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WHEN IS A GIVEN SET OF PDES PART OF AN ELLIPTIC SYSTEM?

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Abstract. We investigate the following question: Given a set of k partial differential equations for m unknowns, where $k < m$, can we find $m - k$ additional equations in such a way that the full set of equations forms an elliptic system? We formulate a maximal rank condition which is obviously necessary. In general, however, the maximal rank condition is sufficient only if we allow the introduction of additional variables, not just additional equations. In particular, the equation $\operatorname{div} u = 0$ can be part of an elliptic system for the components of the vector field u only if the space dimension is 1,2,4, or 8.

1. Introduction

Consider k linear equations for m unknowns:

$$\sum_{j=1}^{m} A_{ij} u_j = 0, \quad i = 1, ..., k, \tag{1.1}$$

where $k < m$. When can we add $m - k$ additional equations so that the resulting system is nonsingular? The answer, of course, is well-known. A necessary and sufficient condition is that the matrix A_{ij}, $1 \leq i \leq k$, $1 \leq j \leq m$ has rank k.

In this paper, we shall consider the analogous question for partial differential equations. Thus, we start with k given partial differential equations for m unknowns, and we ask if it is possible to add $m - k$ additional PDEs such that the resulting system is elliptic (in the sense of Agmon, Douglis and Nirenberg [1]). In analogy to linear algebra, we can formulate a maximal rank condition (details will be given below). This maximal rank condition is obviously necessary, and it is natural to ask whether it is also sufficient.

We shall show that in general the maximal rank condition is sufficient only if we allow additional variables, i.e., the completed system has more

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than m variables. Otherwise, the maximal rank condition is in general not sufficient. Specifically, we shall consider the equation $\operatorname{div} u = 0$ in m space dimensions. We can augment this equation with $u = \nabla\phi$, resulting in a system of $m+1$ equations for $m+1$ variables which is elliptic, and in fact equivalent to Laplace's equation. If, however, only the components of u are allowed as variables, then we shall prove that $\operatorname{div} u = 0$ can be part of an elliptic system if and only if the space dimension is 1,2,4, or 8.

We shall identify two special cases when the maximal rank condition is in fact sufficient, namely if $k = m - 1$ or if the space dimension is 2.

The interest in the problem discussed here may appear to be purely curiosity driven, since the partial differential equations governing specific applications are usually given by the physics, and one would not typically encounter the situation where only some of the equations are known and the rest are at our disposal. Such a situation does arise, however, in the context of boundary conditions, where boundaries are often not physical boundaries, but may result from numerical truncation of the domain. In that context, the question investigated here has indeed come up [3].

2. Definitions and statement of problem

We shall use the common multi-index notation. That is, a multiindex

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \tag{2.1}$$

is a vector whose components are nonnegative integers, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + ... + \alpha_n. \tag{2.2}$$

Moreover, for a vector $\xi \in \mathbb{R}^n$, we define

$$\xi^\alpha = \prod_{i=1}^n \xi_i^{\alpha_i}. \tag{2.3}$$

Let s_i, $i = 1, ..., m$ and t_j, $j = 1, ..., m$ be given integers, and let

$$L_{ij}(D) = \sum_{|\alpha|=s_i+t_j} a_{ij}^\alpha D^\alpha. \tag{2.4}$$

be differential operators of order s_i+t_j with real coefficients a_{ij}^α (if $s_i+t_j < 0$, it is understood that $L_{ij} = 0$). With $L_{ij}(D)$, we associate the symbol

$$L_{ij}(i\xi) = \sum_{|\alpha|=s_i+t_j} a_{ij}^\alpha (i\xi)^\alpha. \tag{2.5}$$

We recall the following definition.

Definition 1. *The system*

$$\sum_{j=1}^{m} L_{ij}(D)u_j = 0, \quad i = 1, ..., m, \tag{2.6}$$

is called elliptic (of order $q = \sum_{i=1}^{m} s_i + \sum_{j=1}^{m} t_j$) if

$$\det(L_{ij}(i\xi)) \neq 0 \tag{2.7}$$

whenever $\xi \neq 0$.

We note that we can factor out i^q from the determinant and put (2.7) in the equivalent form

$$\det(L_{ij}(\xi)) \neq 0. \tag{2.8}$$

The problem discussed in this paper is the following:

Problem 1. *Given $1 \leq k < m$ and k partial differential equations*

$$\sum_{j=1}^{m} L_{ij}(D)u_j = 0, \quad i = 1, ..., k, \tag{2.9}$$

where

$$L_{ij}(D) = \sum_{|\alpha|=s_i+t_j} a_{ij}^{\alpha} D^{\alpha}, \tag{2.10}$$

is it possible to find $m-k$ additional partial differential equations such that the completed system is elliptic?

Obviously, the problem has the following equivalent algebraic form:

Problem 2. *Given $1 \leq k < m$, integers s_i, $i = 1, ..., k$, t_j, $j = 1, ..., m$, and homogeneous polynomials*

$$L_{ij}(\xi) = \sum_{|\alpha|=s_i+t_j} a_{ij}^{\alpha} \xi^{\alpha}, \quad a_{ij}^{\alpha} \in \mathbb{R}, \tag{2.11}$$

is it possible to find s_i, $i = k+1, ..., m$ and homogeneous polynomials $L_{ij}(\xi)$, $i = k+1, ..., m$, $j = 1, ..., m$ of degree $s_i + t_j$ such that

$$\det(L_{ij}(\xi)) \neq 0 \tag{2.12}$$

for every nonzero $\xi \in \mathbb{R}^n$?

From linear algebra, we find an obvious necessary condition, to which we shall refer as the maximal rank condition.

Condition 1. *For every nonzero ξ, the rank of the matrix $L_{ij}(\xi)$, $i = 1, ..., k$, $j = 1, ..., m$ is equal to k.*

3. THE EXAMPLE $\operatorname{div} u = 0$

It is tempting to conjecture that the maximal rank condition is also sufficient. This, however, is not the case.

Theorem 1. *Let $n = m$, $k = 1$ and $L_{1j}(\xi) = \xi_j$. Then Problem 2 above has a solution if and only if $n = 1, 2, 4$ or 8.*

Proof. Suppose Problem 2 does have a solution. Then rows 2 through n of the matrix form $n-1$ vector fields on the sphere which are linearly independent of each other and of the normal (given by row 1). Thus it follows that the sphere has $n-1$ (pointwise) linearly independent tangent fields. Theorem V.2.10 in [2] states that this is the case only if $n = 1, 2, 4$ or 8 (we remark that the well-known "hedgehog" theorem states that in odd dimensions the sphere does not even have one tangent field without zeros).

Thus, in dimensions other than possibly 1,2,4 and 8, the equation

$$\operatorname{div} u = 0, \tag{3.1}$$

where u is a vector field, cannot be part of an elliptic system for the components of u. In dimension 2, an example of an elliptic system is the well-known Cauchy-Riemann system. In dimension 4, the determinant of the matrix

$$\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ -\xi_2 & \xi_1 & -\xi_4 & \xi_3 \\ -\xi_3 & \xi_4 & \xi_1 & -\xi_2 \\ -\xi_4 & -\xi_3 & \xi_2 & \xi_1 \end{pmatrix} \tag{3.2}$$

is $|\xi|^4$. In eight dimensions, we can verify that

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_8 \\ -\xi_2 & \xi_1 & \xi_5 & \xi_8 & -\xi_3 & \xi_7 & -\xi_6 & -\xi_4 \\ -\xi_3 & -\xi_5 & \xi_1 & \xi_6 & \xi_2 & -\xi_4 & \xi_8 & -\xi_7 \\ -\xi_4 & -\xi_8 & -\xi_6 & \xi_1 & \xi_7 & \xi_3 & -\xi_5 & \xi_2 \\ -\xi_5 & \xi_3 & -\xi_2 & -\xi_7 & \xi_1 & \xi_8 & \xi_4 & -\xi_6 \\ -\xi_6 & -\xi_7 & \xi_4 & -\xi_3 & -\xi_8 & \xi_1 & \xi_2 & \xi_5 \\ -\xi_7 & \xi_6 & -\xi_8 & \xi_5 & -\xi_4 & -\xi_2 & \xi_1 & \xi_3 \\ -\xi_8 & \xi_4 & \xi_7 & -\xi_2 & \xi_6 & -\xi_5 & -\xi_3 & \xi_1 \end{vmatrix} = |\xi|^8. \tag{3.3}$$

We note that in all these cases the order of the resulting elliptic system is equal to the space dimension. This is, in fact, the best possible. Because, if not, one row of the matrix would have to be constant. Any constant vector, however, is collinear with the normal at some point on the sphere.

4. Sufficiency of the maximal rank condition

In any number of dimensions, the system

$$\begin{aligned} \operatorname{div} u &= 0, \\ \operatorname{grad} \phi &= u \end{aligned} \tag{4.1}$$

is equivalent to Laplace's equation and elliptic of order 2. It does, however, contain the additional variable ϕ. This construction can in fact be generalized.

Theorem 2. *Suppose that $k < m$ and that, for $1 \leq i \leq k$, $1 \leq j \leq m$, $L_{ij}(\xi)$ is a homogeneous polynomial of degree $s_i + t_j$ such that the maximal rank condition holds. Then Problem 2 can be solved by expanding the number of variables to $k + m$.*

Proof. By adding a positive integer to the weights t_j and subtracting the same integer from the s_i, we can arrange it so that $t_j \geq 0$. Now let $\sigma_i = \tau_i = t_i$ for $1 \leq i \leq m$, $\sigma_i = \tau_i = s_{i-m}$ for $m+1 \leq i \leq m+k$, and consider the $(m+k) \times (m+k)$ matrix with block form

$$M(\xi) = \begin{pmatrix} D(\xi) & -L^T(\xi) \\ L(\xi) & 0 \end{pmatrix}, \tag{4.2}$$

where D is the diagonal matrix with entries $D_{ii} = |\xi|^{2t_i}$. The entries of M are homogeneous polynomials of degree $\sigma_i + \tau_j$, and the determinant of M is

$$\det M = \det D \det (LD^{-1}L^T). \tag{4.3}$$

Since L has maximal rank, this is nonzero for every nonzero ξ.

In the special case $k = m - 1$, it is not necessary to increase the number of variables.

Theorem 3. *Let $k = m - 1$, and assume the maximal rank condition holds. Then Problem 2 has a solution.*

Proof. Simply set $L_{mj}(\xi)$ equal to $|\xi|^{2t_j}$ times its cofactor (as before, we can assume $t_j \geq 0$).

We now show that the maximal rank condition is in fact sufficient in the case of two space dimensions.

Theorem 4. *Assume $n = 2$, and assume the maximal rank condition holds. Then Problem 2 has a solution.*

Proof. We can represent any ξ on the unit circle as $\xi(\phi) = (\cos\phi, \sin\phi)$ with $0 \le \phi < 2\pi$.

Let now $a_j(\phi)$ denote the jth column of the matrix $L_{ij}(\xi(\phi))$, $1 \le i \le k$, $1 \le j \le m$. We shall show that by elementary column operations, we can transform the matrix L_{ij} to an equivalent one which has the following property, henceforth called condition (P):

For $1 \le p \le k$, the rank of the matrix

$$\begin{pmatrix} a_1(\phi) & \dots & a_p(\phi) \end{pmatrix} \tag{4.4}$$

is at least $p-1$ for every ϕ and it is equal to p for all but finitely many ϕ.

We shall construct the a_p inductively. For $p = 1$, we simply have to find a column of the matrix that is not identically zero, and then exchange columns accordingly. Suppose now that the first $p-1$ columns satisfy condition (P), and let $\phi_1, \phi_2, ..., \phi_q$ be the exceptional values for which the rank of

$$\begin{pmatrix} a_1(\phi) & \dots & a_{p-1}(\phi) \end{pmatrix} \tag{4.5}$$

is only $p-2$. Because of the maximal rank condition, there is a $j(r)$ for $1 \le r \le q$ such that $a_{j(r)}(\phi_r)$ is not a linear combination of $a_1(\phi_r)$,...,$a_{p-1}(\phi_r)$. Moreover, we can pick a generic ϕ_0 (not one of the ϕ_r) and a $j(0)$ such that $a_{j(0)}(\phi_0)$ is not a linear combination of $a_1(\phi_0)$,...,$a_{p-1}(\phi_0)$. Let $l(\xi)$ be a linear polynomial in ξ such that $l(\xi(\phi_r)) \neq 0$ for $0 \le r \le q$. Now consider the combination

$$A(\phi) = \sum_{r=0}^{q} \lambda_r l(\xi(\phi))^{t_m - t_{j(r)}} a_{j(r)}(\phi). \tag{4.6}$$

Here t_m is the maximum of the $t_{j(r)}$. For a generic choice of the λ_r, this combination will be linearly independent of $a_1(\phi_r)$,...,$a_{p-1}(\phi_r)$ for every r. We can now use elementary column operations to make $A(\phi)$ the pth column of the matrix. It is easy to see that the thus obtained matrix satisfies Property (P) for its first p columns.

After having thus rearranged the first k columns, we observe that we can use the same construction to make the $(k+1)$st column linearly independent of the first k columns whenever ϕ is one of the exceptional values ϕ_1,...,ϕ_r. After doing so, the matrix $L_{ij}(\xi)$, $1 \le i \le k$, $1 \le j \le k+1$ satisfies the maximal rank condition. We can now use Theorem 3 to add a $(k+1)$st row such that the matrix $L_{ij}(\xi)$, $1 \le i, j \le k+1$ fulfills the requirement of ellipticity. Finally, rows $k+2$ through m can simply be filled by adding the equations $u_j = 0$, $j = k+2, ..., m$.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II,* Comm. Pure Appl. Math., 17 (1964), 35–92.

[2] M. Karoubi, "K-Theory," Springer 1978.

[3] M. Renardy, *An alternative approach to inflow boundary conditions for Maxwell fluids in three space dimensions,* J. Non-Newtonian Fluid Mech., 36 (1990), 419–425.