Differential and Integral Equations Volume 18, Number 2, Pages 169–194

GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SMALL SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS IN 3D

Yuichiro Kawahara
Department of Mathematics, Graduate School of Science
Osaka University, Toyonaka, Osaka 560-0043, Japan

(Submitted by: Tohru Ozawa)

Abstract. We study the global existence and asymptotic behavior in time of small solutions to nonlinear Schrödinger equations with quadratic nonlinearities,

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \mathcal{N}(u,\bar{u}), & (t,x) \in \mathbf{R}\times\mathbf{R}^3, \\ u(0,x) = u_0, & x \in \mathbf{R}^3, \end{cases}$$

where the initial data u_0 are sufficiently small in a suitable norm, $\bar{u}$ is the complex conjugate of u. The nonlinear term $\mathcal{N}$ is a smooth quadratic function in the neighborhood of the origin with respect to u and $\bar{u}$ and does not contain the term $|u|^2$. Our purpose in this paper is to show there exists a unique final state u_+ such that

$$\left\| u(t) - e^{\frac{it}{2}\Delta} u_+ \right\|_{L^2} \le C t^{-\frac{5}{4}}, \quad \text{for small } u_0.$$

1. Introduction

In this paper, we study the global existence and asymptotic behavior in time of small solutions to nonlinear Schrödinger equations with quadratic nonlinearities

$$\begin{cases} \mathcal{L}u = \mathcal{N}(u,\bar{u}), & (t,x) \in \mathbf{R}\times\mathbf{R}^3, \\ u(0,x) = u_0, & x \in \mathbf{R}^3, \end{cases} \tag{1.1}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2}\Delta$, the initial data u_0 are sufficiently small in a suitable norm, $\bar{u}$ is the complex conjugate of u. The nonlinear term $\mathcal{N}$ is a smooth quadratic function in the neighborhood of the origin with respect to u and $\bar{u}$ and satisfies $\frac{\partial^2}{\partial u \partial \bar{u}}\mathcal{N}(0,0) = 0$. The aim of this paper is to improve the results on asymptotic behavior in time of small solutions obtained in [6].

Accepted for publication: July 2004.
AMS Subject Classifications: 35Q55; 35B40.

There are many works on nonlinear Schrödinger equations with quadratic nonlinearities. More precisely, the nonlinear Schrödinger equation of the form

$$\begin{cases} \mathcal{L}u = \mathcal{N}_1(u, \nabla u), & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0,x) = u_0, & x \in \mathbf{R}^n, \end{cases} \tag{1.2}$$

was studied, where the nonlinear term $\mathcal{N}_1$ is a smooth quadratic function in the neighborhood of the origin. In [14], it was proved that global existence of small solutions and solutions satisfy the time-decay estimates $\|u(t)\|_{L^3} \le Ct^{-\frac{n}{6}}$ when $n \ge 4$ and $\mathcal{N}_1$ depends only on u and $\bar{u}$. In [11] and [13], a global-existence theorem for the Cauchy problem (1.2) was shown under the conditions such that $n \ge 5$ and

$$\mathrm{Re}\Big(\frac{\partial \mathcal{N}_1}{\partial \zeta_j}(z,\zeta)\Big) = 0, \quad (z,\zeta) \in \mathbf{C} \times \mathbf{C}^n, \quad j = 1, \dots, n. \tag{1.3}$$

Their method is based on the L^p-L^q time-decay estimates of the fundamental solution to the linear Schrödinger equation and classical energy method. In [11] and [13], the time-decay estimate $\|u(t)\|_{L^4} \le Ct^{-\frac{n}{4}}$ was obtained. By using the time-decay estimate, it is easy to see that there exists a unique final state $u_+ \in L^2$ such that

$$\big\|u(t) - \mathcal{U}(t)u_+\big\|_{L^2} \le Ct^{1-\frac{n}{2}} \quad \text{for } t > 1,$$

where the free Schrödinger evolution group $\mathcal{U}(t)$ is given by

$$\mathcal{U}(t)\phi = \frac{1}{(2\pi i t)^{\frac{n}{2}}} \int e^{\frac{i|x-y|^2}{2t}} \phi(y) dy.$$

In paper [3], lower space dimensions $n = 3, 4$ were considered when the nonlinearity $\mathcal{N}_1$ satisfies (1.3) and $\frac{\partial \mathcal{N}_1}{\partial \zeta_j} \ne 0$ $(j = 1, \dots, n)$, and the global existence of small solutions for (1.2) and the time-decay estimate

$$\|\nabla u(t)\|_{L^\infty} \le Ct^{-\frac{3n}{8}}$$

were proved by the method of the vector field associated with the Schrödinger evolution group. As a product we have a unique final state $u_+ \in L^2$ such that

$$\big\|u(t) - \mathcal{U}(t)u_+\big\|_{L^2} \le Ct^{1-\frac{3n}{8}} \quad \text{for } t > 1.$$

In [4], the condition (1.3) was removed. However, $\mathcal{N}_1$ was supposed to have the derivation of unknown functions. On the other hand, time-decay estimates were improved as follows:

$$\|u(t)\|_{L^\infty} \le Ct^{-\frac{3}{4}}, \quad \|\nabla u(t)\|_{L^\infty} \le Ct^{-\frac{5}{4}}, \; n = 3.$$

By these estimates, we have a unique final state $u_+ \in L^2$ such that

$$\left\|u(t) - \mathcal{U}(t)u_+\right\|_{L^2} \leq Ct^{-\frac{1}{4}} \quad \text{for} \quad t > 1.$$

From the results in [3] and [4], we see that the problem becomes harder when $\mathcal{N}_1$ does not depend on derivatives of unknown functions and lower-dimensional spaces. N. Hayashi and P. Naumkin [5] studied the global existence and asymptotic behavior in time of small solutions to nonlinear Schrödinger equations

$$\begin{cases} \mathcal{L}u = \lambda u^2 + \mu \bar{u}^2, & (t,x) \in \mathbf{R} \times \mathbf{R}^3, \\ u(0,x) = u_0, & x \in \mathbf{R}^3, \end{cases} \tag{1.4}$$

where λ, $\mu \in \mathbf{C}$. They showed the time-decay estimates

$$\|u(t)\|_{L^\infty} \leq Ct^{-1-\gamma}, \quad \gamma \in (0, \frac{1}{30}),$$

and the existence of a unique final state $u_+ \in L^2$ such that

$$\left\|u(t) - \mathcal{U}(t)u_+\right\|_{L^2} \leq C\varepsilon t^{-\gamma}, \ \gamma \in (0, \frac{1}{30}), \quad \text{for} \quad t > 1.$$

In [6], the previous results in [5] were improved. More precisely, the time-decay estimate

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{3}{2}} \tag{1.5}$$

and the existence of a unique final state $u_+ \in L^2$ such that

$$\left\|u(t) - \mathcal{U}(t)u_+\right\|_{L^2} \leq Ct^{-\frac{1}{2}} \quad \text{for} \quad t > 1 \tag{1.6}$$

were obtained. The L^∞ time-decay estimate is the same as that of solutions to the free Schrödinger equations, and (1.6) seems to be optimal since we have by (1.4) and (1.5)

$$\left\|u(t) - \mathcal{U}(t)u_+\right\|_{L^2} \leq \int_t^\infty \left\|\lambda u^2 + \mu \bar{u}^2\right\|_{L^2} d\tau \leq Ct^{-\frac{1}{2}}.$$

Our purpose in this paper is to show that the estimate (1.6) is not optimal by using a special structure of nonlinearities u^2 and $\bar{u}^2$.

There are many papers on nonlinear Schrödinger equations with the gauge-invariant nonlinearity $f(|u|^2)u$. Among them there are some works on the sharp time decay on the second term of the asymptotic expansion of solutions (see [7], [8], [9], [10], and [15]). Asymptotic completeness is the important problem; however, it is still open for nonlinear Schrödinger equations with quadratic nonlinearities, even if the space dimension is three. For a partial answer to the problem, we refer to [1] and [12], in which the asymptotic completeness of nonlinear Schrödinger equations with the quadratic and gauge-invariant nonlinearity $|u|u$ was shown.

We introduce some notation and function spaces which are used in this paper. Let

$$\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ix\xi}\phi(x)dx$$

denote the Fourier transform of ϕ and

$$\mathcal{F}^{-1}\phi = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{ix\xi}\phi(\xi)d\xi$$

denote the inverse Fourier transform of ϕ. We have the following identity:

$$\mathcal{U}(t) = M(t)\mathcal{D}(t)\mathcal{F}M(t),$$

where

$$M(t) = e^{\frac{i|x|^2}{2t}},\ (\mathcal{D}(t)\phi)(x) = (it)^{-\frac{3}{2}}\phi(\frac{x}{t}).$$

By a direct calculation,

$$\mathcal{U}(-t)\phi = -i\bar{M}(t)\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}(t)\phi, \qquad \mathcal{D}(t)^{-1}\phi = -i\mathcal{D}(\frac{1}{t})\phi,$$

$$\mathcal{J} = x + it\nabla = Mit\nabla\bar{M} = \mathcal{U}(t)x\mathcal{U}(-t),\ [\mathcal{L}, \mathcal{J}] = 0.$$

These factorizations were introduced by N. Hayashi and T. Ozawa [2]. The weighted Sobolev space $H_p^{m,k}$ is defined by

$$H_p^{m,k} = \Big\{\phi \in L^p : \|\phi\|_{m,k,p} = \Big\|(1+|x|^2)^{\frac{k}{2}}(1-\Delta)^{\frac{m}{2}}\phi\Big\|_{L^p} < \infty\Big\},$$

where $m, k \in \mathbf{R}_+$ and $1 \le p \le \infty$. For simplicity, we denote $\|\cdot\| = \|\cdot\|_{L^2}$, $\|\cdot\|_p = \|\cdot\|_{L^p}$, $\|\cdot\|_{m,k} = \|\cdot\|_{m,k,2}$, $H^{m,k} = H_2^{m,k}$, $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$, and $\mathcal{K} = \mathcal{F}\mathcal{U}(-t)$.

We state our main result.

Theorem 1.1. *Let $u_0 \in H^{3,0} \cap H^{1,2}$ and $\varepsilon = \|u_0\|_{3,0} + \|u_0\|_{1,2}$. Then there exists an $\varepsilon > 0$ such that* (1.1) *has a unique global solution u satisfying $u \in C\big(\mathbf{R}; H^{3,0} \cap H^{1,2}\big)$ and*

$$\|u(t)\|_\infty \le C\varepsilon^{\frac{1}{2}}\langle t\rangle^{-\frac{3}{2}},\ \|u(t)\| \le C\varepsilon^{\frac{1}{2}}.$$

Moreover, there exists a unique final state $\hat{u}_+ \in L^2 \cap L^\infty$ such that

$$\Big\|\mathcal{F}\mathcal{U}(-t)u(t) - \hat{u}_+ - \lambda 2^{-\frac{3}{2}} i^{-\frac{9}{2}} \int_t^\infty e^{\frac{i\tau|\xi|^2}{4}}\tau^{-\frac{3}{2}}\, d\tau\, \hat{u}_+^2(\frac{\xi}{2})$$

$$-\mu 2^{-\frac{3}{2}} i^{\frac{9}{2}} \int_t^\infty e^{\frac{3i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}}\, d\tau\, \overline{\hat{u}}_+^2(-\frac{\xi}{2})\Big\| \le C\varepsilon t^{\theta-\frac{3}{2}} \tag{1.7}$$

for $t > 1$ *and*

$$\big\|u(t) - \mathcal{U}(t)u_+\big\| \le C\varepsilon t^{-\frac{5}{4}} \quad \textit{for} \quad t > 1, \tag{1.8}$$

where $\theta > 0$,

$$\Big\|\lambda 2^{-\frac{3}{2}} i^{-\frac{9}{2}} \int_t^\infty e^{\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}}\, d\tau\, \hat{u}_+^2(\frac{\xi}{2})\Big\| + \Big\|\mu 2^{-\frac{3}{2}} i^{\frac{9}{2}} \int_t^\infty e^{\frac{3i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}}\, d\tau\, \overline{\hat{u}}_+^2(-\frac{\xi}{2})\Big\|$$
$$\le C\varepsilon t^{-\frac{5}{4}} \quad \textit{for} \quad t > 1.$$

Remark. Using the Taylor expansion, we see that the nonlinear term $\mathcal{N}$ is written as

$$\mathcal{N}(u, \bar{u}) = \lambda u^2 + \mu \bar{u}^2 + \mathcal{G}(u, \bar{u}),$$

where $\lambda, \mu \in \mathbf{C}$. The nonlinear term $\mathcal{G}$ is a smooth, cubic function in the neighborhood of the origin with respect to u and $\bar{u}$. Therefore, we consider the following nonlinear Schrödinger equations with quadratic and cubic nonlinearities from now on:

$$\begin{cases} \mathcal{L}u = \lambda u^2 + \mu \bar{u}^2 + \mathcal{G}(u, \bar{u}), & (t, x) \in \mathbf{R} \times \mathbf{R}^3, \\ u(0, x) = u_0, & x \in \mathbf{R}^3. \end{cases} \tag{1.9}$$

For the convenience of the readers, we state our strategy of the proof of the asymptotic behavior in time of small solutions (1.7).

Multiplying the integral equation associated with (1.9) by the operator $\mathcal{F}\mathcal{U}(-t)$, we find

$$\mathcal{F}\mathcal{U}(-t)u(t) = \hat{u}_+ + i\int_t^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{N}(\tau) d\tau + i\int_t^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{G}(\tau) d\tau, \tag{1.10}$$

where $\mathcal{N}(\tau) = \lambda u^2 + \mu \bar{u}^2$,

$$\hat{u}_+ = \hat{u}_0 - i\int_0^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{N}(\tau) d\tau - i\int_0^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{G}(\tau) d\tau.$$

The third term of the right-hand side of (1.10) is decaying in time faster than the second term of the right-hand side of (1.10). By careful consideration of $\mathcal{F}\mathcal{U}(-t)u^2$ and $\mathcal{F}\mathcal{U}(-t)\bar{u}^2$, we can show

$$\mathcal{F}\mathcal{U}(-t)u(t) = \hat{u}_+ + V(t) + W(t) + O(t^{\theta-\frac{3}{2}}) \ \text{ in } L^2 \text{ as } t \to \infty, \tag{1.11}$$

where $\theta > 0$ and $V(t)$ is the term including the oscillating function $e^{it|\xi|^2}$; more precisely, $V(t)$ and $W(t)$ are written as

$$V(t) = C\int_t^\infty \tau^{-\frac{3}{2}} e^{i\tau|\xi|^2} d\tau (\hat{u}_+)^2,$$

$$W(t) = C\int_t^\infty \tau^{-\frac{3}{2}} e^{i\tau|\xi|^2} \big((\mathcal{F}\mathcal{U}(-\tau)u)^2 - (\hat{u}_+)^2\big) d\tau,$$

with some constant C. For the exact representation of $V(t)$ or $W(t)$, see Section 4.

Integrating by parts with respect to τ, we have

$$\|V(t)\| = O(t^{-\frac{5}{4}}) \text{ as } t \to \infty. \tag{1.12}$$

By the estimate (see Proposition 4.1)

$$\big\|\mathcal{F}\mathcal{U}(-t)u - \hat{u}_+\big\| = O(t^{-\frac{1}{2}}) \text{ as } t \to \infty, \tag{1.13}$$

we have

$$\|W(t)\| = O(t^{-1}) \text{ as } t \to \infty. \tag{1.14}$$

By (1.11), (1.12), and (1.14), we get

$$\big\|\mathcal{F}\mathcal{U}(-t)u - \hat{u}_+\big\| = O(t^{-1}) \text{ as } t \to \infty. \tag{1.15}$$

Using (1.15) instead of (1.13), we have

$$\|W(t)\| = O(t^{-\frac{3}{2}}) \text{ as } t \to \infty. \tag{1.16}$$

By (1.11) and (1.16), we obtain our desired estimate

$$\big\|\mathcal{F}\mathcal{U}(-t)u - \hat{u}_+ - V(t)\big\| = O(t^{\theta-\frac{3}{2}}) \text{ as } t \to \infty.$$

This paper is organized as follows. In Section 2, we prove preliminary estimates. In Section 3, we prove a global existence theorem. In Section 4, the main result is shown.

2. Preliminaries

We use the notation

$$\|u\|_Z = \|u\|_{3,0} + \|\mathcal{J}u\|_{2,0} + \|\mathcal{J}^2 u\|_{1,0},$$

$$\|u\|_{\tilde{Y}} = \|u\|_{3,0} + \|\mathcal{J}u\|_{2,0} + \langle t\rangle^{-\frac{1}{2}}\|\mathcal{J}^2 u\|_{1,0} + \langle t\rangle^{\frac{3}{2}}\|u\|_{1,0,\infty},$$

$$\|u\|_Y = \|u\|_{\tilde{Y}} + \langle t\rangle^{-\theta}\|\mathcal{J}^2 w\|_{1,0},\ \theta > 0,$$

$$w = u - t\lambda\mathcal{A}_{\alpha_1}u^2 - \frac{t}{3}\mu\mathcal{A}_{\alpha_2}\bar{u}^2,\ \alpha_1 = -\frac{3}{2},\ \alpha_2 = -\frac{1}{2},\ \mathcal{A}_\alpha = (i\alpha + \frac{t}{4}\Delta)^{-1}.$$

In this section, we summarize the results obtained in [6] to get the main result. The following lemma is the identity used in order to show Proposition 3.2 of [6].

Lemma 2.1. *Let* $\rho \neq 0,\ E = e^{\frac{it|\xi|^2}{2}}$, *and* ϕ *be an arbitrary function. Then*

$$\mathcal{F}\bar{M}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}\phi = i^{\frac{3}{2}}\mathcal{D}(\rho)E^{\rho^2-\rho}\mathcal{F}\bar{M}^{\frac{1}{\rho}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^{\rho}\phi. \tag{2.1}$$

Proof. For the convenience of the readers we give a proof. By the identities $\mathcal{F}\mathcal{U}(-t)\phi = -i\mathcal{F}\bar{M}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}\phi$ and $\mathcal{F}\mathcal{U}(-t)\phi{=}E\mathcal{F}\phi$, the identity in the lemma is the same as the identity

$$E\mathcal{F}\phi = i^{\frac{1}{2}}\mathcal{D}(\rho)E^{\rho^2-\rho}\mathcal{F}\bar{M}^{\frac{1}{\rho}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^{\rho}\phi,$$

which is written as

$$\mathcal{F}^{-1}\bar{E}^{\rho^2-\rho}\mathcal{D}(\rho)^{-1}E\mathcal{F}\phi = i^{\frac{1}{2}}\bar{M}^{\frac{1}{\rho}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^{\rho}\phi. \tag{2.2}$$

We consider the right-hand side of (2.2). By a simple computation, we have

$$\mathcal{D}(\frac{1}{t})\bar{M}^{\rho}\phi = \mathcal{D}(\frac{1}{t})e^{-\frac{i\rho|x|^2}{2t}}\phi = i^{-\frac{3}{2}}t^{\frac{3}{2}}e^{-\frac{i\rho t|x|^2}{2}}\phi(tx).$$

Applying the inverse Fourier transform to the identity, we get

$$\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^{\rho}\phi = \frac{i^{-\frac{3}{2}}t^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}}\int e^{ix\xi-\frac{i\rho t|\xi|^2}{2}}\phi(t\xi)d\xi.$$

Hence, we obtain

$$\begin{aligned} &i^{\frac{1}{2}}\bar{M}^{\frac{1}{\rho}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^{\rho}\phi = e^{-\frac{i|x|^2}{2t\rho}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^{\rho}\phi \\ &= \frac{i^{-1}t^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}}\int e^{-\frac{i|x|^2}{2t\rho}+ix\xi-\frac{i\rho t|\xi|^2}{2}}\phi(t\xi)d\xi = \frac{i^{-1}t^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}}\int e^{-\frac{i}{2t\rho}|x-t\rho\xi|^2}\phi(t\xi)d\xi. \end{aligned} \tag{2.3}$$

We next consider the left-hand side of (2.2). By $E\mathcal{F}\phi(\xi) = e^{\frac{it|\xi|^2}{2}}\hat{\phi}(\xi)$, it follows that

$$\mathcal{D}(\rho)^{-1}E\mathcal{F}\phi = (i\rho)^{\frac{3}{2}}e^{\frac{it\rho^2|\xi|^2}{2}}\hat{\phi}(\rho\xi).$$

Therefore, we have

$$\bar{E}^{\rho^2-\rho}\mathcal{D}(\rho)^{-1}E\mathcal{F}\phi = e^{-\frac{it(\rho^2-\rho)|\xi|^2}{2}}(i\rho)^{\frac{3}{2}}e^{\frac{it\rho^2|\xi|^2}{2}}\hat{\phi}(\rho\xi) = (i\rho)^{\frac{3}{2}}e^{\frac{it\rho|\xi|^2}{2}}\hat{\phi}(\rho\xi).$$

Applying the inverse Fourier transform to the identity, we get

$$\mathcal{F}^{-1}\bar{E}^{\rho^2-\rho}\mathcal{D}(\rho)^{-1}E\mathcal{F}\phi = (i\rho)^{\frac{3}{2}}(2\pi)^{-\frac{3}{2}}\Big(\mathcal{F}^{-1}[e^{\frac{it\rho|\xi|^2}{2}}] * \mathcal{F}^{-1}[\hat{\phi}(\rho\xi)]\Big)(x)$$
$$= \frac{i^{-1}t^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}}\int e^{-\frac{i}{2t\rho}|x-t\rho\xi|^2}\phi(t\xi)d\xi. \tag{2.4}$$

Thus Lemma 2.1 is proved from (2.3) and (2.4). □

In order to prove the existence of global solutions, we prepare next two lemmas.

Lemma 2.2. ([6], Lemma 2.1) *Let* $t > 0$, $\eta \in \mathbf{R}$, $\alpha \in \mathbf{R}\backslash\{0\}$, *and* $\mathcal{A}_\alpha = (i\alpha + \frac{t}{4}\Delta)^{-1}$. *Then*

$$t^\eta uv = \mathcal{L}t^{\eta+1}\mathcal{A}_\alpha uv + t^\eta\Big(\alpha\mathcal{A}_\alpha - i\Big(\eta - \frac{3}{2} - \alpha\Big)\Big)\mathcal{A}_\alpha uv$$
$$- t^{\eta+1}\mathcal{A}_\alpha\big(u\mathcal{L}v + v\mathcal{L}u\big) + \frac{t^{\eta-1}}{4}\mathcal{A}_\alpha\big(2(\mathcal{J}u)\mathcal{J}v - v\mathcal{J}^2u - u\mathcal{J}^2v\big)$$

and

$$3t^\eta\overline{uv} = \mathcal{L}t^{\eta+1}\mathcal{A}_\alpha\overline{uv} + t^\eta\Big(\alpha\mathcal{A}_\alpha - i\Big(\eta - \frac{3}{2} - 3\alpha\Big)\Big)\mathcal{A}_\alpha\overline{uv}$$
$$+ t^{\eta+1}\mathcal{A}_\alpha\big(\overline{u}\overline{\mathcal{L}v} + \overline{v}\overline{\mathcal{L}u}\big) - \frac{t^{\eta-1}}{4}\mathcal{A}_\alpha\big(2(\overline{\mathcal{J}u})\overline{\mathcal{J}v} - \overline{v}\overline{\mathcal{J}^2u} - \overline{u}\overline{\mathcal{J}^2v}\big).$$

The following lemma is the formula used in order to prove Proposition 3.2 of [6]. Let u be a solution of (1.9). Then by Lemma 2.2, we have

Lemma 2.3.

$$\mathcal{L}\Psi = \sum_{j=1}^{4} I_j + \mathcal{J}\mathcal{G}, \tag{2.5}$$

where

$$\Psi = \mathcal{J}u - \frac{i}{2}t^2\nabla\big(\lambda\mathcal{A}_{\alpha_1}u^2 + \mu\mathcal{A}_{\alpha_2}\overline{u}^2\big),$$
$$I_1 = \lambda u\mathcal{J}u + \mu\overline{u\mathcal{J}u},\ I_2 = -it^2\nabla\big(\lambda\mathcal{A}_{\alpha_1}(u\mathcal{L}u) - \mu A_{\alpha_2}(\overline{u}\overline{\mathcal{L}u})\big),$$
$$I_3 = \frac{i\lambda}{2}\Big(\alpha_1\mathcal{A}_{\alpha_1} + i\Big(\frac{1}{2} + \alpha_1\Big)\Big)t\nabla\mathcal{A}_{\alpha_1}u^2$$
$$+ \frac{i\mu}{2}\Big(\alpha_2\mathcal{A}_{\alpha_2} + i\Big(\frac{1}{2} + 3\alpha_2\Big)\Big)t\nabla\mathcal{A}_{\alpha_2}\overline{u}^2,$$
$$I_4 = \frac{i\lambda}{4}\nabla\mathcal{A}_{\alpha_1}\Big((\mathcal{J}u)^2 - u\mathcal{J}^2u\Big) - \frac{i\mu}{4}\nabla\mathcal{A}_{\alpha_2}\Big((\overline{\mathcal{J}u})^2 - \overline{u}\overline{\mathcal{J}^2u}\Big).$$

$$\mathcal{L}\Phi = I_5 + I_6 + I_7 + \mathcal{J}^2\mathcal{G}, \tag{2.6}$$

where

$$\begin{aligned}
\Phi &= \mathcal{J}^2 u + \Big(\frac{3i}{2}t^2 - it^2\nabla\mathcal{J} - \frac{1}{4}t^3\Delta\Big)\lambda\mathcal{A}_{\alpha_1}u^2 \\
&\quad + \Big(\frac{3i}{2}t^2 - it^2\nabla\mathcal{J} - \frac{3}{4}t^3\Delta\Big)\mu\mathcal{A}_{\alpha_2}\overline{u}^2, \\
I_5 &= \frac{1}{2}\lambda\Big(u\mathcal{J}^2u + (\mathcal{J}u)^2\Big) + \frac{1}{2}\mu\Big(\overline{u}\overline{\mathcal{J}^2u} + (\overline{\mathcal{J}u})^2\Big), \\
I_6 &= -\Big(\frac{3i}{2}t - it\nabla\mathcal{J} - \frac{1}{4}t^2\Delta\Big)\Big(\alpha_1\mathcal{A}_{\alpha_1} + i\Big(\frac{1}{2} + \alpha_1\Big)\Big)\lambda\mathcal{A}_{\alpha_1}u^2 \\
&\quad - \Big(\frac{3i}{2}t - it\nabla\mathcal{J} - \frac{3}{4}t^2\Delta\Big)\Big(\alpha_2\mathcal{A}_{\alpha_2} + i\Big(\frac{1}{2} + 3\alpha_2\Big)\Big)\mu\mathcal{A}_{\alpha_2}\overline{u}^2 \\
&\quad - \frac{i\lambda}{4}t^2\Delta\mathcal{A}_{\alpha_1}u^2 - \frac{3i\mu}{4}t^2\Delta\mathcal{A}_{\alpha_2}\overline{u}^2,
\end{aligned}$$

and

$$\begin{aligned}
I_7 = &\big(3it^2 - 2it^2\nabla\mathcal{J} - \frac{1}{2}t^3\Delta\big)\lambda\mathcal{A}_{\alpha_1}u\mathcal{L}u - \big(3it^2 - 2it^2\nabla\mathcal{J} - \frac{3}{2}t^3\Delta\big)\mu\mathcal{A}_{\alpha_2}\overline{u}\overline{\mathcal{L}u} \\
&- \big(\frac{3i}{4} - \frac{i}{2}\nabla\mathcal{J} - \frac{1}{8}t\Delta\big)\lambda\mathcal{A}_{\alpha_1}\big((\mathcal{J}u)^2 - u\mathcal{J}^2u\big) \\
&+ \big(\frac{3i}{4} - \frac{i}{2}\nabla\mathcal{J} - \frac{3}{8}t\Delta\big)\mu\mathcal{A}_{\alpha_2}\big((\overline{\mathcal{J}u})^2 - \overline{u}\overline{\mathcal{J}^2u}\big).
\end{aligned}$$

We also need the next lemma in order to obtain the large-time asymptotics of the solution.

Lemma 2.4. ([6], Lemma 2.2) *Let* $\overline{M(t)}u \in H_r^{l,0} \cap H_q^{m,0}$ *and* $\frac{1}{p} = a\big(\frac{1}{q} - \frac{m}{3}\big) + (1-a)\big(\frac{1}{r} - \frac{l}{3}\big)$, $l, m = 0, 1, 2$, $a \in [0, 1]$, *and* $1 \le p, q, r < \infty$. *Then*

$$\|u\|_p \le Ct^{\frac{3}{p} - \frac{3a}{q} - \frac{3(1-a)}{r}}\Big(\sum_{|\alpha|=m}\|\mathcal{J}^\alpha u\|_q\Big)^a\Big(\sum_{|\beta|=l}\|\mathcal{J}^\beta u\|_r\Big)^{1-a}$$

for $t > 0$. *Furthermore,*

$$\big\|u(t) - M(t)\mathcal{D}(t)\hat{v}(t)\big\|_\infty = O\Big(t^{-\frac{3}{2}-\theta}\big(\|u(t)\| + \|\mathcal{J}^2u(t)\|\big)\Big)$$

as $t \to \infty$, *where* $\hat{v}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$ *and* $\theta \in (0, \frac{1}{4})$.

We state the estimates involving the operator $\mathcal{A}_\alpha$ in the following lemma.

Lemma 2.5. ([6], Lemma 2.3) *Let* $1 \le q \le p \le \infty$, $l = 0, 1$, *and* $\mathcal{A}_\alpha = (i\alpha + \frac{t}{4}\Delta)^{-1}$.

(i) *Suppose that* $u \in L^q$ *and that* $\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{l}{2}<1$. *Then*

$$\left\|\nabla^l \mathcal{A}_\alpha u\right\|_p \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{l}{2}}\|u\|_q$$

for $t>0$.

(ii) *Suppose* $p \in [2,6)$ *and* $q \in [2,\infty)$. *Then*

$$\left\|\nabla^l \mathcal{A}_\alpha uv\right\| \leq Ct^{-\frac{l}{2}}\Big(\|u\mathcal{J}v\|_p+\|v\mathcal{J}u\|_p+\langle t\rangle^{\frac{1}{2}}\|uv\|_p+\langle t\rangle^{\frac{3}{4}}\|uv\|_q\Big),$$

$$\left\|\nabla^l \mathcal{A}_\alpha u^3\right\| \leq Ct^{-\frac{l}{2}}\Big(\|u^2\mathcal{J}u\|_p+\langle t\rangle^{\frac{1}{2}}\|u^3\|_p+\langle t\rangle^{\frac{3}{4}}\|u^3\|_q\Big),$$

and

$$\left\|\nabla^l \mathcal{A}_\alpha u^2\overline{v}\right\| \leq Ct^{-\frac{l}{2}}\Big(\|u^2\overline{\mathcal{J}v}\|_p+\|u\overline{v}\mathcal{J}u\|_p+\langle t\rangle^{\frac{1}{2}}\|u^2\overline{v}\|_p+\langle t\rangle^{\frac{3}{4}}\|u^2\overline{v}\|_q\Big)$$

for $t>0$ *provided that the right-hand sides are finite.*

3. EXISTENCE OF GLOBAL SOLUTIONS

We apply the method used in [6] to the quadratic nonlinearities.

Proposition 3.1. *Assume that* $u_0 \in H^{3,0}\cap H^{1,2}$ *and* $\|u_0\|_{3,0}+\|u_0\|_{1,2}=\varepsilon$. *Then there exists an* $\varepsilon>0$ *such that* (1.9) *has a unique solution* u *satisfying* $u \in C\big([0,T];H^{3,0}\cap H^{1,2}\big)$ *with* $T>1$ *and*

$$\sup_{t\in[0,T]}\|u(t)\|_Y < \varepsilon^{\frac{1}{2}}.$$

Proof. We consider the linearized version of the Cauchy problem (1.9)

$$\begin{cases} \mathcal{L}u=\lambda v^2+\mu\bar{v}^2+\mathcal{G}(v,\bar{v}), & (t,x)\in \mathbf{R}\times\mathbf{R}^3,\\ u(0,x)=u_0, & x\in\mathbf{R}^3,\end{cases} \tag{3.1}$$

where $v \in Z_{2\varepsilon}=\{w(t)\in C([0,T];L^2) : |||w|||_Z=\sup_{t\in[0,T]}\|w(t)\|_Z \leq 2\varepsilon\}$.
Applying the energy method to (3.1), we find

$$\begin{aligned}\frac{d}{dt}\big(\|u\|_{3,0}+\|\mathcal{J}u\|_{2,0}+\|\mathcal{J}^2u\|_{1,0}\big) &\leq C\big(\|v^2\|_{3,0}+\|\mathcal{G}\|_{3,0}+\|\mathcal{J}(v^2)\|_{2,0}\\ &+\|\mathcal{J}(\bar{v}^2)\|_{2,0}+\|\mathcal{J}\mathcal{G}\|_{2,0}+\|\mathcal{J}^2(v^2)\|_{1,0}+\|\mathcal{J}^2(\bar{v}^2)\|_{1,0}+\|\mathcal{J}^2\mathcal{G}\|_{1,0}\big).\end{aligned} \tag{3.2}$$

Sobolev's inequality gives us

$$\|v^2\|_{3,0}\leq C\varepsilon^2,\ \ \|\mathcal{G}\|_{3,0}\leq C\varepsilon^3.$$

Using the identities

$$\mathcal{J}(uv)=\frac{1}{2}\big(u\mathcal{J}v+v\mathcal{J}u\big)+\frac{1}{2}it\nabla(uv)$$

and

$$\mathcal{J}(\overline{uv}) = \frac{1}{2}\big(\overline{u\mathcal{J}v} + \overline{v\mathcal{J}u}\big) + \frac{3}{2}it\nabla(\overline{uv}),$$

we obtain

$$\begin{aligned}&\|\mathcal{J}(v^2)\|_{2,0} + \|\mathcal{J}^2(\bar{v}^2)\|_{1,0}\\ &\quad \le C\big(\|v\|_{2,0}\|\mathcal{J}v\|_\infty + \|v\|_{1,0,\infty}\|\mathcal{J}v\|_{2,0} + t\|v\|_{3,0}\|v\|_{1,0,\infty}\big) \le C\varepsilon^2\langle t\rangle.\end{aligned}$$

By the formula $xv = \mathcal{J}v - it\nabla v$, we have

$$\|xv\| + \|x\Delta v\| \le C\varepsilon\langle t\rangle.$$

Hence,

$$\begin{aligned}\|\mathcal{J}\mathcal{G}\|_{2,0} &\le C\big(\|xv\|\|v\|^2_{1,0,\infty} + \|x\Delta v\|\|v\|^2_\infty + \|v\|_{1,0}\|v\|^2_\infty + t\|v\|_{3,0}\|v\|^2_{1,0,\infty}\big)\\ &\le C\varepsilon^3\langle t\rangle.\end{aligned}$$

Similarly,

$$\|\mathcal{J}^2(v^2)\|_{1,0} + \|\mathcal{J}^2(\bar{v}^2)\|_{1,0} + \|\mathcal{J}^2\mathcal{G}\|_{1,0} \le C\varepsilon^2\langle t\rangle^2.$$

Here we have used the estimate

$$\|x^2v\| \le C\big(\|\mathcal{J}^2v\| + t\|\mathcal{J}v\|_{1,0} + t\|v\| + t^2\|v\|_{2,0}\big) \le C\varepsilon\langle t\rangle^2.$$

Therefore, by (3.2),

$$\frac{d}{dt}\big(\|u\|_{3,0} + \|\mathcal{J}u\|_{2,0} + \|\mathcal{J}^2u\|_{1,0}\big) \le C\varepsilon^2\langle t\rangle^2.$$

Thus, we obtain

$$\|u\|_{3,0} + \|\mathcal{J}u\|_{2,0} + \|\mathcal{J}^2u\|_{1,0} \le \varepsilon + C\varepsilon^2\langle T\rangle^3 \le 2\varepsilon \tag{3.3}$$

if $C\varepsilon\langle T\rangle^3 \le 1$, which implies $|||u|||_Z \le 2\varepsilon$. Hence it follows that the mapping $\mathcal{M}$ defined by $u = \mathcal{M}v$ transforms a set $Z_{2\varepsilon}$ into itself. In the same way, we have

$$|||\mathcal{M}v_1 - \mathcal{M}v_2|||_Z \le C\varepsilon\langle T\rangle^3|||v_1 - v_2|||_Z \le \frac{1}{2}|||v_1 - v_2|||_Z$$

if we take ε satisfying $C\varepsilon\langle T\rangle^3 \le \frac{1}{2}$. Therefore, we have a unique fixed point $u = \mathcal{M}u$ such that $u \in C\big([0,T]; H^{3,0} \cap H^{1,2}\big)$ with $T > 1$. We next show that

$$\sup_{t\in[0,T]}\|u(t)\|_Y < \varepsilon^{\frac{1}{2}}.$$

It is sufficient to prove that

$$\|u\|_Y \le C\langle T\rangle^{\frac{3}{2}}\|u\|_Z + C\langle T\rangle^{\frac{5}{2}}\|u\|^2_Z \tag{3.4}$$

for $t \in [0,T]$. By a simple calculation, we have $\|u\|_{\tilde{Y}} \le C\langle T\rangle^{\frac{3}{2}}\|u\|_Z$ for $t \in [0,T]$. Thus, we find

$$\|u\|_Y = \|u\|_{\tilde{Y}} + \langle t\rangle^{-\theta}\|\mathcal{J}^2 w\|_{1,0} \le C\langle T\rangle^{\frac{3}{2}}\|u\|_Z + \|\mathcal{J}^2 w\|_{1,0}.$$

By $w = u - t\lambda\mathcal{A}_{\alpha_1}u^2 - \frac{t}{3}\mu\mathcal{A}_{\alpha_2}\bar{u}^2$, we get

$$\mathcal{J}^2 w = \mathcal{J}^2 u - t\lambda\mathcal{J}^2\mathcal{A}_{\alpha_1}u^2 - \frac{t}{3}\mu\mathcal{J}^2\mathcal{A}_{\alpha_2}\bar{u}^2. \tag{3.5}$$

Using the relations $[\mathcal{J},\mathcal{A}_\alpha] = \frac{t}{2}\nabla\mathcal{A}_\alpha^2$ and $[\mathcal{J},\nabla\mathcal{A}_\alpha] = \frac{t}{2}\Delta\mathcal{A}_\alpha^2$, we obtain

$$\mathcal{J}^2\mathcal{A}_\alpha u^2 = \mathcal{A}_\alpha\mathcal{J}^2 u^2 + t\nabla\mathcal{A}_\alpha^2\mathcal{J}u^2 + \frac{t^2}{2}\Delta\mathcal{A}_\alpha^3 u^2. \tag{3.6}$$

The identity $\mathcal{J}(uv) = \frac{1}{2}\big(u\mathcal{J}v + v\mathcal{J}u\big) + \frac{1}{2}it\nabla(uv)$ gives us

$$\begin{aligned}
t^2\|\Delta\mathcal{A}_\alpha^3 u^2\| &\le Ct\|u\|_\infty\|u\| \le C\langle t\rangle\|u\|_Z^2,\\
t\|\nabla\mathcal{A}_\alpha^2\mathcal{J}u^2\| &\le Ct^{\frac{1}{2}}\|\mathcal{J}u^2\| \le Ct^{\frac{1}{2}}\big(\|u\|_\infty\|\mathcal{J}u\| + t\|u\|_{1,0}\|u\|_\infty\big)\\
&\le C\langle t\rangle^{\frac{3}{2}}\|u\|_Z^2,\\
\|\mathcal{A}_\alpha\mathcal{J}^2u^2\| &\le C\big(\|u\|_\infty\|\mathcal{J}^2u\| + \|\mathcal{J}u\|_\infty\|\mathcal{J}u\| + t\|u\|_{1,0,\infty}\|\mathcal{J}u\|_{1,0}\\
&\qquad + t\|u\|_\infty\|u\| + t^{\frac{3}{2}}\|u\|_{1,0,\infty}\|u\|\big)\\
&\le C\langle t\rangle^{\frac{3}{2}}\|u\|_Z^2.
\end{aligned}$$

Thus, we get

$$\|\mathcal{J}^2\mathcal{A}_\alpha u^2\| \le C\langle t\rangle^{\frac{3}{2}}\|u\|_Z^2.$$

Therefore,

$$\sup_{t\in[0,T]}\|\mathcal{J}^2 w(t)\| \le \sup_{t\in[0,T]}\|u(t)\|_Z + C\langle T\rangle^{\frac{5}{2}}\sup_{t\in[0,T]}\|u(t)\|_Z^2.$$

In the same way, we obtain

$$\sup_{t\in[0,T]}\|\mathcal{J}^2 w(t)\|_{1,0} \le \sup_{t\in[0,T]}\|u(t)\|_Z + C\langle T\rangle^{\frac{5}{2}}\sup_{t\in[0,T]}\|u(t)\|_Z^2. \tag{3.7}$$

We have by (3.7)

$$\begin{aligned}
\sup_{t\in[0,T]}\|u(t)\|_Y &\le C\langle T\rangle^{\frac{3}{2}}\sup_{t\in[0,T]}\|u(t)\|_Z + \sup_{t\in[0,T]}\|\mathcal{J}^2 w(t)\|_{1,0}\\
&\le C\langle T\rangle^{\frac{3}{2}}\sup_{t\in[0,T]}\|u(t)\|_Z + C\langle T\rangle^{\frac{5}{2}}\sup_{t\in[0,T]}\|u(t)\|_Z^2\\
&\le C\varepsilon\langle T\rangle^{\frac{3}{2}} + C\varepsilon^2\langle T\rangle^{\frac{5}{2}} \le C\varepsilon\langle T\rangle^{\frac{5}{2}} < \varepsilon^{\frac{1}{2}}
\end{aligned} \tag{3.8}$$

if $C\varepsilon^{\frac{1}{2}}\langle T\rangle^{\frac{5}{2}} < 1$. Proposition 3.1 is proved. □

The global existence of small solutions for the Cauchy problem (1.9) is obtained in the following proposition.

Proposition 3.2. *Assume that* $u_0 \in H^{3,0} \cap H^{1,2}$ *and* $\|u_0\|_{3,0} + \|u_0\|_{1,2} = \varepsilon$. *Then there exists an* $\varepsilon > 0$ *such that* (1.9) *has a unique global solution* u *satisfying* $u \in C\big([0,\infty]; H^{3,0} \cap H^{1,2}\big)$ *and*

$$\sup_{t\geq 0} \|u(t)\|_Y < \varepsilon^{\frac{1}{2}}.$$

We need the next lemma to show Proposition 3.2. The estimates in the following lemma were shown in the proof of Proposition 3.2 of [6].

Lemma 3.1. *Let* I_j $(j = 1, \ldots, 7)$ *be the same ones defined in Lemma* 2.3. *We assume that* $u_0 \in H^{3,0} \cap H^{1,2}$ *and* $\|u_0\|_{3,0} + \|u_0\|_{1,2} = \varepsilon$, *and that there exists a time* T *such that* $\sup_{t\in[0,T]} \|u(t)\|_Y \leq \varepsilon^{\frac{1}{2}}$. *Then, we have the estimates*

$$\begin{aligned}
&\|I_1\|_{2,0} \leq C\varepsilon\langle t\rangle^{-\frac{5}{4}},\ \|I_2\|_{2,0} \leq C\varepsilon\langle t\rangle^{-\frac{3}{2}},\ \|I_3\|_{2,0} \leq C\varepsilon\langle t\rangle^{\theta-\frac{3}{2}},\\
&\|I_4\|_{2,0} \leq C\varepsilon\langle t\rangle^{-\frac{5}{4}},\ \|I_5\|_{1,0} \leq C\varepsilon\langle t\rangle^{-1},\ \|I_6\|_{1,0} \leq C\varepsilon\langle t\rangle^{\theta-1},\\
&\|I_7\|_{1,0} \leq C\varepsilon\langle t\rangle^{\theta-1},\ \|\mathcal{J}^2 w - \Phi\|_{1,0} \leq C\varepsilon\langle t\rangle^{\theta},
\end{aligned}$$

for $t \in [0,T]$.

Proof of Proposition 3.2. We are now in a position to prove Proposition 3.2. By Proposition 3.1, we can find a $T > 1$ such that

$$\sup_{t\in[0,T]} \|u(t)\|_Y < \varepsilon^{\frac{1}{2}}.$$

We assume that there exists a time T such that

$$\sup_{t\in[0,T]} \|u(t)\|_Y \leq \varepsilon^{\frac{1}{2}}.$$

By using a contradiction argument we prove that we can let $T = \infty$. In order to prove that we can let $T = \infty$, we will derive *à priori* estimates which do not depend on T. Due to the energy method we have

$$\begin{aligned}
\frac{d}{dt}\|u\|_{3,0} &\leq C\big(\|u^2\|_{3,0} + \|\mathcal{G}\|_{3,0}\big)\\
&\leq C\big(\|u\|_{1,0,\infty}\|u\|_{3,0} + \|u\|_{3,0}\|u\|_{1,0,\infty}^2\big) \leq C\varepsilon\langle t\rangle^{-\frac{3}{2}}.
\end{aligned}$$

Thus, we get

$$\sup_{t\in[0,T]} \|u(t)\|_{3,0} \le C\varepsilon. \tag{3.9}$$

Now we estimate $\|\mathcal{J}u\|_{2,0}$. By Lemma 2.3, we have

$$\mathcal{L}\Psi = \sum_{j=1}^{4} I_j + \mathcal{J}\mathcal{G}. \tag{3.10}$$

Applying the energy method to (3.10), we find

$$\frac{d}{dt}\|\Psi\|_{2,0} \le C\Big(\sum_{j=1}^{4}\|I_j\|_{2,0} + \|\mathcal{J}\mathcal{G}\|_{2,0}\Big). \tag{3.11}$$

By the formula $xu = \mathcal{J}u - it\nabla u$, we have

$$\|xu\| + \|x\Delta u\| \le C\varepsilon^{\frac{1}{2}}\langle t\rangle.$$

Hence,

$$\begin{aligned}\|\mathcal{J}\mathcal{G}\|_{2,0} &\le C\big(\|xu\|\|u\|_{1,0,\infty}^2 + \|x\Delta u\|\|u\|_{\infty}^2 + \|u\|_{1,0}\|u\|_{\infty}^2 + t\|u\|_{3,0}\|u\|_{1,0,\infty}^2\big)\\ &\le C\varepsilon^{\frac{3}{2}}\langle t\rangle^{-2}.\end{aligned}$$

By the inequality and Lemma 3.1, we have

$$\frac{d}{dt}\|\Psi\|_{2,0} \le C\big(\varepsilon\langle t\rangle^{-\frac{5}{4}} + \varepsilon^{\frac{3}{2}}\langle t\rangle^{-2}\big) \le C\varepsilon\langle t\rangle^{-\frac{5}{4}}.$$

Thus we obtain

$$\|\Psi(t)\|_{2,0} \le C\varepsilon. \tag{3.12}$$

By (3.12) and Lemma 2.5, we have

$$\begin{aligned}\|\mathcal{J}u\|_{2,0} &= \Big\|\Psi + \frac{i}{2}t^2\nabla\big(\lambda\mathcal{A}_{\alpha_1}u^2 + \mu\mathcal{A}_{\alpha_2}\overline{u}^2\big)\Big\|_{2,0}\\ &\le \|\Psi\|_{2,0} + Ct^2\|\nabla\mathcal{A}_{\alpha_1}u^2\|_{2,0} + Ct^2\|\nabla\mathcal{A}_{\alpha_2}\overline{u}^2\|_{2,0}\\ &\le C\varepsilon + Ct^{\frac{3}{2}}\|u\|_{1,0,\infty}\|u\| \le C\varepsilon \end{aligned}\tag{3.13}$$

for $t \in [0,T]$.

Next we estimate $\|\mathcal{J}^2u\|_{1,0}$ and $\|\mathcal{J}^2w\|_{1,0}$. By Lemma 2.3, we get

$$\mathcal{L}\Phi = I_5 + I_6 + I_7 + \mathcal{J}^2\mathcal{G}. \tag{3.14}$$

Applying the energy method to (3.14), we find

$$\frac{d}{dt}\|\Phi\|_{1,0} \le C\big(\|I_5\|_{1,0} + \|I_6\|_{1,0} + \|I_7\|_{1,0} + \|\mathcal{J}^2\mathcal{G}\|_{1,0}\big), \tag{3.15}$$

$$\begin{aligned}\|\mathcal{J}^2\mathcal{G}\|_{1,0} &\le C\big(\|xu\|\|u\|_\infty^2 + \|x^2u\|\|u\|_{1,0,\infty}^2 + t\|u\|_{1,0}\|u\|_\infty^2 \\ &\quad + t\|(x\cdot\nabla)u\|\|u\|_{1,0,\infty}^2 + t\|x\Delta u\|\|u\|_\infty^2 + t^2\|u\|_{2,0}\|u\|_{1,0,\infty}^2\big) \\ &\le C\varepsilon^{\frac{3}{2}}\langle t\rangle^{-1}.\end{aligned}$$

Here, we have used the estimates

$$\begin{aligned}&\|xu\| + \|(x\cdot\nabla)u\| + \|x\Delta u\| \le C\varepsilon^{\frac{1}{2}}\langle t\rangle, \\ &\|x^2u\| \le C\big(\|\mathcal{J}^2u\| + t\|\mathcal{J}u\|_{1,0} + t\|u\| + t^2\|u\|_{2,0}\big) \le C\varepsilon^{\frac{1}{2}}\langle t\rangle^2.\end{aligned}$$

By Lemma 3.1, we have

$$\frac{d}{dt}\|\Phi\|_{1,0} \le C\varepsilon\langle t\rangle^{\theta-1} + C\varepsilon^{\frac{3}{2}}\langle t\rangle^{-1} \le C\varepsilon\langle t\rangle^{\theta-1}.$$

Thus we obtain

$$\|\Phi\|_{1,0} \le C\varepsilon\langle t\rangle^{\theta}. \tag{3.16}$$

Therefore,

$$\|\mathcal{J}^2u\|_{1,0} \le \|\Phi\|_{1,0} + Ct^2\|u^2\| + Ct^{\frac{3}{2}}\|u\mathcal{J}u\| \le C\varepsilon\langle t\rangle^{\frac{1}{2}} \tag{3.17}$$

for $t \in [0,T]$. By Lemma 3.1, we find

$$\|\mathcal{J}^2w\|_{1,0} \le \|\Phi\|_{1,0} + C\varepsilon\langle t\rangle^{\theta} \le C\varepsilon\langle t\rangle^{\theta} \tag{3.18}$$

for $t \in [0,T]$. We next estimate $\|u\|_{1,0,\infty}$. Let us prove the estimate $\|\mathcal{K}w\|_{0,1,\infty} \le C\varepsilon$. If $t \in [0,1]$, we obtain by the identity $\mathcal{K}u^2 = \mathcal{F}\mathcal{U}(-t)u^2 = e^{\frac{it|\xi|^2}{2}}\mathcal{F}u^2$

$$\|\mathcal{K}u^2\|_\infty \le C\|u\|^2 \le C\varepsilon. \tag{3.19}$$

We have

$$\mathcal{K}f = \mathcal{F}\mathcal{U}(-t)f = -i\mathcal{F}\bar{M}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}f.$$

By Lemma 2.1, $\mathcal{K}f$ is written as

$$\begin{aligned}\mathcal{K}f &= i^{\frac{1}{2}}\mathcal{D}(2)E^2\mathcal{F}\bar{M}^{\frac{1}{2}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^2f \\ &= i^{\frac{1}{2}}\mathcal{D}(2)E^2\mathcal{D}(\frac{1}{t})\bar{M}^2f + i^{\frac{1}{2}}\mathcal{D}(2)E^2\mathcal{F}(\bar{M}^{\frac{1}{2}}-1)\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^2f.\end{aligned}$$

We put $f = u^2$; then we find

$$\left\| i^{\frac{1}{2}} \mathcal{D}(2) E^2 \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|_\infty \leq C t^{\frac{3}{2}} \|u\|_{1,0,\infty}^2 \leq C\varepsilon t^{-\frac{3}{2}} \tag{3.20}$$

for $t \in (1, T]$. Using the relation $|\bar{M}^{\frac{1}{2}} - 1| \leq C t^{-\alpha} |x|^{2\alpha}$ $(0 < \alpha \leq 1)$, we get for $t \in (1, T]$

$$\begin{aligned} &\left\| i^{\frac{1}{2}} \mathcal{D}(2) E^2 \mathcal{F}(\bar{M}^{\frac{1}{2}} - 1) \mathcal{F}^{-1} \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|_\infty \leq C \left\| (\bar{M}^{\frac{1}{2}} - 1) \mathcal{F}^{-1} \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|_1 \\ &\leq C t^{-\alpha} \left\| \langle x \rangle^{2\alpha - 2} \langle x \rangle^2 \mathcal{F}^{-1} \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|_1 \\ &\leq C t^{-\alpha} \left\| \langle x \rangle^{2\alpha - 2} \right\| \left\| \langle x \rangle^2 \mathcal{F}^{-1} \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|. \end{aligned} \tag{3.21}$$

Here, we have chosen $0 < \alpha < \frac{1}{4}$, which yields boundedness of $\left\| \langle x \rangle^{2\alpha - 2} \right\|$. Thus we have

$$\left\| i^{\frac{1}{2}} \mathcal{D}(2) E^2 \mathcal{F}(\bar{M}^{\frac{1}{2}} - 1) \mathcal{F}^{-1} \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|_\infty \leq C t^{-\alpha} \left\| \langle x \rangle^2 \mathcal{F}^{-1} \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|.$$

We take $\alpha = \frac{1}{4} - \theta,\ \theta > 0$; then we find

$$\begin{aligned} &\left\| i^{\frac{1}{2}} \mathcal{D}(2) E^2 \mathcal{F}(\bar{M}^{\frac{1}{2}} - 1) \mathcal{F}^{-1} \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|_\infty \leq C t^{\theta - \frac{1}{4}} \left\| (1 - \Delta) \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\| \\ &\leq C t^{\theta - \frac{1}{4}} \left\| (1 - t^2 \Delta) \bar{M}^2 f \right\| \leq C t^{\theta - \frac{1}{4}} \left(\|f\| + \|(-t^2 \Delta) \bar{M}^2 f\| \right). \end{aligned} \tag{3.22}$$

We estimate $\|(-t^2\Delta)\bar{M}^2 f\|$. We put $f = u^2$; then by Sobolev's inequality we have

$$\begin{aligned} \|(-t^2 \Delta)(\bar{M}u)^2\| &\leq C \|t^2 \bar{M} u \Delta \bar{M} u\| + C \|t^2 (\nabla \bar{M} u)^2\| \\ &\leq C t^2 \|u\|_\infty \|\Delta \bar{M} u\| \leq C \|u\|_\infty \|\mathcal{J}^2 u\| \leq C t^{-\frac{3}{2} + \frac{1}{2}} \varepsilon \leq C \varepsilon t^{-1}. \end{aligned}$$

Thus we get

$$\left\| i^{\frac{1}{2}} \mathcal{D}(2) E^2 \mathcal{F}(\bar{M}^{\frac{1}{2}} - 1) \mathcal{F}^{-1} \mathcal{D}(\frac{1}{t}) \bar{M}^2 f \right\|_\infty \leq C t^{\theta - \frac{1}{4}} \left(\varepsilon t^{-\frac{3}{2}} + \varepsilon t^{-1} \right) \leq C \varepsilon t^{\theta - \frac{5}{4}}, \tag{3.23}$$

which implies

$$\|\mathcal{K} u^2\|_\infty \leq C \varepsilon t^{\theta - \frac{5}{4}} \quad \text{for } t \in (1, T]. \tag{3.24}$$

By (3.19) and (3.24), we have

$$\|\mathcal{K} u^2\|_\infty \leq C \varepsilon \langle t \rangle^{\theta - \frac{5}{4}} \quad \text{for } t \in [0, T]. \tag{3.25}$$

Using Lemma 2.1, $\mathcal{K}f$ is written as

$$\begin{aligned}\mathcal{K}f &= i^{\frac{1}{2}}\mathcal{D}(-2)E^6\mathcal{F}M^{\frac{1}{2}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})M^2 f\\ &= i^{\frac{1}{2}}\mathcal{D}(-2)E^6\mathcal{D}(\frac{1}{t})M^2 f + i^{\frac{1}{2}}\mathcal{D}(-2)E^6\mathcal{F}(M^{\frac{1}{2}}-1)\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})M^2 f.\end{aligned}$$

In the same way as in the proof of (3.25), putting $f=\bar{u}^2$, we find

$$\|\mathcal{K}\bar{u}^2\|_\infty \le C\varepsilon\langle t\rangle^{\theta-\frac{5}{4}} \quad \text{for } t\in[0,T]. \tag{3.26}$$

Similarly, we obtain for $t\in[0,T]$

$$\|\mathcal{K}\partial^\beta u^2\|_\infty + \|\mathcal{K}\partial^\beta\bar{u}^2\|_\infty \le C\varepsilon\langle t\rangle^{\theta-\frac{5}{4}} \quad \text{for } \ |\beta|=0,1. \tag{3.27}$$

On the other hand, by the identity $\mathcal{K}\partial^\beta\mathcal{G}=\mathcal{F}\mathcal{U}(-t)\partial^\beta\mathcal{G}=e^{\frac{it|\xi|^2}{2}}\mathcal{F}\partial^\beta\mathcal{G}$, we have for $t\in[0,T]$

$$\|\mathcal{K}\partial^\beta\mathcal{G}\|_\infty \le C\|\partial^\beta\mathcal{G}\|_1 \le C\|u\|_{1,0,\infty}\|u\|^2 \le C\varepsilon^{\frac{3}{2}}\langle t\rangle^{-\frac{3}{2}} \ \text{ for } |\beta|=0,1. \tag{3.28}$$

Applying the operator $\mathcal{K}\partial^\beta$ to equation (1.9), we get

$$i\partial_t\mathcal{K}\partial^\beta u = \lambda\mathcal{K}\partial^\beta u^2 + \mu\mathcal{K}\partial^\beta\bar{u}^2 + \mathcal{K}\partial^\beta\mathcal{G}.$$

Hence, integration with respect to t yields

$$i\mathcal{K}\partial^\beta u(t) = i\mathcal{F}\partial^\beta u_0 + \int_0^t \big(\lambda\mathcal{K}\partial^\beta u^2 + \mu\mathcal{K}\partial^\beta\bar{u}^2\big)(\tau)d\tau + \int_0^t \mathcal{K}\partial^\beta\mathcal{G}(\tau)d\tau.$$

By (3.27) and (3.28), we obtain for $t\in[0,T]$

$$\|\mathcal{K}\partial^\beta u\|_\infty \le C\varepsilon + C\varepsilon\int_0^t\langle\tau\rangle^{\theta-\frac{5}{4}}d\tau + C\varepsilon^{\frac{3}{2}}\langle t\rangle^{-\frac{1}{2}} \le C\varepsilon. \tag{3.29}$$

Let $\widehat{\mathcal{A}_\alpha}=\big(i\alpha-\frac{t|\xi|^2}{4}\big)^{-1}$. Noting that $\mathcal{K}\mathcal{A}_\alpha=\widehat{\mathcal{A}_\alpha}\mathcal{K}$, we have $\mathcal{K}w=\mathcal{K}u-\lambda t\widehat{\mathcal{A}_{\alpha_1}}\mathcal{K}u^2-\frac{t}{3}\mu\widehat{\mathcal{A}_{\alpha_2}}\mathcal{K}\overline{u}^2$, and by (3.27) it follows that

$$\begin{aligned}\|\mathcal{K}\partial^\beta w\|_\infty &\le \|\mathcal{K}\partial^\beta u\|_\infty + C\Big(t\|\widehat{\mathcal{A}_{\alpha_1}}\mathcal{K}\partial^\beta u^2\|_\infty + t\|\widehat{\mathcal{A}_{\alpha_2}}\mathcal{K}\partial^\beta\overline{u}^2\|_\infty\Big)\\ &\le C\varepsilon + C\varepsilon\langle t\rangle^{\theta-\frac{1}{4}} \le C\varepsilon \ \text{ for } t\in[0,T].\end{aligned} \tag{3.30}$$

By virtue of Lemma 2.4 and (3.30), we find

$$\begin{aligned}\|\partial^\beta w\|_\infty &\le \|M\mathcal{D}\mathcal{K}\partial^\beta w\|_\infty + C\big(t^{-\frac{3}{2}-\theta}(\|\partial^\beta w\| + \|\mathcal{J}^2\partial^\beta w\|)\big)\\ &\le Ct^{-\frac{3}{2}}\|\mathcal{K}\partial^\beta w\|_\infty + C\varepsilon t^{-\frac{3}{2}} \le C\varepsilon t^{-\frac{3}{2}} \ \text{ for } t\in(1,T].\end{aligned} \tag{3.31}$$

If $t \in [0,1]$, we obtain by the identity $\partial^\beta w = \partial^\beta u - t\lambda\partial^\beta \mathcal{A}_{\alpha_1} u^2 - \frac{t}{3}\mu\partial^\beta\mathcal{A}_{\alpha_2}\bar{u}^2$

$$\begin{aligned}\|\partial^\beta w\|_\infty &\le \|\partial^\beta u\|_\infty + C\|\partial^\beta \mathcal{A}_{\alpha_1} u^2\|_\infty + C\|\partial^\beta \mathcal{A}_{\alpha_2}\bar{u}^2\|_\infty \\ &\le C\|u\|_{3,0} + C\|\partial^\beta \mathcal{A}_{\alpha_1} u^2\|_\infty + C\|\partial^\beta \mathcal{A}_{\alpha_2}\bar{u}^2\|_\infty \le C\varepsilon. \qquad (3.32)\end{aligned}$$

By (3.31) and (3.32), we find

$$\|\partial^\beta w\|_\infty \le C\varepsilon\langle t\rangle^{-\frac{3}{2}} \quad \text{for } t \in [0,T].$$

Therefore, by virtue of Lemma 2.5, we obtain

$$\|u\|_{1,0,\infty} \le \|w\|_{1,0,\infty} + Ct\|\mathcal{A}_{\alpha_1} u^2\|_{1,0,\infty} + Ct\|\mathcal{A}_{\alpha_2}\overline{u}^2\|_{1,0,\infty} \le C\varepsilon\langle t\rangle^{-\frac{3}{2}} \qquad (3.33)$$

for $t \in [0,T]$. Combining the estimates (3.9), (3.13), (3.17), (3.18), and (3.33), we get

$$\|u(t)\|_Y \le C\varepsilon < \varepsilon^{\frac{1}{2}}$$

for $t \in [0,T]$. Then we have a contradiction. Proposition 3.2 is proved.

4. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need

Proposition 4.1. *Let $u_0 \in H^{3,0} \cap H^{1,2}$ and $\varepsilon > 0$, and let $\varepsilon = \|u_0\|_{3,0} + \|u_0\|_{1,2}$ be sufficiently small. Let u be the solution of* (1.9) *constructed in Proposition* 3.2. *Then there exists a unique final state $\hat{u}_+ = \lim_{t\to\infty} \mathcal{F}\mathcal{U}(-t)u(t) \in L^2 \cap L^\infty$ such that*

$$\left\|\mathcal{F}\mathcal{U}(-t)u(t) - \hat{u}_+\right\|_\infty \le C\varepsilon t^{\theta-\frac{1}{4}}, \ \left\|\mathcal{F}\mathcal{U}(-t)u(t) - \hat{u}_+\right\| \le C\varepsilon t^{-\frac{1}{2}}$$

for $t > 1$ and $\theta > 0$.

Proof of Proposition 4.1. By (3.27), (3.28), and the identity $i\partial_t \mathcal{K}u = \lambda\mathcal{K}u^2 + \mu\mathcal{K}\bar{u}^2 + \mathcal{K}\mathcal{G}$, we have

$$\begin{aligned}\left\|\mathcal{K}u(t) - \mathcal{K}u(s)\right\|_\infty &\le \int_s^t \left\|\lambda\mathcal{K}u^2 + \mu\mathcal{K}\overline{u}^2\right\|_\infty d\tau + \int_s^t \|\mathcal{K}\mathcal{G}\|_\infty d\tau \\ &\le C\varepsilon\langle s\rangle^{\theta-\frac{1}{4}} + C\varepsilon^{\frac{3}{2}}\langle s\rangle^{-\frac{1}{2}} \le C\varepsilon\langle s\rangle^{\theta-\frac{1}{4}} \qquad (4.1)\end{aligned}$$

for $t > s > 1$. Thus $\mathcal{K}u(t)$ is a Cauchy sequence in L^∞, so there exists a unique final state $\hat{u}_+ = \lim_{t\to\infty} \mathcal{K}u(t) \in L^\infty$ such that

$$\left\|\mathcal{K}u(t) - \hat{u}_+\right\|_\infty \le C\varepsilon\langle t\rangle^{\theta-\frac{1}{4}}. \qquad (4.2)$$

On the other hand, using the identity $i\partial_t \mathcal{K}u = \lambda\mathcal{K}u^2 + \mu\mathcal{K}\bar{u}^2 + \mathcal{K}\mathcal{G}$, we obtain

$$\begin{aligned}\big\|\mathcal{K}u(t) - \mathcal{K}u(s)\big\| &\le C\int_s^t \big\|\lambda\mathcal{K}u^2 + \mu\mathcal{K}\bar{u}^2\big\| d\tau + C\int_s^t \|\mathcal{K}\mathcal{G}\| d\tau \\ &\le C\int_s^t \|u\|_\infty \|u\| d\tau + C\int_s^t \|u\|_\infty^2 \|u\| d\tau \\ &\le C\varepsilon\int_s^t \tau^{-\frac{3}{2}} d\tau + C\varepsilon^{\frac{3}{2}}\int_s^t \tau^{-3} d\tau \le C\varepsilon s^{-\frac{1}{2}} \qquad (4.3)\end{aligned}$$

for $t > s > 1$. Thus $\mathcal{K}u(t)$ is a Cauchy sequence in L^2, so there exists a unique final state $\hat{u}_+ = \lim_{t\to\infty} \mathcal{K}u(t) \in L^2$ such that

$$\big\|\mathcal{K}u(t) - \hat{u}_+\big\| \le C\varepsilon t^{-\frac{1}{2}} \quad \text{for } t > 1. \qquad (4.4)$$

This completes the proof of the Proposition 4.1. □

The following lemma gives the estimates to u and w.

Lemma 4.1. *Let $u_0 \in H^{3,0} \cap H^{1,2}$ and $\varepsilon > 0$, and let $\varepsilon = \|u_0\|_{3,0} + \|u_0\|_{1,2}$ be sufficiently small. Let $u \in C(\mathbf{R}; H^{3,0} \cap H^{1,2})$ be a global solution of* (1.9), *and let $w = u - t\lambda\mathcal{A}_{\alpha_1}u^2 - \frac{t}{3}\mu\mathcal{A}_{\alpha_2}\bar{u}^2$. Then*

$$\|\mathcal{F}M\mathcal{U}(-t)w\|_\infty \le C\varepsilon^{\frac{1}{2}}, \ \big\|(\bar{M}^{\frac{1}{2}} - 1)\mathcal{F}^{-1}t^{-\frac{3}{2}}(\mathcal{F}M\mathcal{U}(-t)w)^2\big\| \le C\varepsilon t^{\theta-\frac{5}{2}},$$

$$\|u - w\| \le C\varepsilon t^{\theta-1}, \ \|\mathcal{F}M\mathcal{U}(-t)u\|_\infty \le C\varepsilon^{\frac{1}{2}}$$

for $t > 1$ and $\theta > 0$.

Proof. In the same way as in the proof of (3.21), we have

$$\|\mathcal{F}(M-1)\mathcal{U}(-t)w\|_\infty \le C\varepsilon^{\frac{1}{2}}t^{\theta-\frac{7}{4}}. \qquad (4.5)$$

By (3.30) and (4.5), we get

$$\begin{aligned}\|\mathcal{F}M\mathcal{U}(-t)w\|_\infty &\le \|\mathcal{F}(M-1)\mathcal{U}(-t)w\|_\infty + \|\mathcal{F}\mathcal{U}(-t)w\|_\infty \\ &\le C\varepsilon^{\frac{1}{2}}t^{\theta-\frac{7}{4}} + C\varepsilon \le C\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Thus we obtain the first estimate of the lemma. In the same way as in the proof of (3.22), we obtain the second estimate as follows:

$$\begin{aligned}\big\|(\bar{M}^{\frac{1}{2}} - 1)\mathcal{F}^{-1}t^{-\frac{3}{2}}(\mathcal{F}M\mathcal{U}(-t)w)^2\big\| &\le Ct^{-1}\big(\|w\|_\infty\|w\| + \|w\|_\infty\|\mathcal{J}^2 w\|\big) \\ &\le C\varepsilon t^{\theta-\frac{5}{2}}. \qquad (4.6)\end{aligned}$$

Here we have used

$$\|w\|_\infty \le \|u\|_\infty + Ct\big(\|\mathcal{A}_{\alpha_1}u^2\|_\infty + \|\mathcal{A}_{\alpha_2}\bar{u}^2\|_\infty\big) \le C\varepsilon^{\frac{1}{2}}t^{-\frac{3}{2}},$$

$$\|w\| \le \|u\| + Ct\big(\|\mathcal{A}_{\alpha_1} u^2\| + \|\mathcal{A}_{\alpha_2} \bar{u}^2\|\big) \le C\varepsilon^{\frac{1}{2}}.$$

By the identity $w = u - t\lambda \mathcal{A}_{\alpha_1} u^2 - \frac{t}{3}\mu \mathcal{A}_{\alpha_2}\bar{u}^2$, we find

$$\|u - w\| \le Ct\big(\|\mathcal{A}_{\alpha_1} u^2\| + \|\mathcal{A}_{\alpha_2}\bar{u}^2\|\big).$$

By virtue of Lemma 2.5, letting $p \in [2, 6)$ be close to 6 and $\frac{1}{q} = \frac{1}{p} - \frac{1}{6}$, we get

$$\|\mathcal{A}_\alpha u^2\| \le C\Big(\|u\mathcal{J}u\|_p + t^{\frac{1}{2}}\|u^2\|_p + t^{\frac{3}{4}}\|u^2\|_q\Big) \le C\varepsilon t^{\theta-2}, \tag{4.7}$$

since

$$\begin{aligned}
\|u\|_q &\le \|u\|^{\frac{2}{q}}\|u\|_\infty^{1-\frac{2}{q}} \le C\varepsilon^{\frac{1}{2}} t^{\theta-\frac{3}{2}},\\
\|u\mathcal{J}u\|_p &\le C\|u\|_q\|\mathcal{J}u\|_6 \le Ct^{-1}\|u\|_q\|\mathcal{J}^2 u\| \le C\varepsilon t^{\theta-2},\\
\|u^2\|_p &\le C\|u\|_q\|u\|_6 \le Ct^{-1}\|u\|_q\|\mathcal{J}u\| \le C\varepsilon t^{\theta-\frac{5}{2}},\\
\|u^2\|_q &\le \|u\|_q\|u\|_\infty \le C\varepsilon t^{\theta-3}.
\end{aligned}$$

By (4.7) and the identity $\mathcal{A}_\alpha \bar{u}^2 = \overline{\mathcal{A}_{-\alpha} u^2}$, we get

$$\|u - w\| \le C\varepsilon t^{\theta-1}.$$

Thus we have the third estimate of the lemma. In the same way as in the proof of (4.5), we have

$$\|\mathcal{F}(M-1)\mathcal{U}(-t)u\|_\infty \le C\varepsilon^{\frac{1}{2}} t^{\theta-\frac{5}{4}}. \tag{4.8}$$

By (3.29) and (4.8), we get

$$\begin{aligned}
\|\mathcal{F}M\mathcal{U}(-t)u\|_\infty &\le \|\mathcal{F}(M-1)\mathcal{U}(-t)w\|_\infty + \|\mathcal{F}\mathcal{U}(-t)u\|_\infty\\
&\le C\varepsilon^{\frac{1}{2}} t^{\theta-\frac{5}{4}} + C\varepsilon \le C\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

Thus we obtain the last estimate of the lemma. This completes the proof of Lemma 4.1.

Proof of Theorem 1.1. By the integral equation associated with (1.9), we have

$$u(t) = \mathcal{U}(t)u_0 - i\int_0^t \mathcal{U}(t-\tau)\mathcal{N}(\tau)d\tau - i\int_0^t \mathcal{U}(t-\tau)\mathcal{G}(\tau)d\tau,$$

where $\mathcal{N}(\tau) = \lambda u^2 + \mu\bar{u}^2$. Applying the operator $\mathcal{F}\mathcal{U}(-t)$ to the above identity, we get

$$\mathcal{F}\mathcal{U}(-t)u(t) = \hat{u}_0 - i\int_0^t \mathcal{F}\mathcal{U}(-\tau)\mathcal{N}(\tau)d\tau - i\int_0^t \mathcal{F}\mathcal{U}(-\tau)\mathcal{G}(\tau)d\tau$$

$$= \hat{u}_+ + i\int_t^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{N}(\tau)d\tau + i\int_t^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{G}(\tau)d\tau, \quad (4.9)$$

where

$$\hat{u}_+ = \hat{u}_0 - i\int_0^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{N}(\tau)d\tau - i\int_0^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{G}(\tau)d\tau.$$

By Lemma 2.1, we have the identities

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)u^2 &= -i\mathcal{F}\bar{M}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}u^2 = i^{\frac{1}{2}}\mathcal{D}(2)E^2\mathcal{F}\bar{M}^{\frac{1}{2}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})\bar{M}^2u^2\\ &= i^{-4}\mathcal{D}(2)E^2\mathcal{F}\bar{M}^{\frac{1}{2}}\mathcal{F}^{-1}t^{-\frac{3}{2}}(\mathcal{F}M\mathcal{U}(-t)u)^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)\bar{u}^2 &= i^{\frac{1}{2}}\mathcal{D}(-2)E^6\mathcal{F}M^{\frac{1}{2}}\mathcal{F}^{-1}\mathcal{D}(\frac{1}{t})M^2\bar{u}^2\\ &= i^2\mathcal{D}(-2)E^6\mathcal{F}M^{\frac{1}{2}}\mathcal{F}^{-1}t^{-\frac{3}{2}}(\overline{\mathcal{F}M\mathcal{U}(-t)u})^2.\end{aligned}$$

Using the identities and dividing $(\mathcal{F}M\mathcal{U}(-\tau)u)^2$ into two terms $(\mathcal{F}M\mathcal{U}(-\tau)w)^2$ and $(\mathcal{F}M\mathcal{U}(-\tau)u)^2 - (\mathcal{F}M\mathcal{U}(-\tau)w)^2$, we obtain

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)u(t) &= \hat{u}_+ + \lambda i^{-3}\int_t^\infty \mathcal{D}(2)E^2\mathcal{F}\bar{M}^{\frac{1}{2}}\mathcal{F}^{-1}\tau^{-\frac{3}{2}}(\mathcal{F}M\mathcal{U}(-\tau)w)^2d\tau\\ &+ \mu i^3\int_t^\infty \mathcal{D}(-2)E^6\mathcal{F}M^{\frac{1}{2}}\mathcal{F}^{-1}\tau^{-\frac{3}{2}}(\overline{\mathcal{F}M\mathcal{U}(-\tau)w})^2d\tau + R_1, \quad (4.10)\end{aligned}$$

where

$$\begin{aligned}R_1 &= \lambda i^{-3}\int_t^\infty \mathcal{D}(2)E^2\mathcal{F}\bar{M}^{\frac{1}{2}}\mathcal{F}^{-1}\tau^{-\frac{3}{2}}\big((\mathcal{F}M\mathcal{U}(-\tau)u)^2 - (\mathcal{F}M\mathcal{U}(-\tau)w)^2\big)d\tau\\ &+\mu i^3\int_t^\infty \mathcal{D}(-2)E^6\mathcal{F}M^{\frac{1}{2}}\mathcal{F}^{-1}\tau^{-\frac{3}{2}}\big((\overline{\mathcal{F}M\mathcal{U}(-\tau)u})^2 - (\overline{\mathcal{F}M\mathcal{U}(-\tau)w})^2\big)d\tau\\ &+ i\int_t^\infty \mathcal{F}\mathcal{U}(-\tau)\mathcal{G}(\tau)d\tau.\end{aligned}$$

By Lemma 4.1, we find

$$\begin{aligned}\|R_1\| &\le C\Big(\int_t^\infty \tau^{-\frac{3}{2}}\|\mathcal{F}M\mathcal{U}(-\tau)(u-w)\|\big(\|\mathcal{F}M\mathcal{U}(-\tau)u\|_\infty\\ &\quad + \|\mathcal{F}M\mathcal{U}(-\tau)w\|_\infty\big)d\tau + \int_t^\infty \|u\|_\infty^2\|u\|d\tau\Big)\\ &\le C\Big(\int_t^\infty \varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}\|u-w\|d\tau + \varepsilon^{\frac{3}{2}}\int_t^\infty \tau^{-3}d\tau\Big)\end{aligned}$$

$$\leq C\Big(\int_t^\infty \varepsilon^{\frac{1}{2}}\tau^{-\frac{3}{2}}\varepsilon\tau^{\theta-1}d\tau + \varepsilon^{\frac{3}{2}}t^{-2}\Big) \leq C\varepsilon t^{\theta-\frac{3}{2}}. \tag{4.11}$$

Thus, we have

$$\mathcal{F}\mathcal{U}(-t)u(t) = \hat{u}_+ + \lambda i^{-3}\int_t^\infty \mathcal{D}(2)E^2\tau^{-\frac{3}{2}}(\mathcal{F}M\mathcal{U}(-\tau)w)^2 d\tau \tag{4.12}$$

$$+ \mu i^3\int_t^\infty \mathcal{D}(-2)E^6\tau^{-\frac{3}{2}}(\overline{\mathcal{F}M\mathcal{U}(-\tau)w})^2 d\tau + R_2 + O(\varepsilon t^{\theta-\frac{3}{2}}) \text{ in } L^2,$$

where

$$R_2 = \lambda i^{-3}\int_t^\infty \mathcal{D}(2)E^2\mathcal{F}(\bar{M}^{\frac{1}{2}}-1)\mathcal{F}^{-1}\tau^{-\frac{3}{2}}(\mathcal{F}M\mathcal{U}(-\tau)w)^2 d\tau$$

$$+ \mu i^3\int_t^\infty \mathcal{D}(-2)E^6\mathcal{F}(M^{\frac{1}{2}}-1)\mathcal{F}^{-1}\tau^{-\frac{3}{2}}(\overline{\mathcal{F}M\mathcal{U}(-\tau)w})^2 d\tau.$$

By Lemma 4.1, we find

$$\|R_2\| \leq C\Big(\int_t^\infty \|(\bar{M}^{\frac{1}{2}}-1)\mathcal{F}^{-1}\tau^{-\frac{3}{2}}(\mathcal{F}M\mathcal{U}(-\tau)w)^2\|d\tau$$

$$+ \int_t^\infty \|(M^{\frac{1}{2}}-1)\mathcal{F}^{-1}\tau^{-\frac{3}{2}}(\overline{\mathcal{F}M\mathcal{U}(-\tau)w})^2\|d\tau\Big)$$

$$\leq C\varepsilon\int_t^\infty \tau^{\theta-\frac{5}{2}}d\tau \leq C\varepsilon t^{\theta-\frac{3}{2}}. \tag{4.13}$$

We divided $(\mathcal{F}M\mathcal{U}(-\tau)w)^2$ into two terms, $(\mathcal{F}\mathcal{U}(-\tau)w)^2$ and $(\mathcal{F}M\mathcal{U}(-\tau)w)^2 - (\mathcal{F}\mathcal{U}(-\tau)w)^2$, and the second is considered to be the remainder terms in our function space. Indeed, by virtue of Lemma 4.1,

$$\Big\|\int_t^\infty \mathcal{D}(2)E^2\tau^{-\frac{3}{2}}\big((\mathcal{F}M\mathcal{U}(-\tau)w)^2 - (\mathcal{F}\mathcal{U}(-\tau)w)^2\big)d\tau\Big\|$$

$$\leq C\int_t^\infty \tau^{-\frac{3}{2}}\|\mathcal{F}(M-1)\mathcal{U}(-\tau)w\|\big(\|\mathcal{F}M\mathcal{U}(-\tau)w\|_\infty + \|\mathcal{F}\mathcal{U}(-\tau)w\|_\infty\big)d\tau$$

$$\leq C\int_t^\infty \tau^{-\frac{3}{2}}\varepsilon^{\frac{1}{2}}\tau^{\theta-1}\big(\varepsilon^{\frac{1}{2}}+\varepsilon\big)d\tau \leq C\varepsilon t^{\theta-\frac{3}{2}}. \tag{4.14}$$

Here we have used the estimate

$$\|\mathcal{F}(M-1)\mathcal{U}(-\tau)w\| \leq C\tau^{-1}\|\mathcal{J}^2 w\| \leq C\varepsilon^{\frac{1}{2}}\tau^{\theta-1}.$$

Therefore, it is sufficient to start with

$$\mathcal{F}\mathcal{U}(-t)u(t) = \hat{u}_+ + \lambda i^{-3}\int_t^\infty \mathcal{D}(2)E^2\tau^{-\frac{3}{2}}(\mathcal{F}\mathcal{U}(-\tau)w)^2 d\tau \tag{4.15}$$

$$+\mu i^3\int_t^\infty \mathcal{D}(-2)E^6\tau^{-\frac{3}{2}}(\overline{\mathcal{F}\mathcal{U}(-\tau)w})^2d\tau+O(\varepsilon t^{\theta-\frac{3}{2}}) \text{ in } L^2.$$

Since $\mathcal{F}\mathcal{U}(-t)u$ converges $\hat{u}_+$ by Proposition 4.1, we transform the above identity as follows:

$$\mathcal{F}\mathcal{U}(-t)u(t)=\hat{u}_++\lambda 2^{-\frac{3}{2}}i^{-\frac{9}{2}}\int_t^\infty e^{\frac{i\tau|\xi|^2}{4}}\tau^{-\frac{3}{2}}d\tau\hat{u}_+^2(\frac{\xi}{2}) \\ +\mu 2^{-\frac{3}{2}}i^{\frac{9}{2}}\int_t^\infty e^{\frac{3i\tau|\xi|^2}{4}}\tau^{-\frac{3}{2}}d\tau\overline{\hat{u}}_+^2(-\frac{\xi}{2})+R_3+O(\varepsilon t^{\theta-\frac{3}{2}}) \text{ in } L^2, \tag{4.16}$$

where

$$R_3=\lambda i^{-3}\int_t^\infty \mathcal{D}(2)E^2\tau^{-\frac{3}{2}}\big((\mathcal{F}\mathcal{U}(-\tau)w)^2-\hat{u}_+^2\big)d\tau \\ +\mu i^3\int_t^\infty \mathcal{D}(-2)E^6\tau^{-\frac{3}{2}}\big((\overline{\mathcal{F}\mathcal{U}(-\tau)w})^2-\overline{\hat{u}}_+^2\big)d\tau.$$

By the identities $w=u-t\lambda\mathcal{A}_{\alpha_1}u^2-\frac{t}{3}\mu\mathcal{A}_{\alpha_2}\bar{u}^2$ and $\mathcal{A}_\alpha\bar{u}^2=\overline{\mathcal{A}_{-\alpha}u^2}$, and by Proposition 4.1, we have

$$\begin{aligned}\|\mathcal{F}\mathcal{U}(-\tau)w-\hat{u}_+\| &\le \|\mathcal{F}\mathcal{U}(-\tau)w-\mathcal{F}\mathcal{U}(-\tau)u\|+\|\mathcal{F}\mathcal{U}(-\tau)u-\hat{u}_+\| \\ &\le C\big(\tau(\|\mathcal{A}_{\alpha_1}u^2\|+\|\mathcal{A}_{\alpha_2}\bar{u}^2\|)+\varepsilon\tau^{-\frac{1}{2}}\big) \\ &\le C\varepsilon(\tau^{\theta-1}+\tau^{-\frac{1}{2}})\le C\varepsilon\tau^{-\frac{1}{2}},\end{aligned} \tag{4.17}$$

which implies

$$\begin{aligned}\|R_3\| &\le C\int_t^\infty \tau^{-\frac{3}{2}}\|\mathcal{F}\mathcal{U}(-\tau)w-\hat{u}_+\|\big(\|\mathcal{F}\mathcal{U}(-\tau)w\|_\infty+\|\hat{u}_+\|_\infty\big)d\tau \\ &\le C\varepsilon\int_t^\infty \tau^{-2}d\tau\le C\varepsilon t^{-1}.\end{aligned}$$

Thus we arrive at

$$\mathcal{F}\mathcal{U}(-t)u(t)=\hat{u}_++\lambda 2^{-\frac{3}{2}}i^{-\frac{9}{2}}\int_t^\infty e^{\frac{i\tau|\xi|^2}{4}}\tau^{-\frac{3}{2}}\,d\tau\,\hat{u}_+^2(\frac{\xi}{2}) \\ +\mu 2^{-\frac{3}{2}}i^{\frac{9}{2}}\int_t^\infty e^{\frac{3i\tau|\xi|^2}{4}}\tau^{-\frac{3}{2}}\,d\tau\,\overline{\hat{u}}_+^2(-\frac{\xi}{2})+O(\varepsilon t^{-1}) \text{ in } L^2. \tag{4.18}$$

We next estimate the second and third terms of the right-hand side of (4.18). We have

$$\Big\|\lambda 2^{-\frac{3}{2}}i^{-\frac{9}{2}}\int_t^\infty e^{\frac{i\tau|\xi|^2}{4}}\tau^{-\frac{3}{2}}\,d\tau\,\hat{u}_+^2(\frac{\xi}{2})\Big\|\le C\Big\|\int_t^\infty e^{\frac{i\tau|\xi|^2}{4}}\tau^{-\frac{3}{2}}\,d\tau\,\Big\|\|\hat{u}_+\|_\infty^2$$

$$\leq C\varepsilon \Big\| \int_t^\infty e^{\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} d\tau \Big\|.$$

By the identity

$$e^{\frac{i\tau|\xi|^2}{4}} = \frac{1}{1+\frac{i\tau|\xi|^2}{4}} \partial_\tau \big(\tau e^{\frac{i\tau|\xi|^2}{4}}\big),$$

we get

$$\begin{aligned} e^{\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} &= \frac{1}{1+\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} \partial_\tau \big(\tau e^{\frac{i\tau|\xi|^2}{4}}\big) \\ &= \partial_\tau \Big(\frac{1}{1+\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{1}{2}} e^{\frac{i\tau|\xi|^2}{4}} \Big) + \frac{3}{2} \tau^{-\frac{3}{2}} e^{\frac{i\tau|\xi|^2}{4}} \frac{1}{1+\frac{i\tau|\xi|^2}{4}} \\ &\quad + \tau^{-\frac{1}{2}} e^{\frac{i\tau|\xi|^2}{4}} \frac{i|\xi|^2}{4} \frac{1}{\big(1+\frac{i\tau|\xi|^2}{4}\big)^2}. \end{aligned}$$

Using the above identity, we obtain

$$\begin{aligned} \Big\| \int_t^\infty e^{\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} d\tau \Big\| &\leq C \Big\| t^{-\frac{1}{2}} \frac{1}{1+\frac{it|\xi|^2}{4}} \Big\| + C \int_t^\infty \Big\| \tau^{-\frac{3}{2}} \frac{1}{1+\frac{i\tau|\xi|^2}{4}} \Big\| d\tau \\ &\quad + C \int_t^\infty \Big\| \tau^{-\frac{1}{2}} |\xi|^2 \frac{1}{\big(1+\frac{i\tau|\xi|^2}{4}\big)^2} \Big\| d\tau. \end{aligned} \tag{4.19}$$

Changing the variable $\sqrt{t}\xi = \eta$,

$$\begin{aligned} \Big\| t^{-\frac{1}{2}} \frac{1}{1+\frac{it|\xi|^2}{4}} \Big\| &= t^{-\frac{1}{2}} \Big(\int \frac{1}{\big(1+\frac{t|\xi|^2}{4}\big)^2} d\xi \Big)^{\frac{1}{2}} = t^{-\frac{5}{4}} \Big(\int \frac{1}{\big(1+\frac{|\eta|^2}{4}\big)^2} d\eta \Big)^{\frac{1}{2}} \\ &\leq C t^{-\frac{5}{4}} \int_0^\infty \frac{r^2}{\big(1+\frac{r^2}{4}\big)^2} dr \leq C t^{-\frac{5}{4}}. \end{aligned} \tag{4.20}$$

In the same way, we get

$$\int_t^\infty \Big\| \tau^{-\frac{3}{2}} \frac{1}{1+\frac{i\tau|\xi|^2}{4}} \Big\| d\tau \leq C t^{-\frac{5}{4}}. \tag{4.21}$$

By $\sqrt{\tau}\xi = \eta$, we have

$$\begin{aligned} \Big\| |\xi|^2 \frac{1}{\Big(1+\frac{i\tau|\xi|^2}{4}\Big)^2} \Big\| &= \Big(\int \frac{|\xi|^4}{\big(1+\frac{\tau|\xi|^2}{4}\big)^4} d\xi \Big)^{\frac{1}{2}} = \tau^{-\frac{7}{4}} \Big(\int \frac{|\eta|^4}{\big(1+\frac{|\eta|^2}{4}\big)^4} d\eta \Big)^{\frac{1}{2}} \\ &\leq C \tau^{-\frac{7}{4}} \Big(\int_0^\infty \frac{r^6}{\big(1+\frac{r^2}{4}\big)^4} dr \Big)^{\frac{1}{2}} \leq C \tau^{-\frac{7}{4}}. \end{aligned} \tag{4.22}$$

Hence,

$$\int_t^\infty \left\| \tau^{-\frac{1}{2}} |\xi|^2 \frac{1}{(1+\frac{i\tau|\xi|^2}{4})^2} \right\| d\tau \le C t^{-\frac{5}{4}}. \tag{4.23}$$

Thus, we obtain

$$\begin{aligned} \left\| \lambda 2^{-\frac{3}{2}} i^{-\frac{9}{2}} \int_t^\infty e^{\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} \, d\tau \, \hat{u}_+^2(\frac{\xi}{2}) \right\| &\le C\varepsilon \left\| \int_t^\infty e^{\frac{i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} d\tau \right\| \\ &\le C\varepsilon t^{-\frac{5}{4}}. \end{aligned} \tag{4.24}$$

Similarly, we find

$$\left\| \mu 2^{-\frac{3}{2}} i^{\frac{9}{2}} \int_t^\infty e^{\frac{3i\tau|\xi|^2}{4}} \tau^{-\frac{3}{2}} \, d\tau \, \overline{\hat{u}}_+^2(-\frac{\xi}{2}) \right\| \le C\varepsilon t^{-\frac{5}{4}}. \tag{4.25}$$

Therefore, by virtue of (4.18),

$$\|\mathcal{F}\mathcal{U}(-t)u(t) - \hat{u}_+\| \le C\varepsilon t^{-\frac{5}{4}} + C\varepsilon t^{-1} \le C\varepsilon t^{-1}. \tag{4.26}$$

Here we reconsider the estimate for $\|R_3\|$. In the same way as in the proof of (4.17), we have by (4.26)

$$\begin{aligned} \|\mathcal{F}\mathcal{U}(-\tau)w - \hat{u}_+\| &\le C\tau\big(\|\mathcal{A}_{\alpha_1} u^2\| + \|\mathcal{A}_{\alpha_2} \bar{u}^2\|\big) + \|\mathcal{F}\mathcal{U}(-\tau)u - \hat{u}_+\| \\ &\le C\varepsilon(\tau^{\theta-1} + \tau^{-1}) \le C\varepsilon\tau^{\theta-1}. \end{aligned} \tag{4.27}$$

Using (4.27), we get

$$\begin{aligned} \|R_3\| &\le C \int_t^\infty \tau^{-\frac{3}{2}} \|\mathcal{F}\mathcal{U}(-\tau)w - \hat{u}_+\| \big(\|\mathcal{F}\mathcal{U}(-\tau)w\|_\infty + \|\hat{u}_+\|_\infty\big) d\tau \\ &\le C\varepsilon t^{\theta-\frac{3}{2}}. \end{aligned} \tag{4.28}$$

Collecting estimates (4.11), (4.13), (4.14), and (4.28), we obtain the estimate of (1.7). Then in particular, by virtue of (4.24) and (4.25), we have

$$\|\mathcal{F}\mathcal{U}(-t)u(t) - \hat{u}_+\| \le C\varepsilon t^{-\frac{5}{4}} + C\varepsilon t^{\theta-\frac{3}{2}} \le C\varepsilon t^{-\frac{5}{4}}$$

for t > 1. This completes the proof of Theorem 1.1.

Acknowledgments. The author wishes to express his sincere gratitude to Professor Nakao Hayashi for his valuable advice, support, and constant encouragement. The author would like to thank Professors Soichiro Katayama, Hideo Kubo, Takeshi Wada, and Hideaki Sunagawa for their helpful comments, too.

Finally, the author also thanks the referees for their careful reading of the first draft and many useful suggestions.

REFERENCES

[1] T. Cazenave and F.B. Weissler, *Rapidly decaying solutions of the nonlinear Schrödinger equation,* Comm. Math. Phys., 147 (1992), 75–100.

[2] N. Hayashi and T. Ozawa, *Scattering theory in the weighted $L^2(\mathbf{R}^n)$ spaces for some Schrödinger equations,* Ann. Inst. H. Poincaré Phys. Théor., 48 (1988), 17–37.

[3] N. Hayashi, *Global existence of small solutions to quadratic nonlinear Schrödinger equations,* Comm. Partial Differential Equations, 18 (1993), 1109–1124.

[4] N. Hayashi, C. Miao, and P.I. Naumkin, *Global existence of small solutions to the generalized derivative nonlinear Schrödinger equations,* Asymptot. Anal., 21 (1999), 133–147.

[5] N. Hayashi and P.I. Naumkin, *On the quadratic nonlinear Schrödinger equation in three space dimensions,* Int. Math. Res. Not., 3 (2000), 115–131.

[6] N. Hayashi, T. Mizumachi, and P.I. Naumkin, *Time decay of small solutions to quadratic nonlinear Schrödinger equations in 3D,* Differential Integral Equations, 16 (2003), 159–179.

[7] N. Kita, *Sharp L^r asymptotics of the small solutions to the nonlinear Schrödinger equations,* Nonlinear Anal., 52 (2003), 1365–1377.

[8] N. Kita and T. Ozawa, *Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations with repulsive interactions,* Commun. Contemp. Math., to appear.

[9] N. Kita and T. Wada, *Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations in one space dimension,* Funkcial. Ekvac., 45 (2002), 53–69.

[10] N. Kita and T. Wada, *Sharp asymptotics of the small solutions to the nonlinear Schrödinger equations of derivative type,* Differential Integral Equations, 15 (2002), 367–384.

[11] S. Klainerman and G. Ponce, *Global, small amplitude solutions to nonlinear evolution equations,* Comm. Pure Appl. Math., 36 (1983), 133–141.

[12] K. Nakanishi and T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations,* NoDEA Nonlinear Differential Equations Appl., 9 (2002), 45–68.

[13] J. Shatah, *Global existence of small solutions to nonlinear evolution equations,* J. Differential Equations, 46 (1982), 409–425.

[14] W.A. Strauss, *Nonlinear scattering thory at low energy,* J. Funct. Anal., 41 (1981), 110–133.

[15] T. Wada, *Asymptotic expansion of the solution to the nonlinear Schrödinger equation with nonlocal interaction,* J. Funct. Anal., 180 (2001), 11–30.