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NONLINEAR PSEUDODIFFERENTIAL EQUATIONS ON A SEGMENT

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Abstract. We study the global existence and large-time asymptotic behavior of solutions to the initial/boundary-value problem for the nonlinear nonlocal Whitham equation on a segment $(0,a)$,

$$\left\{ \begin{array}{c} u_t + uu_x + \mathbb{K}u = 0,\ t>0, x\in(0,a) \\ u(x,0) = u_0(x),\ x \in (0,a)\,, \end{array} \right. \tag{0.1}$$

where the pseudodifferential operator $\mathbb{K}u$ on a segment $[0,a]$ is defined by

$$\mathbb{K}u = \theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}K(p)\Big(\widehat{u}(p,t) - \frac{u(0,t)-e^{-pa}u(a,t)}{p}\Big)dp, \tag{0.2}$$

where $K(p) = C_\alpha p^\alpha$, $\alpha \in (\frac{3}{2},2)$, and C_α is chosen by the dissipation conditions. We prove that if the initial data $u_0 \in \mathbf{L}^\infty(0,a)$ have a small norm $\|u_0\|_{\mathbf{L}^\infty} < \varepsilon$, then there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2(0,a)\right) \cap \mathbf{C}\left((0,\infty);\mathbf{H}^1(0,a)\right)$ to problem (0.2). Moreover, there exists a function $A\,(x) \in \mathbf{L}^\infty(0,a)$ such that the solution has the following asymptotics for large time $t\to\infty$:

$$u(x,t) = A\,(x)\,Bt^{-\frac{1}{\alpha}} + O(t^{-\frac{1+\delta}{\alpha}}),$$

uniformly with respect to $x\in(0,a)$, where $\delta \in (0, 2-\alpha)$.

1. Introduction

Our aim in the present paper is to study the global existence and large-time asymptotic behavior of solutions to the initial-/boundary-value problem for the nonlinear Whitham equation on a segment $[0,a]$,

$$\left\{ \begin{array}{c} u_t + u\ u_x + \mathbb{K}u = 0,\ t>0, x\in(0,a)\,, \\ u(x,0) = u_0(x),\ x \in (0,a)\,, \end{array} \right. \tag{1.1}$$

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where the pseudodifferential operator $\mathbb{K}u$ on a segment $[0,a]$ is defined by

$$\mathbb{K}u = \theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}K(p)\Big(\widehat{u}(p,t) - \frac{u(0,t)-e^{-pa}u(a,t)}{p}\Big)dp, \qquad (1.2)$$

$$\theta_a(x) = \begin{cases} 1, & 0 \leqslant x \leqslant a, \\ 0, & x \notin [0,a], \end{cases}$$

and $K(p) = C_\alpha p^\alpha$, $\alpha \in (\frac{3}{2}, 2)$, and C_α will be chosen below.

The nonlinear nonlocal Whitham equation (1.1) is a simple model appearing as the first approximation in the description of the dispersive dissipative nonlinear waves; it has many applications in various fields of physics, biology and engineering (see [13]). In the case of the Cauchy problem, global existence and large-time asymptotics of solutions were obtained in [13], [2], and [12]. In the case of the boundary-value problem for the Whitham equation on a half-line, the large-time asymptotics of solutions was studied in papers [1], [5], [9], [7], and [11]. For the general theory of nonlinear nonlocal equations on a half-line we refer to the book [8]. Recently the nonlinear Schrödinger equation with pseudodifferential operator of order $\alpha \in (0,1)$ was studied in paper [3]. However, as far as we know there are no results in the case of nonlinear pseudodifferential equations of order $\alpha > 1$ on a segment. In this paper we fill this gap, considering the initial-/boundary-value problem for the nonlinear Whitham equation with pseudodifferential operator $\mathbb{K}u$ on a segment of order $\alpha \in (\frac{3}{2}, 2)$. There are many open natural questions which we need to solve in this respect. The first of them is how many boundary data should we pose on the initial-/boundary-value problem with pseudodifferential operator $\mathbb{K}$ for its correct solvability. We believe that the methods proposed in this paper could be applied to a wide class of nonlocal equations with pseudodifferential operators on a segment.

Let us start with the following linear, nonlocal initial-/boundary-value problem

$$\begin{cases} u_t + \mathbb{K}u = f(x,t),\ t>0,\ x \in (0,a)\,, \\ u(x,0) = u_0(x),\ x \in (0,a)\,, \\ \partial_x^j u(0,t) = h_{0j}(t),\ j = 1,\dots,m, \\ \partial_x^l u(a,t) = h_{al}(t),\ l = 1,\dots,n, \end{cases} \qquad (1.3)$$

where the pseudodifferential operator $\mathbb{K}u$ on a segment $[0,a]$ we define by the inverse Laplace transformation

$$\mathbb{K}u = \theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}K(p)\Big(\widehat{u}(p,t) - \sum_{j=1}^{[\alpha]}\frac{\partial_x^{j-1}u(0,t)-e^{-pa}\partial_x^{j-1}u(a,t)}{p^j}\Big)dp.$$

Note that we get the usual differential operator when the symbol $K(p)$ is a polynomial. We make a cut along a contour Γ,

$$\Gamma = \left\{ z \in \mathbb{C},\ z \in \left(\infty e^{i(-2\pi+\beta)}, 0 e^{i(-2\pi+\beta)}\right) \cup \left(0 e^{i\beta}, \infty e^{i\beta}\right) \right\}, \tag{1.4}$$

where $\beta \in (\frac{\pi}{2}, \frac{3\pi}{2})$. That is, we choose $\arg z \in [-2\pi+\beta, \beta)$ for any complex $z \in \mathbb{C}$. Here $[\alpha]$ is the integer part of the number α, and C_α will be chosen by the dissipation condition $\operatorname{Re} K(p) > 0$ for all $\operatorname{Re} p = 0$. Multiplication by the factor $\theta_a(x)$ yields that the operator $\mathbb{K}u$ vanishes outside of the segment $[0, a]$. Thus the solution $u(x,t)$ is considered for all $x \in \mathbf{R}$ extended to be zero outside of the segment $[0, a]$. We will show that similarly to the case of a half-line the numbers n and m of the boundary data are determined by the number of regions $\operatorname{Re} K(p) < 0$.

For the Laplace transform of the operator $\mathbb{K}u$ we get

$$\begin{aligned} \int_0^a e^{-px} \mathbb{K}u\, dx &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{(q-p)a} - 1}{q-p} K(q) \widehat{\widehat{u}}(q,t) dq \\ = \frac{e^{-pa}}{2\pi i} &\int_\Gamma \frac{e^{qa}}{q-p} K(q) \widehat{\widehat{u}}(q,t) dq + K(p) \widehat{\widehat{u}}(p,t), \end{aligned} \tag{1.5}$$

where

$$\widehat{\widehat{u}}(p,t) = \widehat{u}(p,t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1} u(0,t) - e^{-pa} \partial_x^{j-1} u(a,t)}{p^j}.$$

Therefore, applying the Laplace transformation with respect to x to problem (1.3) we get

$$\begin{cases} \widehat{u}_t + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{(q-p)a}-1}{q-p} K(q) \widehat{\widehat{u}}(q,t) dq = \widehat{f}(p,t), \quad t > 0, \\ \widehat{u}(p,0) = \widehat{u}_0(p), \\ \partial_x^j u(0,t) = h_{0j}(t), j = 1, \ldots, n, \\ \partial_x^l u(a,t) = h_{al}(t), l = 1, \ldots, m. \end{cases}$$

Integrating with respect to time t in view of (1.5) we obtain for the Laplace transform $\widehat{u}(p,t)$

$$\widehat{u}(p,t) = e^{-K(p)t} \hat{u}_0(p) + \int_0^t e^{-K(p)(t-\tau)} f_1(p,\tau) d\tau, \tag{1.6}$$

where

$$f_1(p,t) = \widehat{f}(p,t) + K(p) \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1} u(0,t) - e^{-pa} \partial_x^{j-1} u(a,t)}{p^j}$$

$$- \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{(q-p)a}}{q-p} K(q) \widehat{\widehat{u}}(q,\tau) dq.$$

In order to get the integral formula for solutions of (1.3), we need to know the boundary values $\partial_x^{j-1} u(0,t)$ and $\partial_x^{j-1} u(a,t)$. Some of the boundary values we put in the problem as given boundary data, and the rest of the boundary values we will find from the equation (1.6) using the growth condition

$$|\hat{u}(p,t)| \leq M(1+|p|)^{\beta} \left(1+|e^{-pa}|\right) \text{ for all } |p| \geq 1, \tag{1.7}$$

with some $M > 0$ and $\beta > 0$ which guarantee that the inverse Laplace transform $u(x,t)$ vanishes for all $x < 0$ and $x > a$. It is easy to see from (1.6) that condition (1.7) is fulfilled in domains where $\operatorname{Re} K(p) > 0$. In domains where $\operatorname{Re} K(p) < 0$, we rewrite formula (1.6) as follows:

$$\widehat{u}(p,t) = e^{-K(p)t} \Big(\widehat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau} f_1(p,\tau) d\tau \Big) - \int_t^{+\infty} e^{-K(p)(t-\tau)} f_1(p,\tau) d\tau.$$

Clearly the last integral

$$\int_t^{+\infty} e^{-K(p)(t-\tau)} f_1(p,\tau) d\tau$$

satisfies condition (1.7) in the regions Re $K(p) < 0$. However, the first summand does not satisfy condition (1.7) due to the exponentially growing factor $e^{-K(p)t}$; therefore, we have to set the additional conditions,

$$\hat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau} f_1(p,\tau) d\tau = 0, \tag{1.8}$$

for all $|p| > 1$ in the domains where Re $K(p) < 0$. We use equation (1.8) to find some of the boundary values $\partial_x^j u(0,t)$ and $\partial_x^j u(a,t)$ involved in formula (1.6). Making a change of the independent variable $K(p) = -\xi$, we transform the domains $\operatorname{Re} K(p) < 0$ to the half complex plane $\operatorname{Re} \xi > 0$ by N different roots $\phi_1(\xi)$, $\phi_2(\xi)$, $\ldots$, $\phi_N(\xi)$, each of which is an analytic function in the region $\operatorname{Re} \xi > 0$. Then condition (1.8) can be written as a system of N equations in the half complex plane Re $\xi > 0$

$$\begin{aligned} & \widehat{u}_0(\phi_l) + \widehat{\widehat{f}}(\phi_l,\xi) - \xi \sum_{j=1}^{[\alpha]} \int_0^{+\infty} e^{-\xi\tau} \frac{\partial_x^{j-1} u(0,\tau) - e^{-\phi_l a} \partial_x^{j-1} u(a,\tau)}{\phi_l^j} d\tau \\ = \quad & -\frac{1}{2\pi i} \int_{\Gamma} \frac{e^{(q-\phi_l(\xi))a}}{q-\phi_l(\xi)} K(q) \int_0^{+\infty} e^{-\xi\tau} \end{aligned}$$

$$\times\Big(\widehat{u}(q,\tau)-\sum_{j=1}^{[\alpha]}\frac{\partial_x^{j-1}u(0,\tau)-e^{-qa}\partial_x^{j-1}u(a,\tau)}{q^j}\Big)d\tau\,dq, \tag{1.9}$$

for $l=1,2,\ldots,N$, where $\widehat{u}(q,\tau)$ is the solution of problem (1.3) and

$$\widehat{u}_0(\phi_l)=\int_0^a e^{-\phi_l y}u_0(y)dy,\ \widehat{\widehat{f}}(\phi_l,\xi)=\int_0^{+\infty}\int_0^a e^{-(\phi_l y+\xi t)}f(y,t)dy\,dt.$$

Therefore, we obtain a system (1.9) of N equations with $2\,[\alpha]$ unknown boundary values $u_x^{(j-1)}(0,t)$ and $u_x^{(j-1)}(a,t)$, which permits us to determine N boundary values, and the remaining $2\,[\alpha]-N$ boundary data we need to put in problem (1.3). Note that the integer N depends on the symbol $K\,(p)$ of the pseudodifferential operator $\mathbb{K}u$.

In the present paper we restrict our attention to the case $\alpha\in\left(\frac{3}{2},2\right)$. We will prove that there exist $N=2$ different roots $\phi_1(\xi)$ and $\phi_2(\xi)$ of equation $K(p)=-\xi$, so all the boundary values $u(0,t)$ and $u(a,t)$, which are involved in the definition of the pseudodifferential operator $\mathbb{K}u$, are defined by the following system:

$$\widehat{u}_0(\phi_l)+\widehat{\widehat{f}}(\phi_l,\xi)-\xi\frac{\widehat{u}(0,\xi)-e^{-\phi_l a}\widehat{u}(a,\xi)}{\phi_l}=-\frac{1}{2\pi i}\int_\Gamma\frac{e^{(q-\phi_l(\xi))a}}{q-\phi_l(\xi)}K(q)$$
$$\times\int_0^{+\infty}e^{-\xi\tau}\Big(\widehat{u}(q,\tau)-\sum_{j=1}^{[\alpha]}\frac{\partial_x^{j-1}u(0,t)-e^{-qa}\partial_x^{j-1}u(a,t)}{q^j}\Big)d\tau\,dq \tag{1.10}$$

(here $\widehat{u}(0,\xi)$ and $\widehat{u}(a,\xi)$ are the Laplace transforms with respect to time of the boundary values $u(0,t)$ and $u(a,t)$). Thus we do not need to put any boundary data into the system (1.9). The main difficulty in the study of system (1.10) consists in the presence of the integral term involving an unknown solution $u(x,t)$ of problem (1.1). In this work we propose a general method for solving system (1.10) by introducing special projectors, which do not alter the solution on the segment $[0,a]$.

We define the usual Sobolev space by

$$\mathbf{H}^1(0,a)=\left\{\varphi\in\mathbf{L}^2\,(0,a)\,;\|\varphi\|_{\mathbf{H}^1}=\|\varphi\|_{\mathbf{L}^2}+\|\varphi_x\|_{\mathbf{L}^2}<+\infty\right\}.$$

By the same letter C we denote different positive constants. We consider the initial data for problem (1.1) belonging to the Lebesgue space $\mathbf{L}^\infty$. For obtaining $\mathbf{L}^p$ estimates of the Green's function we will use the methods of our previous papers [5] and [9].

We now state the main result of this paper.

Theorem 1. *Let the initial data $u_0\in\mathbf{L}^\infty(0,a)$ have the norm $\|u_0\|_{\mathbf{L}^\infty}\le\varepsilon$, where $\varepsilon>0$ is sufficiently small. Then there exists a unique solution $u\in$*

$\mathbf{C}\left([0,\infty);\mathbf{L}^2(0,a)\right)\cap\mathbf{C}\left((0,\infty);\mathbf{H}^1(0,a)\right)$ *of problem* (1.1). *Moreover, there exists a function* $A(x)\in\mathbf{L}^\infty(0,a)$ *such that the solution has the following asymptotics:*

$$u(x,t)=A(x)\,Bt^{-\frac{1}{\alpha}}+O(t^{-\frac{1+\delta}{\alpha}})$$

for $t\to+\infty$ *uniformly with respect to* $x\in(0,a)$, *where*

$$B=\frac{2\sin\frac{2\pi}{\alpha}}{\pi a}e^{i\frac{2\pi}{\alpha}}\int_0^{i\infty}e^{-K(z)}dz$$

and $\delta\in(0,2-\alpha)$.

Remark 1. Note that the decay rate of the solution is power-like, whereas in the case $\alpha=2$ the solutions decay exponentially. The coefficient B vanishes in the limiting case $\alpha=2$. Moreover the number of boundary data changes discontinuously at $\alpha=2$: for $\alpha\in\left(\frac{3}{2},2\right)$ there are no boundary data in the problem (1.1), and for $\alpha=2$ it is well known that two boundary data should be added into the problem (1.1). The case of $\alpha\in\left(1,\frac{3}{2}\right]$ will be considered separately in the forthcoming paper.

We organize the rest of the paper as follows. In Section 2 we solve the linear initial-/boundary-value problem corresponding to (1.1) and prove some preliminary estimates in Lemma 4. Section 3 is devoted to the proof of Theorem 1.

2. Linear problem

We consider the following linear initial-/boundary-value problem:

$$\begin{cases} u_t+\mathbb{K}u=f(x,t),\ t>0,\ x\in(0,a), \\ u(x,0)=u_0(x),\ x\in(0,a), \end{cases} \tag{2.1}$$

where the pseudodifferential operator $\mathbb{K}u$ on a segment $[0,a]$ is defined by (1.2). We make a cut along a contour Γ,

$$\Gamma=\left\{z\in\mathbb{C},\ z\in\left(\infty e^{i(-2\pi+\beta)},0e^{i(-2\pi+\beta)}\right)\cup\left(0e^{i\beta},\infty e^{i\beta}\right)\right\}' \tag{2.2}$$

that is, we choose $\arg z\in[-2\pi+\beta,\beta)$ for any complex $z\in\mathbb{C}$, where $\beta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ will be chosen below.

2.1. **Preliminary lemmas.** Denote by $\{x\}=x-[x]$ the fractional part of $x\in\mathbf{R}$.

Lemma 1. *Let* $K(p)=C_\alpha p^\alpha$, $\alpha\in\left(\frac{3}{2},2\right)$, *and*

$$\frac{3-3\alpha}{4}<\frac{\arg C_\alpha}{2\pi}\le\frac{\alpha-3}{4}.$$

Then there exist $m = 2$ *different inverse functions* $\phi_j(\xi) = K^{-1}(-\xi)$ *such that* $\phi_j(\xi)$ *are analytic for* $\operatorname{Re}\xi \geq 0$ *and*

$$\operatorname{Re}\phi_1(\xi) > 0,\ \operatorname{Re}\phi_2(\xi) < 0.$$

Proof. The function $\xi = -C_\alpha p^\alpha$ defined in $\arg p \in [-2\pi + \beta, \beta)$ has different inverse functions,

$$\phi_l(\xi) = \left|\frac{\xi}{C_\alpha}\right|^{\frac{1}{\alpha}} \exp\Big(\frac{i(\pi + 2\pi l)}{\alpha} + \frac{i}{\alpha}\arg\xi - \frac{i}{\alpha}\arg C_\alpha\Big),$$

in the domain $\operatorname{Re}\xi \geq 0$, where l is an integer such that

$$-2\pi + \beta \leq \frac{1}{\alpha}\left(\pi + 2\pi l + \arg\xi - \arg C_\alpha\right) < \beta \tag{2.3}$$

for all $-\frac{\pi}{2} \leq \arg\xi \leq \frac{\pi}{2}$. From (2.3) we get the following estimate:

$$-\alpha + \psi - \frac{1}{2} \leq l < \psi$$

for

$$\psi = -\frac{\arg\xi}{2\pi} + \frac{\arg C_\alpha}{2\pi} + \frac{\beta\alpha}{2\pi} - \frac{1}{2}.$$

So for $l_1 = l + [\alpha]$ we obtain

$$-\{\alpha\} + \psi \leq l_1 < \psi + [\alpha]\,. \tag{2.4}$$

Since $\{\alpha\} > \frac{1}{2}$ we choose β such that

$$\begin{cases} \beta > \frac{2\pi}{\alpha}\left(\frac{1}{4} - \frac{\arg C_\alpha}{2\pi} + \frac{1}{2}\right) \\ \beta \leq \frac{2\pi}{\alpha}\left(\{\alpha\} + \frac{1}{4} - \frac{\arg C_\alpha}{2\pi}\right) \end{cases} \tag{2.5}$$

for all $\frac{\arg\xi}{2\pi} \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. Therefore, we have $0 < \psi \leq \{\alpha\}$. It implies from (2.4) that there exist $[\alpha] + 1 = 2$ integer numbers l which satisfy (2.3), and therefore there exist two different inverse functions $\phi_l(\xi) = K^{-1}(-\xi)$:

$$\begin{aligned} \phi_1(\xi) &= \left|\frac{\xi}{C_\alpha}\right|^{\frac{1}{\alpha}} \exp\Big(-\frac{i\pi}{\alpha} + \frac{i}{\alpha}\arg\xi - \frac{i}{\alpha}\arg C_\alpha\Big) \\ \phi_2(\xi) &= \left|\frac{\xi}{C_\alpha}\right|^{\frac{1}{\alpha}} \exp\Big(+\frac{i\pi}{\alpha} + \frac{i}{\alpha}\arg\xi - \frac{i}{\alpha}\arg C_\alpha\Big). \end{aligned}$$

To prove that these functions are analytic in $\operatorname{Re}\xi > 0$, we need to prove that there exists $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, which satisfies the condition (2.5). The value $\arg C_\alpha$ is defined by the dissipation condition $\operatorname{Re}K(p) > 0$ for $\operatorname{Re}p = a$ when $|p|$ is large; i.e., $\cos\left(\arg C_\alpha \pm \frac{\pi}{2}\alpha\right) > 0$, which implies

$$\begin{cases} -\frac{\alpha+1}{4} \leq \frac{\arg C_\alpha}{2\pi} \leq -\frac{\alpha+1}{4} + \frac{1}{2} \\ -\frac{\alpha+1}{4} + \left\{\frac{\alpha+1}{2}\right\} - \frac{1}{2} \leq \frac{\arg C_\alpha}{2\pi} \leq -\frac{\alpha+1}{4} + \left\{\frac{\alpha+1}{2}\right\}. \end{cases} \tag{2.6}$$

Also, for the existence of $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, from condition (2.5) we get

$$\begin{cases} \frac{\arg C_\alpha}{2\pi} > -\frac{3\alpha}{4} + \frac{1}{4} + \frac{1}{2} \\ \frac{\arg C_\alpha}{2\pi} \leq \{\alpha\} + \frac{1}{4} - \frac{\alpha}{4}. \end{cases}$$

Therefore, we see that if

$$\max\Big(-\frac{\alpha+3}{4} + \Big\{\frac{\alpha+1}{2}\Big\}, -\frac{3(\alpha-1)}{4}\Big) < \frac{\arg C_\alpha}{2\pi} \leq -\frac{\alpha+1}{4} + \Big\{\frac{\alpha+1}{2}\Big\},$$

then there exist two different inverse functions $\phi_l(\xi) = K^{-1}(-\xi)$, which are analytic in $\operatorname{Re}\xi \geq 0$. Lemma 1 is proved. □

We denote the operator

$$\mathbb{P}\left[\Phi(p,t)\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(q-p)a}-1}{q-p}\Phi(q,t)dq.$$

In the next lemma we will find the properties of the operator $\mathbb{P}$. Denote the inverse Laplace transformation by $\mathcal{L}^{-1}$.

Lemma 2. *Let the function $\Phi(p)$ be analytic for all complex p except $p \in \Gamma$, and suppose it satisfies the following estimate:*

$$|\Phi(p)| < \frac{1+|e^{-pa}|}{(1+|p|)^\delta},$$

where $\delta > 0$. Then

$$\mathbb{P}\left[\Phi(p)\right] = \Phi(p) + \int_\Gamma \frac{e^{(q-p)a}}{q-p}\Phi(q,t)dq \tag{2.7}$$

and

$$\mathcal{L}^{-1}\left\{\mathbb{P}\left[\Phi(p,t)\right]\right\} = \theta_a(x)\mathcal{L}^{-1}\left\{\Phi(p,t)\right\}. \tag{2.8}$$

Proof. Suppose that the function $\Phi(p)$ is analytic for all $\operatorname{Re}p \geq 0$. Let us consider the case $\operatorname{Re}p > 0$. Using the Cauchy theorem we obtain

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(q-p)a}}{q-p}\Phi(q,t)dq = \frac{1}{2\pi i}\int_\Gamma \frac{e^{(q-p)a}}{q-p}\Phi(q,t)dq$$

and

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-p}\Phi(q,t)dq = -\Phi(p,t).$$

In the same way we have for $\operatorname{Re}p \leq 0$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(q-p)a}}{q-p}\Phi(q,t)dq = \Phi(p,t) + \frac{1}{2\pi i}\int_\Gamma \frac{e^{(q-p)a}}{q-p}\Phi(q,t)dq$$

and

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-p}\Phi(q,t)dq = 0.$$

Therefore statement (2.7) is proved. By direct calculation using the Cauchy theorem we have for $x > a$

$$\int_{-i\infty}^{i\infty} e^{px}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}\frac{e^{(q-p)a}-1}{q-p}\Phi(q,t)dq\,dp = 0.$$

Also, for $x \le a$,

$$\int_{-i\infty}^{i\infty} dpe^{p(x-a)}\int_{\Gamma}\frac{e^{qa}}{q-p}\Phi(q,t)dq = 0.$$

Therefore, using representation (2.7) we obtain formula (2.8). Lemma 2 is proved. □

Lemma 3. *Let the function $\Phi(p)$ be analytic for all complex p except $p \in \Gamma$. Suppose that function $\Phi(p)$ has the following asymptotics for $|p| > 1$:*

$$\Phi(p) = \frac{a - e^{-pa}b}{p} + O\Big(\frac{1+|e^{-pa}|}{|p|^{2+\gamma}}\Big). \tag{2.9}$$

Then the function $\Phi_1(p) = \mathbb{P}\left[\Phi(p)\right]$ has the same main term of the asymptotics as the function $\phi(p)$; i.e.,

$$\Phi_1(p) = \mathbb{P}\left[\Phi(p)\right] = \frac{a - e^{-pa}b}{p} + O\Big(\frac{1+|e^{-pa}|}{|p|^{2}}\Big). \tag{2.10}$$

Moreover,

$$\mathbb{P}\Big[K(p)\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}\Phi(q)dq\Big] = 0. \tag{2.11}$$

Proof. We have

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}\frac{e^{-qa}b}{q}dq = -\frac{e^{-pa}}{p}b.$$

Therefore, using the asymptotic formula for the function $\phi(p)$ and applying the Cauchy theorem we obtain

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}\Phi(q)dq = \frac{1}{2\pi i}\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}\Big(\Phi(q)+\frac{e^{-qa}b}{q}\Big)dq + \frac{e^{-pa}}{p}b \tag{2.12}$$
$$= \frac{1}{2\pi i}\frac{e^{-pa}}{p}\int_{\Gamma}e^{qa}\Big(\Phi(q)+\frac{e^{-qa}b}{q}\Big)dq + \frac{e^{-pa}}{p}b$$
$$-\frac{1}{2\pi i}\frac{e^{-pa}}{p}\int_{\Gamma}\frac{e^{qa}}{q-p}q\Big(\Phi(q)+\frac{e^{-qa}b}{q}\Big)dq.$$

Using

$$\lim_{x\to a-0}\mathcal{L}^{-1}\left\{\Phi(p)\right\} = \lim_{x\to a-0}\int_{-i\infty}^{i\infty} e^{px}\Big(\frac{a-e^{-pa}b}{p}+O\Big(\frac{1+|e^{-pa}|}{|p|^{1+\gamma}}\Big)\Big)dp = -b,$$

via formula (2.8) we have

$$\begin{aligned}
&\lim_{x\to a-0}\mathcal{L}^{-1}\left\{\mathbb{P}\Phi\right\} \qquad (2.13)\\
&= \lim_{x\to a-0}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} e^{px}\frac{1}{2\pi i}\int\frac{e^{(q-p)a}-1}{q-p}\Big(\Phi(q,t)+\frac{e^{-qa}}{q}b(t)\Big)dq\\
&= \lim_{x\to a-0}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} e^{p(x-a)}\frac{1}{2\pi i}\int_\Gamma\frac{e^{qa}}{q-p}\Big(\Phi(q,t)+\frac{e^{-qa}}{q}b(t)\Big)dq\\
&+ \lim_{x\to a-0}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} e^{px}\Big(\Phi(p,t)+\frac{e^{-pa}}{p}b(t)\Big)dp\\
&= \frac{1}{2\pi i}\int_\Gamma e^{qa}\Big(\Phi(q,t)+\frac{e^{-qa}}{q}b(t)\Big)dq = -b.
\end{aligned}$$

Putting (2.13) into (2.12), we obtain

$$\frac{1}{2\pi i}\int_\Gamma\frac{e^{(q-p)a}}{q-p}\Phi(q,t)dq = -\frac{1}{2\pi i}\frac{e^{-pa}}{p}\int_\Gamma\frac{e^{qa}}{q-p}q\Big(\Phi(q,t)+\frac{e^{-qa}b(t)}{q}\Big)dq. \qquad (2.14)$$

Since for $q\in\Gamma$

$$\Phi(q)+\frac{e^{-qa}b}{q} = O\Big(\frac{|e^{-qa}|}{|q|^{2+\gamma}}\Big),$$

we can prove that

$$\frac{e^{-pa}}{p}\int_\Gamma\frac{e^{qa}}{q-p}q\Big(\Phi(q)+\frac{e^{-qa}b}{q}\Big)dq = O\Big(\frac{|e^{-pa}|}{|p|^2}\Big),$$

and as a consequence of (2.14)

$$\frac{1}{2\pi i}\int_\Gamma\frac{e^{(q-p)a}}{q-p}\Phi(q,t)dq = O\Big(\frac{|e^{-pa}|}{|p|^2}\Big). \qquad (2.15)$$

Therefore via the representation (2.7) we have

$$\Phi_1(p) = \Phi(p)+\frac{1}{2\pi i}\int_\Gamma\frac{e^{(q-p)a}}{q-p}\Phi(q)dq = \frac{a-e^{-pa}b}{p}+O\Big(\frac{1+|e^{-pa}|}{|p|^2}\Big).$$

The first statement of the lemma is proved. Via estimate (2.15) we have

$$\Big|K(p)\frac{1}{2\pi i}\int_\Gamma\frac{e^{(q-p)a}}{q-p}\Phi(q)dq\Big| < \frac{1+|e^{-pa}|}{(1+|p|)^\delta}.$$

Therefore, from (2.7) and formulas (2.14) and (2.15) we see that

$$\begin{aligned}
&\mathcal{L}^{-1}\mathbb{P}\Big[K(p)\frac{1}{2\pi i}\int_\Gamma \frac{e^{(q-p)a}}{q-p}\Phi(q)dq\Big]\\
&=\theta_a(x)\mathcal{L}^{-1}\Big\{\frac{e^{-pa}}{p}K(p)\frac{1}{2\pi i}\int_\Gamma \frac{e^{qa}}{q-p}q\Big(\Phi(q)+\frac{e^{-qa}}{q}b\Big)dq\Big\}\\
&=\theta_a(x)\frac{1}{2\pi i}\int_\Gamma \frac{e^{qa}}{q-p}q\Big(\Phi(q)+\frac{e^{-qa}}{q}b\Big)dq\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{p(x-a)}K(p)}{p(p-q)}dp=0.
\end{aligned}$$

As a consequence we get (2.11). Lemma 3 is proved. □

Remark 2. If the function $\Phi(p)$ is analytic for all $p\in\mathbb{C}$, then

$$\mathbb{P}\left[\Phi(p)\right]=\Phi(p).$$

2.2. **Green's function.** To derive an integral representation for solutions of the problem (2.1) we suppose that there exists a solution $u(x,t)$ of problem (2.1), which is extended to be zero outside of the interval $(0,a)$; that is, $u(x,t)=0$ for all $x\notin[0,a]$. We have for the Laplace transforms of the operator $\mathbb{K}u$ and functions $u(x,t)$ and $f(x,t)$

$$\int_0^a e^{-px}\mathbb{K}u\,dx=\mathbb{P}\Big[K(p)\Big(\widehat{u}(p,t)-\frac{u(0,t)-e^{-qa}u(a,t)}{p}\Big)\Big],$$

$$\widehat{u}(p,t)=\mathbb{P}\left[\widehat{u}(p,t)\right],\ \ \widehat{f}(p,t)=\mathbb{P}[\widehat{f}(p,t)].$$

Applying the Laplace transformation with respect to x to problem (2.1) we obtain

$$\begin{cases}\mathbb{P}\Big[\widehat{u}_t+K(p)\left(\widehat{u}(p,t)-\frac{u(0,t)-e^{-qa}u(a,t)}{q}\right)-\widehat{f}(p,t)\Big]=0,\ t>0,x\in(0,a)\,,\\ \widehat{u}(p,0)=u_0(p).\end{cases}\tag{2.16}$$

We look for the solution of (2.16) in the form

$$\widehat{u}(p,t)=\mathbb{P}\left[u_1(p,t)\right].\tag{2.17}$$

The substitution of the representation (2.17) into (2.16) yields

$$\begin{cases}\mathbb{P}\left[u_{1t}+K(p)\left(u_1(p,t)-\frac{u(0,t)-e^{-pa}u(a,t)}{p}\right)-\widehat{f}(p,t)\right]\\ +\mathbb{P}\left[\frac{1}{2\pi i}\int_\Gamma\frac{e^{(q-p)a}}{q-p}(u_{1t}(q,t)+K(p)u_1(q,t))dq\right]=0,\ t>0,\\ \widehat{u}_1(p,0)=u_0(p).\end{cases}\tag{2.18}$$

Now we prove that under some conditions (see (2.23) below) we can define the function $u_1(p,t)$ as the solution to the following problem:

$$\begin{cases} \widehat{u}_{1t} + K(p)\left(u_1(p,t) - \frac{u(0,t)-e^{-pa}u(a,t)}{p}\right) = \widehat{f}(p,t) \\ u_1(p,0) = \widehat{u}_0(p). \end{cases} \tag{2.19}$$

Indeed, integrating with respect to time equation (2.19), we write $u_1(p,t)$ as

$$\widehat{u}_1(p,t) = e^{-K(p)t}\widehat{u}_0(p) + \int_0^t e^{-K(p)(t-\tau)} f_1(p,\tau)d\tau, \tag{2.20}$$

where

$$f_1(p,\tau) = \widehat{f}(p,t) - K(p)\Big(u_1(p,t) - \frac{u(0,t)-e^{-pa}u(a,t)}{p}\Big).$$

In order to get the integral formula for solutions of (2.18), we need to know the boundary values $u\,(0,t)$ and $u(a,t)$. We will find them using the growth condition

$$|\hat{u}_1(p,t)| \le M(1+|p|)^{-\delta}\left(1+\left|e^{-pa}\right|\right) \text{ for all } |p| \ge 1, \tag{2.21}$$

with some $M, \delta > 0$ which guarantee us that $u_1(p,t)$ has the following asymptotics for large $|p| > 1$,

$$\widehat{u}_1(p,t) = \frac{u(0,t)-e^{-pa}u(a,t)}{p} + t^{-1+\gamma}O\Big(\frac{|1+e^{-pa}|}{|p|\,|K(p)|^{1-\gamma}}\Big), \tag{2.22}$$

and as consequence of Lemma 3

$$\mathbb{P}\Big[\frac{1}{2\pi i}\int_\Gamma \frac{e^{(q-p)a}}{q-p}(u_{1t}(q,t)+K(p)u_1(q,t))dq\Big] = 0.$$

Also under condition (2.21) the inverse Laplace transform $u(x,t)$ vanishes for all $x<0$ and $x>a$ (see Lemma 2). Note that condition (2.21) is easily fulfilled in domains $\operatorname{Re} K(p) > 0$. In domains where $\operatorname{Re} K(p) < 0$, we rewrite formula (2.20) as follows:

$$\widehat{u}(p,t) = e^{-K(p)t}\Big(\widehat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau} f_1(p,\tau)d\tau\Big) - \int_t^{+\infty} e^{-K(p)(t-\tau)} f_1(p,\tau)d\tau.$$

Clearly, the last integral,

$$\int_t^{+\infty} e^{-K(p)(t-\tau)} f_1(p,\tau)d\tau,$$

satisfies condition (2.21) for all $|p| \ge 1$ such that Re $K(p) < 0$. However, the first summand with exponentially growing factor $e^{-K(p)t}$ does not satisfy condition (2.21). Therefore, we have to set the following conditions,

$$\hat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau} f_1(p,\tau)d\tau = 0, \tag{2.23}$$

for all $|p| > 1$ in the domains where Re $K(p) < 0$.

We use the equation (2.23) to find the boundary values $u(0,t)$ and $u(a,t)$ involved in formula (2.20).

Considering two different roots $\phi_1(\xi)$ and $\phi_2(\xi)$ of equation $K(p) = -\xi$ which are analytic functions for all $\operatorname{Re}\xi > 0$, we transform the half complex plane Re $\xi > 0$ to domains where Re $K(p) < 0$ (see Lemma 1). Also, we see from Lemma 1 that

$$\phi_1(\xi) = |\xi|^{\frac{1}{\alpha}} e^{i(-\frac{\pi}{\alpha} + \frac{Arg\xi}{\alpha} - \frac{ArgC_\alpha}{\alpha})}, \quad \phi_2(\xi) = |\xi|^{\frac{1}{\alpha}} e^{i(\frac{\pi}{\alpha} + \frac{Arg\xi}{\alpha} - \frac{ArgC_\alpha}{\alpha})}. \tag{2.24}$$

Note that in the right-half complex plane $\operatorname{Re}\xi > 0$,

$$\operatorname{Re}\phi_1 > 0,\ \operatorname{Re}\phi_2 < 0.$$

The condition (2.23) can be written as a system of two equations in the half complex plane Re $\xi > 0$:

$$\widehat{u}_0(\phi_l) + \widehat{\widehat{f}}(\phi_l,\xi) - \xi\frac{\widehat{u}(0,\xi) - e^{-\phi_l a}\widehat{u}(a,\xi)}{\phi_l} = 0,\ l = 1,2, \tag{2.25}$$

where functions $\widehat{u}(0,\xi)$ and $\widehat{u}(a,\xi)$ are Laplace transforms of boundary data $u(0,t)$ and $u(a,t)$ with respect to time, and

$$\widehat{u}_0(\phi) = \int_0^a e^{-\phi y}u_0(y)dy,\ \widehat{\widehat{f}}(\phi,\xi) = \int_0^{+\infty}\int_0^a e^{-(\phi y+\xi t)}f(y,t)dy\,dt.$$

From the system (2.25) we obtain

$$\widehat{u}(0,\xi) = \frac{e^{-\phi_1 a}\phi_2(\widehat{u}_0(\phi_2) + \widehat{\widehat{f}}(\phi_2,\xi)) - e^{-\phi_2 a}\phi_1(\widehat{u}_0(\phi_1) + \widehat{\widehat{f}}(\phi_1,\xi))}{\xi(e^{-\phi_1 a} - e^{-\phi_2 a})} \tag{2.26}$$

and

$$\widehat{u}(a,\xi) = \frac{\phi_2(\widehat{u}_0(\phi_2) + \widehat{\widehat{f}}(\phi_2,\xi)) - \phi_1(\widehat{u}_0(\phi_1) + \widehat{\widehat{f}}(\phi_1,\xi))}{\xi(e^{-\phi_1 a} - e^{-\phi_2 a})}. \tag{2.27}$$

The Laplace transforms $\widehat{u}(0,\xi)$ and $\widehat{u}(a,\xi)$ satisfy the growth condition

$$|\hat{u}(\cdot,\xi)| \le M(1+|\xi|)^\beta \text{ for all } |\xi| \ge 1, \tag{2.28}$$

with some $M, \beta > 0$ which are sufficient for the existence of the inverse Laplace transform $u(\cdot,t)$.

Taking the inverse Laplace transform of (2.26) and (2.27) we obtain

$$u(0,t) = \tag{2.29}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\frac{e^{-\phi_1 a}\phi_2(\widehat{u}_0(\phi_2)+\widehat{\widehat{f}}(\phi_2,\xi))-e^{-\phi_2 a}\phi_1(\widehat{u}_0(\phi_1)+\widehat{\widehat{f}}(\phi_1,\xi))}{\xi(e^{-\phi_1 a}-e^{-\phi_2 a})}d\xi$$

and

$$u(a,t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\frac{\phi_2(\widehat{u}_0(\phi_2)+\widehat{\widehat{f}}(\phi_2,\xi))-\phi_1(\widehat{u}_0(\phi_1)+\widehat{\widehat{f}}(\phi_1,\xi))}{\xi(e^{-\phi_1 a}-e^{-\phi_2 a})}d\xi. \tag{2.30}$$

Thus, supposing the existence of solutions to the problem (2.1), we get the following integral representation for these solutions:

$$u(x,t)=\theta_a(x)\mathcal{L}^{-1}\{u_1\}=\theta_a(x)\frac{1}{2\pi i}\Big(\int_{-i\infty}^{i\infty} dpe^{px}e^{-K(p)t}\widehat{u}_0(p) \tag{2.31}$$
$$+\int_{-i\infty}^{i\infty} dpe^{px}\int_0^t e^{-K(p)(t-\tau)}\Big(f(p,\tau)+\frac{K(p)}{p}\Big(u(0,\tau)-e^{-pa}u(a,\tau)\Big)\Big)d\tau\Big),$$

where functions $u(0,\tau)$ and $u(a,\tau)$ we defined by formulas (2.29) and (2.30).

Now we prove that the function $u(x,t)$ defined by formula (2.31) gives us a solution to problem (2.1). Taking the Laplace transform of (2.31) and using the asymptotic representation (2.22) of $u_1(p,t)$, we get

$$\widehat{u}(p,t)=\mathbb{P}\left[u_1(p,t)\right], \tag{2.32}$$

where the function $u_1(p,t)$ is defined by (2.20); i.e.,

$$\widehat{u}_1(p,t)=e^{-K(p)t}\widehat{u}_0(p)+\int_0^t e^{-K(p)(t-\tau)}f_1(p,\tau)d\tau.$$

By virtue of formula (2.22) and Lemma 3 the Laplace transform $\widehat{u}(p,t)$ has asymptotic representation for large $|p|>1$

$$\widehat{u}(p,t)=\frac{u(0,t)-e^{-pa}u(a,t)}{p}+t^{-1+\gamma}O\Big(\frac{1+|e^{-pa}|}{|p|^2}\Big).$$

Therefore, substituting (2.32) into the definition of the pseudodifferential operator $\mathbb{K}u$ (see formula (1.2)) and using

$$\mathbb{P}\Big[K(p)\int_\Gamma \frac{e^{(q-p)a}}{q-p}u_1(q,t)dq\Big]=0$$

we obtain

$$\mathbb{K}u=\mathcal{L}^{-1}\mathbb{P}\Big[K(p)\Big(\mathbb{P}\left[u_1(p,t)\right]-\frac{u(0,t)-e^{-pa}u(a,t)}{p}\Big)\Big]$$
$$=\mathcal{L}^{-1}\mathbb{P}\Big[K(p)\Big(u_1-\frac{u(0,t)-e^{-pa}u(a,t)}{p}\Big)\Big]+\mathbb{P}\Big[K(p)\int_\Gamma \frac{e^{(q-p)a}}{q-p}u_1(q,t)dq\Big]$$

$$= \mathcal{L}^{-1}\mathbb{P}\Big[K(p)\Big(u_1 - \frac{u(0,t) - e^{-pa}u(a,t)}{p}\Big)\Big].$$

Since $\mathcal{L}^{-1}\mathbb{P} = \theta_a(x)\mathcal{L}^{-1}$ and

$$u_{1t} + K(p)u_1 = f(p,t) + \frac{K(p)}{p}\Big(\frac{u(0,t) - e^{-pa}u(a,t)}{p}\Big),$$

we find

$$\begin{aligned} \mathbb{K}u &= \theta_a(x)\mathcal{L}^{-1}\Big\{K(p)\Big(u_1 - \frac{u(0,t) - e^{-pa}u(a,t)}{p}\Big)\Big\} \\ &= \theta_a(x)\mathcal{L}^{-1}\Big\{-u_{1t} + \widehat{f}(p,t)\Big\} = -u_t(x,t) + f(x,t). \end{aligned}$$

So the function $u\,(x,t)$ given by (2.31) satisfies the equation $u_t(x,t) + \mathbb{K}u = f(x,t)$. Also, it is clear that the initial condition of problem (2.1) is fulfilled:

$$u(x,0) = \theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\widehat{u}_0(p) = u_0(x).$$

Thus there exists a solution to the problem (2.1), which is given by formula (2.31). The uniqueness follows from the fact that all solutions have representation (2.31).

Using representations (2.29) we have (for simplicity we put $f(x,t) = 0$)

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\int_0^t e^{-K(p)(t-\tau)}K(p)\frac{u(0,\tau)}{p}d\tau \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p}e^{-K(p)t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-\phi_1 a}\phi_2\widehat{u}_0(\phi_2) - e^{-\phi_2 a}\phi_1\widehat{u}_0(\phi_1)}{\xi(e^{-\phi_1 a} - e^{-\phi_2 a})}d\xi \\ &\times \int_0^t d\tau e^{(K(p)+\xi)\tau}. \end{aligned}$$

Integrating with respect to τ, substituting the Laplace transform $\widehat{u}_0(\phi_1)$ and $\widehat{u}_0(\phi_2)$, and using

$$\int_{-i\infty}^{i\infty}\frac{e^{-\phi_1 a}\phi_2\widehat{u}_0(\phi_2) - e^{-\phi_2 a}\phi_1\widehat{u}_0(\phi_1)}{\xi(e^{-\phi_1 a} - e^{-\phi_2 a})}\frac{1}{K(p)+\xi}d\xi = 0,$$

we obtain

$$\begin{aligned} I_1 &= -\frac{1}{4\pi^2}\int_0^a dyu_0(y)\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{e^{-\phi_1 a - \phi_2(y)}\phi_2 - e^{-\phi_2 a - \phi_1 y}\phi_1}{\xi(e^{-\phi_1 a} - e^{-\phi_2 a})} \\ &\times \int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p(K(p)+\xi)}d\xi. \end{aligned}$$

We have by the Cauchy theorem

$$\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p(K(p)+\xi)}d\xi = 2\pi i e^{\phi_2(\xi)x}\frac{K(\phi_2)}{\phi_2(K'(\phi_2))} + \int_\Gamma dpe^{px}\frac{K(p)}{p(K(p)+\xi)}.$$

Therefore, using

$$K(\phi_2) = -\xi,\ K'(\phi_2) = -\frac{1}{\phi_2'}$$

we get

$$\begin{aligned} I_1 = \frac{1}{2\pi i}\int_0^a dyu_0(y)\Bigg(&\int_{-i\infty}^{i\infty} e^{\xi t}\frac{e^{-\phi_1 a-\phi_2 y}\phi_2 - e^{-\phi_2 a-\phi_1 y}\phi_1}{(e^{-\phi_1 a}-e^{-\phi_2 a})}e^{\phi_2(\xi)x}\frac{\phi_2'}{\phi_2} \\ &+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{e^{-\phi_1 a-\phi_2 y}\phi_2 - e^{-\phi_2 a-\phi_1 y}\phi_1}{\xi(e^{-\phi_1 a}-e^{-\phi_2 a})}\int_\Gamma dpe^{px}\frac{K(p)}{p(K(p)+\xi)}\Bigg). \end{aligned} \tag{2.33}$$

In the same way, from the representation (2.30) we have

$$\begin{aligned} I_2 &= -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\int_0^t e^{-K(p)(t-\tau)}K(p)\frac{e^{-pa}u(a,\tau)}{p}d\tau \\ &= -\frac{1}{4\pi^2}\int_0^a dyu_0(y)\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{e^{-\phi_2 y}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{-\phi_1 a}-e^{-\phi_2 a})} \\ &\times\int_{-i\infty}^{i\infty} dpe^{p(x-a)}\frac{K(p)}{p(K(p)+\xi)}d\xi. \end{aligned}$$

Since for $x\in[0,a]$

$$\begin{aligned} \int_{-i\infty}^{i\infty} dpe^{p(x-a)}\frac{K(p)}{p(K(p)+\xi)}d\xi &= -2\pi i e^{\phi_1(\xi)(x-a)}\frac{K(\phi_1)}{\phi_1(K'(\phi_1))} \\ &= -2\pi i e^{\phi_1(\xi)(x-a)}\frac{\xi\phi_1'}{\phi_1} \end{aligned}$$

we obtain

$$I_2 = -\frac{1}{2\pi i}\int_0^a dyu_0(y)\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{e^{-\phi_2 y}\phi_2 - e^{-\phi_1 y}\phi_1}{(e^{-\phi_1 a}-e^{-\phi_2 a})}e^{\phi_1(\xi)(x-a)}\frac{\phi_1'}{\phi_1}. \tag{2.34}$$

From (2.33) and (2.34) by direct calculations we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\int_0^t e^{-K(p)(t-\tau)}K(p)\frac{u(0,\tau)}{p}d\tau \\ &-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\int_0^t e^{-K(p)(t-\tau)}K(p)\frac{e^{-pa}u(a,\tau)}{p}d\tau \end{aligned}$$

$$\begin{aligned} = \quad & \frac{1}{2\pi i}\int_0^a dy u_0(y)\int_{-i\infty}^{i\infty} e^{\xi t} H(\xi,x,y)d\xi \\ & -\frac{1}{4\pi^2}\int_0^a dy u_0(y)\int_{-i\infty}^{i\infty} e^{\xi t}\frac{e^{-\phi_1 a+\phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a}-1)}d\xi \\ & \times\int_\Gamma dp e^{px}\frac{K(p)}{p(K(p)+\xi)}, \end{aligned}$$

where using $\phi_2 = \beta\phi_1$ for $\beta = e^{i\frac{2\pi}{\alpha}}$ we get

$$\begin{aligned} H(\xi,x,y) \quad = \quad & \frac{e^{-\phi_1 a}e^{\phi_2(a+x-y)}\beta - e^{-\phi_1 y}e^{\phi_2 x}}{e^{(\phi_2-\phi_1)a}-1}\phi_1' \\ & +\frac{e^{\phi_2(a-y)}e^{\phi_1(x-a)}\beta - e^{\phi_2 a}e^{-\phi_1(a+y-x)}}{e^{(\phi_2-\phi_1)a}-1}\phi_1'. \end{aligned} \tag{2.35}$$

We rewrite for $x > y$

$$\begin{aligned} H(\xi,x,y) = e^{\phi_2(x-y)}\phi_2' + \frac{e^{\phi_2(x-y)}\beta - e^{-\phi_1 y}e^{\phi_2 x}}{(e^{(\phi_2-\phi_1)a}-1)}\phi_1' \\ +\frac{e^{\phi_2(a-y)}e^{\phi_1(x-a)}\beta - e^{(\phi_2-\phi_1)a}e^{\phi_1(x-y)}}{(e^{(\phi_2-\phi_1)a}-1)}\phi_1' \end{aligned}$$

and for $x < y$

$$\begin{aligned} H(\xi,x,y) = -e^{-\phi_1(x-y)}\phi_1' + \frac{e^{(\phi_2-\phi_1)a}e^{\phi_2(x-y)}\beta - e^{-\phi_1(y-x)}}{(e^{(\phi_2-\phi_1)a}-1)}\phi_1' \\ +\frac{-e^{-\phi_1 y}e^{\phi_2 x} + e^{\phi_2(a-y)}e^{\phi_1(x-a)}\beta}{(e^{(\phi_2-\phi_1)a}-1)}\phi_1'. \end{aligned}$$

Since for $x > y$

$$\int_{-i\infty}^{i\infty} dp e^{-K(p)t+p(x-y)} = -\int_{-i\infty}^{i\infty} e^{\xi t}e^{\phi_2(x-y)}\phi_2' d\xi$$

and for $x < y$

$$\int_{-i\infty}^{i\infty} dp e^{-K(p)t+p(x-y)} = \int_{-i\infty}^{i\infty} e^{\xi t}e^{\phi_1(x-y)}\phi_1' d\xi,$$

we obtain the following integral representation for solutions $u(x,t)$ of problem (2.1):

$$u(x,t) = \int_0^a u_0(y)G(x,y,t)dy + \int_0^t d\tau\int_0^a f(y,\tau)G(x,y,t-\tau)d\tau, \tag{2.36}$$

where the Green's function $G(x,y,t)$ is defined by

$$G(x,y,t) =$$
$$\theta_a(x)\frac{1}{2\pi i}\Big(\theta_x(y)\int_{-i\infty}^{i\infty} e^{\xi t}\mathcal{H}_1(\xi,x,y)d\xi + \theta_y(x)\int_{-i\infty}^{i\infty} e^{\xi t}\mathcal{H}_2(\xi,x,y)d\xi$$
$$+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\frac{e^{-\phi_1 a+\phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a}-1)}d\xi\int_{\Gamma} dpe^{px}\frac{K(p)}{p(K(p)+\xi)}\Big)$$

with

$$\mathcal{H}_1(\xi,x,y) = \frac{e^{\phi_2(x-y)}\phi_2' - e^{-\phi_1 y}e^{\phi_2 x}\phi_1'}{(e^{(\phi_2-\phi_1)a}-1)} + \frac{e^{\phi_2(a-y)}e^{\phi_1(x-a)}\phi_2' - e^{\phi_2 a}e^{-\phi_1(a+y-x)}\phi_1'}{(e^{(\phi_2-\phi_1)a}-1)} \tag{2.37}$$

and

$$\mathcal{H}_2(\xi,x,y) = \frac{-\phi_1' e^{-\phi_1(y-x)} + e^{-\phi_1 a}e^{\phi_2(a+x-y)}\phi_2'}{(e^{(\phi_2-\phi_1)a}-1)} + \frac{-e^{-\phi_1 y}e^{\phi_2 x}\phi_1' + e^{\phi_2(a-y)}e^{\phi_1(x-a)}\phi_2'}{(e^{(\phi_2-\phi_1)a}-1)}. \tag{2.38}$$

Thus we have proved the following result.

Theorem 2. *Let the initial data* $u_0 \in \mathbf{L}^1(0,a)$ *and a source* $f(x,t) \in \mathbf{L}^1_{loc}\left(0,\infty;\mathbf{L}^1(0,a)\right)$ *be given. Then there exists a unique solution* $u(x,t)$ *of the initial-/boundary-value problem* (2.1)*, which has representation* (2.36)*.*

2.3. **Asymptotics of the Green's function.** In the next lemma we estimate the kernel $G(x,y,t)$. Denote

$$\Lambda(x,y) = \theta_a(x)\left(y - a\theta_y(x)\right).$$

Lemma 4. *We have the following asymptotics for large time* $t > 1$ *and* $\delta \in (0,1)$ *,*

$$G(x,y,t) = \frac{(e^{i\frac{4\pi}{\alpha}}-1)}{\pi i a}t^{-\frac{1}{\alpha}}\Lambda(x,y)\int_0^{i\infty} e^{-K(z)}dz + a^{\delta}(t^{-\frac{1+\delta}{\alpha}}), \tag{2.39}$$

and the estimates for all $t > 0$,

$$\left\|\int_0^a G(\cdot,y,t)\phi(y)dy\right\|_{\mathbf{L}^2} \le C\left\|\phi\right\|_{\mathbf{L}^\infty}, \tag{2.40}$$

and for $p = 2,\infty,\gamma > 0$,

$$\left\|\int_0^a G_x(\cdot,y,t)\phi(y)dy\right\|_{\mathbf{L}^2} \le Ct^{-\frac{1}{\alpha}-\frac{\gamma}{p}}\left\|\phi\right\|_{\mathbf{L}^p}, \tag{2.41}$$

and for all $t > 1$ *and* $n = 0, 1,$

$$\left\| \int_0^a G_x^{(n)}(\cdot, y, t)\phi(y)dy \right\|_{\mathbf{L}^2} \leq Ct^{-\frac{n+1}{\alpha}} \|\phi\|_{\mathbf{L}^1} \tag{2.42}$$

$$\sup_{x,y\in[0,a]} |G_t(x, y, t)| \leq Ct^{-\frac{1}{\alpha}-1}. \tag{2.43}$$

Proof. We have

$$G(x, y, t) = \theta_a(x)\frac{1}{2\pi i}\Big(I_1 + \frac{1}{2\pi i}I_2\Big), \tag{2.44}$$

where

$$I_1(x, y, t) = \theta_x(y)\int_{-i\infty}^{i\infty} e^{\xi t}\mathcal{H}_1(\xi, x, y)d\xi + \theta_y(x)\int_{-i\infty}^{i\infty} e^{\xi t}\mathcal{H}_2(\xi, x, y)d\xi \tag{2.45}$$

with $\mathcal{H}_1(\xi, x, y)$ and $\mathcal{H}_2(\xi, x, y)$ defined in formulas (2.37) and (2.38) and

$$I_2(x, y, t) = \int_{-i\infty}^{i\infty} e^{\xi t}\frac{e^{-\phi_1 a+\phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a} - 1)}d\xi \int_\Gamma dpe^{px}\frac{K(p)}{p(K(p) + \xi)}. \tag{2.46}$$

First we consider I_2 in the formula (2.44). We change the contour of integration and rewrite I_2 in the following form:

$$\begin{aligned} I_2 &= \int_\Gamma e^{\xi t}\frac{e^{-\phi_1 a+\phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a} - 1)}d\xi \int_\Gamma dpe^{px}\frac{K(p)}{p(K(p) + \xi)} \\ &\quad -2\pi i\int_\Gamma dpe^{px-K(p)t}\frac{e^{-\phi_1(-K(p))a+\phi_2(-K(p))(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{p(e^{(\phi_2-\phi_1)a} - 1)} \\ &\quad -2\pi i\frac{1}{a} + O(e^{\operatorname{Re}\xi_1 t}) \qquad (2.47) \\ &= J_1 - 2\pi iJ_2 - 2\pi i\frac{1}{a} + O(e^{\operatorname{Re}\xi_1 t}), \end{aligned}$$

where $\Gamma = \{\xi \in (\infty e^{i(\beta+2\pi)}, 0) \cup (0, \infty e^{i\beta})\} = \Gamma_- \cup \Gamma_+$ $\frac{\pi}{2} < \beta < \pi$ and ξ_1 is first root of the equation

$$e^{(\phi_2-\phi_1)a} - 1 = 0$$

such that $\operatorname{Re}\xi_1 < 0$. Since for $|\xi| \leq 1$

$$\begin{aligned} &\frac{e^{-\phi_1 a+\phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a} - 1)} \qquad (2.48) \\ &= \frac{1}{\xi a}(1 + \phi_2 a - \phi_2 y - \phi_1 y + \frac{1}{2}(\phi_2 - \phi_1)a) + O((\phi_2 - \phi_1)^{1+\delta}), \end{aligned}$$

we estimate first term in the formula (2.47) as

$$J_1 = \int_\Gamma e^{\xi t} \frac{e^{-\phi_1 a + \phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a} - 1)} d\xi \int_\Gamma dp e^{px} \frac{K(p)}{p(K(p)+\xi)} = M + R_1, \tag{2.49}$$

where

$$M = \int_\Gamma e^{\xi t} \frac{1}{\xi a}(1 + \phi_2(a-y) - \phi_1 y + \frac{1}{2}(\phi_2 - \phi_1)a) d\xi \int_\Gamma dp e^{px} \frac{K(p)}{p(K(p)+\xi)}$$

and

$$\begin{aligned} R_1 &= \int_{\Gamma, |\xi|>1} e^{\xi t} \frac{e^{-\phi_1 a + \phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a} - 1)} d\xi \int_\Gamma dp e^{px} \frac{K(p)}{p(K(p)+\xi)} \\ &\quad + \int_{\Gamma, |\xi|\le 1} e^{\xi t} O((\phi_2 - \phi_1)^{1+\delta} a^\delta) d\xi \int_\Gamma dp e^{px} \frac{K(p)}{p(K(p)+\xi)} \\ &= a^\delta O(t^{-\frac{1+\delta}{\alpha}}). \end{aligned}$$

Since $\phi_2(\xi) = \beta\phi_1(\xi)$ and $\phi_2 \big|_{\xi\in\Gamma_+} = \phi_1 \big|_{\xi\in\Gamma_-}$, we get

$$M = \Big(\frac{4 - \beta - 3\overline{\beta}}{2} - \frac{y(\beta - \overline{\beta})}{a}\Big) \int_{\Gamma_+} e^{\xi t} \phi_2' d\xi \int_\Gamma dp e^{px} \frac{K(p)}{p(K(p)+\xi)} + 2\pi i \frac{1}{a}.$$

Making the change of variables $\xi t = q$ and $K(p)t = K(z)$ we have

$$M = \Big(\frac{4 - \beta - 3\overline{\beta}}{2} - \frac{y(\beta - \overline{\beta})}{a}\Big) t^{-\frac{1}{\alpha}} \int_{\Gamma_+} e^{q} \phi_2'(q) dq \int_\Gamma dz e^{z\widetilde{x}} \frac{K(z)}{z(K(z)+q)} + 2\pi i \frac{1}{a}.$$

Since by the Cauchy theorem

$$\begin{aligned} &\int_{\Gamma_+} e^{q} \phi_2'(q) dq \int_\Gamma dp e^{z\widetilde{x}} \frac{K(z)}{z(K(z)+q)} \\ &= 2\pi i \int_{\Gamma_+} e^{q} \phi_2'(q) dq - \int_{\Gamma_+} e^{q} \phi_2(q) dq \int_\Gamma dp e^{z\widetilde{x}} \frac{1}{z(K(z)+q)} \\ &= 2\pi i \int_{\Gamma_+} e^{q} \phi_2'(q) dq - \int_{\Gamma_+} e^{q} \phi_2(q) dq \int_\Gamma dz (e^{z\widetilde{x}} - 1) \frac{1}{z(K(z)+q)} \\ &- \int_{\Gamma_+} e^{q} \phi_2(q) dq \int_\Gamma dz \frac{1}{z(K(z)+q)} \\ &= 2\pi i \int_{\Gamma_+} e^{q} \phi_2'(q) dq - 2\pi i \int_{\Gamma_+} e^{q} \phi_2(q) \Big(\frac{\phi_1'}{\phi_1} + \frac{\phi_2'}{\phi_2}\Big) dq + a^\delta O(t^{-\frac{\delta}{\alpha}}) \\ &= -2\pi i \beta \int_0^{i\infty} e^{-K(z)} dz + a^\delta O(t^{-\frac{\delta}{\alpha}}), \end{aligned}$$

we obtain for J_1 in (2.47)

$$\begin{aligned} J_1 &= M + 2\pi i \frac{1}{a} + a^{\delta} O(t^{-\frac{1+\delta}{\alpha}}) \qquad (2.50)\\ &= -2\pi i \Big(\frac{-\beta^2 + 4\beta - 3}{2} + \frac{y(1-\beta^2)}{a} \Big) t^{-\frac{1}{\alpha}} \int_0^{i\infty} e^{-K(z)} dz \\ &\quad + 2\pi i \frac{1}{a} + a^{\delta} O(t^{-\frac{1+\delta}{\alpha}}). \end{aligned}$$

Now we estimate the second term J_2 in the formula (2.47). Since

$$\begin{aligned} &\frac{e^{-\phi_1(-K(p))a + \phi_2(-K(p))(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{p(e^{(\phi_2-\phi_1)a} - 1)} \\ = \ &\frac{\phi_2 - \phi_1 + \phi_2 a(\phi_2 - \phi_1) - \phi_2^2 y + \phi_1^2 y + \frac{(\phi_2-\phi_1)^2}{2} a}{p(\phi_2 - \phi_1)a} + O(|p|^{\alpha(1+\delta)}), \end{aligned}$$

we have

$$\begin{aligned} J_2 &= \int_{\Gamma, |p|<1} dp e^{px - K(p)t} \Big(\frac{\phi_2 - \phi_1 + \phi_2 a(\phi_2 - \phi_1)}{p(\phi_2 - \phi_1)a} \\ &+ \frac{-\phi_2^2 y + \phi_1^2 y + \frac{(\phi_2-\phi_1)^2}{2} a}{p(\phi_2 - \phi_1)a} \Big) + O(t^{-\frac{1+\delta}{\alpha}}) \\ &= \int_{\Gamma_+, |p|<1} dp e^{px - K(p)t} \frac{\phi_2 a(\phi_2 - \phi_1) - \phi_2^2 y + \phi_1^2 y + \frac{(\phi_2-\phi_1)^2}{2} a}{p(\phi_2 - \phi_1)a} \\ &+ \int_{\Gamma_-, |p|<1} dp e^{px - K(p)t} \frac{\phi_2 a(\phi_2 - \phi_1) - \phi_2^2 y + \phi_1^2 y + \frac{(\phi_2-\phi_1)^2}{2} a}{p(\phi_2 - \phi_1)a} \\ &+ 2\pi i \frac{1}{a} + O(t^{-\frac{1+\delta}{\alpha}}) \\ &= L_1 + L_2 + 2\pi i \frac{1}{a} + O(t^{-\frac{1+\delta}{\alpha}}). \qquad (2.51) \end{aligned}$$

Changing the variable $K(p)t = K(z)$ and using (2.24) we obtain for the first summand L_1 in formula (2.51)

$$\begin{aligned} L_1 &= \int_0^{i\infty} dp e^{px - K(p)t} \frac{1}{pa} \Big(\phi_2 a - (\phi_2 + \phi_1) y + \frac{\phi_2 - \phi_1}{2} a \Big) \\ = \ &t^{-\frac{1}{\alpha}} \Big(\beta - (\beta + 1)\frac{y}{a} + \frac{(\beta - 1)}{2} + \frac{x}{a} \Big) \int_0^{i\infty} dz e^{-K(z)} + O(t^{-\frac{1+\delta}{\alpha}}), \end{aligned}$$

where $\beta = e^{i\frac{2\pi}{\alpha}}$. Therefore, we get

$$\begin{aligned} J_2 &= t^{-\frac{1}{\alpha}}\Big(\beta - (\beta+1)\frac{y}{a} + \frac{(\beta-1)}{2} + \frac{x}{a}\Big)\int_0^{i\infty} dz e^{-K(z)} \\ &\quad + \int_{\Gamma_-,|p|<1} dp e^{px-K(p)t}\frac{\phi_2 a(\phi_2-\phi_1) - \phi_2^2 y + \phi_1^2 y + \frac{(\phi_2-\phi_1)^2}{2}a}{p(\phi_2-\phi_1)a} \\ &\quad + 2\pi i\frac{1}{a} + O(t^{-\frac{1+\delta}{\alpha}}). \end{aligned} \tag{2.52}$$

Therefore, from formulas (2.47), (2.50), and (2.52) we obtain an asymptotic representation for the second summand for the Green's function (see (2.44))

$$I_2 = -2\pi i\Big(\frac{-\beta^2+7\beta-4}{2} + \frac{y(-\beta^2-\beta)}{a} + \frac{x}{a}\Big)t^{-\frac{1}{\alpha}}\int_0^{i\infty} e^{-K(z)}dz \tag{2.53}$$

$$- 2\pi i\int_{\Gamma_-,|p|<1} dp e^{px-K(p)t}\frac{\phi_2 a(\phi_2-\phi_1) - \phi_2^2 y + \phi_1^2 y + \frac{(\phi_2-\phi_1)^2}{2}a}{p(\phi_2-\phi_1)a}$$

$$+ 4\pi^2\frac{1}{a} + O(t^{-\frac{1+\delta}{\alpha}}).$$

Substituting (2.53) into (2.44) we obtain

$$\begin{aligned} G(x,y,t) &= \theta_a(x)\frac{1}{2\pi i}\Big(\widetilde{I}_1 - \Big(\frac{-\beta^2+7\beta-4}{2} + \frac{y(-\beta^2-\beta)}{a} + \frac{x}{a}\Big) \\ &\quad \times t^{-\frac{1}{\alpha}}\int_0^{i\infty} e^{-K(z)}dz + \frac{2}{a}\int_0^{i\infty} dz e^{-K(z)}\frac{1}{z} - 2\pi i\frac{1}{a}\Big) + O(t^{-\frac{1+\delta}{\alpha}}), \end{aligned} \tag{2.54}$$

where $\widetilde{I}_1$ may be rewritten as

$$\begin{aligned} \widetilde{I}_1(x,y,t) &= 2\pi i\frac{1}{a} + \theta_x(y)\Big(\int_{\Gamma_+} e^{\xi t}\mathcal{H}_{11}(\xi,x,y)d\xi + \int_{\Gamma_-} e^{\xi t}\mathcal{H}_{12}(\xi,x,y)\Big) \\ &\quad + \theta_y(x)\Big(\int_{\Gamma_+} e^{\xi t}\mathcal{H}_{21}(\xi,x,y)d\xi + \int_{\Gamma_-} e^{\xi t}\mathcal{H}_{22}(\xi,x,y)\Big), \end{aligned} \tag{2.55}$$

with

$$\begin{aligned} &\mathcal{H}_{11}(\xi,x,y) \\ &= \frac{\phi_2'}{a}\big(\frac{1}{\phi_2} - \overline{\beta}y - \overline{\beta}(a+y-x) + a - x + \big(1-\overline{\beta}\big)a + 2x - 2y\big) + O(\xi^{\frac{1+\delta}{\alpha}}), \\ &\mathcal{H}_{12}(\xi,x,y) \\ &= \frac{\phi_1'}{a}\Big(\frac{1}{\phi_1} + \beta(x-y) + \beta(a-y) - a - \beta(a-y+x) + \frac{\beta-1}{2}a + x - y\Big) \\ &\quad + O(\xi^{\frac{1+\delta}{\alpha}}), \end{aligned}$$

$$\mathcal{H}_{21}(\xi,x,y)$$
$$=\frac{\phi_2'}{a}(\frac{1}{\phi_2}-\overline{\beta}y+(2-\overline{\beta})a+x-2y-\overline{\beta}(y-x))+O(\xi^{\frac{1+\delta}{\alpha}}),$$
$$\mathcal{H}_{22}(\xi,x,y)=\frac{\phi_1'}{a}(\frac{1}{\phi_1}+\beta(a-y)+\frac{\beta-1}{2}a+x-y)+O(\xi^{\frac{1+\delta}{\alpha}}).$$

Using $\phi_2\left|_{\xi\in\Gamma_+}=\phi_1\right|_{\xi\in\Gamma_-}$, we obtain

$$\int_{\Gamma_+}e^{\xi t}\mathcal{H}_{11}(\xi,x,y)d\xi+\int_{\Gamma_-}e^{\xi t}\mathcal{H}_{12}(\xi,x,y)d\xi \tag{2.56}$$
$$=\int_{\Gamma_+,|\xi|\le 1}e^{\xi t}\frac{\phi_2'}{a}(x\overline{\beta}+y(-2\overline{\beta}-1+\beta)+a(-2\overline{\beta}+3-\frac{\beta-1}{2}))d\xi$$
$$+\int_{\Gamma_-,|\xi|\le 1}e^{\xi t}a^{\delta}O(\xi^{\frac{1+\delta}{\alpha}})d\xi+\int_{\Gamma_+,|\xi|>1}e^{\xi t}\mathcal{H}_{11}(\xi,x,y)d\xi+\int_{\Gamma_-,|\xi|>1}e^{\xi t}\mathcal{H}_{12}(\xi,x,y)d\xi$$
$$=\int_{\Gamma_+}e^{\xi t}\frac{\phi_2'}{a}(x\overline{\beta}+y(-2\overline{\beta}-1+\beta)+a(-2\overline{\beta}+3-\frac{\beta-1}{2}))d\xi+O\left(t^{-\frac{1+\delta}{\alpha}}\right)$$
$$=\Big(\frac{7\beta-\beta^2-4}{2}+\frac{y(\beta^2-2-\beta)}{a}+\frac{x}{a}\Big)t^{-\frac{1}{\alpha}}\int_0^{i\infty}e^{-K(z)}dz+O(t^{-\frac{1+\delta}{\alpha}})$$

and

$$\int_{\Gamma_+}e^{\xi t}\mathcal{H}_{21}(\xi,x,y)d\xi+\int_{\Gamma_-}e^{\xi t}\mathcal{H}_{22}(\xi,x,y)d\xi \tag{2.57}$$
$$=\int_{\Gamma_+,|\xi|\le 1}e^{\xi t}\frac{\phi_2'}{a}\Big(-\overline{\beta}y+(2-\overline{\beta})a+x-2y-\overline{\beta}(y-x)$$
$$-\frac{1}{\phi_1}-\beta(a-y)-\frac{\beta-1}{2}a-x+y\Big)d\xi$$
$$+\int_{\Gamma_-,|\xi|\le 1}e^{\xi t}a^{\delta}O(\xi^{\frac{1+\delta}{\alpha}})d\xi+\int_{\Gamma_+,|\xi|>1}e^{\xi t}\mathcal{H}_{21}(\xi,x,y)d\xi$$
$$+\int_{\Gamma_-,|\xi|>1}e^{\xi t}\mathcal{H}_{22}(\xi,x,y)d\xi$$
$$=\int_{\Gamma_+}d\xi e^{\xi t}\frac{\phi_2'}{a}(y(-2\overline{\beta}-1+\beta)+a(-\frac{3\beta-7+2\overline{\beta}}{2})+\overline{\beta}x)+O\big(t^{-\frac{1+\delta}{\alpha}}\big)$$
$$=\Big(\frac{7\beta-3\beta^2-2}{2}+\frac{y(\beta^2-2-\beta)}{a}+\frac{x}{a}\Big)t^{-\frac{1}{\alpha}}\int_0^{i\infty}e^{-K(z)}dz+O\big(t^{-\frac{1+\delta}{\alpha}}\big).$$

Substituting (2.56) and (2.57) into (2.54) we obtain formula (2.39).

Now we prove the second part of the lemma. We consider the functions I_1 and I_2, which were defined by formulas (2.45) and (2.46). We have

$$\begin{aligned}
|I_2| &= \Big| \int_{-i\infty}^{i\infty} e^{\xi t} \frac{e^{-\phi_1 a+\phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a}-1)} d\xi \int_\Gamma dp e^{px} \frac{K(p)}{p(K(p)+\xi)} \Big| \\
&\le \int_{-i\infty}^{i\infty} \frac{e^{-\operatorname{Re}\phi_1 a}|\phi_2|}{\left|\xi(e^{(\phi_2-\phi_1)a}-1)\right|} |d\xi| \int_\Gamma |dp|\, e^{\operatorname{Re} px} \Big| \frac{K(p)}{p(K(p)+\xi)} \Big| \\
&\quad + \int_{-i\infty}^{i\infty} \frac{e^{-\operatorname{Re}\phi_1 y}|\phi_1|}{\left|\xi(e^{(\phi_2-\phi_1)a}-1)\right|} |d\xi| \int_\Gamma |dp|\, e^{\operatorname{Re} px} \Big| \frac{K(p)}{p(K(p)+\xi)} \Big| \\
&\le C.
\end{aligned}$$

Therefore, for $t > 0$

$$\Big\| \int_0^a I_2(x,y,t)\phi(y)dy \Big\|_{\mathbf{L}^2} \le C \|\phi\|_{\mathbf{L}^\infty}. \tag{2.58}$$

Also, we obtain for $\delta_1+\delta_2+\delta_3>1$ in the case $x>y$

$$\begin{aligned}
&\int_{-i\infty}^{i\infty} e^{\xi t}\mathcal{H}_1(\xi,x,y)d\xi \\
&= \int_{-i\infty}^{i\infty} e^{\xi t} \frac{e^{\phi_2(x-y)}\phi_2'}{(e^{(\phi_2-\phi_1)a}-1)} d\xi + t^{-\frac{\delta_3}{\alpha}} O(\max(x^{-\delta_1}y^{-\delta_2}, (a-y)^{-\delta_1}(x-a)^{-\delta_2})) \\
&= \int_{-i\infty}^{i\infty} e^{-K(p)t} e^{p(x-y)} dp + t^{-\frac{\delta_3}{\alpha}} O(\max(x^{-\delta_1}y^{-\delta_2}, (a-y)^{-\delta_2}(x-a)^{-\delta_1}))
\end{aligned}$$

and in the case $x<y$

$$\begin{aligned}
&\int_{-i\infty}^{i\infty} e^{\xi t}\mathcal{H}_2(\xi,x,y)d\xi \\
&= \int_{-i\infty}^{i\infty} e^{\xi t} \frac{-e^{\phi_1(x-y)}\phi_1'}{(e^{(\phi_2-\phi_1)a}-1)} d\xi + t^{-\frac{\delta_3}{\alpha}} O(\max(x^{-\delta_1}y^{-\delta_2}, (a-y)^{-\delta_1}(x-a)^{-\delta_2})) \\
&= \int_{-i\infty}^{i\infty} e^{-K(p)t} e^{p(x-y)} dp + t^{-\frac{\delta_3}{\alpha}} O(\max(x^{-\delta_1}y^{-\delta_2}, (a-y)^{-\delta_2}(x-a)^{-\delta_1})).
\end{aligned}$$

Since $\operatorname{Re} K(p) > 0$ for $\operatorname{Re} p \ge 0$ we have

$$\begin{aligned}
\Big\| \int_0^a dy f(y) \int_{-i\infty}^{i\infty} e^{-K(p)t} e^{p(x-y)} dp \Big\|_{\mathbf{L}^2} &= \big\| e^{-K(p)t}\widehat{f} \big\|_{\mathbf{L}^2} \\
&\le C \big\| \widehat{f} \big\|_{\mathbf{L}^2} = C \|f\|_{\mathbf{L}^2},
\end{aligned} \tag{2.59}$$

for any $f \in \mathbf{L}^2(0,a)$. So we get

$$\begin{aligned}&\Big\| \int_0^a I_1(x,y,t)\phi(y)dy \Big\|_{\mathbf{L}^2} \\ &\le C\|\phi\|_{\mathbf{L}^2} + Ct^{-\frac{\delta_3}{\alpha}}\Big(\int_0^a \max(x^{-2\delta_1},(x-a)^{-2\delta_1})dx\Big)^{\frac{1}{2}} \\ &\times \int_0^a \max(y^{-\delta_2},(y-a)^{-\delta_2})\phi(y)dy.\end{aligned}$$

Choosing $\delta_1 = \frac{1}{2} - \gamma$, $\delta_2 = 1-\gamma$, and $\delta_3 = 0$ we obtain for $\gamma > 0$

$$\Big\| \int_0^a I_1(\cdot,y,t)\phi(y)dy \Big\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^\infty}. \tag{2.60}$$

Therefore, using (2.44), (2.58), and (2.60) we easily get

$$\Big\| \int_0^a G(\cdot,y,t)\phi(y)dy \Big\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^\infty}.$$

From (2.39) we have for $t > 1$

$$\Big\| \int_0^a G(\cdot,y,t)\phi(y)dy \Big\|_{\mathbf{L}^2} \le Ct^{-\frac{1}{\alpha}}\|\phi\|_{\mathbf{L}^1}.$$

Therefore we obtain the following estimate for $p = 1, \infty$:

$$\Big\| \int_0^a G(\cdot,y,t)\phi(y)dy \Big\|_{\mathbf{L}^2} \le Ct^{-\frac{p}{\alpha}}\|\phi\|_{\mathbf{L}^p}.$$

Since

$$\begin{aligned}|I_{2x}| &= \Big| \int_{-i\infty}^{i\infty} e^{\xi t}\frac{e^{-\phi_1 a+\phi_2(a-y)}\phi_2 - e^{-\phi_1 y}\phi_1}{\xi(e^{(\phi_2-\phi_1)a}-1)}d\xi \int_\Gamma dp e^{px}\frac{K(p)}{(K(p)+\xi)}\Big| \\ &\le \int_{-i\infty}^{i\infty} \frac{e^{-\operatorname{Re}\phi_1 a}|\phi_2|}{|\xi(e^{(\phi_2-\phi_1)a}-1)|}|d\xi| \int_\Gamma |dp|\, e^{\operatorname{Re}px}\Big|\frac{K(p)}{p(K(p)+\xi)}\Big| \\ &+ \int_{-i\infty}^{i\infty} \frac{e^{-\operatorname{Re}\phi_1 y}|\phi_1|}{|\xi(e^{(\phi_2-\phi_1)a}-1)|}|d\xi| \int_\Gamma |dp|\, e^{\operatorname{Re}px}\Big|\frac{K(p)}{(K(p)+\xi)}\Big| \le C,\end{aligned}$$

and for $\delta_1 + \delta_2 + \delta_3 > 2$

$$I_{1x} = \int_{-i\infty}^{i\infty} e^{-K(p)t}e^{p(x-y)}pdp + t^{-\frac{\delta_3}{\alpha}}O(\max(x^{-\delta_1}y^{-\delta_2},(a-y)^{-\delta_2}(x-a)^{-\delta_1})),$$

we can estimate that for $p = 2, \infty$

$$\Big\| \int_0^a I_{1x}(\cdot,y,t)\phi(y)dy \Big\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^p}$$

and

$$\left\| \int_0^a I_{2x}(\cdot,y,t)\phi(y)dy \right\|_{\mathbf{L}^2} \le Ct^{-\frac{1}{\alpha}-\frac{\gamma}{p}} \|\phi\|_{\mathbf{L}^p}.$$

Therefore

$$\left\| \int_0^a G_x(\cdot,y,t)\phi(y)dy \right\|_{\mathbf{L}^2} \le Ct^{-\frac{1}{\alpha}-\frac{\gamma}{p}} \|\phi\|_{\mathbf{L}^p}.$$

From formula (2.39) we have for $t > 1$

$$\left\| \int_0^a G_x(\cdot,y,t)\phi(y)dy \right\|_{\mathbf{L}^2} \le Ct^{-\frac{2}{\alpha}} \|\phi\|_{\mathbf{L}^1}.$$

In the same way we can obtain the following estimate:

$$\sup_{x,y\in[0,a]} |G_t(x,y,t)| \le Ct^{-\frac{1}{\alpha}-1}.$$

Lemma 4 is proved. □

3. Global existence

We now prove Theorem 1. Consider the linearized version of problem (1.1)

$$\begin{cases} u_t + \mathbb{K}u = -vv_x, \quad t > 0,\, x \in (0,a), \\ u(x,0) = u_0(x),\, x \in (0,a). \end{cases} \tag{3.1}$$

We suppose that $\|u_0\|_{\mathbf{L}^\infty} \le \varepsilon_1$ and $v \in X_\varepsilon$, where $\varepsilon_1 > 0$ is small enough, $\varepsilon = 100C\varepsilon_1$ with some positive constant C, and $\mathbf{X}_\varepsilon = \{v \in \mathbf{X},\ \|v\|_{\mathbf{X}} \le \varepsilon\}$, $\mathbf{X} = \left\{v \in \mathbf{C}([0,+\infty)\,;\mathbf{L}^2(0,a)), \|v\|_{\mathbf{X}} < +\infty\right\}$, where

$$\|v\|_{\mathbf{X}} = \sup_{t>0} \left(\langle t\rangle^{\frac{1}{\alpha}} \|v(t)\|_{\mathbf{L}^2} + t^{\frac{1}{\alpha}} \|v_x(t)\|_{\mathbf{L}^2} \right).$$

Via (2.36) we have for $n = 0, 1$

$$\partial_x^n u(x,t) = \int_0^a u_0(y)\partial_x^n G(x,y,t)dy + \int_0^t d\tau \int_0^a v_y v(y,\tau)\partial_x^n G(x,y,t-\tau)d\tau, \tag{3.2}$$

where the Green's function $G(x,y,t)$ was defined by (2.44). Via $v \in X_\varepsilon$ we obtain

$$\|v_y v(\tau)\|_{\mathbf{L}^1} = \|v_y(\tau)\|_{\mathbf{L}^2} \|v(\tau)\|_{\mathbf{L}^2} \le C\varepsilon^2 \tau^{-\frac{1}{\alpha}}(1+\tau)^{-\frac{1}{\alpha}}. \tag{3.3}$$

By virtue of Lemma 2 we have

$$\left\| \int_0^a G(\cdot,y,t_1)\phi(y,t_2)dy \right\|_{\mathbf{L}^2} \le C \|\phi(\cdot,t_2)\|_{\mathbf{L}^\infty}$$

and

$$\left\| \int_0^a G(\cdot,y,t_1)\phi(y,t_2)dy \right\|_{\mathbf{L}^2} \le Ct_1^{-\frac{1}{\alpha}} \|\phi(\cdot,t_2)\|_{\mathbf{L}^1}.$$

Applying the $\mathbf{L}^2$ norm to equation (3.2) and using (3.3) we get

$$\begin{aligned}
&\|u(t)\|_{\mathbf{L}^2} \le C\Big\| \int_0^a u_0(y)G(\cdot,y,t)dy\Big\|_{\mathbf{L}^2} \\
&+ C\Big\| \int_0^t d\tau \int_0^a v_y v(y,\tau)G(\cdot,y,t-\tau)d\tau\Big\|_{\mathbf{L}^2} \\
&\le C\left(1-\theta(t-1)\right)\|u_0\|_{\mathbf{L}^\infty} + C\theta(t-1)t^{-\frac{1}{\alpha}}\|u_0\|_{\mathbf{L}^1} \\
&+ \int_0^t d\tau \, \|v_y v(\tau)\|_{\mathbf{L}^1} (t-\tau)^{-\frac{1}{\alpha}} d\tau \qquad (3.4) \\
&\le \varepsilon_1 (1+t)^{-\frac{1}{\alpha}} + \varepsilon^2 C \int_0^t \tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{-\frac{1}{\alpha}} (t-\tau)^{-\frac{1}{\alpha}} d\tau \le \varepsilon (1+t)^{-\frac{1}{\alpha}}.
\end{aligned}$$

Since $v \in X_\varepsilon$ we obtain

$$\begin{aligned}
\|v_y v(\tau)\|_{\mathbf{L}^2} &= \|v_y(\tau)\|_{\mathbf{L}^2} \|v(\tau)\|_{\mathbf{L}^\infty} \\
&\le C\|v_y(\tau)\|_{\mathbf{L}^2}^{\frac{3}{2}} \|v(\tau)\|_{\mathbf{L}^2}^{\frac{1}{2}} \le C\varepsilon^2 \tau^{-\frac{3}{2\alpha}} (1+\tau)^{-\frac{1}{2\alpha}}.
\end{aligned}$$

Via Lemma 2, we have

$$\begin{aligned}
\Big\| \int_0^a G_x(\cdot,y,t_1)\phi(y,t_2)dy\Big\|_{\mathbf{L}^2} &\le Ct_1^{-\frac{1}{\alpha}} \|\phi(\cdot,t_2)\|_{\mathbf{L}^\infty}, \\
\Big\| \int_0^a G_x(\cdot,y,t_1)\phi(y,t_2)dy\Big\|_{\mathbf{L}^2} &\le Ct_1^{-\frac{1+\gamma}{\alpha}} \|\phi(\cdot,t_2)\|_{\mathbf{L}^2} \\
\Big\| \int_0^a G_x(\cdot,y,t_1)\phi(y,t_2)dy\Big\|_{\mathbf{L}^2} &\le Ct_1^{-\frac{2}{\alpha}} \|\phi(\cdot,t_2)\|_{\mathbf{L}^1}.
\end{aligned}$$

Therefore, using the integral representation (3.2) for the derivative u_x we obtain for small $\gamma > 0$

$$\begin{aligned}
&\|u_x(t)\|_{\mathbf{L}^2} \le C\Big\| \int_0^a u_0(y)G_x(\cdot,y,t)dy\Big\|_{\mathbf{L}^2} \\
&+ C\Big\| \int_0^t d\tau \int_0^a v_y v(y,\tau)G_x(\cdot,y,t-\tau)d\tau\Big\|_{\mathbf{L}^2} \\
&\le Ct^{-\frac{1}{\alpha}}\|u_0\|_{\mathbf{L}^\infty} + \int_0^{\frac{t}{2}} d\tau \, \|v_y v(\tau)\|_{\mathbf{L}^1} (t-\tau)^{-\frac{2}{\alpha}} d\tau \\
&+ \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\gamma}{\alpha}} \|v_y v(\tau)\|_{\mathbf{L}^2} d\tau \\
&\le \varepsilon_1 t^{-\frac{1}{\alpha}} + \varepsilon^2 C\Big(\int_0^{\frac{t}{2}} \tau^{-\frac{1}{\alpha}} \langle\tau\rangle^{-\frac{1}{\alpha}} (t-\tau)^{-\frac{2}{\alpha}} d\tau \Big) \qquad (3.5)
\end{aligned}$$

$$+ C\int_{\frac{t}{2}}^{t}\left\|v_y\right\|_{\mathbf{L}^2}\left\|v(\tau)\right\|_{\mathbf{L}^\infty}(t-\tau)^{-\frac{1+\gamma}{\alpha}}d\tau$$
$$\le \varepsilon_1 t^{-\frac{1}{\alpha}} + C\varepsilon^2 t^{-\frac{1}{\alpha}} + C\varepsilon^2\int_{\frac{t}{2}}^{t}\tau^{-\frac{3}{2\alpha}}\langle\tau\rangle^{-\frac{1}{2\alpha}}(t-\tau)^{-\frac{1+\gamma}{\alpha}}d\tau \le \varepsilon t^{-\frac{1}{\alpha}}.$$

Therefore, by (3.4) and (3.5) we get

$$\sup_{t\ge 0}\left((1+t)^{\frac{1}{\alpha}}\left\|u(t)\right\|_{\mathbf{L}^2} + t^{\frac{1}{\alpha}}\left\|u_x(t)\right\|_{\mathbf{L}^2}\right) \le \varepsilon. \tag{3.6}$$

We introduce the distance in $\mathbf{X}$

$$d(f,g) = \sup_{t>0}\left((1+t)^{\frac{1}{\alpha}}\left\|f-g\right\|_{\mathbf{L}^2} + t^{\frac{1}{\alpha}}\left\|f_x - g_x\right\|_{\mathbf{L}^2}\right).$$

Then in the same way as in the proof of (3.4) we have

$$d(u_1,u_2) = d(\mathbb{M}v_1 - \mathbb{M}v_2) \le \frac{1}{2}d(v_1 - v_2), \tag{3.7}$$

where

$$\begin{cases} \partial_t u_j + \mathbb{K}u_j = -iv_{xj}v_j, \quad t>0,\ x\in(0,a), \\ u_j(x,0) = u_0(x),\ x\in(0,a). \end{cases}$$

The estimates (3.4) and (3.7) show that $\mathbb{M}$ is a contraction mapping from $\mathbf{X}$ into itself. Therefore there exists a unique solution $u(x,t)\in\mathbf{X}$ satisfying the estimate $\|u\|_{\mathbf{X}} \le \varepsilon$. This completes the proof of the first part of Theorem 1.

Now using estimate (3.6) we prove that the solution has the following asymptotics:

$$u(x,t) = t^{-\frac{1}{\alpha}}BA(x) + O\left(t^{-\frac{1+\delta}{\alpha}}\right)$$

for $t \longrightarrow +\infty$ uniformly with respect to $x\in[0,a]$, where $\delta\in(0,2-\alpha)$,

$$B = \frac{2\sin\frac{2\pi}{\alpha}}{\pi a}e^{i\frac{2\pi}{\alpha}}\int_0^{i\infty}e^{-K(z)}dz,$$

and

$$A(x) = \theta_a(x)\int_0^a(y-\theta_y(x)a)u_0(y)dy$$
$$+ \theta_a(x)\int_0^{+\infty}d\tau\int_0^a(y-\theta_y(x)a)u_y(y,\tau)u(y,\tau)dy < +\infty.$$

Indeed, in view of asymptotics (2.39) of Lemma 4 we have

$$u(x,t) = t^{-\frac{1}{\alpha}}BA(x) + R(x,t), \tag{3.8}$$

where

$$|R(x,t)| \le O(t^{-\frac{1+\delta}{\alpha}}) \int_0^a |u_0(y)|\, dy + O(t^{-\frac{1+\delta}{\alpha}}) \int_0^t d\tau \int_0^a |u_y u|\, dy$$
$$+ t^{-\frac{1}{\alpha}} \Lambda A \int_t^{+\infty} d\tau \Big(\int_0^a y\, |u_y(y,\tau) u(y,\tau)|\, dy + a \int_x^a |u_y(y,\tau)u(y,\tau)|\, dy \Big)$$
$$+ \int_0^t d\tau \int_0^a |u(y,\tau)|^3 \, |G(x,y,t-\tau) - G(x,y,t)|\, dy$$
$$\le Ct^{-\frac{1+\delta}{\alpha}} \|u_0\|_{\mathbf{L}^1} + Ct^{-\frac{1+\delta}{\alpha}} \int_0^{+\infty} \|u_y\|_{\mathbf{L}^2} \|u\|_{\mathbf{L}^2} d\tau + Ct^{-\frac{1}{\alpha}} \int_t^{+\infty} \|u_y\|_{\mathbf{L}^2} \|u\|_{\mathbf{L}^2} d\tau$$
$$+ \int_0^t d\tau \int_0^a |u(y,\tau)|\, |u_y(y,\tau)|\, |G(x,y,t-\tau) - G(x,y,t)|\, dy. \tag{3.9}$$

We have

$$\|u_y\|_{\mathbf{L}^2} \, \|u\|_{\mathbf{L}^2} \le Ct^{-\frac{1}{\alpha}} (1+t)^{-\frac{1}{\alpha}};$$

therefore,

$$\int_0^t \|u_y\|_{\mathbf{L}^2} \, \|u\|_{\mathbf{L}^2} \, d\tau \le C \int_0^\infty \tau^{-\frac{1}{\alpha}} (1+\tau)^{-\frac{1}{\alpha}} d\tau \le C \tag{3.10}$$

and

$$\int_t^{+\infty} \|u_y\|_{\mathbf{L}^2} \, \|u\|_{\mathbf{L}^2} \, d\tau \le C \int_t^\infty \tau^{-\frac{1}{\alpha}} (1+\tau)^{-\frac{1}{\alpha}} d\tau \le Ct^{-\frac{\delta}{\alpha}}. \tag{3.11}$$

From Lemma 4 we have

$$\sup_{x,y \in [0,a]} |G_t(x,y,t)| \le Ct^{-\frac{1}{\alpha}-1}.$$

Therefore we obtain for $\mu \in (0,1)$

$$|G(x,y,t-\tau) - G(x,y,t)| \le Ct^{-\mu \frac{1+\alpha}{\alpha}} \tau^{\mu} (t-\tau)^{-\frac{1-\mu}{\alpha}} \tau^{-\frac{1-\mu}{\alpha}}.$$

So choosing $\mu \in \left(\frac{1+\delta}{1+\alpha}, 1\right)$ we get

$$\int_0^t d\tau \int_0^a |u(y,\tau)|\, |u_y(y,\tau)|\, |G(x,y,t-\tau) - G(x,y,t)|\, dy$$
$$\le Ct^{-\mu \frac{1+\alpha}{\alpha}} \int_0^\infty \tau^{-\frac{2-\mu}{\alpha}+\mu} (1+\tau)^{-\frac{1}{\alpha}} (t-\tau)^{-\frac{1-\mu}{\alpha}} d\tau \le Ct^{-\frac{1+\delta}{\alpha}} \tag{3.12}$$

for $0 < \delta < 2 - \alpha$ and $t > 1$. Hence by virtue of (3.8)–(3.12) we have asymptotics of the solution. Theorem 1 is proved.

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