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SYMMETRY AND NONUNIFORMLY ELLIPTIC OPERATORS

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Abstract. The goal of this paper is to study the symmetry properties of nonnegative solutions of elliptic equations involving a nonuniformly elliptic operator. We consider on a ball the solutions of

$$\Delta_p u + f(u) = 0$$

with zero Dirichlet boundary conditions, for $p > 2$, where Δ_p is the p-Laplace operator and f a continuous nonlinearity. The main tools are a comparison result for weak solutions and a local moving-plane method which has been previously used in the $p = 2$ case. We prove local and global symmetry results when u is of class $C^{1,\gamma}$ for γ large enough, under some additional technical assumptions.

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1. Introduction and main results

The goal of this paper is to prove local and global symmetry results for the nonnegative solutions of

$$\begin{cases} \Delta_p u + f(u) = 0 & \text{in} \quad B \\ u \geq 0 & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator or p-Laplacian, and $B = B(0,1)$ is the unit ball in $\mathbb{R}^d$, $d \geq 1$. We will focus on the nonuniformly elliptic case, $p > 2$, and consider nonlinearities f which are continuous on $\mathbb{R}^+$.

In earlier works of two of the authors [14, 15, 16], it has been noted that for $p = 2$ the C^2 regularity of u is apparently a threshold in order to apply the moving-plane technique. Here we will see that the threshold for $p > 2$ is rather $C^{1,1/(p-1)}$. As in [16], additional properties of f are required at least close to its zeros. The main tools of this paper are the maximum principle when the nonlinearity is nonincreasing or, based on a local inversion, when $\nabla u \neq 0$, and a local moving-plane technique.

Our results are not entirely satisfactory, since the statements are rather technical, but on the other hand, they are a first achievement in obtaining symmetry results by comparison techniques in the case of nonuniformly elliptic operators. Let us start with a simple case, where we assume the following additional conditions on f.

(H1) *Let f be a real function defined on $\mathbb{R}^+$ such that*
- a) $f \in C^\alpha(\mathbb{R}^+)$ *for some* $\alpha \in (0,1)$,
- b) $\exists\, \theta > 0$ *such that* $f > 0$ *on* $(0,\theta)$, $f < 0$ *on* $(\theta, +\infty)$,
- c) f *is nonincreasing on* $(\theta - \eta, \theta + \eta)$ *for some* $\eta > 0$.

Theorem 1. (Symmetry result for the p-Laplacian with $p > 2$) *Let $p > 2$, $\gamma \in (1/(p-1), 1)$. Consider a function f satisfying* (H1) *and a solution $u \in C^{1,\gamma}(B)$ of* (1.1). *Then u is positive and radially symmetric on B.*

Remark 1. A consequence of the $C^{1,\gamma}$ regularity of the solution on the whole ball is that the maximum of this solution needs to be reached at the level $u = \theta$ (see Lemma 8). Does such a solution exist? Given a function f and using a shooting method, we obtain a solution on a ball of radius large enough. Then a rescaling gives a solution on B for an appropriately rescaled nonlinearity f, which provides an example of existence of a solution.

A result of symmetry similar to that of Theorem 1 also holds under weaker regularity assumptions on f.

(H2) *$f \in C^0(\mathbb{R}^+)$ and $f > 0$ on $(0, +\infty)$.*

Theorem 2. (Symmetry result with f only continuous) *Let $p > 2$ and $\gamma \in (1/(p-1), 1)$. Assume that* (H2) *holds, and consider a solution $u \in W^{1,p}(B) \cap L^\infty(B)$ of* (1.1) *such that, with $M = \|u\|_{L^\infty(B)}$,*

$$u \in C^2\left(B \setminus (\nabla u)^{-1}(0)\right) \cap C^{1,\gamma}\left(u^{-1}([0, M))\right).$$

If f is nonincreasing in a neighborhood of M, then u is positive and radially symmetric on B.

Remark 2. If we assume that u belongs to $C^{1,\gamma}(B)$ with $\gamma > 1/(p-1)$, then $f(M) = 0$. See Lemma 8. Reciprocally, if $f(M) \neq 0$, as we shall see in Example 1, a solution like the one of Theorem 2 has a regularity which is not better than $C^{1,1/(p-1)}$ at the points x such that $u(x) = M$. This is the reason why the $C^{1,\gamma}$ regularity cannot be assumed at such points.

The starting point of the moving-plane technique is the celebrated paper by Gidas, Ni, and Nirenberg [20] based on [1, 2, 26]. An improvement was made in [19, 5, 27] using the monotonicity properties of the nonlinearity. This led to a local moving-plane technique [14, 15, 16] in the case of the Laplacian, which is well suited for nonlinearities with low regularity. The corresponding notion of *local symmetry* also seems appropriate in the case of the p-Laplacian. Further interesting results were obtained in [17, 18].

For nonlinear elliptic operators, the story is shorter [8, 9, 10]. It strongly relies on comparison methods [29, 6, 7] but covers the case of the p-Laplace operator only when $p < 2$. For completeness, let us mention that related results have been obtained by rearrangement techniques, for instance in [23, 21], and that in [3, 4], F. Brock has been able to get symmetry results for the p-Laplace operator by a completely different approach based on a continuous Steiner symmetrization.

Our goal is to get local symmetry results for $p > 2$ using an adapted version of the maximum principle [11, 24, 25] and regularity properties [12, 28, 13, 22], with a *local moving-plane* method which is essentially adapted from [15, 16]. The paper is organized as follows. In the next section, we state a local symmetry result (Theorem 3). Theorem 2 is a simple corollary of Theorem 3. In the third section we start with two examples which motivate the choice of the class of regularity of the solutions and then state three useful lemmas. The fourth section is devoted to the proof of Theorem 3 by the *local moving-plane* method. Proofs of Theorems 1 and 2 are contained in the proof of Theorem 3.

2. A general result

In this section we present a local symmetry result, generalizing Theorem 2 to sign-changing functions f.

(H) *The function* $f \in C^0(\mathbb{R}^+)$ *has a finite number of zeros and* $f(0) \geq 0$. *Moreover, if* $f(u_0) = 0$ *for some* $u_0 \geq 0$, *then there exists* $\eta > 0$ *such that*
(a) *either* f *is nonincreasing on* $(u_0 - \eta, u_0 + \eta) \cap \mathbb{R}^+$,
(b) *or* $f(u) \geq 0$ *on* $(u_0, u_0 + \eta)$.

We say that $u_0 \in f^{-1}(0)$ belongs to Z_a in case (a) and to Z_b in case (b). Note that $f^{-1}(0) = Z_a \cup Z_b$.

Theorem 3. (Local symmetry result with f only continuous) *Let* $p > 2$ *and* $\gamma \in (1/(p-1), 1)$. *Assume that* f *satisfies* (H), *and consider a* $C^1(\overline{B})$ *solution* u *of* (1.1) *such that, with* $M = \|u\|_{L^\infty(B)}$,

$$u \in C^2\left(B \setminus (\nabla u)^{-1}(0)\right) \cap C^{1,\gamma}\left(u^{-1}([0, M))\right) \ .$$

If f *is nonincreasing in a neighborhood of* M *and if the set* $J := u^{-1}(Z_b) \cap (\nabla u)^{-1}(0)$ *is empty, then* u *is* locally radially symmetric *and the set* $u^{-1}([0, M)) \cap (\nabla u)^{-1}(0)$ *is contained in* Z_a.

Here *locally radially symmetric* [4] means that there exists an at-most-countable family $(u_i)_{i \in I}$ of radial functions with supports in balls B_i, $i \in I$, such that $u_{i|B_i}$ is a nonnegative, radial, nonincreasing solution of $\Delta_p u_i + f(u_i + C_i) = 0$ on B_i, where $u_{i|(B \setminus B_i)} \equiv 0$ and $C_i \geq 0$ is a constant satisfying $f(C_i) = 0$, and such that

$$u = \sum_{i \in I} u_i \ .$$

Remark 3. In the case of Theorem 3, I is finite. Monotonicity holds on balls or annuli. As a consequence, in Theorems 1 and 2, monotonicity holds along any radius. The monotonicity is even strict in case of Theorem 2, while there might be a plateau at level θ in case of Theorem 1.

The regularity of locally radially symmetric solutions can be studied by elementary methods, which, under an additional condition on f in case (a) of (H), allows us to give a sufficient condition under which all sufficiently smooth solutions are globally radially symmetric.

Corollary 4. *Let* $\alpha \in (0,(p-2)/2)$, *and consider* f *such that for any* $u_0 \in Z_a$,

$$0 < \liminf_{u \to u_0} \frac{|f(u)|}{|u-u_0|^\alpha} < +\infty \ .$$

Under the same assumptions as in Theorem 3, *if* $\frac{\alpha+1}{p-\alpha-1} < \gamma < 1$, *then* u *is globally radially symmetric and decreasing along any radius.*

This corollary is a consequence of Proposition 5, which will be stated in the next section.

3. Preliminaries

Let us start with two examples and a statement on the regularity of locally radially symmetric solutions of (1.1). Consider a radial solution in a ball centered at 0 given as a solution, with $r = |x|$, of the ordinary differential equation

$$\begin{cases} \frac{1}{r^{d-1}} \left(r^{d-1} \, |u'|^{p-2} \, u' \right)' + f(u) = 0 \ , \\ u(0) = u_0 > 0 \quad \text{and} \quad u'(0) = 0 \ . \end{cases} \tag{3.1}$$

Example 1. Let $f(u_0) \neq 0$ and $f(u) \equiv f(u_0)$ on a neighborhood of u_0. Then $u(r) = u_0 - A \operatorname{sgn}(f(u_0)) \, r^{\beta+1}$ is the solution of (3.1) in a neighborhood of $r = 0$, with $\beta = 1/(p-1)$ and $A = (|f(u_0)|/d)^\beta/(\beta+1)$. The same formula for u is true up to lower-order terms if f is smooth but not constant. In any case, u is exactly in $C^{1,1/(p-1)}$ at $r = 0$, and this is why the level $u = M$ has to be excluded in our results if $f(M) \neq 0$.

Example 2. Let $f(u) = |u_0 - u|^{\alpha-1}(u_0 - u)$. Then exactly as above, $u(r) = u_0 \pm A \, r^{\beta+1}$ is a solution of (3.1) in a neighborhood of $r = 0$, with $\beta = (\alpha+1)/(p-\alpha-1)$ and for some $A > 0$, which depends on d, α, and p. In this case, u is exactly in $C^{1,\beta}$ at $r = 0$ with $\beta > 1/(p-1)$ if $\alpha > 0$. However, there is no uniqueness since $u \equiv u_0$ is also a solution. Actually we may find a continuum of solutions as follows: take $u \equiv u_0$ on $(0, r_0)$ for some $r_0 > 0$, and $u(r) = u_0 \pm A \, (r - r_0)^{\beta+1}(1 + o(r - r_0))$, for r in a neighborhood of r_0^+. This solution also has exactly a $C^{1,\beta}$ regularity.

Remark 4. If $u_0 \in Z_b$, then there is no radial solution with $u(0) > u_0$ and $\nabla u = 0$ on $\{u = u_0\}$, because of Lemma 9 (strong maximum principle and Hopf's lemma; see below).

These two examples unveil regularity, not only for radially symmetric solutions of (1.1) but also for locally radially symmetric solutions, since

(1.1) is invariant under translation. This can be stated as follows (the proof is left to the reader).

Proposition 5. *Assume that f is continuous. Any locally radially symmetric solution of* (1.1) *is of class C^2 outside of its critical set. At a critical point $x_0 \in B$, u is exactly in $C^{1,1/(p-1)}$ if $u_0 = u(x_0) \notin f^{-1}(0)$. If we further assume that f satisfies the hypotheses of Corollary* 4*, then u is either locally constant or at most of class $C^{1,\beta}$ with $\beta = (\alpha+1)/(p-\alpha-1)$.*

The proof of Corollary 4 is a straightforward consequence of Theorem 3 and Proposition 5. The next result relates the assumption $u \geq 0$ with the fact that, in the definition of the local radial symmetry, $u_i \geq 0$ on B_i.

Corollary 6. *Assume that f is continuous, and let u be a solution of*

$$\begin{cases} \Delta_p u + f(u) = 0 & \text{in} \quad B\ , \\ u = 0 & \text{on} \quad \partial B\ , \end{cases}$$

having the following property: there exists an at-most-countable family $(u_i)_{i\in I}$ of radial monotone functions with supports in balls $B_i \subset B$ such that $u = \sum_{i\in I} u_i$, where u_i satisfies $\Delta_p u_i + f(u_i + C_i) = 0$ on B_i, $u_{i|\partial B_i} \equiv 0$, the constants C_i are all nonnegative, and $f(C_i) = 0$ for all $i \in I$.

If $d \geq 2$ and $u \geq 0$, then $u_i > 0$ in B_i for any $i \in I$, and u is locally radially symmetric in the sense of the definition given in Section 2*. Moreover, for any $d \geq 1$, if there exists an $i \in I$ such that $\inf_{B_i} u_i < 0$, then $\inf_{B_i} u < 0$.*

Proof. Under the assumptions of Theorem 3, this is a straightforward consequence of the proof. The general case is an extension to $p > 2$ of a result which has been proved for $p = 2$ in [14], Proposition 1 (also see [16]). □

Before proving Theorem 3, we shall state three lemmata. Let us start with a weak comparison result due to Montenegro [24] (also see [11, 25] for related results).

Lemma 7. (Weak comparison principle) *Let u and v be solutions in $W^{1,p} \cap L^\infty(\omega)$ of*

$$\Delta_p u + f(u) = 0 \quad \text{and} \quad \Delta_p v + f(v) = 0\ ,$$

respectively, where ω is a bounded, open, connected set in $\mathbb{R}^d$ with a C^1-by-parts boundary. Assume that f is a nonincreasing function on $u(\omega) \cup v(\omega)$. If $u \geq v$ on $\partial\omega$ almost everywhere, then $u \geq v$ in ω almost everywhere.

Lemma 8. (Characterization of the critical set) *Consider a domain ω in $\mathbb{R}^d$. Let $u \in C^{1,\gamma}(\omega)$ be a solution of*

$$\Delta_p u + f(u) = 0 \quad \text{in} \quad \omega,$$

and consider $x_0 \in \overline{\omega}$ *such that* $\nabla u(x_0) = 0$*. If* $x_0 \in \partial\omega$*, assume moreover that* $u \in C^{1,\gamma}(\overline{\omega})$*,* $\partial\omega$ *is Lipschitz, and* $A(x_0) = \lim_{\epsilon\to 0} \epsilon^{-d}\,\mathrm{Vol}(\{x \in \omega\ :\ |x - x_0| < \epsilon\}) > 0$*. If* $\gamma > 1/(p-1)$*, then* $f(u(x_0)) = 0$*.*

Proof. Assume for simplicity that $x_0 = 0$ and $u(x_0) = 0$. Consider then $u^\epsilon(x) = \epsilon^{-1} u(\epsilon x)$. A straightforward computation gives

$$\epsilon^{-1}\Delta_p u^\epsilon + f(\epsilon\, u^\epsilon) = 0\ .$$

Because of the $C^{1,\gamma}$ regularity of u,

$$|\nabla u^\epsilon(x)|^{p-1} \le C\,|x|^{\gamma(p-1)}\epsilon^{\gamma(p-1)}$$

for some nonnegative constant C. Let ϕ be a nonnegative, nonzero, radial test function; then

$$\begin{aligned} f(0)\,A(x_0)\int_{\mathbb{R}^d}\phi\,dx &= \lim_{\epsilon\to 0}\int_{\epsilon^{-1}\omega} f(\epsilon u^\epsilon)\,\phi\,dx \\ &= \lim_{\epsilon\to 0}\epsilon^{-1}\Big(\int_{\epsilon^{-1}\omega}|\nabla u^\epsilon|^{p-2}\nabla u^\epsilon\cdot\nabla\phi\,dx\Big) \\ &\le C\,\lim_{\epsilon\to 0}\epsilon^{\gamma(p-1)-1}\int_{\mathbb{R}^d}|x|^{\gamma(p-1)}|\nabla\phi|\,dx = 0\ , \end{aligned}$$

which proves that $f(0) = 0$. Here we assume $A(x_0) = |S^{d-1}|/d$ if $x_0 \in \omega$. □

For the completeness of this paper, we finally recall a result due to Vázquez [29].

Lemma 9. (Strong maximum principle and Hopf's lemma) *Let* ω *be a domain in* $\mathbb{R}^d$*,* $d \ge 1$*, and let* $u \in C^1(\omega)$ *be such that* $\Delta_p u \in L^2_{\mathrm{loc}}(\omega)$*,* $u \ge 0$ *almost everywhere in* ω*. Assume that*

$$-\Delta_p u + \beta(u) \ge 0 \quad a.e. \quad in \quad \omega$$

with $\beta : [0, +\infty) \to \mathbb{R}$ *continuous, nondecreasing,* $\beta(0) = 0$ *and such that either there exists an* $\epsilon > 0$ *for which* $\beta \equiv 0$ *on* $(0, \epsilon)$ *or* $\beta(s) > 0$ *for any* $s > 0$ *and* $\int_0^1 (s\,\beta(s))^{-1/p}\,ds = +\infty$*. Then if* u *does not vanish identically on* ω*, it is positive everywhere in* ω*. Moreover, if* $u \in C^1(\omega \cup \{x_0\})$ *for an* $x_0 \in \partial\omega$ *that satisfies an interior sphere condition and* $u(x_0) = 0$*, then the derivative along the unit outgoing normal vector at* x_0 *satisfies* $\frac{\partial u}{\partial \nu}(x_0) > 0$*.*

4. Proof of Theorem 3

We will split the proof of Theorem 3 into five steps which are more or less classical in moving-plane techniques. In the first one, we prove that one can start the method. Then we characterize the obstructions and get the global symmetry result when there is no obstruction. In the third and fourth steps, we prove that obstructions are due only to balls on which the solution is radially symmetric. Finally, in the fifth step we establish the local symmetry result. We also indicate how the special cases of Theorems 1 and 2 can be treated.

Consider $e_1 \in S^{d-1}$, and denote by x_1 the coordinate along the direction given by e_1, and by x' the coordinate in the orthogonal hyperplane identified with $\mathbb{R}^{d-1}$. Let $\lambda > 0$ and

$$T_\lambda = \{x \in B \ : \ x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, \ x_1 = \lambda\} \ ,$$
$$\Sigma_\lambda = \{x \in B \ : \ x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, \ x_1 > \lambda\} \ ,$$
$$x_\lambda = (2\lambda - x_1, x') \quad \text{if} \quad x = (x_1, x') \ , \quad u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda = u_\lambda - u \ .$$

Step 1: Starting the moving-plane method. Consider $\lambda_0 = \sup\{x_1 \in (0,1] \ : \ \exists\, x' \text{ such that } (x_1, x') \in \overline{B} \text{ and } u(x_1, x') > 0\}$. Up to a change of coordinates, we may therefore assume that $\lambda_0 > 0$. We claim that the moving-plane method can be started; that is, there is a $\overline{\lambda} < \lambda_0$ such that $w_\lambda \geq 0$ in Σ_λ for any $\lambda \in (\overline{\lambda}, \lambda_0)$.

Let us assume first that $0 \in Z_b$ or $f(0) > 0$. In this case we notice that for some $\eta > 0$,

$$-\Delta_p u \geq 0 \quad \text{on } \{x \in B \ : \ 0 < u(x) < \eta\} \ .$$

We can use Lemma 9 to conclude that $u > 0$ in B, and $\frac{\partial u}{\partial x_1}(1, 0) < 0$. Thus $\lambda_0 = 1$ and the claim is proved.

Next we assume that $0 \in Z_a$; that is, f is nonincreasing on $(0, \eta)$. Assume for the sake of contradiction that there exists a sequence $\lambda_n \nearrow 1$, such that for each $n \in \mathbb{N}$, there exists $x_n \in \Sigma_{\lambda_n}$ such that

$$w_{\lambda_n}(x_n) < 0 \ .$$

For n large, we can certainly have that $\eta > \sup_{x \in \Sigma_{\lambda_n}} u(x)$. Noticing that on $\partial \Sigma_{\lambda_n}$ we have $u_{\lambda_n} \geq u$, we may apply Lemma 7 with $\omega = \Sigma_{\lambda_n}$ and get a contradiction that proves the claim.

Step 2: Characterizing the obstructions. Let $\overline{\lambda} = \inf\{\lambda > 0 \ : \ w_\lambda(x) \geq 0 \quad \forall\, x \in \Sigma_\lambda\}$. We assume that $\overline{\lambda} > 0$: there exists a sequence $\lambda_n \nearrow \overline{\lambda}$ and a sequence $x_n \in \Sigma_{\lambda_n}$ such that $w_{\lambda_n}(x_n) < 0$. We may assume that x_n is a minimum point of w_{λ_n}, so that $\nabla w_{\lambda_n}(x_n) = 0$. Up to the extraction of a subsequence, x_n converges as $n \to +\infty$ to some $\overline{x} \in \overline{\Sigma_{\overline{\lambda}}}$, which satisfies $w_{\overline{\lambda}}(\overline{x}) = 0$ and $\nabla w_{\overline{\lambda}}(\overline{x}) = 0$. We call this point $\overline{x}$ *an obstruction.*

We observe that in case $0 \in Z_b$ or $f(0) > 0$, u is decreasing in the direction x_1 near $\partial B \cap \overline{\Sigma}_\lambda$ for λ close to $\overline{\lambda}$ and so that $\overline{x} \notin \partial B$. Then one of the following cases holds:

1) $\overline{x} \in \Sigma_{\overline{\lambda}}$. At this point we may have $\nabla u(\overline{x}) = 0$, which implies that $f(u(\overline{x})) = 0$ by Lemma 8 if $u(\overline{x}) < M$, or $w_{\overline{\lambda}} \equiv 0$ on $B(\overline{x}, \epsilon)$ for some $\epsilon > 0$. In fact, if $\nabla u(\overline{x}) \neq 0$, we can use a local inversion method and the strong maximum principle as in [15] to conclude that $w_{\overline{\lambda}} \equiv 0$ near $\overline{x}$, since $w_{\overline{\lambda}}$ has a local minimum at $\overline{x}$.
2) $\overline{x} \in T_{\overline{\lambda}} \cap B$. Since $\overline{x} \in B$ it is standard that $\frac{\partial u}{\partial x_1}(\overline{x}) = -\frac{1}{2}\frac{\partial w_{\overline{\lambda}}}{\partial x_1}(\overline{x}) = 0$, which implies that $\nabla w_{\overline{\lambda}}(\overline{x}) = \nabla u(\overline{x}) = 0$. Actually, if $\nabla u(\overline{x}) \neq 0$, we can use a local inversion method as in [15] and get a contradiction with Hopf's lemma for usual elliptic operators. Now we have two possibilities: either $u(\overline{x}) = M$, or we can use Lemma 8 to find that $f(u(\overline{x})) = 0$. In summary, either $u(\overline{x}) = M$ or $f(u(\overline{x})) = 0$, and $\nabla u(\overline{x}) = 0$.
3) In case $0 \in Z_a$ it may occur that $\overline{x} \in \partial B \cap \overline{\Sigma}_{\overline{\lambda}}$. If $\overline{x} \in \partial B \cap \overline{\Sigma}_{\overline{\lambda}}$ then $u_{\overline{\lambda}}(\overline{x}) = u(\overline{x}) = 0$, and so $\nabla u_{\overline{\lambda}}(\overline{x}) = 0$, since $\overline{x}_{\overline{\lambda}} \in B$. Consequently, if $\nabla u(\overline{x}) \neq 0$ we have that $\nabla w_{\overline{\lambda}}(\overline{x}) \cdot e_1 < 0$, which is a contradiction. Thus we must have $\nabla u(\overline{x}) = 0$. If $\overline{x} \in \overline{T}_{\overline{\lambda}} \cap B$ then $\partial w_{\overline{\lambda}}/\partial x_1(\overline{x}) = -2\partial u/\partial x_1(\overline{x}) = 0$, and we also have $\nabla u(\overline{x}) = 0$.

Step 3: The obstruction set and the critical points of u. Let $\overline{\lambda}$ be the first value of λ for which an obstruction point $\overline{x}$ appears in applying the moving-plane method. Remark that all possible obstruction points $\overline{x}$ are contained in the set

$$K := \left\{x \in \partial\overline{\Sigma}_{\overline{\lambda}} \ : \ \nabla u(x) = 0\right\} \cup \left\{x \in \overline{\Sigma}_{\overline{\lambda}} \setminus \overline{T}_{\overline{\lambda}} \ : \ u(x) = u_{\overline{\lambda}}(x)\right\} \ .$$

We claim that the set $K_1 = \{x \in K \ : \ \nabla u(x) \neq 0\}$ is not empty.

Assume for the sake of contradiction that $K_1 = \emptyset$: $\nabla u(x) = 0$ for all $x \in K$. Let us denote by $Q_\varepsilon(x)$ the set $\{y \in \mathbb{R}^d \ : \ \max_{i=1,2,\ldots,d} |y_i - x_i| < \varepsilon\}$, where $\varepsilon > 0$ is taken small. From the covering $\cup_{x\in K} Q_\varepsilon(x)$ of the compact set K if $0 \in Z_b$ or $f(0) > 0$, and of the compact set $K \cup \left(\overline{K} \cap \overline{T}_{\overline{\lambda}} \cap \partial B\right)$ if $0 \in Z_a$, we can extract a finite subcovering corresponding to points $x_j \in K$, $j = 1, 2, \ldots, L$ for some number L, and define the open set

$$\omega_\varepsilon := \bigcup_{j=1}^{L} Q_\varepsilon(x_j) \supset K \ .$$

If $0 \in Z_b$ or by Lemma 9 if $f(0) > 0$, we can choose ε small enough so that $\omega_\varepsilon \subset B$ and observe that by construction ω_ε has a boundary of class C^1 by parts. Next we define $\Gamma = \partial\omega_\varepsilon$. We can find $\delta_1 > 0$ and $\delta > 0$ such that

$$u_{\overline{\lambda}} \geq u + \delta \quad \text{on} \quad \overline{\Sigma}_{\overline{\lambda}+\delta_1} \cap \Gamma \; .$$

Next we claim that

$$\frac{\partial u}{\partial x_1} < -\delta \quad \text{on} \quad T_{\overline{\lambda}} \cap \Gamma \; ,$$

for a possibly smaller $\delta > 0$. If the claim were false, then there would be a point $x_0 \in T_{\overline{\lambda}} \cap \Gamma$ with $\frac{\partial u}{\partial x_1}(x_0) = 0$. But then, since $x_0 \in \Gamma$ and $u_{\overline{\lambda}}(x_0) = u(x_0)$, we have $\nabla u(x_0) \neq 0$. Thus, we can use a local inversion method and the strong maximum principle as in [15] to prove that $u_{\overline{\lambda}} \equiv u$ in a neighborhood of x_0 in $\Sigma_{\overline{\lambda}}$, and then $K_1 \neq \emptyset$, which is a contradiction with our assumption.

If $0 \in Z_a$, we take δ_1 smaller than $\operatorname{dist}\left(\overline{T}_{\overline{\lambda}} \cap \partial B, \partial\omega_\varepsilon\right)$, and we replace ω_ε by $\omega_\varepsilon \cap B$ and Γ by $\partial\omega_\varepsilon \cap \overline{B}$. Let us consider $\eta > 0$ small. First we analyze the behavior of u and u_λ in $A_1 = (\overline{B} \setminus B(0, 1-\eta)) \cap \overline{\Sigma}_{\overline{\lambda}+\delta_1}$. For any $x_0 \in \Gamma \cap \partial B \cap \overline{\Sigma}_{\overline{\lambda}+\delta_1}$, by definition of K, $u_{\overline{\lambda}}(x_0) = u_{\overline{\lambda}}(x_0) - u(x_0) \geq \delta_0 > 0$.

By continuity, for x in a neighborhood of x_0 and for $\overline{\lambda} - \lambda > 0$ small enough, we have $u_\lambda(x_0) - u(x_0) \geq \delta_0/2 > 0$. Taking η smaller if necessary, this implies that

$$u_\lambda \geq u \quad \text{in} \quad \Gamma \cap A_1,$$

for $\overline{\lambda} - \lambda > 0$ small enough. Next we consider $A_2 = \overline{B}(0, 1-\eta) \cap \Sigma_{\overline{\lambda}+\delta_1}$. Then by continuity, the property

$$u_{\overline{\lambda}} \geq u + \delta \quad \text{on} \quad \Gamma \cap A_2$$

can be extended to

$$u_\lambda \geq u + \delta/2 \quad \text{on} \quad \Gamma \cap A_2 \; ,$$

for $\overline{\lambda} - \lambda > 0$ small enough. The property

$$\frac{\partial u}{\partial x_1} < -\delta \quad \text{on} \quad T_{\overline{\lambda}} \cap \Gamma$$

holds for the same reasons as in the first case.

Thus, in both cases, using continuity again, for possibly smaller δ and δ_1, we see that

$$\frac{\partial u}{\partial x_1} < -\delta \quad \text{on} \quad \left(\overline{\Sigma}_{\overline{\lambda}} \backslash \Sigma_{\overline{\lambda}+\delta_1}\right) \cap \Gamma$$

and, for $\eta > 0$ small enough, we also have

$$u_\lambda \geq u + \delta/2 \quad \text{on} \quad \overline{\Sigma}_{\overline{\lambda}+\delta_1} \cap \Gamma \; , \quad \text{for} \quad \lambda \in (\overline{\lambda} - \eta, \overline{\lambda}) \; ,$$

$$\frac{\partial u}{\partial x_1} \le -\delta/2 \quad \text{on} \quad \left(\overline{\Sigma}_\lambda \backslash \Sigma_{\overline{\lambda}+\delta_1}\right) \cap \Gamma\ , \quad \text{for} \quad \lambda \in (\overline{\lambda} - \eta, \overline{\lambda})\ .$$

From the last inequality, we get

$$u_\lambda \ge u \quad \text{on} \quad \left(\overline{\Sigma}_\lambda \backslash \Sigma_{\overline{\lambda}+\delta_1}\right) \cap \Gamma\ ,$$

from which it follows that

$$u_\lambda \ge u \quad \text{on} \quad \Gamma \cap \Sigma_\lambda\ .$$

But we also have $u_\lambda = u$ on $T_\lambda \cap \omega_\varepsilon$ and $u_\lambda \ge u$ in $\partial B \cap \Gamma$. As a consequence, for $\omega_\varepsilon^\lambda = \omega_\varepsilon \cap \Sigma_\lambda$, we get

$$u_\lambda \ge u \quad \text{on} \quad \partial\omega_\varepsilon^\lambda\ .$$

By Lemma 8 we have $f(u(K) \setminus \{M\}) \subset \{0\}$, and then, by the assumptions of Theorem 3, $u(K) \subset Z_a \cup \{M\}$. We recall that in a neighborhood of every point in $Z_a \cup \{M\}$, f is nonincreasing, by hypothesis. Thus, for ε and η small enough, this guarantees that f is nonincreasing on $u\left(\omega_\varepsilon^\lambda\right) \cup u_\lambda\left(\omega_\varepsilon^\lambda\right)$. Then Lemma 7 applies and gives that

$$u_\lambda \ge u \quad \text{on} \quad \omega_\varepsilon^\lambda\ ,$$

for $\lambda \in (\overline{\lambda} - \eta, \overline{\lambda})$. This proves that there is no obstruction in $\omega_\varepsilon^\lambda$, a contradiction: $K_1 \ne \emptyset$.

Step 4: Getting a local symmetry result. Let us notice that $K_1 \subset \Sigma_{\overline{\lambda}}$. Because of the local inversion method and the strong maximum principle as in [15] again, K_1 is an open set. Let C^+ be one of its connected components:

$$\nabla u = 0 \quad \text{on} \quad \Gamma^+ = \partial C^+ \cap \left(\overline{\Sigma}_{\overline{\lambda}} \backslash T_{\overline{\lambda}}\right)\ .$$

By Lemma 8 and the assumptions of Theorem 3, $u\left(\Gamma^+\right) \subset Z_a \cup \{M\}$ and u is equal to a constant on each component of $\partial C^+ \cap \Sigma_{\overline{\lambda}}$. This in particular implies that $\overline{C^+} \cap T_{\overline{\lambda}} \ne \emptyset$; otherwise, we would have $\overline{C^+} \subset \overline{\Sigma}_{\overline{\lambda}} \backslash T_{\overline{\lambda}}$: the fact that u is constant in each connected component of ∂C^+ would contradict the fact that u is nondecreasing and $\nabla u \ne 0$ in C^+.

Define $C = \overline{C^+ \cup C^-}$, where C^- is the reflection of C^+ with respect to $T_{\overline{\lambda}}$. The set C is connected and symmetric with respect to $T_{\overline{\lambda}}$, and by construction the function u is symmetric on C with respect to $T_{\overline{\lambda}}$.

Step 5: Getting a global symmetry result. Let us define $u_1 = \inf_C u$ and $u_2 = \sup_C u$. By monotonicity of u in the x_1 direction, we know that u takes either the value $u_1 > 0$ on the boundary of C, where $\nabla u = 0$, or $u_1 = 0$ on $\partial C \cap \partial B$. Consequently, by Lemma 8, we have $f(u_1) = 0$ if $u_1 \ne 0$. Similarly, either u takes the value u_2 on another connected component of the boundary of C and then $f(u_2) = 0$, or u reaches its maximum in an interior point $\overline{x} \in C$ that we may take in $T_{\overline{\lambda}} \cap C$.

Because u_1 and u_2 belong to $\{0\} \cup Z_a \cup \{M\}$, which is a finite set, and $|\nabla u|$ is uniformly bounded in B, we deduce that C contains a ball of some radius $r_0 > 0$ which depends only on f and u. There are therefore only a finite number of such components C. Moreover, if we consider all possible sets C, symmetric with respect to a certain hyperplane, along which u is increasing and where $\nabla u = 0$ on ∂C (except maybe for $u_1 = 0$), we still have a finite number, say $\mathcal{N}_0$.

Now we consider $\mathcal{N} = d\mathcal{N}_0 + 1$ directions $\gamma_i \in S^{d-1}$, $i = 1, 2, \ldots, \mathcal{N}$, such that the angle (γ_i, γ_j) is 2π-irrational for any (i, j) with $i \neq j$, and such that any family of d such unit vectors generates $\mathbb{R}^d$. It is not hard to prove that a set which is symmetric with respect to those directions is actually radially symmetric, with respect to some point in it. See [16] for more details. Thus, if we apply the moving-plane procedure in all the directions given above, we find a connected component C which is radially symmetric, i.e., an annulus (or a ball) $C = B_{r_1}(x_0) \backslash B_{r_2}(x_0)$ with $r_1 > r_2 \geq 0$, and such that $u = u_1$ on $\partial B_{r_1}(x_0)$ and $u = u_2$ on $\partial B_{r_2}(x_0)$. Then we can consider the following modifications of the function u. First we consider v_1 defined in B as $v_1 \equiv u_1$ in $B_{r_1}(x_0)$ and $v_1 = u$ in $B \setminus B_{r_1}(x_0)$. Second, in case $r_2 > 0$, we define $v_2 = u$ in $B_{r_2}(x_0)$.

At this point, under the hypothesis of Theorem 1, the function u is radially symmetric, and $M = \theta$. We observe that the set $\{x : u(x) = M\}$ is a closed ball of radius $r \geq 0$.

Under the hypothesis of Theorem 2, we also see that the function u is radially symmetric.

In the case of Theorem 3, we reapply steps 1–5 to the function $x \mapsto v_1(x)$ in B, and to the function $x \mapsto v_2(x) - u_2$ in $B_{r_2}(x_0)$. We remark that $u \geq u_2$ on $B_{r_2}(x_0)$ because of the monotonicity of the solution in x_1. Thus all assumptions of Theorem 3 hold with B replaced by $B_{r_2}(x_0)$. A finite iteration provides the result. □

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