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THE SUB-SUPERSOLUTION METHOD FOR AN EVOLUTIONARY REACTION-DIFFUSION AGE-DEPENDENT PROBLEM

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Abstract. In this work, we analyze a nonlinear population-dynamics model with age dependence and spatial diffusion, and where we are assuming the influence of a reaction term. Using a sub-supersolution method we derive existence and uniqueness results. We apply this method to study the existence and uniqueness of a positive solution of a generalized logistic and of a Holling-Tanner-type age-dependent model as well as its large-time behaviour.

1. Introduction

In this paper we consider a nonlinear model describing the dynamics of a single-species population with age dependence and spatial structure.

Let $u(x,a,t)$ denote the population density of age $a>0$ at time $t>0$ and at position $x\in\Omega$, Ω being a bounded domain in $\mathbb{R}^N$, having a suitably smooth boundary $\partial\Omega$.

Following Gurtin [6], the evolution of u is governed by the balance law,

$$u_t+u_a=-div\ q+s,$$

where q is the flux of population due to dispersal and s represents the supply of individuals.

We assume diffusion lineal, the so-called random spatial diffusion (see for instance [7]); i.e., the flux of population takes the form $q:=-\nabla u$, with ∇ the gradient vector with respect to the spatial variable. Further, we suppose

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that the supply of individuals contains a death term and a term of reaction; i.e.,

$$s := -\mu(x,a,t)u + f(x,a,t,u),$$

where $\mu(x,a,t)$ is the natural death rate of individuals and f describes the effect of the environment on the population, and is positive when the environment is favorable and negative when it is hostile.

Let $A_\dagger$ be the highest age attained by the individuals in the population. We suppose that birth is described by the "renewal equation" (see e.g. [8])

$$u(x,0,t) = \int_0^{A_\dagger} \beta(x,a,t)u(x,a,t)da,$$

where $\beta(x,a,t)$ represents the natural fertility rate. Finally, the boundary $\partial\Omega$ of the domain Ω is supposed to be extremely inhospitable.

Then, the system of equations governing the dynamics of the population is given as

$$\begin{cases} u_t + u_a - \Delta u + \mu(x,a,t)u = f(x,a,t,u) & \text{in } \Omega \times \mathcal{O}, \\ u(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ u(x,a,0) = u_0(x,a) & \text{in } \Omega \times (0,A_\dagger), \\ u(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta(x,a,t)u(x,a,t)da & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

where $T > 0$ and $\mathcal{O} := (0,A_\dagger) \times (0,T)$.

For results on existence and uniqueness of the solution of models with linear diffusion, in the framework of semigroups we refer to [10] and [4], where the authors applied their results to a logistic model which is nonlinear in the total population.

In other frameworks different from semigroups we refer to [2, 5, 9, 11], among others. All of these models suppose that the supply s is due only to deaths, at the most, a linear factor, $f(x,a,t)$, that takes into account possible external increase of the population. To our knowledge, not many results about a model with a term of reaction depending on the population density are available presently.

More recently, results about existence of solutions for age-dependent models with diffusion appear in papers about control theory (see for instance [1, 3]).

In this paper, first we give a general result of existence and uniqueness of the solution of (1.1) assuming that f is globally Lipschitz in the variable u. Second, we prove that, assuming the existence of a sub-supersolution of (1.1) and the Lipschitz continuity of f, there exists a unique solution between the subsolution and the supersolution. The advantage of using this method lies in the fact that it allows us analyze the large-time behaviour of the solution. We would like to point out that, although comparison results have been studied for this kind of problem when $f(x,a,t,u) \equiv f(x,a,t)$ (see, for instance, [12, Corollary 4.2]), this result generalizes the classical sub-supersolution method in these ways: (1.1) is a first-order problem with a singular potential and a nonlocal initial condition. We note here that several difficulties come from the fact $\mu(x,a,t) \to +\infty$ as $a \to A_\dagger$, which is a well-known condition to assure that the solution of (1.1) vanishes when $a = A_\dagger$.

The paper is organized as follows. In Section 2, we introduce the notion of solution of (1.1), and we enunciate some preliminaries. In Section 3 we prove that the sub-supersolution method works for equations such as (1.1). In the last section we apply these results to different ecological models: to a generalized logistic problem and to a Holling-Tanner-type model (see for instance [13]); a convenient choice of sub- supersolutions provides us some information about the large-time behaviour of the solution of these models.

2. Preliminaries

We suppose

$(\mathcal{H}_\mu)$

$$\mu \in \mathcal{C}^0(\overline{\Omega} \times [0,A_\dagger) \times [0,T]), \quad \mu \geq 0 \tag{2.1}$$

and that its behaviour at $a = A_\dagger$ is given by the divergence condition (see [5])

$$\begin{cases} 0 < t < A_\dagger, \quad x \in \Omega, \quad \lim_{a \to A_\dagger} \displaystyle\int_0^t \mu(x, a-t+\tau, \tau)d\tau = +\infty, \\ A_\dagger < t < T, \quad x \in \Omega, \quad \lim_{a \to A_\dagger} \displaystyle\int_0^a \mu(x,\alpha,t-a+\alpha)d\alpha = +\infty. \end{cases} \tag{2.2}$$

$(\mathcal{H}_\beta)$ $\beta \in L^\infty(\Omega \times \mathcal{O})$, $\beta \geq 0$ nontrivial. We denote

$$\bar{\beta} := \sup\{\beta(x,a,t) : (x,a,t) \in \Omega \times \mathcal{O}\}.$$

$(\mathcal{H}_0)$ $u_0 \in L^2(\Omega \times (0,A_\dagger))$.

Remark 2.1. Note that the condition $\mu \geq 0$ is not particularly restrictive. Indeed, under the change of variable $w = e^{-kt}u$, $k > 0$, we obtain that w satisfies

$$w_t + w_a - \Delta w + (\mu + k)w = g(x,a,t,w) := e^{-kt} f(x,a,t,e^{kt}w),$$

and so, thanks to $(\mathcal{H}_\mu)$, we can take k large such that $\mu + k \geq 0$.

Remark 2.2. Condition (2.2) ensures that, under some conditions for f, the solution of the problem (1.1) vanishes at $a = A_\dagger$ (see [5, Theorem 3]).

For (1.1), we use the notion of solution considered in [5].

Definition 2.3. *A function* $u(x, a, t)$, $x \in \Omega$, $a \in (0, A_\dagger)$, $t > 0$ *is a solution of problem* (1.1) *if* $u : \Omega \times \mathcal{O} \to \mathbb{R}$ *is a measurable function such that* $u \in L^2(\mathcal{O}; H^1_0(\Omega))$, $u_t + u_a + \mu u \in L^2(\mathcal{O}; H^{-1}(\Omega))$, $f(\cdot, \cdot, \cdot, u) \in L^2(\mathcal{O}; H^{-1}(\Omega))$, *and for any* $w \in L^2(\mathcal{O}; H^1_0(\Omega))$ *one has*

$$\iint_{\mathcal{O}} \langle u_t + u_a + \mu u, w\rangle da\, dt + \iiint_{\Omega\times\mathcal{O}} \nabla u \cdot \nabla w\, dx\, da\, dt = \iint_{\mathcal{O}} \langle f(\cdot, a, t, u), w\rangle da\, dt \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $H^1_0(\Omega)$ *and its dual space* $H^{-1}(\Omega)$. *Then, the initial conditions make sense in* $L^2(\Omega \times (0, A_\dagger))$ *and* $L^2(\Omega \times (0, T))$; *hence, u must satisfy*

$$u(x, a, 0) = u_0(x, a), \text{ in } L^2(\Omega \times (0, A_\dagger))$$
$$u(x, 0, t) = \int_0^{A_\dagger} \beta(x, a, t)u(x, a, t)da, \text{ in } L^2(\Omega \times (0, T)).$$

We enunciate a minimum principle. The proof is completed by applying the same technique as [5].

Lemma 2.4. *Assume* $(\mathcal{H}_\mu)$ *and* $(\mathcal{H}_\beta)$. *Let z be a function in* $L^2(\mathcal{O}; H^1(\Omega))$ *such that* $z_t + z_a + \mu z \in L^2(\mathcal{O}; (H^1(\Omega))')$, *and satisfies*

$$\begin{cases} z_t + z_a - \Delta z + \mu(x, a, t)z \leq L|z| & \text{in } \Omega \times \mathcal{O}, \\ z(x, a, t) \leq 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ z(x, a, 0) \leq 0 & \text{in } \Omega \times (0, A_\dagger), \\ z(x, 0, t) \leq \displaystyle\int_0^{A_\dagger} \beta(x, a, t)z(x, a, t)da & \text{in } \Omega \times (0, T), \end{cases} \tag{2.4}$$

for some $L > 0$. *Then we have* $z \leq 0$ *in* $\Omega \times \mathcal{O}$.

3. The sub-supersolution method

Definition 3.1. *A function* $\underline{u} \in L^2(\mathcal{O}; H^1(\Omega))$ *is called a subsolution of* (1.1) *if* $\underline{u}_t + \underline{u}_a + \mu\underline{u} \in L^2(\mathcal{O}; [H^1(\Omega)]')$, $f(\cdot, \cdot, \cdot, \underline{u}) \in L^2(\Omega \times \mathcal{O})$, *and satisfies*

a) $\underline{u}_t + \underline{u}_a - \Delta\underline{u} + \mu\underline{u} \le f(x,a,t,\underline{u})$ *in* $\Omega \times \mathcal{O}$ *on the weak sense; i.e.,* $\forall v \in L^2_+(\mathcal{O}; H^1_0(\Omega))$,

$$\iint_{\mathcal{O}} \langle (\partial_t + \partial_a)\,\underline{u} + \mu\underline{u}, v\rangle da\,dt + \iiint_{\Omega\times\mathcal{O}} \nabla\underline{u}\cdot\nabla v\,dx\,da\,dt \le \iint_{\mathcal{O}} f(x,a,t,\underline{u})\,v\,da\,dt \tag{3.1}$$

b) $\underline{u}(x,a,t) \le 0$ *on* $\partial\Omega \times \mathcal{O}$, *in the weak sense,*
c) $\underline{u}(x,0,t) \le \int_0^{A_\dagger} \beta(x,a,t)\underline{u}(x,a,t)da$ *in* $\Omega \times (0,T)$,
d) $\underline{u}(x,a,0) \le u_0(x,a)$ *in* $\Omega \times (0, A_\dagger)$.

Similarly, we define a supersolution, $\overline{u}$, *by reversing the above inequalities.*

The following result assures that a pair of sub-supersolutions is really ordered.

Theorem 3.2. *Assume that there exists a pair of sub-supersolutions of* (1.1), $\underline{u}$ *and* $\overline{u}$, *and that* f *satisfies*

$$|f(x,a,t,s_1) - f(x,a,t,s_2)| \le L|s_1 - s_2| \quad \forall s_1, s_2 \in [u_*, u^*]\,, \tag{3.2}$$

with

$$\begin{aligned} u_* &= \inf_{(x,a,t)\in\Omega\times\mathcal{O}} \{\underline{u}(x,a,t), \overline{u}(x,a,t)\} \\ u^* &= \sup_{(x,a,t)\in\Omega\times\mathcal{O}} \{\underline{u}(x,a,t), \overline{u}(x,a,t)\}. \end{aligned} \tag{3.3}$$

Then $\underline{u} \le \overline{u}$.

Proof. Let $\omega = \underline{u} - \overline{u}$; hence, ω satisfies

$$\begin{cases} \omega_t + \omega_a - \Delta\omega + \mu\omega \le f(x,a,t,\underline{u}) - f(x,a,t,\overline{u}) & \text{in } \Omega \times \mathcal{O}, \\ \omega(x,a,t) \le 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ u(x,a,0) \le 0 & \text{in } \Omega \times (0, A_\dagger), \\ u(x,0,t) \le \displaystyle\int_0^{A_\dagger} \beta(x,a,t)\omega(x,a,t)da & \text{in } \Omega \times (0,T). \end{cases} \tag{3.4}$$

From (3.2) we have $f(x,a,t,\underline{u}) - f(x,a,t,\overline{u}) \le L|\omega|$. It follows easily, from Lemma 2.4, that $\omega \le 0$ and so that $\underline{u} \le \overline{u}$. □

Remark 3.3. In particular, under the assumptions of Theorem 3.2, if u is a solution of (1.1), we have $\underline{u} \le u \le \overline{u}$.

Theorem 3.4. *Assume* $(\mathcal{H}_\mu)$, $(\mathcal{H}_\beta)$, *and* $(\mathcal{H}_0)$. *If there exists a pair of sub-supersolutions of* (1.1), $\underline{u}$ *and* $\overline{u}$, *and* f *satisfies*

$$|f(x,a,t,s_1) - f(x,a,t,s_2)| \le L|s_1 - s_2| \quad \forall s_1, s_2 \in [u_*, u^*], \tag{3.5}$$

with u_* *and* u^* *defined in* (3.3), *then* (1.1) *admits a unique solution*, u, *such that* $\underline{u} \le u \le \overline{u}$.

Proof. Note that thanks to Theorem 3.2, we have $\underline{u} \le \overline{u}$. We consider the operator $\widetilde{\Lambda}$ as

$$\widetilde{\Lambda} : \left[e^{-\alpha t}\underline{u}, e^{-\alpha t}\overline{u}\right] \longrightarrow L^2(\mathcal{O}; H^1(\Omega)), \quad v \longrightarrow w, \tag{3.6}$$

where w is the solution of

$$\begin{cases} w_t + w_a - \Delta w(x,a,t) + (M+\alpha+\mu)w & \\ \qquad\qquad = e^{-\alpha t} f(x,a,t,e^{\alpha t}v) + Mv & \text{in } \Omega \times \mathcal{O}, \\ w(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ w(x,a,0) = u_0(x,a) & \text{in } \Omega \times (0, A_\dagger), \\ w(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta(x,a,t) v(x,a,t)\,da & \text{in } \Omega \times (0,T), \end{cases} \tag{3.7}$$

with $M > 0$ to be chosen later, α large enough, and

$$\begin{aligned} &\left[e^{-\alpha t}\underline{u}, e^{-\alpha t}\overline{u}\right] := \{u \in L^2(\mathcal{O}; H^1(\Omega)) : \\ &e^{-\alpha t}\underline{u}(x,a,t) \le u(x,a,t) \le e^{-\alpha t}\overline{u}(x,a,t) \text{ a.e. } (x,a,t) \in \Omega \times \mathcal{O}\}. \end{aligned}$$

Analysis similar to the proof of Lemma 3 in [5] shows that there exists a unique v, a fixed point of $\widetilde{\Lambda}$. Therefore, $u = e^{\alpha t}v$ is the solution of problem (1.1). Thus, it remains to prove that $\widetilde{\Lambda} : [e^{-\alpha t}\underline{u}, e^{-\alpha t}\overline{u}] \to [e^{-\alpha t}\underline{u}, e^{-\alpha t}\overline{u}]$. To this end, consider $v \in \left[e^{-\alpha t}\underline{u}, e^{-\alpha t}\overline{u}\right]$, and we have to show that $e^{-\alpha t}\overline{u} - \widetilde{\Lambda}v := \omega \ge 0$. It is clear that ω satisfies

$$\begin{cases} \omega_t + \omega_a - \Delta\omega + (M+\alpha+\mu)\omega & \\ \ge e^{-\alpha t}\left(f(x,a,t,\overline{u}) - f(x,a,t,e^{\alpha t}v)\right) + M\left(e^{-\alpha t}\overline{u} - v\right) & \text{in } \Omega \times \mathcal{O}, \\ \omega(x,a,t) \ge 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ \omega(x,a,0) \ge 0 & \text{in } \Omega \times (0, A_\dagger), \\ \omega(x,0,t) \ge \displaystyle\int_0^{A_\dagger} \beta(x,a,t)(e^{-\alpha t}\overline{u} - v)(x,a,t)\,da \ge 0 & \text{in } \Omega \times (0,T). \end{cases}$$

From (3.5) and $e^{\alpha t}v \in [\underline{u}, \overline{u}]$, we have

$$L(e^{\alpha t}v - \overline{u}) \le f(x,a,t,\overline{u}) - f(x,a,t,e^{\alpha t}v) \le L(\overline{u} - e^{\alpha t}v).$$

Hence, if we define

$$h(x,t,a) := e^{-\alpha t}\left(f(x,a,t,\overline{u}) - f(x,a,t,e^{\alpha t}v)\right) + M\left(e^{-\alpha t}\overline{u} - v\right)$$

we have

$$h(x,t,a) \geq (M-L)\left(e^{-\alpha t}\overline{u} - v\right).$$

Thus, taking $M > L$ we obtain that $h \geq 0$ and, consequently, $\omega \geq 0$ (see [5, Theorem 2]). The same reasoning applies to the case $\widetilde{\Lambda} v \geq e^{-\alpha t}\underline{u}$. $\square$

4. Application to different ecological models

In this section, we are going to apply the sub-supersolution method to prove the existence and uniqueness of a positive solution and some properties of the large-time behaviour of different problems.

Let us first consider the linear problem

$$\begin{cases} \omega_t + \omega_a - \Delta\omega + q(a)\omega = \lambda\omega & \text{in } \Omega \times \mathcal{O}, \\ \omega(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ \omega(x,a,0) = u_0(x,a) & \text{in } \Omega \times (0,A_\dagger), \\ \omega(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta(a)\omega(x,a,t)da & \text{in } \Omega \times (0,T), \end{cases} \tag{4.1}$$

where β satisfies $(\mathcal{H}_\beta)$, $\lambda \in \mathbb{R}$, and u_0 and q satisfy

$(\mathcal{H}_0^*)$ $u_0 \in L^\infty(\Omega \times (0,A_\dagger))$ and $u_0(x,a) \geq 0$,

$(\mathcal{H}_q)$ q is a function such that $q \in L^\infty(0,r)$ for $r < A_\dagger$ and

$$\int_0^{A_\dagger} q(a)da = +\infty. \tag{4.2}$$

Observe that (4.2) is equivalent to (2.2) when $\mu \equiv \mu(a)$.

Following the notation used in [12], denote by

$$\pi(a) = \exp\Big(-\int_0^a q(\sigma)d\sigma\Big),$$

and by r_q the unique real solution of

$$\int_0^{A_\dagger} \beta(a)\pi(a)e^{-ra}da = 1. \tag{4.3}$$

Finally, λ_1 stands for the first eigenvalue of the homogeneous Dirichlet problem for $-\Delta$. The results about the large-time behaviour of (4.1) are the following:

Theorem 4.1. *Suppose* $(\mathcal{H}_\beta)$, $(\mathcal{H}_0^*)$, *and* $(\mathcal{H}_q)$, *and assume that there exists* $0 \leq A_0 \leq A_\dagger$ *such that supp*$(\beta) \subset [0, A_0]$. *Then, there exists a unique positive solution,* ω_λ, *of* (4.1). *Moreover, for fixed* $T > 0$, ω_λ *is bounded in* $\Omega \times \mathcal{O}$. *Furthermore,*

a) *if supp*$(u_0) \subset \bar{\Omega} \times (A_0, A_\dagger)$ *the solution of* (4.1) *satisfies* $\omega_\lambda(x, a, t) = 0$ *for* $t > a$ *and* $x \in \Omega$; *therefore, for each* $A > 0$

$$\omega_\lambda(\cdot, \cdot, t) \to 0 \text{ uniformly on } \bar{\Omega} \times [0, A] \text{ as } t \to +\infty.$$

b) *Assume that the initial distribution* u_0 *satisfies*

$$supp(u_0) \cap (\Omega \times (0, A_0)) \neq \emptyset;$$

then

i) *if* $\lambda < \lambda_1 - r_q$, *the solution of* (4.1) *satisfies*

$$\omega_\lambda(\cdot, \cdot, t) \to 0 \text{ uniformly on } \bar{\Omega} \times [0, A_\dagger] \text{ as } t \to +\infty;$$

ii) *if* $\lambda = \lambda_1 - r_q$, *the solution of* (4.1) *satisfies*

$$\omega_\lambda(x, a, t) \to g(x, a) \text{ in } L^2(\Omega \times (0, A_\dagger)) \text{ as } t \to +\infty,$$

for some $g \in L^2_+(\Omega \times (0, A_\dagger))$;

iii) *if* $\lambda > \lambda_1 - r_q$, *the solution of* (4.1) *satisfies*

$$\omega_\lambda(x, a, t) \to +\infty \text{ in } L^2(\Omega \times (0, A_\dagger)) \text{ as } t \to +\infty.$$

A proof is given in [12, Sections 3 and 4].

4.1. **A generalized logistic problem.** Our first application is the following generalized logistic problem:

$$\begin{cases} u_t + u_a - \Delta u + q(a)u = \lambda u - g(u) & \text{in } \Omega \times \mathcal{O}, \\ u(x, a, t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ u(x, a, 0) = u_0(x, a) & \text{in } \Omega \times (0, A_\dagger), \\ u(x, 0, t) = \displaystyle\int_0^{A_\dagger} \beta(a) u(x, a, t) da & \text{in } \Omega \times (0, T), \end{cases} \tag{4.4}$$

where β, u_0, and q satisfy $(\mathcal{H}_\beta)$, $(\mathcal{H}_0^*)$, and $(\mathcal{H}_q)$, respectively, $\lambda \in \mathbb{R}$, and g satisfies

$(\mathcal{H}_g)$ is locally Lipschitz, $g(0) = 0$, and $g(s) \geq 0$ for all $s \in \mathbb{R}_+$ and satisfies

$$\lim_{s \to 0} \frac{g(s)}{s} = 0 \tag{4.5}$$

$$\lim_{s \to +\infty} \frac{g(s)}{s} = +\infty. \tag{4.6}$$

In the next result, we show the existence and the uniqueness of the solution of (4.4), and using adequate sub-supersolutions, that for $\lambda < \lambda_1 - r_q$, system (4.4) goes to extinction, but the species persists if $\lambda > \lambda_1 - r_q$.

Theorem 4.2. *Suppose* $(\mathcal{H}_\beta)$, $(\mathcal{H}_0^*)$, $(\mathcal{H}_q)$, *and* $(\mathcal{H}_g)$. *There exists a unique positive solution,* u, *of problem* (4.4). *Assume that there exists* $0 \leq A_0 \leq A_\dagger$ *such that* $supp(\beta) \subset [0, A_0]$; *then*

i) *if* $supp(u_0) \subset \bar{\Omega} \times (A_0, A_\dagger)$ *the solution of* (4.4) *satisfies* $u(x,a,t) = 0$ *for* $t > a$ *and* $x \in \Omega$; *therefore, for each* $A > 0$

$$u(\cdot,\cdot,t) \to 0 \textit{ uniformly on } \bar{\Omega} \times [0, A] \textit{ as } t \to +\infty.$$

ii) *Assume that the initial distribution* u_0 *satisfies*

$$supp(u_0) \cap (\Omega \times (0, A_0)) \neq \emptyset;$$

then

I) *if* $\lambda < \lambda_1 - r_q$, *the solution of* (4.4) *satisfies*

$$u(\cdot,\cdot,t) \to 0 \textit{ uniformly on } \bar{\Omega} \times [0, A_\dagger] \textit{ as } t \to +\infty;$$

II) *if* $\lambda > \lambda_1 - r_q$ *and* $u_0(x,a) > 0$, *nontrivial, in* $\Omega \times (0, A_\dagger)$, *then the model* (4.4) *is permanent, in the following sense: there exist a subsolution* $\underline{u}$ *and a supersolution* $\overline{u}$ *of* (4.4) *such that* $\underline{u}(x,a) \leq u(x,a,t) \leq \overline{u}(x,a)$ *for almost every* $(x,a,t) \in \Omega \times (0, A_\dagger) \times (0, +\infty)$.

Remark 4.3. A detailed study of the stationary problem related to (4.4) will allow us to analyze the particular case $\lambda = \lambda_1 - r_q$ and the exact behaviour of $u(x,a,t)$ as $t \to +\infty$.

Proof. Using the definition of sub-supersolution we can assert that $\underline{u} \equiv 0$ and $\overline{u} \equiv \omega_\lambda$ is a pair of sub-supersolutions of (4.4), respectively, where ω_λ is the solution of (4.1). Thanks to Theorem 4.1, we have that ω_λ is bounded and, on the other hand, $f(x,a,t,s) \equiv \lambda s - g(s)$ is globally Lipschitz to $s \in [0, \omega^*]$, with

$$\omega^* := \sup_{(x,a,t) \in \Omega \times \mathcal{O}} \{\omega_\lambda(x,a,t)\}.$$

Consequently, applying Theorem 3.4, we obtain that there exists a unique solution of (4.4), u, such that

$$0 \leq u(x,a,t) \leq \omega_\lambda(x,a,t) \text{ in } \Omega \times \mathcal{O}.$$

Hence, we prove the uniqueness of the positive solution to the problem (4.4). Moreover, the proofs of i) and ii) I) are straightforward, using Theorem 4.1. Now, we will prove ii) II).

First, we will find a subsolution of (4.4). Let $\varepsilon > 0$; we take

$$\underline{u}(x,a) := \varepsilon \exp\Big(- r_q a - \int_0^a q(s)ds\Big)\varphi_1(x),$$

with φ_1 a positive eigenfunction associated to λ_1. We will show that $\underline{u}$ is a subsolution of (4.4) for ε sufficiently small. It easy to check that $\underline{u} = 0$ on $\partial\Omega$, $\underline{u}(x,0) = \int_0^{A_\dagger} \beta(a)\underline{u}(x,a,t)da$, and that $\underline{u}(x,a) \leq u_0(x,a)$. To finish, we have to prove that

$$\underline{u}_t + \underline{u}_a - \Delta\underline{u} + q(a)\underline{u} \leq \lambda\underline{u} - g(\underline{u}).$$

But this is equivalent to

$$\frac{g(\underline{u})}{\underline{u}} \leq \lambda - (\lambda_1 - r_q) \text{ in } \Omega \times (0, A_\dagger),$$

and, for ε small, we have this condition thanks to (4.5).

Now, we will build another supersolution of (4.4). Fix $\lambda > \lambda_1 - r_q$, and take a domain $\widetilde{\Omega}$ such that $\Omega \subset \widetilde{\Omega}$. Let $\widetilde{\lambda_1}$ be the first eigenvalue associated to $\widetilde{\Omega}$. Thanks to the hypothesis $(\mathcal{H}_q)$, we can consider a function m such that $m \in \mathcal{C}^0([0, A_\dagger])$ and $m(a) \leq q(a)$ for all $a \in [0, A_\dagger]$. Fixing $K > 0$, we consider

$$\overline{u}(x,a) := K \exp\Big(- r_m a - \int_0^a m(s)ds\Big)\widetilde{\varphi_1}(x),$$

where $\widetilde{\varphi_1}$ is the positive eigenfunction associated to $\widetilde{\lambda_1}$. We claim that $\overline{u}$ is a supersolution of (4.4) for K large enough. Indeed, we have that $\overline{u} > 0$ on $\partial\Omega$, $\overline{u}(x,0) = \int_0^{A_\dagger} \beta(a)\overline{u}(x,a,t)da$, and $\overline{u}(x,a) \geq u_0(a,x)$, for K large. In this last inequality we have used that $\widetilde{\varphi_1}(x) \geq c_0 > 0$ in $\overline{\Omega}$. Then, it remains to prove that

$$\overline{u}_t + \overline{u}_a - \Delta\overline{u} + q(a)\overline{u} \geq \lambda\overline{u} - g(\overline{u}),$$

but this is equivalent to proving that

$$\frac{g(\overline{u})}{\overline{u}} \geq \lambda - (\widetilde{\lambda_1} - r_m) + (m(a) - q(a)).$$

By (4.6), for K large, this inequality holds. □

Remark 4.4. The typical example for g is $g(s) := s^p,$ with $p > 1$.

4.2. **A Holling-Tanner-type model.** Now, we are going to apply the sub-supersolution method to a Holling-Tanner-type model

$$\begin{cases} u_t + u_a - \Delta u + q(a)u = \lambda u + \dfrac{u}{1+u} & \text{in } \Omega \times \mathcal{O}, \\ u(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ u(x,a,0) = u_0(x,a) & \text{in } \Omega \times (0, A_\dagger), \\ u(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta(a)u(x,a,t)da & \text{in } \Omega \times (0,T), \end{cases} \tag{4.7}$$

where β satisfies $(\mathcal{H}_\beta)$, $\lambda \in \mathbb{R}$, and u_0 and q satisfy $(\mathcal{H}_0^*)$ and $(\mathcal{H}_q)$, respectively.

Before proving the main result of this section we need a continuity result on the map $q \mapsto r_q$, where we recall that r_q is the unique solution of (4.3).

Lemma 4.5. *Assume that there exists a sequence $q_n \in L^\infty(0, A_\dagger)$, $n \in \mathbb{N}$, such that the sequence defined by*

$$Q_n(a) := \int_0^a q_n(s)ds$$

is increasing in $n \in \mathbb{N}$ and satisfies

$$Q_n(a) \nearrow Q(a) := \int_0^a q(s)ds, \quad \text{as } n \to \infty \text{ a.e. } a \in (0, A_\dagger).$$

Then, $r_{q_n} \searrow r_q$.

Proof. Since Q_n is increasing, r_{q_n} is decreasing, and since $Q_n(a) \nearrow Q(a)$, it follows that $r_{q_n} \geq r_q$. Then, there exists $r_* \geq r_q$ such that $r_{q_n} \searrow r_*$. We will prove that $r^* = r_q$. Consider the map $G_n(a) := \exp\left(-r_{q_n}a - Q_n(a)\right)$. It is not hard to prove that $G_n(a) \to \exp\left(-r_*a - Q(a)\right)$ for almost every $a \in (0, A_\dagger)$ as $n \to +\infty$ and $|G_n(a)| \leq e^{-r_*a}$, and by the dominated convergence theorem it follows that

$$1 = \int_0^{A_\dagger} \exp\Big(- r_*a - \int_0^a q(s)ds\Big) da;$$

hence, thanks to the definition of r_q, we have that $r_* = r_q$. □

The main result of this section reads as follows:

Theorem 4.6. *Suppose $(\mathcal{H}_\beta)$, $(\mathcal{H}_0^*)$, $(\mathcal{H}_q)$, and that $q \in C^0([0, A_\dagger))$ is increasing. Then, there exists a unique positive solution u of (4.7). Assume that there exists $0 \leq A_0 \leq A_\dagger$ such that supp$(\beta) \subset [0, A_0]$; then*

i) *if* $supp(u_0) \subset \bar{\Omega} \times (A_0, A_\dagger)$, *the solution of* (4.7) *satisfies* $u(x,a,t) = 0$ *for* $t > a$ *and* $x \in \Omega$; *therefore, for each* $A > 0$

$$u(\cdot,\cdot,t) \to 0 \text{ uniformly on } \bar{\Omega} \times [0, A] \text{ as } t \to +\infty.$$

ii) *Assume that the initial distribution* u_0 *satisfies*

$$supp(u_0) \cap (\Omega \times (0, A_0)) \neq \emptyset;$$

then

I) *if* $\lambda < \lambda_1 - r_q - 1$, *the solution of* (4.7) *satisfies*

$$u(\cdot,\cdot,t) \to 0 \text{ uniformly on } \bar{\Omega} \times [0, A_\dagger] \text{ as } t \to +\infty;$$

II) *if* $\lambda > \lambda_1 - r_q$, *the solution of* (4.7) *satisfies*

$$u(x,a,t) \to +\infty \text{ in } L^2(\Omega \times (0, A_\dagger)) \text{ as } t \to +\infty;$$

III) *if* $\lambda \in (\lambda_1 - r_q - 1, \lambda_1 - r_q)$ *and* $u_0(x,a) > 0$, *nontrivial, in* $\Omega \times (0, A_\dagger)$, *then* (4.7) *is permanent, in the sense of the definition of Theorem* 4.2.

Remark 4.7. Again, a detailed study of the stationary problem related to (4.7) would provide us the behaviour of the model for $\lambda = \lambda_1 - r_q$.

Proof. First, observe that

$$\lambda u + \frac{u}{1+u} = (\lambda + 1)u - \frac{u^2}{1+u}.$$

Note that $s^2/(1+s)$ does not satisfy (4.6); thus, we do not apply the results of the logistic problem.

Again $\underline{u} \equiv 0$ and $\overline{u} \equiv \omega_{\lambda+1}$ is a pair of sub-supersolutions of (4.7). Moreover, paragraph I) follows by Theorem 4.1 b) i).

On the other hand, since u is a supersolution of (4.1), we get that $\omega_\lambda \leq u$, whence we obtain II).

Fix $\lambda \in (\lambda_1 - r_q - 1, \lambda_1 - r_q)$. As in Theorem 4.2 it can be proved that

$$\underline{u}(x,a) := \varepsilon \exp\Big(- r_q a - \int_0^a q(s)ds \Big) \varphi_1(x)$$

is a subsolution of (4.7).

Now, fix $\lambda < \lambda_1 - r_q$ and take $\delta > 0$ such that $\lambda < \lambda_1 - r_q - \delta$. Thanks to the continuity of λ_1 with respect to the domain, there exists a domain Ω_1, $\Omega \subset \Omega_1$ such that $\lambda < \overline{\lambda}_1 - r_q - \delta$, where $\overline{\lambda}_1$ is the first eigenvalue associated to Ω_1. Finally, we claim that there exists $q_n \in \mathcal{C}^0([0, A_\dagger + 1/n))$, $q_n \leq q$, satisfying Lemma 4.5, and so that

$$\lambda < \overline{\lambda}_1 - r_{q_n} - \delta \quad \text{for } n \geq n_0 \text{ for some } n_0 \geq 0. \tag{4.8}$$

Indeed, for each $n \in \mathbb{N}$ define

$$q_n(s) := \begin{cases} q(0) & 0 \leq s \leq 1/n, \\ q(s - 1/n) & 1/n < s < A_\dagger + 1/n. \end{cases}$$

It is not hard to show that $q_n \leq q$ and that it satisfies the hypotheses of Lemma 4.5. Now, take

$$\overline{u}(x, a) := K \exp\Big(- r_{q_n} a - \int_0^a q_n(s) ds \Big) \overline{\varphi}_1(x),$$

where $\overline{\varphi}_1$ is the positive eigenfunction associated to $\overline{\lambda}_1$. We will show that $\overline{u}$ is a supersolution of (4.7) provided that K is sufficiently large. Indeed, observe that $\overline{u} > 0$ on $\partial\Omega$, $\overline{u}(x, 0) = \int_0^{A_\dagger} \beta(a)\overline{u}(x, a, t) da$ in Ω, and $\overline{u}(x, a) \geq u_0(a, x)$, for K large. In this last inequality we have used, again, that $\overline{\varphi}_1(x) \geq c_0 > 0$ in $\overline{\Omega}$ and that, since q_n is finite in $[0, A_\dagger]$,

$$\overline{u} \geq KC_0 > 0 \quad \text{in } \overline{\Omega} \times [0, A_\dagger]. \tag{4.9}$$

Finally,

$$\overline{u}_t + \overline{u}_a - \Delta\overline{u} + q(a)\overline{u} \geq (\lambda + 1)\overline{u} - \frac{\overline{u}^2}{1 + \overline{u}}$$

is equivalent to

$$\frac{\overline{u}}{1 + \overline{u}} \geq \lambda + 1 - \overline{\lambda}_1 + r_{q_n} + q_n(a) - q(a). \tag{4.10}$$

But, since $q_n \leq q$ and because of (4.8), we have that

$$\lambda + 1 - \overline{\lambda}_1 + r_{q_n} + q_n(a) - q(a) \leq 1 - \delta,$$

and by (4.9)

$$\frac{\overline{u}}{1 + \overline{u}} \geq \frac{KC_0}{1 + KC_0},$$

and so for K large (4.10) holds. The proof is complete. □

Remark 4.8. From a biological point of view, the hypothesis that q increasing makes perfect sense. Anywhere, it helps to clarify the exposure.

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