathcal{M}_b(Q)^+$  with  $\nu_\ell(Q) \rightarrow 0$  as  $\ell \rightarrow \infty$  such that for each  $\ell \geq 1$  and  $\phi \in \mathcal{T}_\ell$ ,

$$\int_Q S_0^-(u \vee (-\ell) - \phi) (f - \phi_t - \operatorname{div} \mathbf{F}(\phi)) dxdt + \int_Q S_0^-(u_* \vee (-\ell) - \phi) d\nu_\ell \geq 0 \quad (2.4)$$

and

$$(u(t, \cdot) \vee \ell - u_0 \vee \ell)^- \rightarrow 0 \text{ in } L^1_{loc}(\mathbb{R}^N) \text{ as } t \rightarrow 0 \text{ essentially.}$$

(iii) We say  $u \in L^1(Q)$  is a renormalized dissipative solution of (CP) if  $u$  is a renormalized dissipative subsolution of (CP) and also a renormalized dissipative supersolution of (CP).

Then we obtain the following main result.

**Theorem 2.3.** *Suppose that  $u \in L^1(Q)$ , and  $u^*(t, x) < \infty$  and  $u_*(t, x) > -\infty$  for almost every  $(t, x) \in Q$ . Then  $u$  is a renormalized entropy subsolution (respectively supersolution) of (CP) if and only if  $u$  is a renormalized dissipative subsolution (respectively supersolution) of (CP).*

### 3. PROOF OF THEOREM 2.3

**Claim 1:** *If  $u \in L^1(Q)$  and  $u^*(t, x) < \infty$  (respectively  $u_*(t, x) > -\infty$ ) for almost every  $(t, x) \in Q$ , then a renormalized entropy subsolution (respectively supersolution)  $u$  of (CP) implies a renormalized dissipative subsolution (respectively supersolution).*

The proof of Claim 1 will be divided into several parts.

**Step 1:** It follows from [1; Proposition 2.7] that there exists a sequence  $\{\mu_\ell\} \subset \mathcal{M}_b(Q)^+$  such that  $\mu_\ell(Q) \rightarrow 0$  as  $\ell \rightarrow \infty$  and  $\mu_{k, \ell} = \mu_\ell - \mu_k - \chi_{\{u > \ell\}} f$  for  $k < \ell$ , where  $\chi_A$  denotes the indicator function of  $A$ . Then we have

$$\int_{u^* \wedge \ell < k} \theta d\mu_\ell = \int_{u^* \wedge \ell < k} \theta d\mu_k$$

for each  $\theta \in C_0^\infty(Q)$ . Indeed, since  $u^* \wedge \ell$  is upper semicontinuous,  $\{u^* \wedge \ell < k\}$  is open, and hence for any  $\varphi \in C_0^\infty(\{u^* \wedge \ell < k\})$ ,

$$\begin{aligned} & \int_Q \varphi d\mu_k - \int_Q \varphi d\mu_\ell = - \int_Q \varphi d\mu_{k,\ell} \\ & = \int_Q S_0^+(u \wedge \ell - k) \left\{ (u \wedge \ell - k) \varphi_t + f \varphi + (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k)) \cdot \nabla \varphi \right\} dx dt. \end{aligned}$$

By the partition of unity (see [12]) we get

$$\int_{u^* \wedge \ell < k} \theta d\mu_\ell = \int_{u^* \wedge \ell < k} \theta d\mu_k$$

for each  $\theta \in C_0^\infty(Q)$ . Then, from (2.1) again we obtain for any  $\theta \in C_0^\infty(Q)^+$  and  $k < \ell$ ,

$$\begin{aligned} & \int_Q S_0^+(u \wedge \ell - k) \left\{ (u \wedge \ell - k) \theta_t + f \theta + (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k)) \cdot \nabla \theta \right\} dx dt \\ & - \int_{u^* \wedge \ell \geq k} \theta d\mu_\ell + \int_{u^* \wedge \ell \geq k} \theta d\mu_k + \int_{u^* \wedge \ell \geq k} \chi_{\{u > \ell\}} f \theta dx dt \\ & \geq - \int_{u^* \wedge \ell \geq k} \theta d\mu_\ell + \int_{u^* \wedge \ell \geq k} \chi_{\{u > \ell\}} f \theta dx dt, \end{aligned}$$

which implies

$$\begin{aligned} & \int_Q S_0^+(u \wedge \ell - k) \left\{ (u \wedge \ell - k) \theta_t + f \theta + (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k)) \cdot \nabla \theta \right\} dx dt \\ & + \int_Q S_0^+(u^* \wedge \ell - k) \left\{ \theta d\mu_\ell - \chi_{\{u > \ell\}} f \theta dx dt \right\} \geq 0 \end{aligned} \quad (3.1)$$

for any  $\theta \in C_0^\infty(Q)^+$  and  $k, \ell \in \mathbb{R}$ .

**Step 2:** Let  $\eta_\varepsilon$  and  $\rho_\lambda$  be standard mollifiers on  $\mathbb{R}^N$  and  $\mathbb{R}$ , respectively, and let  $\zeta_n$  be a nonnegative smooth function satisfying

$$\zeta_n(t, x) := \begin{cases} 1 & \text{if } |x| \leq n, \\ 0 & \text{if } |x| \geq 2n, \end{cases}$$

and  $|\nabla \zeta_n| \leq C/n$  with positive constant  $C$ . Take  $\phi \in \mathcal{I}_\ell$ . Then we put  $\theta = \eta_\varepsilon(x - y) \rho_\lambda(t - s) \zeta_n(t, x)$  and  $k = \phi(s, y)$  in (3.1), and integrate in  $s$  and  $y$  over  $Q$  to obtain

$$\begin{aligned} 0 \leq & \int_{Q^2} S_0^+(u \wedge \ell - \phi) \left\{ (u \wedge \ell - \phi) (\eta_\varepsilon \rho_\lambda \zeta_n)_t + f \eta_\varepsilon \rho_\lambda \zeta_n \right. \\ & \left. + (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \cdot \nabla_x (\eta_\varepsilon \rho_\lambda \zeta_n) \right\} dy ds dx dt \end{aligned}$$

$$\begin{aligned}
& + \int_{Q^2} S_0^+(u^* \wedge \ell - \phi) \left\{ \eta_\varepsilon \rho_\lambda \zeta_n d\mu_\ell - \chi_{\{u>\ell\}} f \eta_\varepsilon \rho_\lambda \zeta_n dx dt \right\} dy ds \\
= & \int_{Q^2} S_0^+(u \wedge \ell - \phi) \left[ (u \wedge \ell - \phi) \eta_\varepsilon \rho_\lambda (\zeta_n)_t - \eta_\varepsilon \rho_\lambda \zeta_n \phi_s \right. \\
& - ((u \wedge \ell - \phi) \eta_\varepsilon \rho_\lambda \zeta_n)_s + f \eta_\varepsilon \rho_\lambda \zeta_n \\
& - \operatorname{div}_y \mathbf{F}(\phi) \eta_\varepsilon \rho_\lambda \zeta_n + \eta_\varepsilon \rho_\lambda (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \cdot \nabla_x \zeta_n \\
& \left. - \operatorname{div}_y \{ (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \eta_\varepsilon \rho_\lambda \zeta_n \} \right] dy ds dx dt \\
& + \int_{Q^2} S_0^+(u^* \wedge \ell - \phi) \left\{ \eta_\varepsilon \rho_\lambda \zeta_n d\mu_\ell - \chi_{\{u>\ell\}} f \eta_\varepsilon \rho_\lambda \zeta_n dx dt \right\} dy ds \\
= & \sum_{j=1}^9 I_j^{\varepsilon, \lambda, n}. \tag{3.3}
\end{aligned}$$

We begin with  $I_4^{\varepsilon, \lambda, n}$ . For  $p, q > 0$  we set

$$\Phi(p, q) := \sup \{ |\phi(t, x) - \phi(s, y)|; (t, x), (s, y) \in Q, |t - s| \leq p, |x - y| \leq q \}.$$

Then we have

$$\begin{aligned}
I_4^{\varepsilon, \lambda, n} & = \int_{f \geq 0} S_0^+(u \wedge \ell - \phi(s, y)) f \eta_\varepsilon \rho_\lambda \zeta_n dy ds dx dt \\
& \quad + \int_{f < 0} S_0^+(u \wedge \ell - \phi(s, y)) f \eta_\varepsilon \rho_\lambda \zeta_n dy ds dx dt \\
& \leq \int_{f \geq 0} S_0^+(u \wedge \ell - \phi(t, x) + \Phi(\lambda, \varepsilon)) f \eta_\varepsilon \rho_\lambda \zeta_n dy ds dx dt \\
& \quad + \int_{f < 0} S_0^+(u \wedge \ell - \phi(t, x) - \Phi(\lambda, \varepsilon)) f \eta_\varepsilon \rho_\lambda \zeta_n dy ds dx dt,
\end{aligned}$$

which implies

$$\begin{aligned}
\limsup_{\varepsilon, \lambda \downarrow 0} I_4^{\varepsilon, \lambda, n} & \leq \int_{f \geq 0} S_1^+(u \wedge \ell - \phi) f \zeta_n dx dt + \int_{f < 0} S_0^+(u \wedge \ell - \phi) f \zeta_n dx dt \\
& = \int_Q S_0^+(u \wedge \ell - \phi) f \zeta_n dx dt + \int_{u \wedge \ell = \phi} f^+ \zeta_n dx dt.
\end{aligned}$$

In a similar way we also have that

$$\limsup_{\varepsilon, \lambda \downarrow 0} I_1^{\varepsilon, \lambda, n} \leq \int_Q S_0^+(u \wedge \ell - \phi) (u \wedge \ell - \phi) (\zeta_n)_t dx dt,$$

$$\begin{aligned} \limsup_{\varepsilon, \lambda \downarrow 0} I_2^{\varepsilon, \lambda, n} &\leq - \int_Q S_0^+(u \wedge \ell - \phi) \phi_t \zeta_n \, dx \, dt + \int_{u \wedge \ell = \phi} (\phi_t)^- \zeta_n \, dx \, dt, \\ \limsup_{\varepsilon, \lambda \downarrow 0} I_5^{\varepsilon, \lambda, n} &\leq - \int_Q S_0^+(u \wedge \ell - \phi) \operatorname{div} \mathbf{F}(\phi) \zeta_n \, dx \, dt + \int_{u \wedge \ell = \phi} (\operatorname{div} \mathbf{F}(\phi))^- \zeta_n \, dx \, dt, \end{aligned}$$

$$\begin{aligned} \limsup_{\varepsilon, \lambda \downarrow 0} I_6^{\varepsilon, \lambda, n} &\leq \int_Q S_0^+(u \wedge \ell - \phi) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \cdot \nabla \zeta_n \, dx \, dt \\ &\quad + \int_{u \wedge \ell = \phi} ((\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \cdot \nabla \zeta_n)^+ \, dx \, dt \\ &= \int_Q S_0^+(u \wedge \ell - \phi) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \cdot \nabla \zeta_n \, dx \, dt, \end{aligned}$$

$$\limsup_{\varepsilon, \lambda \downarrow 0} I_8^{\varepsilon, \lambda, n} \leq \int_Q S_0^+(u^* \wedge \ell - \phi) \zeta_n \, d\mu_\ell + \int_{u^* \wedge \ell = \phi} \zeta_n \, d\mu_\ell,$$

and

$$\begin{aligned} \limsup_{\varepsilon, \lambda \downarrow 0} I_9^{\varepsilon, \lambda, n} \\ \leq \int_Q S_0^+(u^* \wedge \ell - \phi) \chi_{\{u > \ell\}} f \zeta_n \, dx \, dt + \int_{u^* \wedge \ell = \phi} (\chi_{\{u > \ell\}} f)^+ \zeta_n \, dx \, dt. \end{aligned}$$

As to  $I_7^{\varepsilon, \lambda, n}$ , we introduce a sequence  $\{\alpha_m\} \subset C^1(\mathbb{R})$  with  $0 \leq \alpha'_m(r) \leq Cm \chi_{\{|r| \leq 1/m\}}$  which approximates  $S_0^+(r)$ . Then we have

$$\begin{aligned} I_7^{\varepsilon, \lambda, n} &= - \int_{Q^2} S_0^+(u \wedge \ell - \phi) \operatorname{div}_y \{(\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \eta_\varepsilon \rho_\lambda \zeta_n\} \, dy \, ds \, dx \, dt \\ &= - \lim_{m \rightarrow \infty} \int_{Q^2} \alpha_m(u \wedge \ell - \phi) \operatorname{div}_y \{(\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \eta_\varepsilon \rho_\lambda \zeta_n\} \, dy \, ds \, dx \, dt \\ &= - \lim_{m \rightarrow \infty} \int_{Q^2} \alpha'_m(u \wedge \ell - \phi) \eta_\varepsilon \rho_\lambda \zeta_n (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \cdot \nabla_y \phi \, dy \, ds \, dx \, dt. \end{aligned}$$

Let us denote by  $L_\ell$  the Lipschitz constant of  $\mathbf{F}$  on  $[-\ell, \ell]$ . Then, for any  $(t, x) \in Q$  and  $m$  large enough, we get

$$\begin{aligned} &|\alpha_m(u \wedge \ell - \phi) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi))| |\nabla_y \phi| \\ &\leq Cm \chi_{\{|u \wedge \ell - \phi| \leq 1/m\}} L_\ell |u \wedge \ell - \phi| |\nabla_y \phi| \\ &\leq CL_\ell \chi_{\{|u \wedge \ell - \phi| \leq 1/m\}} |\nabla_y (u \wedge \ell - \phi)| \\ &\rightarrow CL_\ell \chi_{\{|u \wedge \ell - \phi| = 0\}} |\nabla_y (u \wedge \ell - \phi)| = 0 \quad \text{as } m \rightarrow \infty, \end{aligned}$$

and hence  $I_7^{\varepsilon, \lambda, n} = 0$ . Similarly, we also get  $I_3^{\varepsilon, \lambda, n} = 0$ . Passing to the limit in (3.3) as  $\varepsilon, \lambda \downarrow 0$  gives

$$\begin{aligned}
0 &\leq \int_Q S_0^+(u \wedge \ell - \phi) (u \wedge \ell - \phi) (\zeta_n)_t dx dt \\
&\quad - \int_Q S_0^+(u \wedge \ell - \phi) \phi_t \zeta_n dx dt + \int_{u \wedge \ell = \phi} (\phi_t)^- \zeta_n dx dt \\
&\quad + \int_Q S_0^+(u \wedge \ell - \phi) f \zeta_n dx dt + \int_{u \wedge \ell = \phi} f^+ \zeta_n dx dt \\
&\quad - \int_Q S_0^+(u \wedge \ell - \phi) \operatorname{div} \mathbf{F}(\phi) \zeta_n dx dt + \int_{u \wedge \ell = \phi} (\operatorname{div} \mathbf{F}(\phi))^- \zeta_n dx dt \\
&\quad + \int_Q S_0^+(u \wedge \ell - \phi) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \cdot \nabla \zeta_n dx dt \\
&\quad + \int_Q S_0^+(u^* \wedge \ell - \phi) \zeta_n d\mu_\ell + \int_{u^* \wedge \ell = \phi} \zeta_n d\mu_\ell \\
&\quad + \int_Q S_0^+(u^* \wedge \ell - \phi) \chi_{\{u > \ell\}} f \zeta_n dx dt + \int_{u^* \wedge \ell = \phi} (\chi_{\{u > \ell\}} f)^+ \zeta_n dx dt.
\end{aligned}$$

Passing to the limit as  $n \rightarrow \infty$ , we have

$$\begin{aligned}
0 &\leq \int_Q S_0^+(u \wedge \ell - \phi) (f - \phi_t - \operatorname{div} \mathbf{F}(\phi)) dx dt \\
&\quad + \int_Q S_0^+(u^* \wedge \ell - \phi) \left\{ d\mu_\ell + \chi_{\{u > \ell\}} f dx dt \right\} \\
&\quad + \int_{u \wedge \ell = \phi} \left\{ f^+ + (\phi_t)^- + (\operatorname{div} \mathbf{F}(\phi))^- \right\} dx dt \\
&\quad + \int_{u^* \wedge \ell = \phi} \left\{ d\mu_\ell + (\chi_{\{u > \ell\}} f)^+ dx dt \right\} \\
&\quad + \limsup_{n \rightarrow \infty} \int_Q S_0^+(u \wedge \ell - \phi) (u \wedge \ell - \phi) (\zeta_n)_t dx dt \\
&\quad + \limsup_{n \rightarrow \infty} \int_Q S_0^+(u \wedge \ell - \phi) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi)) \cdot \nabla \zeta_n dx dt. \tag{3.4}
\end{aligned}$$

**Step 3:** We calculate the last term on the right hand in (3.4). Since  $S_0^+(u \wedge \ell - \phi) \mathbf{F}(u \wedge \ell) \in L^1(Q)^N$ , we see that

$$\lim_{n \rightarrow \infty} \int_Q S_0^+(u \wedge \ell - \phi) \mathbf{F}(u \wedge \ell) \cdot \nabla \zeta_n dx dt = 0.$$



Suppose that  $\phi(t, x) \equiv k$  for large  $|x|$ . If  $k = 0$ , then  $\phi \in L^1(Q)$ , and hence

$$\lim_{n \rightarrow \infty} \int_Q S_0^+(u \wedge \ell - \phi) \mathbf{F}(\phi) \cdot \nabla \zeta_n \, dx \, dt = 0.$$

We have

$$\begin{aligned} & \int_Q S_0^+(u \wedge \ell - \phi) \mathbf{F}(\phi) \cdot \nabla \zeta_n \, dx \, dt \\ &= \int_Q S_0^+(u \wedge \ell - \phi) (\mathbf{F}(\phi) - \mathbf{F}(k)) \cdot \nabla \zeta_n \, dx \, dt \\ &+ \int_Q S_0^+(u \wedge \ell - \phi) \mathbf{F}(k) \cdot \nabla \zeta_n \, dx \, dt \\ &= \int_Q S_0^+(u \wedge \ell - \phi) (\mathbf{F}(\phi) - \mathbf{F}(k)) \cdot \nabla \zeta_n \, dx \, dt + \int_{u \wedge \ell > \phi} \mathbf{F}(k) \cdot \nabla \zeta_n \, dx \, dt. \end{aligned}$$

The first integral converges to 0 as  $n \rightarrow \infty$  since  $\mathbf{F}(\phi) - \mathbf{F}(k) \in L^1(Q)^N$ . As to the second integral, we first note that for fixed  $k \in (-\ell, \ell)$  and  $\xi \in C_0^\infty(Q)$  we may write  $\phi = \xi + k$ . Assume  $0 < k < \ell$ . Then by Chebyshev's inequality we get

$$\mathcal{L}^{N+1}(\{u \wedge \ell > \phi\}) = \mathcal{L}^{N+1}(\{u \wedge \ell - \xi > k\}) \leq \frac{1}{k} \int_Q |u \wedge \ell - \xi| \, dx \, dt < \infty,$$

where  $\mathcal{L}^{N+1}$  denotes the  $(N+1)$ -dimensional Lebesgue measure on  $Q$ . Hence this integral converges to 0 as  $n \rightarrow \infty$ .

Next assume that  $-\ell < k < 0$ . Since the second integral equals

$$- \int_{u \wedge \ell \leq \phi} \mathbf{F}(k) \cdot \nabla \zeta_n \, dx \, dt,$$

we have

$$\mathcal{L}^{N+1}(\{u \wedge \ell \leq \phi\}) = \mathcal{L}^{N+1}(\{u \wedge \ell - \xi \leq k\}) \leq \frac{1}{|k|} \int_Q |u \wedge \ell - \xi| \, dx \, dt < \infty.$$

Therefore, the second integral also converges to 0 as  $n \rightarrow \infty$ . Thus we obtain

$$\lim_{n \rightarrow \infty} \int_Q S_0^+(u \wedge \ell - \phi) \mathbf{F}(\phi) \cdot \nabla \zeta_n \, dx \, dt = 0$$

for  $\phi \in \mathcal{T}_\ell$ . In a similar way, we also see that

$$\lim_{n \rightarrow \infty} \int_Q S_0^+(u \wedge \ell - \phi) (u \wedge \ell - \phi) (\zeta_n)_t \, dx \, dt = 0$$

for  $\phi \in \mathcal{T}_\ell$ .

**Step 4:** Recall that we may write  $\phi = \xi + k$  for  $k \in (-\ell, \ell)$  and  $\xi \in C_0^\infty(Q)$ . Then we see that the set  $\{k \in (-\ell, \ell); \mu(\{u \wedge \ell = \xi + k\}) = 0\}$  is dense in  $(-\ell, \ell)$  because  $\sum_{k \in C} |k| \mu(\{u \wedge \ell = \xi + k\})$  is finite for any countable set  $C \subset (-\ell, \ell)$ , where  $\mu$  denotes the  $(N+1)$ -dimensional Lebesgue measure  $\mathcal{L}^{N+1}$  or  $\mu_\ell$ . Hence the cardinality of the set  $\{k \in (-\ell, \ell); \mu(\{u \wedge \ell = \xi + k\})\}$  is at most countable.

We now fix any  $k \in (-\ell, \ell)$  and choose a sequence  $\{k_n^+\}$  such that  $k_n^+ \downarrow k$  as  $n \rightarrow \infty$  and  $\mu(\{u \wedge \ell = \xi + k_n^+\}) = 0$  for any  $n \geq 1$ . It follows from (3.4) with  $\phi = \xi + k_n^+$  that

$$\begin{aligned} 0 &\leq \int_Q S_0^+(u \wedge \ell - \phi) (f - \phi_t - \operatorname{div} \mathbf{F}(\phi)) \, dx \, dt \\ &\quad + \int_Q S_0^+(u^* \wedge \ell - \phi) \left\{ d\mu_\ell + \chi_{\{u > \ell\}} f \, dx \, dt \right\}, \end{aligned}$$

which means that (2.3) holds with  $d\mu_\ell$  replaced by  $d\mu_\ell + \chi_{\{u > \ell\}} f \, dx \, dt$ .

**Claim 2:** *If  $u \in L^1(Q)$  and  $u^*(t, x) < \infty$  for almost every  $(t, x) \in Q$ , then a renormalized dissipative subsolution  $u$  of (CP) implies a renormalized entropy subsolution.*

The proof of Claim 2 will be also divided into several parts.

**Step 5:** Let  $k \in (-\ell, \ell)$  and  $\theta \in C_0^\infty(Q)^+$ . For each  $\delta, \varepsilon > 0$ , choose a function  $\psi_{\delta, \varepsilon} \in C_0^\infty(Q)$  such that

$$\psi_{\delta, \varepsilon}(t, x) := \begin{cases} 0 & \text{if } (t, x) \in B_\delta(0, 0), \\ 1/\varepsilon & \text{if } (t, x) \in Q \setminus B_{\delta+\varepsilon}(0, 0), \end{cases}$$

where  $B_\delta(0, 0) = \{(t, x) \in Q; t^2 + |x|^2 \leq \delta^2\}$ . Putting  $\phi_{\delta, \varepsilon} := k + \psi_{\delta, \varepsilon}(t - s, x - y)$  in (2.3) for each  $(s, y) \in Q$ , multiplying (2.3) by an arbitrary function  $\theta(s, y) \in C_0^\infty(Q)^+$ , and integrating in  $s$  and  $y$  over  $Q$ , we obtain

$$\begin{aligned} &\int_{Q^2} S_0^+(u \wedge \ell - \phi_{\delta, \varepsilon}) \left\{ f - (\phi_{\delta, \varepsilon})_t - \operatorname{div}_x \mathbf{F}(\phi_{\delta, \varepsilon}) \right\} \theta \, dy \, ds \, dx \, dt \\ &\quad + \int_{Q^2} S_0^+(u^* \wedge \ell - \phi_{\delta, \varepsilon}) \theta \, dy \, ds \, d\mu_\ell \geq 0. \end{aligned} \tag{3.5}$$

Note that  $S_0^+(u \wedge \ell - \phi_{\delta, \varepsilon}) \rightarrow S_0^+(u \wedge \ell - k) \chi_{B_\delta(t, x)}(s, y)$  and  $S_0^+(u^* \wedge \ell - \phi_{\delta, \varepsilon}) \rightarrow S_0^+(u^* \wedge \ell - k) \chi_{B_\delta(t, x)}(s, y)$  as  $\varepsilon \downarrow 0$ , and also note that

$$- \int_{Q^2} S_0^+(u \wedge \ell - \phi_{\delta, \varepsilon}) (\phi_{\delta, \varepsilon})_t \theta \, dy \, ds \, dx \, dt$$

$$\begin{aligned}
&= \int_{Q^2} S_0^+(u \wedge \ell - \phi_{\delta,\varepsilon}) (u \wedge \ell - \phi_{\delta,\varepsilon}) \theta_s dy ds dx dt \\
&\quad - \int_{Q^2} S_0^+(u \wedge \ell - \phi_{\delta,\varepsilon}) ((u \wedge \ell - \phi_{\delta,\varepsilon}) \theta)_s dy ds dx dt
\end{aligned}$$

and

$$\begin{aligned}
&- \int_{Q^2} S_0^+(u \wedge \ell - \phi_{\delta,\varepsilon}) \operatorname{div}_x \mathbf{F}(\phi_{\delta,\varepsilon}) \theta dy ds dx dt \\
&= \int_{Q^2} S_0^+(u \wedge \ell - \phi_{\delta,\varepsilon}) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi_{\delta,\varepsilon})) \cdot \nabla_y \theta dy ds dx dt \\
&\quad - \int_{Q^2} S_0^+(u \wedge \ell - \phi_{\delta,\varepsilon}) \operatorname{div}_y \{(\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi_{\delta,\varepsilon})) \theta\} dy ds dx dt. \quad (3.6)
\end{aligned}$$

**Step 6:** We first compute the second integral on the right hand in (3.6). As in the argument above, by using again the approximating functions  $\{\alpha_m\} \subset C^1(\mathbb{R})$  we see that this integral vanishes. As to the first integral on the right hand in (3.6), note that for  $k \in (-\ell, \ell)$  and  $\varepsilon > 0$  sufficiently small,

$$|S_0^+(u \wedge \ell - \phi_{\delta,\varepsilon}) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi_{\delta,\varepsilon}))| \leq L_\ell (u \wedge \ell - \phi_{\delta,\varepsilon})^+,$$

which implies that  $S_0^+(u \wedge \ell - \phi_{\delta,\varepsilon}) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(\phi_{\delta,\varepsilon})) \cdot \nabla_y \theta \in L^1(Q^2)$ . Therefore, passing to the limit in (3.5) as  $\varepsilon \downarrow 0$  yields

$$\begin{aligned}
&\int_Q \int_{B_\delta(t,x)} S_0^+(u \wedge \ell - k) \{f\theta + (u \wedge \ell - k)\theta_s + (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k)) \cdot \nabla_y \theta\} dy ds dx dt \\
&\quad + \int_Q \int_{B_\delta(t,x)} S_0^+(u^* \wedge \ell - k) \theta dy ds d\mu_\ell \geq 0.
\end{aligned}$$

Dividing by the volume of the ball  $B_\delta(t, x)$  and passing to the limit as  $\delta \downarrow 0$ , we obtain

$$\begin{aligned}
&\int_Q S_0^+(u \wedge \ell - k) \left\{ f\theta + (u \wedge \ell - k)\theta_t + (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k)) \cdot \nabla \theta \right\} dx dt \\
&\quad + \int_Q S_0^+(u^* \wedge \ell - k) \theta d\mu_\ell \geq 0
\end{aligned}$$

for each  $\theta \in C_0^\infty(Q)^+$ . This means that

$$\begin{aligned}
&(u \wedge \ell - k)_t^+ + \operatorname{div} \{S_0^+(u \wedge \ell - k) (\mathbf{F}(u \wedge \ell) - \mathbf{F}(k))\} \\
&\quad - S_0^+(u \wedge \ell - k) f - S_0^+(u^* \wedge \ell - k) \mu_\ell
\end{aligned}$$

is a Radon measure on  $Q$ , and hence

$$\mu_{k,\ell} = (u \wedge \ell - k)_t^+ + \operatorname{div} \{S_0^+(u \wedge \ell - k)(\mathbf{F}(u \wedge \ell) - \mathbf{F}(k))\} - S_0^+(u \wedge \ell - k) f$$

is also a Radon measure on  $Q$ . Moreover,

$$\mu_{k,\ell}^+(Q) \leq \int_Q S_0^+(u^* \wedge \ell - k) d\mu_\ell \leq \mu_\ell(Q) \rightarrow 0 \quad \text{as } \ell \rightarrow \infty,$$

and we also see that for each  $\ell \in \mathbb{R}$ ,  $(u(t, \cdot) \wedge \ell - u_0 \wedge \ell)^+ \rightarrow 0$  in  $L_{loc}^1(\mathbb{R}^N)$  as  $t \rightarrow 0$  essentially. Thus we complete the proof of the theorem.

#### 4. RELAXATION

We prove the existence of renormalized dissipative solutions of (CP) via relaxation methods.

Let  $\omega_i > 0$  and suppose that for  $k = 1, 2, \dots$  and  $i = 1, 2, \dots, N$ , the  $V_{k,i}$  satisfy the conditions

$$1 + \sum_{i=1}^N \frac{1}{V_{k,i}} \inf_{|u| \leq k} F_i'(u) > 0,$$

$$\frac{1 + \Omega}{1 + \sum_{j=1}^N \frac{1}{V_{k,j}} \inf_{|u| \leq k} F_j'(u)} \frac{1}{V_{k,i}} \sup_{|u| \leq k} F_i'(u) < \omega_i,$$

where  $\Omega = \sum_{i=1}^N \omega_i$ . It is proved in [5; Lemma 4.1] that there are a strictly increasing function  $r_k : [-k, k] \rightarrow \mathbb{R}$  defined by

$$w = r_k(u) := \frac{1}{1 + \Omega} \left( u + \sum_{i=1}^N \frac{F_i(u)}{V_{k,i}} \right)$$

and functions  $h_{k,i} : [r_k(-k), r_k(k)] \rightarrow \mathbb{R}$ , satisfying the conditions  $dh_{k,i}/dw < 0$  and  $h_{k,i}(0) = 0$ , such that

$$w - \sum_{i=1}^N h_{k,i}(w) = u,$$

$$\omega_i w + h_{k,i}(w) = \frac{F_i(u)}{V_{k,i}}, \quad u \in [-k, k].$$

We consider the following family of relaxation systems for  $w^\varepsilon$  and  $\mathbf{z}^\varepsilon = (z_1^\varepsilon, \dots, z_N^\varepsilon)$ :

$$(CRS) \quad \begin{cases} \frac{\partial w^\varepsilon}{\partial t} + \sum_{i=1}^N \omega_i V_{k,i} \frac{\partial w^\varepsilon}{\partial x_i} = \frac{1}{\varepsilon} \sum_{i=1}^N (h_{k,i}(w^\varepsilon) - z_i^\varepsilon), \\ \frac{\partial z_i^\varepsilon}{\partial t} - V_{k,i} \frac{\partial z_i^\varepsilon}{\partial x_i} = \frac{1}{\varepsilon} (h_{k,i}(w^\varepsilon) - z_i^\varepsilon), \quad i = 1, \dots, N, \quad \varepsilon > 0 \end{cases}$$

with the initial conditions

$$w^\varepsilon(0, x) = w_0(x), \quad \mathbf{z}^\varepsilon(0, x) = \mathbf{z}_0(x), \quad x \in \mathbb{R}^N, \quad (4.1)$$

$$a \leq w_0 \leq b, \quad h_{k,i}(b) \leq z_{0i} \leq h_{k,i}(a), \quad (4.2)$$

where  $a < 0$  and  $b > 0$  are constants such that

$$-k \leq a + \sum_{i=1}^N h_{k,i}(b) \leq b + \sum_{i=1}^N h_{k,i}(a) \leq k.$$

The following result is obtained in [5; Theorem 4.2].

**Proposition 4.1.** *Let  $k \geq 1$  and  $u^\varepsilon = w^\varepsilon - \sum_{i=1}^N z_i^\varepsilon$ , and let  $u_0 = w_0 - \sum_{i=1}^N z_{0i} \in L^1(\mathbb{R}^N)$ . Then  $\bar{u}_k = \lim_{\varepsilon \downarrow 0} u^\varepsilon$  exists in  $L^1(Q)$  and  $\bar{u}_k$  is an entropy solution of (CP) with  $f = 0$  satisfying  $-k \leq \bar{u}_k \leq k$ .*

Now, let  $u_0 \in L^1(\mathbb{R}^N)$  and choose sequences of functions  $\{w_{0,k}\}_{k \geq 1}$  and  $\{\mathbf{z}_{0,k}\}_{k \geq 1}$  which satisfy condition (4.2). Moreover, we assume that  $u_{0,k} = w_{0,k} - \sum_{i=1}^N z_{0i,k}$  converges as  $k \rightarrow \infty$  to  $u_0$  in  $L^1(\mathbb{R}^N)$ . Since the function  $\bar{u}_k$  is a bounded entropy solution of (CP) with  $f = 0$  the comparison property of entropy solutions leads to

$$\|\bar{u}_k(t) - \bar{u}_{k'}(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_{0,k} - u_{0,k'}\|_{L^1(\mathbb{R}^N)}$$

for  $t \in [0, T]$  and  $k, k' \geq 1$ . Therefore,  $\{\bar{u}_k\}$  converges as  $k \rightarrow \infty$  to some function  $\bar{u}$  in  $L^1(Q)$ . In fact, we can prove

**Theorem 4.2.** *The limit function  $\bar{u}$  above is a unique renormalized dissipative solution of (CP) with  $f = 0$ .*

**Proof.** We shall show inequalities (2.3) and (2.4) with  $\mu_\ell = \nu_\ell = 0$ . To this end we fix  $\ell \geq 1$ . Define  $t_0$  by  $t_0 = 0$  if  $u(t) \leq \ell$  for all  $t \geq 0$  and by  $t_0 = \inf\{t > 0; u(t) = \ell\}$  otherwise. We take any test function  $\phi \in \mathcal{T}_\ell$  and let  $\zeta = r_k(\phi)$  and  $\psi_i = h_{k,i}(\zeta)$ . Choose  $\beta > 0$  such that  $\beta - \sum_{i=1}^N h_{k,i}(\beta) = \ell$ . This choice is possible if  $k$  is taken sufficiently large. Since the constant

functions  $w \equiv \beta$  and  $z_i = h_{k,i}(\beta)$  satisfy the contractive relaxation system (CRS), we have (see (2.11) in [5])

$$0 \leq \int_{(0,t_0) \times \mathbb{R}^N} \left\{ S_0^+(\beta - \zeta) \left[ -\zeta_t - \sum_{i=1}^N \omega_i V_{k,i} \zeta_{x_i} + \frac{1}{\varepsilon} \sum_{i=1}^N (h_{k,i}(\zeta) - \psi_i) \right] \right. \\ \left. + \sum_{i=1}^N S_0^-(h_{k,i}(\beta) - h_{k,i}(\zeta)) \left[ -(\psi_i)_t + V_{k,i}(\psi_i)_{x_i} + \frac{1}{\varepsilon} (h_{k,i}(\zeta) - \psi_i) \right] \right\} dx dt.$$

We notice that  $-S_0^-(h_{k,i}(\beta) - h_{k,i}(\zeta)) = S_0^+(\beta - \zeta) = S_0^+(r_k(\ell) - r_k(\phi)) = S_0^+(\ell - \phi)$ ,  $\beta = r_k(\ell)$ ,  $\zeta - \sum_{i=1}^N \psi_i = \phi$ , and  $\omega_i V_{k,i} \zeta + V_{k,i} \psi_i = F_i(\phi)$ . Thus, the inequality becomes

$$0 \leq \int_{(0,t_0) \times \mathbb{R}^N} S_0^+(\ell - \phi) (-\phi_t - \operatorname{div} \mathbf{F}(\phi)) dx dt.$$

On the other hand, the comparison property for (CRS) yields that  $u(t) \leq \ell$  for  $t \in [t_0, T]$ . A similar argument as in [11; Theorem 2.1] shows that

$$\int_{(t_0, T) \times \mathbb{R}^N} S_0^+(\bar{u}_k - \phi) (-\phi_t - \operatorname{div} \mathbf{F}(\phi)) dx dt \geq 0.$$

Notice, however, that

$$\lim_{\lambda \downarrow 0} \frac{\|(f + \lambda g)^+\|_{L^1} - \|f^+\|_{L^1}}{\lambda} = \inf_{\lambda > 0} \frac{\|(f + \lambda g)^+\|_{L^1} - \|f^+\|_{L^1}}{\lambda} \\ = \int_{f > 0} S_0^+(f) g dx + \int_{f=0} g^+ dx, \quad f, g \in L^1(\mathbb{R}^N).$$

Thus, we have that for any  $\lambda > 0$ ,

$$0 \leq \frac{1}{\lambda} \int_{(t_0, T) \times \mathbb{R}^N} \left\{ (\bar{u}_k - \phi - \lambda \phi_t - \lambda \operatorname{div} \mathbf{F}(\phi))^+ - (\bar{u}_k - \phi)^+ \right\} dx dt.$$

Passing to the limit as  $k \rightarrow \infty$  first and then as  $\lambda \downarrow 0$  yields

$$0 \leq \int_{\bar{u} - \phi > 0} S_0^+(\bar{u} - \phi) (-\phi_t - \operatorname{div} \mathbf{F}(\phi)) dx dt \\ + \int_{\bar{u} - \phi = 0} (-\phi_t - \operatorname{div} \mathbf{F}(\phi))^+ dx dt \\ = \int_{(t_0, T) \times \mathbb{R}^N} S_0^+(\bar{u} - \phi) (-\phi_t - \operatorname{div} \mathbf{F}(\phi)) dx dt.$$

Consequently, we conclude that

$$\int_Q S_0^+(\bar{u} \wedge \ell - \phi) (-\phi_t - \operatorname{div} \mathbf{F}(\phi)) \, dx \, dt \geq 0.$$

The inequality (2.4) can be proved similarly. Therefore,  $\bar{u}$  is a renormalized dissipative solution of (CP), and hence by Theorem 2.3 it is a renormalized entropy solution of (CP). By virtue of the uniqueness theorem in [1]  $\bar{u}$  is a unique solution.  $\square$

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