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ASYMPTOTIC BEHAVIORS OF STAR-SHAPED CURVES EXPANDING BY $V = 1 - K$

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Abstract. We consider asymptotic behaviors of star-shaped curves expanding by $V = 1 - K$, where V denotes the outward-normal velocity and K curvature. In this paper, we show the followings. The difference of the radial functions between an expanding curve and circle has its asymptotic shape as $t \to +\infty$. For two curves, if the asymptotic shapes are identical, then the curves are also. The set of all asymptotic shapes is dense in $C(S^1)$.

1. Introduction

We consider the asymptotic behaviors of expanding curves in the plane governed by the interface equation

$$V = 1 - K, \tag{1.1}$$

where V and K are the outward-normal velocity and curvature of a smooth embedded closed curve in $\mathbf{R}^2$, respectively. It expands where the curvature is smaller than 1, and shrinks where it is larger. The motion of smooth closed curves $(x(\theta,t), y(\theta,t)) \in \mathbf{R}^2$ by (1.1) is the gradient flow for the energy

$$E := (\text{length}) - (\text{area}) = \oint ds - \frac{1}{2}\oint x\,dy - y\,dx$$

with respect to the metric $\langle u, v\rangle := \oint u \cdot v\,ds$, where ds denotes the line element for a smooth curve with respect to the Euclidean metric $ds^2 = dx^2 + dy^2$ on the plane.

We take the family $\{R(t)\}_{t\in(-\infty,+\infty)}$ of the radii of the circle expanding by (1.1). That is, $R(t)$ satisfies the ODE

$$\frac{dR}{dt} = 1 - \frac{1}{R}.$$

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Further, we require

$$R(0) = 2$$

in order to normalize the definition of $R(t)$. Then, we have $R(-\infty) = 1$, $R(+\infty) = +\infty$ and $\frac{dR}{dt}(t) > 0$. For star-shaped curves, we have the following from results of Chow, Liou, and Tsai [3, 5]. (For expanding hypersurfaces and boundaries of level-sets in $\mathbf{R}^N$, see e.g. [2]. For convex curves evolving under more general flows in $\mathbf{R}^2$, also see [1].)

Proposition 1. *Suppose that γ_0 is a smooth curve, star-shaped with respect to $y \in \mathbf{R}^2$ and the curvature of γ_0 is smaller than* 1. *Then, there exists a unique smooth solution $\{\gamma_t\}_{t\in[0,+\infty)}$ to the equation* (1.1). *Further, γ_t is star-shaped with respect to y, the curvature of γ_t is smaller than* 1 *for all $t \in [0, +\infty)$, and the radial function $\{r(\theta, t)\}_{\theta\in S^1}$ of γ_t with y as the origin satisfies the estimate*

$$\sup_{t\in[0,+\infty)} \|r(\theta, t) - R(t)\|_{C^2(S^1)} < +\infty. \tag{1.2}$$

Proof. Take a positive function $F \in C^\infty(\mathbf{R})$ satisfying

$$F(K) = 1 - K \quad \Big(K \leq \frac{1 + \max_{\theta\in S^1} K(\theta, 0)}{2} \Big)$$

and

$$\frac{dF}{dK}(K) < 0,$$

where $K(\theta, t)$ denotes the curvature of the star-shaped curve γ_t. Then, by the proof of Corollary 2.2 in [3], we see that

$$\max_{(\theta,t)\in S^1\times[0,T)} K(\theta, t) = \max_{\theta\in S^1} K(\theta, 0) \; < \; 1$$

for the smooth solution $\{\gamma_t\}_{t\in[0,T)}$ to the equation $V = F(K)$. Hence, solving (1.1) is equivalent to $V = F(K)$ for the initial condition γ_0. Now, Theorem 1.1 of [5] and Theorem 4.1 of [3] can work for the solution to (1.1) as well as $V = F(K)$, and we obtain the conclusion of Proposition 1 (see the remark after Lemma 3.4 of [3] for the estimate $\sup_{(\theta,t)\in S^1\times[0,+\infty)} |r(\theta, t) - R(t)| < +\infty$). □

While we see the difference between $r(\theta, t)$ and $R(t)$ is bounded uniformly in $t \in [0, +\infty)$ from (1.2), the following states that there exists the limiting function $l(\theta)$ of the difference $(r - R)(\theta, t)$ as $t \to +\infty$.

Theorem 2. *Suppose that a smooth solution γ_t to* (1.1) *is star-shaped with respect to $y \in \mathbf{R}^2$ and the curvature of γ_t is smaller than* 1 *for all $t \in$*

$[0,+\infty)$. *Let* $r(\theta,t)$ *be the radial function of* γ_t *with* y *as the origin. Then, there exists* $l \in C^\infty(S^1)$ *such that*

$$\lim_{t\to+\infty} \|r(\theta,t) - (R(t) + l(\theta))\|_{C^k(S^1)} = 0 \tag{1.3}$$

for all $k \in \mathbf{N}$.

Remark. If we use a different solution $\tilde{R}(t)$ to the same ODE $\frac{d\tilde{R}}{dt} = 1 - \frac{1}{\tilde{R}}$ with an initial value $\tilde{R}(0) = c \in (1,+\infty) \setminus \{2\}$, then $\lim_{t\to+\infty}(\tilde{R} - R)(t)$ exists. Hence, as long as the initial value $c \in (1,+\infty)$, it does not matter which solution to the ODE we use.

The following indicates that for two expanding curves γ_t^1 and γ_t^2, if the limits $l^1(\theta)$ and $l^2(\theta)$ of the differences between the radius $R(t)$ and the radial functions $r^1(\theta,t)$ and $r^2(\theta,t)$ of γ_t^1 and γ_t^2 as $t \to +\infty$, respectively, are identical, then the curves γ_t^1 and γ_t^2 are so.

Theorem 3. *Suppose that two smooth solutions* γ_t^1 *and* γ_t^2 *to* (1.1) *are star-shaped and their curvatures are smaller than* 1 *for all* $t \in [0,+\infty)$. *Let* $r^1(\theta,t)$ *and* $r^2(\theta,t)$ *be the radial functions of* γ_t^1 *and* γ_t^2, *respectively. Then,* $\lim_{t\to+\infty} \|r^1(\theta,t) - r^2(\theta,t)\|_{C(S^1)} = 0$ *implies* $\gamma_t^1 \equiv \gamma_t^2$ *for all* $t \in [0,+\infty)$.

The following theorem shows the set of limiting functions $l \in C^\infty(S^1)$ of differences as $t \to +\infty$ is a dense one in $C(S^1)$.

Theorem 4. *Let* $l \in C^\infty(S^1)$ *and* $\varepsilon > 0$. *Then, there exists* $M \in \mathbf{R}$ *such that for any* $t_0 > M$, *the following holds: Let* γ_{t_0} *be the star-shaped curve with the radial function* $R(t_0) + l(\theta)$. *Then, the curvature of* γ_{t_0} *is smaller than* 1, *and the radial function* $r(\theta,t)$ *of the smooth solution* $\{\gamma_t\}_{t\in[t_0,+\infty)}$ *to* (1.1) *satisfies the inequality*

$$\sup_{t\in[t_0,+\infty)} \|r(\theta,t) - (R(t) + l(\theta))\|_{C(S^1)} < \varepsilon. \tag{1.4}$$

In Section 2, we rewrite the equation of the function r in $t \in [0,+\infty)$ to a quasi-linear parabolic equation of the function $r - R$ in $\tau \in [-\log 2, 0)$ by changing the time variable t to τ. Using the quasi-linear parabolic equation, we prove Theorems 2, 3, and 4 in Sections 3, 4, and 5, respectively.

2. Changing the time variable and an equation of the function $u := r - R$

In this section, we introduce a new time variable τ and lead a parabolic equation satisfied by the function $u(\theta,\tau) := (r - R)(\theta,t)$.

From (4) of [3], the radial function $r \in C^\infty(S^1 \times [0,+\infty))$ solves the equation

$$r_t = \frac{(r^2+r_\theta^2)^{1/2}}{r}\Big(1 - \frac{r^2 - rr_{\theta\theta} + 2r_\theta^2}{(r^2+r_\theta^2)^{3/2}}\Big)$$
$$= \Big(1+\frac{r_\theta^2}{r^2}\Big)^{1/2} - \Big(\frac{1}{r} - \frac{r_{\theta\theta}}{r^2} + \frac{2r_\theta^2}{r^3}\Big)\Big(1+\frac{r_\theta^2}{r^2}\Big)^{-1}.$$

Hence, by $\big(1+\frac{r_\theta^2}{r^2}\big)^{1/2} - 1 = \frac{r_\theta^2}{r^2}\big(\big(1+\frac{r_\theta^2}{r^2}\big)^{1/2}+1\big)^{-1}$ and $\frac{1}{R} - \frac{1}{r}\big(1+\frac{r_\theta^2}{r^2}\big)^{-1} = \frac{1}{Rr}\big(r - R + \frac{r_\theta^2}{r}\big)\big(1+\frac{r_\theta^2}{r^2}\big)^{-1}$, we get the equation

$$r_t - 1 + \frac{1}{R} = \frac{r_\theta^2}{r^2}\Big(\Big(1+\frac{r_\theta^2}{r^2}\Big)^{1/2}+1\Big)^{-1} + \frac{1}{Rr}\Big(r - R + \frac{r_\theta^2}{r}\Big)\Big(1+\frac{r_\theta^2}{r^2}\Big)^{-1}$$
$$+ \frac{1}{r^2}\Big(r_{\theta\theta} - \frac{2r_\theta^2}{r}\Big)\Big(1+\frac{r_\theta^2}{r^2}\Big)^{-1}. \tag{2.1}$$

We set a new time variable $\tau(t)$ by the ODE

$$\frac{d\tau}{dt} = \frac{1}{R^2}, \quad \tau(0) = -\log 2. \tag{2.2}$$

Then, because of

$$\tau(t) = -\log 2 + \int_0^t \frac{1}{R(s)^2}ds$$
$$= -\log 2 + \int_{R(0)}^{R(t)} \frac{1}{R(R-1)}dR = \log\big(1 - \frac{1}{R(t)}\big),$$
$$R(t) = \frac{1}{1-e^{\tau(t)}} \tag{2.3}$$

holds. From $R(-\infty) = 1$ and $R(+\infty) = +\infty$, we also see

$$\tau(-\infty) = -\infty, \quad \tau(+\infty) = 0. \tag{2.4}$$

Putting $u(\theta,\tau) = r(\theta,t) - R(t)$, the function $u(\theta,\tau)$ satisfies

$$u_\tau = \frac{dt}{d\tau}\big(r_t - \frac{dR}{dt}\big).$$

Then, from (2.1) and (2.2), we get

$$u_\tau = (\frac{R}{r})^2 r_\theta^2\Big(\Big(1+\frac{r_\theta^2}{r^2}\Big)^{1/2}+1\Big)^{-1} + \frac{R}{r}\Big(r - R + \frac{r_\theta^2}{r}\Big)\Big(1+\frac{r_\theta^2}{r^2}\Big)^{-1}$$
$$+ \Big(\frac{R}{r}\Big)^2\Big(r_{\theta\theta} - \frac{2r_\theta^2}{r}\Big)\Big(1+\frac{r_\theta^2}{r^2}\Big)^{-1}$$

$$= \Big(\frac{R}{r}\Big)^2 \bigg(\Big(\Big(1+\frac{r_\theta^2}{r^2}\Big)^{-1} r_{\theta\theta} + \bigg(\Big(1+\frac{r_\theta^2}{r^2}\Big)^{1/2}+1\bigg)^{-1} r_\theta^2$$
$$+ \Big(\frac{r}{R}(r-R) - \frac{1}{R}\Big(2\frac{R}{r}-1\Big) r_\theta^2\Big)\Big(1+\frac{r_\theta^2}{r^2}\Big)^{-1}\bigg).$$

Therefore, by $R > 0$, $r > 0$, (2.3), and (2.4), the function u solves the quasi-linear parabolic equation

$$u_\tau = \alpha(\tau,u)(\beta(\tau,u,u_\theta)u_{\theta\theta} + \gamma(\tau,u,u_\theta)) \tag{2.5}$$

in $(\theta,\tau) \in S^1 \times [-\log 2, 0)$, where

$$\begin{aligned} \alpha(\tau,u) &:= (1+(1-e^\tau)u)^{-2}, \\ \beta(\tau,u,p) &:= (1+(1-e^\tau)^2\alpha(\tau,u)p^2)^{-1}, \\ \gamma(\tau,u,p) &:= \Big(\beta(\tau,u,p)^{-1/2}+1\Big)^{-1} p^2 \\ &\quad + \Big(\alpha(\tau,u)^{-1/2}u - (1-e^\tau)(2\alpha(\tau,u)^{1/2}-1)p^2\Big)\beta(\tau,u,p). \end{aligned}$$

Here, we note that α, β, and γ are smooth functions defined where

$$1 + (1-e^\tau)u \neq 0. \tag{2.6}$$

3. Proof of Theorem 2

As $\sup_{-\log 2 \le \tau < 0} \|u(\tau)\|_{C(S^1)} < +\infty$ holds from (1.2), we see $\lim_{\tau\to -0} (1-e^\tau)\|u(\tau)\|_{C(S^1)} = 0$. From this and $1+(1-e^\tau)u = \frac{r}{R} > 0$,

$$c := \inf_{(\theta,\tau)\in S^1\times[-\log 2,0)} (1+(1-e^\tau)u(\theta,\tau)) > 0$$

holds. Here, take a positive function $\rho \in C^\infty(\mathbf{R})$ satisfying

$$\rho(s) = s^{-2} \quad (\, s \ge c\,).$$

We put smooth functions

$$\begin{aligned} \bar\alpha(\tau,u) &:= \rho(1+(1-e^\tau)u), \\ \bar\beta(\tau,u,p) &:= (1+(1-e^\tau)^2\bar\alpha(\tau,u)p^2)^{-1}, \\ \bar\gamma(\tau,u,p) &:= \Big(\bar\beta(\tau,u,p)^{-1/2}+1\Big)^{-1} p^2 \\ &\quad + \Big(\bar\alpha(\tau,u)^{-1/2}u - (1-e^\tau)(2\bar\alpha(\tau,u)^{1/2}-1)p^2\Big)\bar\beta(\tau,u,p). \end{aligned}$$

Then, from (2.5), the function u also solves the equation

$$u_\tau = \bar\alpha(\tau,u)(\bar\beta(\tau,u,u_\theta)u_{\theta\theta} + \bar\gamma(\tau,u,u_\theta)). \tag{3.1}$$

Because $\sum_{i=0}^{2} \|\frac{\partial^i u}{\partial \theta^i}\|_{C(S^1\times[-\log 2,0))} < +\infty$ holds from (1.2), the equation (3.1) is uniformly parabolic. Therefore, by using the standard parabolic theory, we can obtain

$$\|u\|_{C^k(S^1\times[-\log 2,0))} < +\infty \tag{3.2}$$

for all $k \geq 0$. This completes the proof. □

4. Proof of Theorem 3

First, we set the functions

$$u^i(\theta,\tau) = r^i(\theta,t) - R(t) \quad (i = 1,2).$$

Because of $\sup_{-\log 2\leq\tau<0} \|u^i(\tau)\|_{C(S^1)} < +\infty$ from (1.2), we see

$$c^i := \inf_{(\theta,\tau)\in S^1\times[-\log 2,0)} (1 + (1 - e^\tau)u^i(\theta,\tau)) > 0.$$

Take a positive function $\rho \in C^\infty(\mathbf{R})$ satisfying

$$\rho(s) = s^{-2} \quad (s \geq \min_{i=1,2} c^i).$$

Then, by putting functions $\bar{\alpha} \in C^\infty(\mathbf{R}^2)$ and $\bar{\beta}$, $\bar{\gamma} \in C^\infty(\mathbf{R}^3)$ as in the proof of Theorem 2, the functions u^1 and $u^2 \in C^\infty(S^1 \times [-\log 2, 0))$ solve the parabolic equation

$$u_\tau = \bar{\alpha}(\tau,u)(\bar{\beta}(\tau,u,u_\theta)u_{\theta\theta} + \bar{\gamma}(\tau,u,u_\theta)).$$

As we put $v := u^1 - u^2 \in C^\infty(S^1 \times [-\log 2,0))$, the function v satisfies the linear evolution equation $\frac{dv}{d\tau} + A(\tau)v = 0$, where $A(\tau)$ is a bounded operator from $H^2(S^1)$ to $L^2(S^1)$ with the form $A(\tau) = a_2(\theta,\tau)\frac{\partial^2}{\partial\theta^2} + a_1(\theta,\tau)\frac{\partial}{\partial\theta} + a_0(\theta,\tau)$. Here, by $\bar{\alpha}(\tau,u)\bar{\beta}(\tau,u,p) > 0$ for all $(\tau,u,p) \in \mathbf{R}^3$ and (3.2), we see

$$\sup_{(\theta,\tau)\in S^1\times[-\log 2,0)} a_2(\theta,\tau) < 0 \tag{4.1}$$

and

$$\|a_i\|_{C^k(S^1\times[-\log 2,0))} < +\infty \tag{4.2}$$

for all $k \geq 0$ and $0 \leq i \leq 2$. By virtue of (4.1) and (4.2), there exist $c > 0$ and $C_1 > 0$ such that

$$\langle A(\tau)v(\tau), v(\tau)\rangle_{L^2} \geq c\|v_\theta(\tau)\|^2_{L^2} - C_1\|v(\tau)\|^2_{L^2} \tag{4.3}$$

for all $\tau \in [-\log 2, 0)$. By

$$2\int_{S^1} a_2(a_{2\theta} - a_1)v_{\theta\theta}v_\theta = -\int_{S^1} (a_2(a_{2\theta} - a_1))_\theta v_\theta^2$$

and (4.2), we also see that there exists $C_2 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\|(A^*(\tau)+A(\tau))v(\tau)\|_{L^2}^2 + \frac{d}{d\tau}\langle A(\tau)v(\tau), v(\tau)\rangle_{L^2} \\
&= \frac{1}{2}\langle (A^*(\tau)+A(\tau))v(\tau), (A^*(\tau)-A(\tau))v(\tau)\rangle_{L^2} + \langle A_\tau(\tau)v(\tau), v(\tau)\rangle_{L^2} \\
&\le C_2(\|v_\theta(\tau)\|_{L^2}^2 + \|v(\tau)\|_{L^2}^2) \qquad (4.4)
\end{aligned}$$

for all $\tau \in [-\log 2, 0)$. From (4.3) and (4.4), we can confirm the assumptions of Theorem II.18.1 of [4]. Hence, $\lim_{\tau\to -0}\|v(\tau)\|_{L^2(S^1)} = 0$ implies $v \equiv 0$ by the corollary of Theorem II.18.1 of [4]. This completes the proof. □

5. Proof of Theorem 4

Take a negative constant τ_0 satisfying (2.6) for all

$$(\tau, u) \in [\tau_0, 0] \times [-\|l\|_{C(S^1)} - 1, \|l\|_{C(S^1)} + 1].$$

Then, there exists $C > 0$ such that

$$|\alpha(\tau, u)(\beta(\tau, u, p)q + \gamma(\tau, u, p))| \le C$$

for all $(\tau, u, p, q) \in [\tau_0, 0]\times[-\|l\|_{C(S^1)}-1, \|l\|_{C(S^1)}+1]\times[-\|l_\theta\|_{C(S^1)}, \|l_\theta\|_{C(S^1)}]$ $\times[-\|l_{\theta\theta}\|_{C(S^1)}, \|l_{\theta\theta}\|_{C(S^1)}]$. Here, because of (2.4), we can take $M_0 \in \mathbf{R}$ such that

$$\tau(M_0) = \max\Big\{-\frac{\varepsilon}{2C}, -\frac{1}{C}, \tau_0\Big\}.$$

So, for any $t_0 > M_0$, the functions $\underline{u}$ and $\overline{u} \in C(S^1 \times [\tau(t_0), 0])$ defined by

$$\underline{u}(\theta, \tau) := l(\theta) - C(\tau - \tau(t_0))$$

and

$$\overline{u}(\theta, \tau) := l(\theta) + C(\tau - \tau(t_0))$$

are sub- and supersolutions to the equation (2.5), respectively. Because of $\underline{u}(\theta, \tau(t_0)) = \overline{u}(\theta, \tau(t_0)) = l(\theta)$ and $l(\theta) - \frac{\varepsilon}{2} \le \underline{u}(\theta, \tau) \le \overline{u}(\theta, \tau) \le l(\theta) + \frac{\varepsilon}{2}$, we see that the solution $u(\theta, \tau)$ to (2.5) with initial data $u(\theta, \tau(t_0)) = l(\theta)$ satisfies

$$\|u(\theta, \tau) - l(\theta)\|_{C(S^1)} \le \frac{\varepsilon}{2}$$

for all $\tau \in [\tau(t_0), 0)$. By a straightforward calculation, we also see that there exists $M_1 \in \mathbf{R}$ such that for any $t_0 > M_1$, the curvature of γ_{t_0} is smaller than 1. Hence, we get the conclusion by setting $M = \max\{M_0, M_1\}$. □

Remark on Theorem 4. We would expect that the same argument could work to prove a result similar to Theorem 4 on hypersurfaces in $\mathbf{R}^N$ $(N \ge 3)$.

On the other hand, we also have a similar result on fronts of bistable reaction-diffusion equations in $\mathbf{R}^N$, and its proof is rather difficult (see [6]).

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