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EXISTENCE RESULTS FOR A NONLINEAR ELLIPTIC EQUATION WITH CRITICAL SOBOLEV EXPONENT

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Abstract. In this paper we study the following nonlinear elliptic problem with Dirichlet boundary condition: $-\Delta u = K(x)u^p$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded, smooth domain of $\mathbb{R}^n$, $n \geq 4$ and $p+1 = 2n/(n-2)$ is the critical Sobolev exponent. Using dynamical and topological methods involving the study of the critical points at infinity of the associated variational problem, we prove some existence results.

1. Introduction and the main results

Let Ω be a bounded and regular domain of $\mathbb{R}^n$, $n \geq 3$. Let K be a C^3 positive function in $\overline{\Omega}$. We look for conditions on K to ensure the existence of a positive solution of the following problem:

$$(1) \qquad \begin{cases} -\Delta u &= \frac{n-2}{4(n-1)} K(x) u^{(n+2)/(n-2)}, \quad u > 0 \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\qquad\qquad \text{on} \quad \partial\Omega. \end{cases}$$

The interest in this equation comes from its resemblance to the scalar curvature problem in differential geometry, which consists in finding suitable conditions on a given function K defined on M such that K is the scalar curvature for a metric $\tilde{g}$ conformally equivalent to g, where (M, g) is an n-dimensional Riemannian manifold without boundary.

The special nature of problem (1) appears when we consider it from the variational viewpoint. The Euler functional associated to (1) does not satisfy the Palais-Smale condition. This means that there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. This is due to the noncompactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$.

In the case of manifolds without boundary, this problem has been widely studied in various works (see for example the monographs [1] and [15], and

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the references therein). A phenomenon appears in dimension $n \geq 5$, due to the fact that the self-interaction of the functions failing the Palais-Smale condition dominates the interaction of two masses of those functions [5]. While in dimension 3 the reverse happens [7]. In dimension 4, we have a balance phenomenon; that is, the self-interaction and the interaction are of the same size [9].

Problem (1) was studied by Ben Ayed-Hammami [12] in dimension 4. They characterized the critical points at infinity for the associated variational problem. Using an Euler-Poincaré characteristic argument, they gave an existence result. For large dimension, $n \geq 5$, their theorem cannot be extended (see page I.13 of [3]).

In this paper, we will focus our attention on the case where n is bigger than 4, and we will characterize the critical points at infinity of the associated variational problem. We will use this characterization in order to give some existence results. Our approach follows the ideas developed in Bahri [5], Aubin-Bahri [2], and Ben Ayed-Chtioui-Hammami [10], where the problem of prescribing the scalar curvature problem on closed manifolds was studied using some algebraic topological tools.

In order to state our results, we need to introduce some notation, recall some known facts, and state the assumptions that we are using in our paper.

We denote by G the Green's function of the Laplacian with Dirichlet boundary condition on Ω and by H its regular part; that is,

$$\begin{aligned} G(x,y) &= \mid x-y \mid^{2-n} - H(x,y) \quad \text{for} \quad (x,y) \in \Omega^2 \\ \Delta_x H &= 0 \text{ in } \Omega^2, \quad G=0 \quad \text{on} \quad \partial(\Omega^2). \end{aligned}$$

We assume throughout the whole paper that K has only nondegenerate critical points $y_0, y_1, \dots, y_l$ such that y_0 is the unique absolute maximum and

$$(A_1) \qquad -\frac{\Delta K(y_i)}{3K(y_i)} + 8H(y_i,y_i) \neq 0 \text{ if } n=4, \quad \Delta K(y_i) \neq 0 \text{ if } n \geq 5$$

for each $i = 0, 1, \dots, l$. We assume also

$$(A_2) \qquad \frac{\partial K}{\partial \nu} < 0 \text{ on } \partial\Omega,$$

where ν denotes the outward-normal vector on $\partial\Omega$.

Now, let I_1 be the subset of the critical points of K satisfying

$$-\frac{\Delta K(y_i)}{3K(y_i)} + 8H(y_i,y_i) > 0 \text{ if } n=4, \quad -\Delta K(y_i) > 0 \text{ if } n \geq 5, \tag{1.1}$$

and let Z be a pseudo gradient of K of Morse-Smale type (that is, the intersections of the stable and unstable manifolds of the critical points of K are transverse). In this paper we assume that

(A_3) $\qquad W_s(y_i) \cap W_u(y_j)$ is empty for each $y_i \in I_1$ and $y_j \notin I_1$.

In the first part of the present work, we establish some existence results when $n = 4$. We have the following results:

Theorem 1.1. *Let $n = 4$. Suppose the following assumptions hold:*
(H_0) *There exists $y_{i_1} \in I_1 \backslash \{y_0\}$ of Morse index equal to $n-k$, for $k \geq 1$, and satisfying $K(y_{i_1}) = \max\{K(y)/\ y \in I_1 \backslash \{y_0\}\}$.*

$$(H_1)\quad \big(-\frac{\Delta K(y_0)}{3K(y_0)} + 8H(y_0,y_0)\big)\big(-\frac{\Delta K(y_{i_1})}{3K(y_{i_1})} + 8H(y_{i_1},y_{i_1})\big) < 64G(y_0,y_{i_1})$$

$$(H_2)\qquad \frac{1}{K(y_0)} + \frac{1}{K(y_{i_1})} < \frac{1}{K(y)} \quad \forall\, y \in I_1 \backslash \{y_0, y_{i_1}\}.$$

Then (1) *has a solution of Morse index k or $k+1$.*

Remark 1. The result of Theorem 1.1 is true if we change the assumption (H_2) to the following assumption:
(H_2') For each $y \in I_1 \backslash \{y_{i_1}\}$ such that $K(y)^{-1} \leq K(y_0)^{-1} + K(y_{i_1})^{-1}$, we have $i(y) \notin \{n-k, n-k-1\}$.

Here $i(y)$ denotes the Morse index of the function K at y.

In contrast to Theorem 1.1, we have the following result based on a topological invariant denoted μ, introduced by A. Bahri [5]. To state this result, we need to introduce some notation. Set

$$X := \overline{W_s(y_{i_1})} = W_s(y_{i_1}) \cup W_s(y_0).$$

We denote by $C_{y_0}(X)$ the following set:

$$C_{y_0}(X) = \{\alpha\delta_{y_0} + (1-\alpha)\delta_x \ /\ \alpha \in [0,1],\, x \in X\}, \tag{1.2}$$

where δ_x is the Dirac mass at x. For λ large enough, we introduce a map $f_\lambda : C_{y_0}(X) \to \Sigma^+$ defined by

$$\alpha\delta_{y_0} + (1-\alpha)\delta_x \rightarrowtail$$
$$\frac{(\alpha/K(y_0)^{(n-2)/4})P\delta_{(y_0,\lambda)} + ((1-\alpha)/K(x)^{(n-2)/4})P\delta_{(x,\lambda)}}{|(\alpha/K(y_0)^{(n-2)/4})P\delta_{(y_0,\lambda)} + ((1-\alpha)/K(x)^{(n-2)/4})P\delta_{(x,\lambda)}|_{H_0^1}},$$

where $P\delta_{(a,\lambda)}$ will be defined in the next section. Then $C_{y_0}(X)$ and $f_\lambda(C_{y_0}(X))$ are manifolds in dimension $k+1$; that is, their singularities arise in dimension $k-1$ and lower; see [5]. Now, we define the intersection number (modulo 2) of $f_\lambda(C_{y_0}(X))$ with $W_s(y_0, y_{i_1})_\infty$:

$$\mu(y_{i_1}) = f_\lambda(C_{y_0}(X)) \cdot W_s(y_0, y_{i_1})_\infty, \tag{1.3}$$

where $W_s(y_0, y_{i_1})_\infty$ is the stable manifold of the critical point at infinity $(y_0, y_{i_1})_\infty$ for a decreasing pseudo gradient Z for J which is transverse to $f_\lambda(C_{y_0}(X))$. In fact we will prove later that such a critical point has a Morse index equal to $k+1$; thus, this number is well defined (see [18]). We then have

Theorem 1.2. *Let $n=4$. We assume that (H_0) is satisfied and the following two conditions hold:*

(T_1) $\quad \big(-\dfrac{\Delta K(y_0)}{3K(y_0)} + 8H(y_0,y_0)\big)\big(-\dfrac{\Delta K(y_{i_1})}{3K(y_{i_1})} + 8H(y_{i_1},y_{i_1})\big) > 64G(y_0,y_{i_1}).$

(T_2) $\qquad K(y_0) > 2K(y)$ *for each* $y \in I_1 \backslash \{y_0\}$.

If $\mu(y_{i_1}) = 0$, then (1) has a solution of Morse index k or $k+1$.

Remark 2. Theorem 1.2 is also true if we change the assumption (T_2) to the following assumption:
(T_2') For each $y \in I_1$ such that $K(y_0) \le 2K(y)$, we have $i(y) > n-k$.

In the second part of our paper, our aim is to give existence results in higher dimension ($n \ge 5$). For this purpose, we will assume that I_1 is not reduced to $\{y_0\}$. Let $k \in \{1,\dots,n-1\}$, and let $y_i \in I_1$ such that $i(y_i) = n-k$; we define $X = \overline{W_s(y_i)}$.
(A_4) Assume that X is without boundary.

For λ large enough, we define $\mu(y_i)$ (as in (1.3)) to be the intersection number of $f_\lambda(C_{y_0}(X))$ with $W_s(y_0, y_i)_\infty$. Then we have the following result:

Theorem 1.3. *Let $n \ge 5$. Under the assumptions (A_4) and that I_1 is not reduced to $\{y_0\}$, if we have*
(A_5) $K(y_0) > 2^{2/(n-2)} K(y)$ $\forall\, y \in I_1 \backslash \{y_0\}$ *and*
(A_6) $\mu(y_i) = 0$,
then (1) has a solution of Morse index k or $k+1$.

Remark 3. Theorem 1.3 is also true if we change the assumption (A_5) to the following assumption:
(A_5') For each $y \in I_1$ such that $K(y_0) \le 2^{2/(n-2)} K(y)$ we have $i(y) > n-k$.

Before stating another existence result in higher dimension, we introduce some assumptions which were first introduced by Aubin-Bahri [2]. Assume the following:

(B_1) K has only nondegenerate critical points $y_0, \dots, y_l$ satisfying

$$K(y_0) \geq \dots \geq K(y_m) > K(y_{m+1}) \geq \dots \geq K(y_l) \text{ and } I_1 = \{y_0, \dots, y_m\}$$

(we recall that, for $n \geq 5$, $I_1 = \{y / \nabla K(y) = 0 \text{ and } - \Delta K(y) > 0\}$). Let $X = \overline{\cup_{y \in I_1} W_s(y)}$.

(B_2) X is not contractible; denote by k the dimension of the first nontrivial reduced homological group.

(B_3) There exists a positive constant $\overline{c}$, $\overline{c} < K(y_m)$, such that X is contractible in $K^{\overline{c}} := \{x \in \Omega / \ K(x) \geq \overline{c}\}$.

We then have

Theorem 1.4. *Let $n \geq 5$. Under the assumptions (A_1), (B_1), (B_2), and (B_3), there exists a positive constant c_0 that is independent of K such that if $K(y_0)/\overline{c} \leq 1 + c_0$, then (1) has a solution of an augmented Morse index $\geq k$.*

The plan of the present paper is the following. In Section 2, we set up the variational structure, we recall some preliminaries, and we perform an expansion of the Euler functional associated to (1) and its gradient near the potential critical points at infinity. In Section 3, we prove a Morse lemma at infinity which allows us to characterize the critical points at infinity, while Section 4 is devoted to the proof of our results.

2. Known results

We recall the variational framework. Problem (1) has a variational structure. The functional is

$$J(u) = \Big(\int_{\Omega} K \mid u \mid^{\frac{2n}{n-2}} \Big)^{-\frac{n-2}{n}}$$

defined on

$$\Sigma = \Big\{ u \in H_0^1(\Omega) / |u|^2_{H_0^1(\Omega)} := \int_{\Omega} \mid \nabla u \mid^2 = 1 \Big\}.$$

The positive critical points of J are solutions to (1) (up a multiplicative constant). Due to the noncompactness of the embedding $H_0^1(\Omega)$ into $L^{\frac{2n}{n-2}}(\Omega)$, the functional J does not satisfy the Palais-Smale condition ((P.S.) for short) on $\Sigma^+ = \{u \in \Sigma / \ u \geq 0\}$. This means that there exist sequences along which J is bounded, its gradient goes to zero, and which do not converge. The failure of (P.S.) has been studied by various authors (see Brezis-Coron [13], Lions [16], and Struwe [20]). In order to characterize the sequences failing (P.S.), we need to introduce some notation.

For $a \in \Omega$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = c_n \Big(\frac{\lambda}{1+\lambda^2 \mid x-a \mid^2} \Big)^{\frac{(n-2)}{2}}; \tag{2.1}$$

c_n is chosen so that

$$-\Delta \delta_{(a,\lambda)} = \delta_{(a,\lambda)}^{\frac{n+2}{n-2}}.$$

Let P be the projection from $H^1(\Omega)$ into $H^1_0(\Omega)$; that is, $u = Pf$ is the solution of

$$\Delta u = \Delta f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2.2}$$

We introduce now the set of potential critical points at infinity.

For any $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let $V(p,\varepsilon)$ be the subset of Σ of the following functions: $u \in \Sigma$ such that there is $(a_1, \dots, a_p) \in \Omega^p$, $(\lambda_1, \dots, \lambda_p) \in (\varepsilon^{-1}, +\infty)^p$, and $(\alpha_1, \dots, \alpha_p) \in (0, +\infty)^p$ such that

$$\Big| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)} \Big|_{H^1_0} < \varepsilon, \quad \lambda_i d(a_i, \partial\Omega) > \varepsilon^{-1}, \quad \Big| \frac{\alpha_i^{\frac{4}{n-2}} K(a_i)}{\alpha_j^{\frac{4}{n-2}} K(a_j)} - 1 \Big| < \varepsilon,$$

and $\varepsilon_{ij} < \varepsilon$ for $i \neq j$, where

$$\varepsilon_{ij} = \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j \mid a_i - a_j \mid^2 \Big)^{-\frac{n-2}{2}}. \tag{2.3}$$

The failure of the Palais-Smale condition can be described as follows

Proposition 2.1. ([13], [16], [20]) *Assume that J has no critical point in Σ^+. Let $(u_k) \in \Sigma^+$ be a sequence such that $(\partial J(u_k))$ tends to zero and $(J(u_k))$ is bounded. Then, after possibly having extracted a subsequence, there exists $p \in N^*$, a sequence (ε_k), ε_k tending to zero, such that $u_k \in V(p, \varepsilon_k)$.*

Following [6] and [16], we consider the following minimization problem for a function $u \in V(p,\varepsilon)$ with ε small enough:

$$(*) \qquad \min \Big\{ \Big| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)} \Big|_{H^1_0}, \, \alpha_i > 0, \, a_i \in \Omega, \, \lambda_i > 0 \Big\},$$

and we state the following proposition, which defines a parametrization of the set $V(p,\varepsilon)$.

Proposition 2.2. [6] *For any $p \in \mathbb{N}^*$, there exists $\varepsilon_p > 0$ such that for any $u \in V(p,\varepsilon)$, with $\varepsilon \leq \varepsilon_p$, the minimization problem $(*)$ has a unique* *solution* $(\overline{\alpha}_i, \overline{a}_i, \overline{\lambda}_i)$ *(up to permutation). Thus we can write* u *as follows:* $u = \sum_{i=1}^{p} \overline{\alpha}_i P\delta_{(\overline{a}_i, \overline{\lambda}_i)} + v$, *where* v *belongs to* $H_0^1(\Omega)$ *and satisfies*

$$(V_0) \qquad (v, P\delta_i)_{H_0^1} = (v, \partial P\delta_i/\partial\lambda_i)_{H_0^1} = 0, \quad (v, \partial P\delta_i/\partial a_i)_{H_0^1} = 0$$

for each $i = 1, \dots, p$, *where* δ_i *denotes* $\delta_{(a_i,\lambda_i)}$.

In order to search for a critical point of J, as a first step, we minimize J in the v-space, and we have the following proposition:

Proposition 2.3. [3, 6] *There exists a* C^1 *map which, to each* (α, a, λ) *satisfying* $\sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V(p, \varepsilon)$, *with* ε *small enough, associates* $\overline{v} = \overline{v}(\alpha, a, \lambda)$ *satisfying* (V_0) *such that* $\overline{v}$ *is unique, minimizes* $J(\sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} + v)$ *with respect to* v *satisfying* (V_0), *and we have the following estimate:*

$$|\overline{v}|_{H_0^1} \le c \sum_{i=1}^{p} \begin{cases} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{(\log \lambda_i d_i)^{\frac{n+2}{2n}}}{(\lambda_i d_i)^{\frac{n+2}{2}}} + \sum_{j\neq i} \varepsilon_{ij}^{\frac{n+2}{2n-4}} (\log \varepsilon_{ij}^{-1})^{\frac{n+2}{2n}} & \\ \qquad\qquad \text{if } n \ge 6 & \\ \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{n-2}} + \sum_{j\neq i} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{n-2}{n}} & \\ \qquad\qquad \text{if } n \le 5. & \end{cases}$$

The next propositions are devoted to a useful expansions of J and its gradient in the set $V(p, \varepsilon)$. Those expansions are extracted from [3].

Proposition 2.4. [3] *For* $\varepsilon > 0$ *small enough,* $p \in \mathbb{N}^*$, *and* $u = \sum_{i=1}^{p} \alpha_i P\delta_i + v \in V(p, \varepsilon)$, v *satisfying* (V_0), *we have*

$$\begin{aligned} J(u) &= \frac{S_n^{2/n} \sum_{i=1}^{p} \alpha_i^2}{(\sum_{i=1}^{p} \alpha_i^{2n/(n-2)} K(a_i))^{(n-2)/n}} (1 + o(1)) \\ &= S_n^{2/n} \Big(\sum_{i=1}^{p} \frac{1}{K(a_i)^{(n-2)/2}} \Big)^{2/n} (1 + o(1)). \end{aligned}$$

Proposition 2.5. [3] *Let* $n \ge 4$. *For* ε *small enough and* $u = \alpha_1 P\delta_1 + \alpha_2 P\delta_2 \in V(2, \varepsilon)$, *we have*

$$\begin{aligned} &\Big(\partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \Big)_{H_0^1} = J(u) \Big[2\overline{c}_1 \alpha_i \frac{\Delta K(a_i)}{K(a_i)\lambda_i^2} - (n-2)\overline{c}_2 \alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}} \\ &- 2\overline{c}_2 \alpha_j \Big(\lambda_i \frac{\partial \varepsilon_{12}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_1, a_2)}{(\lambda_1 \lambda_2)^{(n-2)/2}} \Big) + o\Big(\varepsilon_{12} + \sum \frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^{n-2}} \Big) \Big] \end{aligned}$$

$$\Big(\partial J(u), \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \Big)_{H_0^1}$$

$$= J(u)\Big[-\overline{c}_4 \alpha_i^{\frac{n+2}{n-2}} J(u)^{\frac{n}{n-2}} \frac{\nabla K(a_i)}{\lambda_i}(1+o(1)) + \overline{c}_2 \frac{\alpha_i}{\lambda_i^{n-1}} \frac{\partial H(a_i,a_i)}{\partial a_i}(1+o(1))$$
$$+ 2\overline{c}_2 \alpha_j \Big(\frac{1}{\lambda_i}\frac{\partial \varepsilon_{12}}{\partial a_i} - \frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} \frac{1}{\lambda_i} \frac{\partial H(a_1,a_2)}{\partial a_i}\Big)\Big(1 - J(u)^{\frac{n}{n-2}} \sum \alpha_i^{\frac{4}{n-2}} K(a_i)\Big)$$
$$+ O\Big(\frac{1}{\lambda_i^2} + \sum_k \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1}) + \lambda_j |a_1 - a_2| \varepsilon_{12}^{\frac{n+1}{n-2}}\Big)\Big],$$

where

$$\overline{c}_1 = \frac{n-2}{2n^2} c_n^{2n/(n-2)} \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^n} dx, \quad \overline{c}_2 = c_n^{2n/(n-2)} \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{\frac{n+2}{2}}} dx,$$

c_n is defined in (2.1)*, and $\overline{c}_4$ is a positive constant.*

3. Characterization of the critical points at infinity

A critical point at infinity is the orbits of the flow that remain in $V(p,\varepsilon(s))$ (see [3]). Here $\varepsilon(s)$ is a given function such that $\varepsilon(s)$ tends to zero when s tends to $+\infty$. The following propositions characterize the critical points at infinity in the sets $V(1,\varepsilon)$ and $V(2,\varepsilon)$.

Proposition 3.1. *Let $n \geq 4$. For ε small enough, the only critical points at infinity in $V(1,\varepsilon)$ correspond to $P\delta_{(y,\infty)}$, where y is a critical point of K satisfying*

$$-\Delta K(y) > 0 \ (\text{if } n \geq 5) \ \text{and} \ \frac{-\Delta K(y)}{3K(y)} + 8H(y,y) > 0 \ (\text{if } n = 4).$$

Furthermore, such a critical point at infinity has a Morse index equal to $n - index(K,y)$. Here $index(K,y)$ denotes the Morse index of the function K at y.

Before giving the analogous proposition in $V(2,\varepsilon)$ we need to introduce the following 2×2 matrix. Let y_i and y_j be two different critical points of K, and define a matrix $M(y_i,y_j) = (m_{rq})$ with

$$m_{rr} = \frac{-\Delta K(y_r)}{3K(y_r)} + 8H(y_r,y_r) \text{ for } r = i,j, \quad m_{rq} = -8G(y_r,y_q) \text{ for } r \neq q \tag{3.1}$$

Proposition 3.2. *Let $n \geq 4$. For ε small enough, the only critical points at infinity in $V(2,\varepsilon)$ correspond to $P\delta_{(y_i,\infty)}/K(y_i)^{\frac{n-2}{4}} + P\delta_{(y_j,\infty)}/K(y_j)^{\frac{n-2}{4}}$, where y_i and y_j are two different critical points of K satisfying $-\Delta K(y_r) > 0$ for $r = i,j$ (if $n \geq 5$) and $M(y_i,y_j)$ is positive definite (if $n = 4$).*

Furthermore, such a critical point at infinity has a Morse index equal to $2n - index(K, y_i) - index(K, y_j) + 1$.

For $n = 4$, the proofs are completed in [12]. It remains to prove those propositions when $n \geq 5$. Our argument involves the construction of a special pseudo-gradient for the associated variational problem. The Palais-Smale condition will be satisfied along the decreasing flow lines of the pseudo-gradient as long as these flow lines do not enter the neighborhood of a finite number of different critical points of K such that the related Laplacian $(-\Delta K)$ is positive. This construction involves the proof of a quite difficult and technical Morse lemma at infinity.

Lemma 3.3. *Let* $n \geq 5$. *There exists a pseudogradient* W_1 *so that the following holds: There is a constant* $c > 0$ *independent of* $u = P\delta_i / |P\delta_i|_{H_0^1}$ *in* $V(1, \varepsilon)$ *so that*

(a) $$\big(-\partial J(u), W_1\big) \geq c\Big(\frac{1}{\lambda^2} + \frac{1}{(\lambda d)^{n-1}} + \frac{|\nabla K(a)|^2}{\lambda}\Big)$$

(b) $$\Big(-\partial J(u + \overline{v}), W_1 + \frac{\partial \overline{v}}{\partial(a, \lambda)}(W_1)\Big) \geq c\Big(\frac{1}{\lambda^2} + \frac{1}{(\lambda d)^{n-1}} + \frac{|\nabla K(a)|^2}{\lambda}\Big)$$

(c) W_1 *is bounded and the only region where* λ *increases along the flow lines is when the point* a *is near a critical point of* K *satisfying* $-\Delta K > 0$.
(d) *The distance of* a *to the boundary only increases if it is small enough.*

Proof. Using the assumption (A_2) and the fact that Ω is bounded, there exist two positive constants c_0 and d_0 such that for each $x \in \Omega$, with $d_x \leq 2d_0$, we have $\nabla K(x) \cdot \nu_x < -c_0$, where ν_x is the outward-normal vector to $\partial\Omega_x = \{z \in \Omega / \ d_z = d_x\}$ at the point x. The construction of W_1 will depend on a and λ. We distinguish two cases.

If $d_a < 2d_0$, we define Z_1 as $Z_1 = \lambda^{-1}(\partial P\delta/\partial a)(-\nu_a)$. Using Proposition 2.5 we derive

$$\begin{aligned}(-\partial J(u), Z_1)_{H_0^1} &= -\frac{c}{\lambda}\frac{\partial K(a)}{\partial \nu_a} + \frac{c}{\lambda^{n-1}}\frac{\partial H(a,a)}{\partial \nu_a} + o\Big(\frac{1}{\lambda} + \frac{1}{(\lambda d)^{n-1}}\Big) \qquad (3.2)\\ &\geq \frac{c}{\lambda} + \frac{c}{(\lambda d)^{n-1}} \geq c\Big(\frac{1}{\lambda^2} + \frac{1}{(\lambda d)^{n-1}} + \frac{|\nabla K(a)|^2}{\lambda}\Big)\end{aligned}$$

(since $\nabla H(a,a) \cdot \nu_a \sim c/d^{n-1}$ (see [19])).

If $d_a > d_0$, we note that H and its gradient are bounded. When a is away from the critical points of K, we define $Z_2 = \lambda^{-1}(\partial P\delta/\partial a)(\nabla K(a))$. In the other case, i.e., a is near a critical point y of K, we decrease λ

if $-\Delta K(y) < 0$ and we increase λ if $-\Delta K(y) > 0$. Thus we define $Z_3 = \lambda(\partial P\delta/\partial\lambda)(sign(-\Delta K(y)))+mZ_2$, where m is a fixed small constant. Using Proposition 2.5, the fact that $d_a > d_0$, and $n-2 \geq 3$, we derive that

$$(-\partial J(u), Z_i)_{H_0^1} \geq c\Big(\frac{1}{\lambda^2} + \frac{|\nabla K(a)|^2}{\lambda}\Big), \quad \text{for } i = 2, 3. \tag{3.3}$$

W_1 will be a convex combination of all Z_i's. The proof of (a) follows. The proofs of (c) and (d) are immediate from the definition of W_1. The proof of (b) is the same as in [5] and [9]. □

Lemma 3.4. *Let $n \geq 5$. For $u = P\delta_{(a,\lambda)}/|P\delta_{(a,\lambda)}|_{H_0^1}$ in $V(1,\varepsilon)$ so that a is close to a critical point y of K satisfying $-\Delta K(y) > 0$, there exists a change of variable $(a,\lambda) \to (\tilde{a}, \tilde{\lambda})$ such that*

$$J(u+\overline{v}) = \Psi(\tilde{a},\tilde{\lambda}) := \frac{S_n^{2/n}}{K(\tilde{a})^{(n-2)/n}}\Big(1 - \frac{\overline{c}_1}{S_n K(y)}(1-\eta)\frac{\Delta K(y)}{\tilde{\lambda}^2}\Big),$$

where η is a fixed small positive constant.

The proof of this lemma is omitted here since it is very similar to the proof of Lemma 2 of [5].

Proof of Proposition 3.1 Using Lemma 3.3, the concentration λ is bounded when a is away from the critical points of K satisfying $-\Delta K > 0$. If a is close to a critical point y with $-\Delta K(y) > 0$, Lemma 3.4 gives a split of the variables a and λ, and it is easy to see that if a is equal to y, we have to increase λ to decrease J. Thus $P\delta_{(y,\infty)}$ is a critical point at infinity. By Lemma 3.4, we can easily compute the Morse index of such a critical point at infinity. Since $-\Delta K(y)$ is positive, we derive that the Morse index is equal to $n - index(K, y)$. □

It remains to prove Proposition 3.2. For this effect, we need the following lemma.

Lemma 3.5. *Let $n \geq 5$. There exists a pseudo-gradient W_2 so that the following holds: There is a constant $c > 0$ independent of $u = \sum_{i=1}^2 \alpha_i P\delta_i$ in $V(2,\varepsilon)$ so that the following hold:*

(a) $$\big(-\partial J(u), W_2\big) \geq c\Big(\varepsilon_{12}^{\frac{n-1}{n-2}} + \sum \frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{n-1}} + \frac{|\nabla K(a_i)|^2}{\lambda_i}\Big)$$

(b)

$$\Big(-\partial J(u+\overline{v}), W_2 + \frac{\partial \overline{v}}{\partial(\alpha,a,\lambda)}(W_2)\Big) \geq c\Big(\varepsilon_{12}^{\frac{n-1}{n-2}} + \sum \frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{n-1}} + \frac{|\nabla K(a_i)|^2}{\lambda_i}\Big)$$

(c) W_2 is bounded, and the only region where the maximum of the λ_i's increases along the flow lines is when the points a_i are near two different critical points of K satisfying $-\Delta K > 0$.
(d) The minimal distance to the boundary only increases if it is small enough.

Proof. Without loss of generality, we can assume that $\lambda_1 \leq \lambda_2$. We divide the set $V(2, \varepsilon)$ into three subsets, and we will define a vector field on each subset. After, we define W_2 as a convex combination.

1st subset: $F_1 = \{u \,/\, d_i \leq 2d_0 \text{ for } i = 1, 2\}$. Let M be a fixed large constant; we have three cases.

1st case: If $\lambda_2 \leq 2M\lambda_1$. In this case we define

$$Z_1^1 = \frac{1}{\lambda_2} \sum \frac{\partial P\delta_i}{\partial a_i}(-\alpha_i \nu_i).$$

Using Proposition 2.5, we derive

$$\begin{aligned}
\big(-\partial J(u), Z_1^1\big) \geq{}& \frac{c}{\lambda_2}\Big(1 + \sum \frac{1}{\lambda_i^{n-2} d_i^{n-1}}\Big) \qquad (3.4)\\
&- 2\frac{\overline{c}_2}{\lambda_2}\alpha_1\alpha_2\Big(\frac{\partial \varepsilon_{12}}{\partial a_1}(\nu_1 - \nu_2) - \frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} \sum \frac{\partial H}{\partial \nu_i}(a_1, a_2)\Big)(1 + o(1))\\
&+ O\Big(\varepsilon_{12}^{\frac{n}{n-2}} \log(\varepsilon_{12}^{-1}) + \lambda_1 |a_1 - a_2| \varepsilon_{12}^{\frac{n+1}{n-2}}\Big).
\end{aligned}$$

As in [11] (see (4.15)), we have either $\frac{\partial H}{\partial \nu_i}(a_i, a_j) > 0$ or $\frac{\partial H}{\partial \nu_i}(a_i, a_j) = o(d_i^{-1}(d_i d_j)^{\frac{2-n}{2}})$. Observe now, since λ_1 and λ_2 are of the same order,

$$\frac{\partial \varepsilon_{12}}{\partial a_1}(\nu_1 - \nu_2) = O(\varepsilon_{12}) = o(1)\ , \quad \lambda_1 |a_1 - a_2| \varepsilon_{12}^{\frac{n+1}{n-2}} = O(\varepsilon_{12}^{n/(n-2)}).$$

Thus (3.4) becomes

$$\begin{aligned}
&\big(-\partial J(u), Z_1^1\big) \qquad (3.5)\\
&\geq c\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{(\lambda_1 d_1)^{n-1}} + \frac{1}{(\lambda_2 d_2)^{n-1}}\Big) + O(\varepsilon_{12}^{n/(n-2)} \log(\varepsilon_{12}^{-1})).
\end{aligned}$$

We define now $Z_1^2 = -\varepsilon_{12}^{1/(n-2)} \sum \lambda_i \partial P\delta_i / \partial \lambda_i$. Using Proposition 2.5 we derive

$$\big(-\partial J(u), Z_1^2\big) \geq c\varepsilon_{12}^{\frac{n-1}{n-2}} + o\Big(\sum \frac{1}{\lambda_i^2}\Big) + O\Big(\varepsilon_{12}^{\frac{1}{n-2}} \sum \frac{1}{(\lambda_i d_i)^{n-2}}\Big). \qquad (3.6)$$

Observe that $\varepsilon_{12}^{\frac{1}{n-2}}\sum(\lambda_i d_i)^{2-n} = o(\varepsilon_{12}^{\frac{n-1}{n-2}}) + O(\sum(\lambda_i d_i)^{1-n})$. Thus, for m_1 a fixed large constant, we define $Z_1 = m_1 Z_1^1 + Z_1^2$. Using (3.5) and (3.6) we derive the desired estimate.

2nd case: If $\lambda_2 \geq M\lambda_1$ and $d_1 \geq \sqrt{M} d_2$. In this case we have $d_2 = o(|a_1 - a_2|)$ and $d_1 \sim |a_1 - a_2|$. Thus,

$$R_i := \frac{1}{\lambda_i}\Big|\frac{\partial\varepsilon_{12}}{\partial a_i}\Big| + \lambda_i|a_1 - a_2|\varepsilon_{12}^{\frac{n+1}{n-2}} + \varepsilon_{12}^{\frac{n-1}{n-2}} + \frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\frac{1}{\lambda_i}\Big|\frac{\partial H}{\partial \nu_i}(a_1, a_2)\Big|$$
$$= o\Big(\sum\frac{1}{(\lambda_i d_i)^{n-1}}\Big). \tag{3.7}$$

Therefore, we define $Z_2 = \sum\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}(-\alpha_i\nu_i)$. We derive

$$(-\partial J(u), Z_2) \geq c\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{(\lambda_1 d_1)^{n-1}} + \frac{1}{(\lambda_2 d_2)^{n-1}}\Big) + O(R_1 + R_2). \tag{3.8}$$

(3.7) and (3.8) imply the desired estimate.

3rd case: If $\lambda_2 \geq M\lambda_1$ and $d_1 \leq 2\sqrt{M} d_2$. In this case we have $2\sqrt{M}\lambda_1 d_1 \leq \lambda_2 d_2$. We define

$$Z_3 = \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial a_1}(-\nu_1) + \lambda_1\frac{\partial P\delta_1}{\partial\lambda_1} - m_2\lambda_2\frac{\partial P\delta_2}{\partial\lambda_2},$$

where m_2 is a fixed large constant satisfying $m_2/\sqrt{M}$ is small. Using Proposition 2.5, we derive

$$\begin{aligned}
&(-\partial J(u), Z_3) \qquad\qquad (3.9)\\
&\geq \Big(\frac{c}{\lambda_1} + \frac{c}{(\lambda_1 d_1)^{n-1}} + O(\varepsilon_{12})\Big) + \Big(\frac{c}{(\lambda_1 d_1)^{n-2}} + O(\varepsilon_{12} + \frac{1}{\lambda_1^2})\Big)\\
&+ m_2\Big(c\varepsilon_{12} + O\Big(\frac{1}{\lambda_1^2} + \frac{1}{(\lambda_2 d_2)^{n-2}} + \frac{1}{(\lambda_1 d_1\lambda_2 d_2)^{(n-2)/2}}\Big)\Big)\\
&\geq c\Big(\varepsilon_{12} + \sum\frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_i)^{n-2}}\Big).
\end{aligned}$$

The vector field Y_1 will be a convex combination of Z_1, Z_2 and Z_3.

2nd subset: $F_2 = \{u/\ d_1 \geq d_0 \text{ and } d_2 \geq d_0\}$. In this subset, the regular part of the Green's function H and its gradient are bounded. Thus the expansions of J and its gradient (since $n - 2 \geq 3$) becomes as in [4]. Then we can use the vector field defined in Appendix 2 of [4] in the case where $p = 2$, which we denote Y_2, and we derive

$$(-\partial J(u), Y_2) \geq c\Big(\varepsilon_{12} + \sum\frac{1}{\lambda_i^2} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big). \tag{3.10}$$

Since $(\lambda_i d_i)^{1-n} = o(\lambda_i^{-2})$ we derive the desired estimate.

3rd subset: $F_2 = \{u \,/\, d_i \leq d_0$ and $d_j \geq 2d_0\}$. In this subset we have $|a_1 - a_2| \geq d_0$. Since $n \geq 5$, we derive $\varepsilon_{12} = O(\lambda_1^{-3} + \lambda_2^{-3})$. Furthermore, since $d_j \geq 2d_0$, we have $H(a_j, a_k) \leq c$ for $k = 1, 2$. In this case we define

$$Z_4 = \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}(-\nu_i) + \frac{1}{\lambda_j}\frac{\partial P\delta_j}{\partial a_j}(\nabla K(a_j)).$$

Using Proposition 2.5, we derive

$$\big(-\partial J(u), Z_4\big) \geq \frac{c}{\lambda_i} + \frac{c}{(\lambda_i d_i)^{n-1}} + \frac{|\nabla K(a_j)|^2}{\lambda_j} + O(\frac{1}{\lambda_j^2}). \tag{3.11}$$

If $\lambda_i \leq M\lambda_j$ (for M a fixed large constant) or a_j is far away from any critical point of K, i.e., $|\nabla K(a_j)| \geq c$, (3.11) implies the desired estimate. In the other case, i.e., a_j is near a critical point y and $\lambda_i \geq M\lambda_j$, we use another vector field, which is

$$Z_5 = Z_4 + m_3 sign(-\Delta K(y))\lambda_j \frac{\partial P\delta_j}{\partial \lambda_j},$$

where m_3 is a fixed large constant. Using also Proposition 2.5 and (3.11), we derive the desired estimate in this case also. □

Lemma 3.6. *Let $n \geq 5$. For $u = \sum_{i=1}^2 \alpha_i P\delta_i$ in $V(2,\varepsilon)$ so that the points a_i are close to two different critical points y_{i_1} and y_{i_2} of K satisfying $-\Delta K > 0$, there exists a change of variable $(a_1, a_2, \lambda_1, \lambda_2) \to (\tilde a_1, \tilde a_2, \tilde\lambda_1, \tilde\lambda_2)$ such that*

$$J(u + \overline{v}) = \Psi(\alpha, \tilde a, \tilde\lambda)$$

$$:= \frac{S_n^{2/n}\sum_{i=1}^2 \alpha_i^2}{(\sum_{i=1}^2 \alpha_i^{\frac{2n}{n-2}} K(\tilde a_i))^{\frac{n-2}{n}}}\Big(1 - \frac{\overline{c}_1}{S_n}(1-\eta)\Big(\frac{\Delta K(y_{i_1})}{\tilde\lambda_1^2 K(y_{i_1})} + \frac{\Delta K(y_{i_2})}{\tilde\lambda_2^2 K(y_{i_2})}\Big)\Big),$$

where η is a fixed small positive constant, $\alpha = (\alpha_1, \alpha_2)$, $\tilde a = (\tilde a_1, \tilde a_2)$, and $\tilde\lambda = (\tilde\lambda_1, \tilde\lambda_2)$.

The proof is also skipped since it is also very similar to the proof of Lemma 2 of [5].

4. Proofs of theorems

Proof of Theorem 1.1 We argue by contradiction: We suppose that J has no critical point in Σ^+. Let $c_\infty(y_0, y_{i_1}) = S_4^{1/2}(K(y_0)^{-1} + K(y_{i_1})^{-1})^{1/2}$. Observe that under the assumption (H_1), (y_0, y_{i_1}) is not a critical point at infinity. Using Propositions 3.1 and 3.2, it follows that under the assumption

(H_2), the only critical points at infinity of J under the level $c_1 = c_\infty(y_0, y_{i_1})+\gamma$, for γ small enough, are $P\delta_{(y_0,\infty)}$ and $P\delta_{(y_{i_1},\infty)}$. The unstable manifolds of such critical points, $W_u(y_0)_\infty$ and $W_u(y_{i_1})_\infty$, can be described, using Lemma 3.4 as the product of $W_s(y_0)$ and $W_s(y_{i_1})$ (for a pseudo-gradient of K) by $[A,\infty)$ (domain of the variable λ) for some positive number A large enough. Let

$$X = \overline{W_s(y_{i_1})} = W_s(y_{i_1}) \cup W_s(y_0).$$

Using the assumption (A_2), X cannot dominate the boundary of Ω. Thus it is a compact manifold of Ω in dimension k without boundary, with $k \geq 1$. Since J has no critical point in Σ^+, it follows that $J_{c_1} = \{u \in \Sigma^+ /\ J(u) \leq c_1\}$ retracts by deformation onto $X_\infty = W_u(y_0)_\infty \cup W_u(y_{i_1})_\infty$ (see Sections 7 and 8 of [8]), which can be parametrized as we said before by $X \times [A,\infty)$.

From another part, we have X_∞ is contractible in J_{c_1}. Indeed, let

$$h : [0,1] \times X_\infty \to \Sigma^+, \qquad (t,x,\lambda) \to \frac{tP\delta_{(x,\lambda)} + (1-t)P\delta_{(y_0,\lambda)}}{|tP\delta_{(x,\lambda)} + (1-t)P\delta_{(y_0,\lambda)}|_{H_0^1}} .$$

Let us recall the following expansion of J (see [5]):

Lemma 4.1. [5] *Let a_1 and a_2 be in Ω, $\alpha_1, \alpha_2 > 0$, and let λ be large enough. Let $u = \alpha_1 P\delta_1 + \alpha_2 P\delta_2$; we have*

$$J(u) \leq S_4^{1/2}\Big(\frac{1}{K(a_1)} + \frac{1}{K(a_2)}\Big)^{1/2}(1+o(1)).$$

Using Lemma 4.1 and the fact that, for each $x \in X$, $K(x) \geq K(y_{i_1})$, it follows that the contraction h is performed under the level c_1, so X_∞ is contractible in J_{c_1}, which retracts by deformation onto X_∞. Therefore X_∞ is contractible, leading to the contractibility of X, which is a contradiction, since X is a compact manifold in dimension k without boundary. Hence (1) has a solution.

It remains to compute the Morse index. Using a dimension argument, since $h([0,1] \times X_\infty)$ is a manifold in dimension $k+1$, then the Morse index of the solution provided by Theorem 1.1 is less than $k+1$.

From another part, assume that the Morse index of the solution is less than $k-1$. Perturbing if necessary J, we may assume that all the critical points of J are nondegenerate and have their Morse index less than $k-1$. Such critical points do not change the homological group in dimension k of the level sets of J. Since X defines a homological class in dimension k which is nontrivial in $J_{c_\infty(y_{i_1})+\gamma}$ for γ small enough, where $c_\infty(y_{i_1}) = S_4^{1/2} K(y_{i_1})^{-1/2}$, but trivial in J_{c_1}, our theorem follows. □

If we change the assumption (H_2) to the assumption (H_2'), under the level c_1 we can find other critical points at infinity but of index $\notin \{k, k+1\}$. Using the same argument above, Remark 1 follows.

Proof of Theorem 1.2 First, we notice that assumption (T_1) implies that $(y_0, y_{i_1})_\infty$ is a critical point at infinity. Assume that (1) has no solution; we claim that $f_\lambda(C_{y_0}(X))$ retracts by deformation onto $X_\infty \cup W_u(y_0, y_{i_1})_\infty$. Indeed, let

$$u = \frac{(\alpha/K(x)^{1/2})P\delta_{(x,\lambda)} + ((1-\alpha)/K(y_0)^{1/2})P\delta_{(y_0,\lambda)}}{|(\alpha/K(x)^{1/2})P\delta_{(x,\lambda)} + ((1-\alpha)/K(y_0)^{1/2})P\delta_{(y_0,\lambda)}|_{H_0^1}} \in f_\lambda(C_{y_0}(X)).$$

The action of the flow of the pseudo-gradient is essentially on α. If $\alpha < 1/2$, the flow brings α to zero, and thus u goes in this case to $W_u(y_0)_\infty \equiv \{y_0\}$. If $\alpha > 1/2$, the flow brings α to 1, and thus u goes to $W_u(y_{i_1})_\infty \equiv X_\infty$. If $\alpha = 1/2$, since only x can move, y_0 remains one of the concentration points of u and x goes to $W_s(y_i)$, where $y_i = y_{i_1}$ or $y_i = y_0$, and only those two cases may occur.

In the first case, i.e., $y_i = y_{i_1}$, u goes to $W_u(y_0, y_{i_1})_\infty$. In the second case, i.e., $y_i = y_0$, there exists s_0 such that $x(s_0)$ is close to y_0. Thus, using Lemma 4.1, we have the following estimate:

$$J(u(s_0)) \leq C_\infty(y_0, y_0) + \gamma$$

for γ small enough. It follows from Propositions 3.1 and 3.2, under the assumption (T_2), that $J_{c_\infty(y_0,y_0)+\gamma}$ retracts by deformation onto $W_u(y_0)_\infty \equiv \{y_0\}$, and thus u goes to $W_u(y_0)_\infty$. Therefore $f_\lambda(C_{y_0}(X))$ retracts by deformation onto $X_\infty \cup W_u(y_0, y_{i_1})_\infty$. Since $\mu(y_{i_1}) = 0$ it follows that this strong retract does not intersect $W_u(y_0, y_{i_1})_\infty$, and thus it is contained in X_∞. Therefore X_∞ is contractible; it follows that X is contractible. This yields a contradiction since X is a compact manifold in dimension k without boundary. Hence, (1) has a solution. Arguing as in the proof of Theorem 1.1, we claim that the index of the solution is k or $k+1$. Our theorem follows. □

Proof of Theorem 1.3 We argue by contradiction: We assume that (1) has no solution. Using the same arguments as those used in the proof of Theorem 1.2, it follows that $f_\lambda(C_{y_0}(X))$ retracts by deformation onto $X_\infty \cup W_u(y_0, y_i) \cup D$, where we have that $D \subset \sigma$ is a stratified set and $\sigma = \cup_{y_j \in X \setminus \{y_0, y_i\}} W_u(y_0, y_j)_\infty$. The dimension of σ is at most k. Since $\mu(y_i) = 0$, we can be more precise. This strong retract does not intersect $W_u(y_0, y_i)_\infty$, so $f_\lambda(C_{y_0}(X))$ retracts by deformation onto $X_\infty \cup D$.

Therefore, $H_*(X_\infty \cup D) = 0$ for all $* \in \mathbb{N}^*$, since $f_\lambda(C_{y_0}(X))$ is a contractible set. Using the exact homological sequence of $(X_\infty \cup D, X)$, we have $\cdots \to H_{k+1}(X_\infty \cup D) \to H_{k+1}(X_\infty \cup D, X_\infty) \to H_k(X_\infty) \to H_k(X_\infty \cup D) \to \cdots$.

Since $H_m(X_\infty \cup D) = 0$ for all $m \in \mathbb{N}^*$, then $H_k(X_\infty) \equiv H_{k+1}(X_\infty \cup D, X_\infty)$. In addition $(X_\infty \cup D, X_\infty)$ is a stratified set of dimension at most k, so $H_{k+1}(X_\infty \cup D, X_\infty) = 0$. Therefore, $H_k(X_\infty) = 0$. This yields a contradiction since X_∞ can be parameterized by $X \times [A, +\infty)$ and X is a manifold in dimension k without boundary. Hence, (1) has a solution. Arguing as the proof of Theorem 1.1, we claim that the index of the solution is k or $k+1$. The result follows. □

Proof of Theorem 1.4. Our proof follows the algebraic topological argument introduced in [2]. Arguing by contradiction, we suppose that (1) has no solution. Choosing c_0 small enough, it follows that every critical point at infinity having more than two masses in its description is above the level $c_1 = S_n^{2/n}/K(y_m)^{(n-2)/n} + \varepsilon$, for ε small enough. In addition, since $K(y_m) = \min\{K(y)/\ y \in I_1\}$, it follows that the critical points at infinity under the level c_1 are in one-to-one correspondence with the critical points of K in I_1. So J_{c_1} retracts by deformation onto $X_\infty = \cup_{y \in I_1} W_u(y)_\infty$ (see Sections 7 and 8 of [8]), which can be parametrized by $X \times [A, \infty)$. From another part, we have X_∞ is contractible in $J_{c_2+\varepsilon}$, where $c_2 = S_n^{2/n}/\overline{c}^{(n-2)/n}$. Indeed, from the assumption (B_3), it follows that there exists a contraction $h : [0,1] \times X \to K^{\overline{c}}$, h continuous, such that for any $a \in X$, $h(0,a) = a$ and $h(1,a) = a_0$, a fixed point of X. Such a contraction gives rise to the following contraction:

$$\tilde{h} : [0,1] \times X_\infty \to \Sigma^+ \ , \qquad (t, a, \lambda) \to \frac{P\delta_{(h(t,a),\lambda)} + \overline{v}}{P\delta_{(h(t,a),\lambda)} + \overline{v}} \ .$$

Observe that $\tilde{h}(0, a, \lambda) = P\delta_{(a,\lambda)} + \overline{v}$ and $\tilde{h}(1, a, \lambda) = P\delta_{(a_0,\lambda)} + \overline{v}$. Now using Lemma 3.4, we derive

$$J(P\delta_{(h(t,a),\lambda)} + \overline{v}) = \frac{S_n^{2/n}}{K(h(t,a))^{\frac{n-2}{n}}}(1 + o(1)).$$

By construction we have $K(h(t,a)) \geq \overline{c}$; therefore, such a contraction is performed under $c_2 + \varepsilon$ for A large enough. So X_∞ is contractible in $J_{c_2+\varepsilon}$.

In addition, choosing c_0 small enough, between the levels $c_2+\varepsilon$ and c_1 there are no critical points at infinity; thus, $J_{c_2+\varepsilon}$ retracts by deformation onto J_{c_1}, which retracts by deformation onto X_∞. Therefore, X_∞ is contractible, leading to the contractibility of X, which is a contradiction, since X is a compact manifold in dimension k without boundary. Hence, (1) has a solution. Arguing as the proof of Theorem 1.1, we claim that the index of the solution is bigger than k. Our theorem follows. □

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