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SMOOTH BIFURCATION FOR AN OBSTACLE PROBLEM

Jan Eisner and Milan Kučera
Mathematical Institute of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Prague 1, Czech Republic

Lutz Recke
Institute of Mathematics of the Humboldt University of Berlin
Unter den Linden 6, 10099 Berlin, Germany

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Abstract. The existence of smooth families of solutions bifurcating from the trivial solution for a two-parameter bifurcation problem for a class of variational inequalities is proved. As an example, a model of an elastic beam compressed by a force λ and supported by a unilateral connected fixed obstacle at the height h is studied. In the language of this example, we show that nontrivial solutions touching the obstacle on connected intervals bifurcate from the trivial solution and form smooth families parametrized by λ and h. In particular, the corresponding contact intervals depend smoothly on λ and h.

1. Introduction

Let us denote by H the Sobolev space $W^{2,2}(-1,1) \cap W_0^{1,2}(-1,1)$ with the inner product $\langle u, v\rangle := \int_{-1}^{1} u''v''dx$, and set

$$K_h := \{u \in H : \, u(x) \le h \text{ for all } x \in [-1,1]\}.$$

Let U be an open set in H and $F : \mathbb{R} \times U \to H$ a C^1-smooth map such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. We will consider the bifurcation problem for a variational inequality of the type

$$\lambda \in \mathbb{R},\, h \ge 0,\, u \in U \cap K_h : \, \langle F(\lambda, u), \varphi - u\rangle \ge 0 \text{ for all } \varphi \in K_h \qquad (1.1)$$

with bifurcation parameters λ and h. The assumptions about F will be specified later. Roughly speaking, F will be generated by a quasilinear fourth-order ordinary differential operator in a divergence form. For a particular choice of this differential operator, (1.1) will describe the equilibrium

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states of a unilaterally supported elastic beam which is simply fixed at its ends, compressed by a force corresponding to the parameter λ and supported unilaterally on the whole interval by a fixed connected obstacle given by h.

Our goal is to show that smooth families of solutions touching the obstacle in an interval and parametrized by λ and h, bifurcate from the trivial solution $u = 0$ at any $\lambda > a_0\pi^2$ and $h = 0$, where a_0 will be specified later. Roughly speaking, we will prove the following result. For arbitrary positive integers k and ℓ there is a family of solutions $(\lambda, h, \overline{u}_{k\ell}(\lambda, h))$, where $\overline{u}_{k\ell}$ is a C^1 mapping on a certain relatively open subset $\mathcal{U}_{k\ell}$ of the closed halfplane $\{(\lambda, h) \in \mathbb{R}^2 : h \geq 0\}$,

$$\{(\lambda, h) \in \mathcal{U}_{k\ell} : \ h = 0\} = \Big\{(\lambda, 0) \in \mathbb{R}^2 : \ \lambda > a_0\Big(\frac{k+\ell}{2}\pi\Big)^2\Big\},$$

such that $\overline{u}_{k\ell}(\lambda, 0) = 0$ and

$$\{x \in [-1, 1] : (\overline{u}_{k\ell}(\lambda, h))\,(x) = h\} = \big[\overline{\alpha}_{k\ell}(\lambda, h), \overline{\beta}_{k\ell}(\lambda, h)\big] \ \text{ for } h > 0$$

with C^1 maps $\overline{\alpha}_{k\ell}$ and $\overline{\beta}_{k\ell}$ satisfying

$$\overline{\alpha}_{k\ell}(\lambda, 0) = -1 + \sqrt{\frac{a_0}{\lambda}}k\pi, \ \ \overline{\beta}_{k\ell}(\lambda, 0) = 1 - \sqrt{\frac{a_0}{\lambda}}\ell\pi \ \text{ for all } \lambda > a_0\Big(\frac{k+\ell}{2}\pi\Big)^2.$$

Moreover, we have $\overline{u}_{k\ell}(\lambda, h) = hu_{k\ell}(\lambda) + o(h)$ for $h \to 0$, where $u_{k\ell}(\lambda)$ is a solution of the "linearized inequality"

$$u \in K_1 : \ \big\langle F'_u(\lambda, 0)u, \varphi - u\big\rangle \geq 0 \ \text{ for all } \varphi \in K_1 \tag{1.2}$$

such that $\{x \in [-1, 1] : (u_{k\ell}(\lambda))\,(x) = 1\} = \big[-1 + \sqrt{\frac{a_0}{\lambda}}k\pi, 1 - \sqrt{\frac{a_0}{\lambda}}\ell\pi\big]$.

Let us recall that our variational inequality is a nonsmooth nonlinearizable problem, and the "linearization" (1.2) is in fact a nonlinear problem again. In particular, a direct use of theorems of analytic bifurcation theory is impossible.

The basic idea of the proof of our result is similar to that in the paper [11]. However, in [11] we discussed no bifurcation but a smooth dependence of solutions u on the parameters in a neighborhood of a given solution (λ_0, h_0, u_0), where u_0 touches the obstacle in an interval. In fact, the existence of such a solution is rigorously proved only in the present paper. We show that under our assumptions, the set of all nontrivial solutions to (1.1) near a given $(\lambda_0, 0, 0)$, $\lambda_0 > a_0\pi^2$, for which the contact set is an interval close to a given one, is diffeomorphic to the set of all solutions to a certain C^1-smooth operator equation. Simple geometrical ideas together with a hard regularity analysis lead to this equivalence result. For the local description of the bifurcating solutions in a neighbourhood of λ_0, the classical implicit function theorem and a scaling technique are then applied to the operator equation mentioned. Unifying such local information for all λ_0 in the unbounded interval under consideration, we obtain families defined globally on the sets $\mathcal{U}_{k\ell}$ mentioned above.

The implicit function theorem in a nonstandard way was used for obtaining smooth families of bifurcating solutions of general variational inequalities already in [10]. However, we considered there particular assumptions guaranteeing that the set of contact with the obstacle of nontrivial solutions in a neighbourhood of a bifurcation point is a priori constant. The application to a model of a beam with a finite number of unilateral obstacles at zero was given. In that case it was essentially simpler to find a suitable operator equation locally equivalent to our variational inequality. Cf. also [4], where an application to a bifurcation of spatial patterns in a reaction–diffusion system with nonlocal unilateral boundary conditions was given. In the paper [2], a smooth continuation of solutions to variational inequalities was given in the same framework.

Various methods were developed for the study of bifurcation for variational inequalities (see e.g. [7, 3, 9, 1, 12, 6]; see [5] for a survey). However, as far as we know, no result about a smoothness (differentiability) of bifurcation branches has been published with the exception of the papers mentioned above.

The precise formulation of the problem and the main result (Theorem 2.2) as well as a related local assertion (Theorem 2.5) and an explanation of basic ideas including the equivalence result mentioned above (Theorem 2.7) are given in Section 2. Section 3 contains technical tools necessary for their proofs, which are the subject of Section 4.

2. Main results and ideas

We will work in the Hilbert space $H := W^{2,2}(-1,1) \cap W_0^{1,2}(-1,1)$ with the inner product $\langle u, v\rangle = \int_{-1}^{1} u''v''\,\mathrm{d}x$ and the corresponding norm $\|\cdot\|$.

We consider an open interval J containing zero and functions

$$a \in C^3(J),\ f \in C^2(\mathbb{R}\times J\times\mathbb{R}), \tag{2.1}$$

$$a(\xi) > 0 \text{ for all } \xi \in J, \tag{2.2}$$

$$f(\lambda,0,0) = \lambda \text{ for all } \lambda \in \mathbb{R}. \tag{2.3}$$

We will assume that for any $(\lambda,\xi) \in \mathbb{R}\times J$ there exists $c > 0$ such that for all $\eta, \eta_1, \eta_2 \in \mathbb{R}$ we have

$$|f(\lambda,\xi,\eta)| + |\partial_1 f(\lambda,\xi,\eta)| + |\partial_2 f(\lambda,\xi,\eta)| \le c(1+|\eta|^2), \tag{2.4}$$

$$|\partial_3 f(\lambda,\xi,\eta_1) - \partial_3 f(\lambda,\xi,\eta_2)| \le c\,|\eta_1-\eta_2|\,. \tag{2.5}$$

Moreover, let us assume that for any $(\lambda,\xi)\in\mathbb{R}\times J$ and $\varepsilon>0$ there exists $\delta>0$ such that for all $\tilde\lambda,\tilde\xi,\eta\in\mathbb{R}$ with $|\lambda-\tilde\lambda|+|\xi-\tilde\xi|\le\delta$ we have

$$|\partial_1 f(\lambda,\xi,\eta)-\partial_1 f\ (\tilde\lambda,\tilde\xi,\eta)|+|\partial_2 f(\lambda,\xi,\eta)-\partial_2 f(\tilde\lambda,\tilde\xi,\eta)|\le\varepsilon(1+|\eta|^2), \tag{2.6}$$

$$|\partial_3 f(\lambda,\xi,\eta)-\partial_3 f(\tilde\lambda,\tilde\xi,\eta)|\le\varepsilon(1+|\eta|). \tag{2.7}$$

Let us introduce an open set U and closed convex sets K_h in H defined by

$$U:=\{u\in H:\ u'(x)\in J \text{ for all } x\in[-1,1]\}, \tag{2.8}$$

$$K_h:=\{\varphi\in H:\ \varphi\le h \text{ on } [-1,1]\}\quad(\text{for } h\ge 0). \tag{2.9}$$

We will study the variational inequality (1.1) with the operator $F:\mathbb{R}\times U\to H$,

$$\langle F(\lambda,u),v\rangle:=\int_{-1}^{1}\big(a(u')u''v''-f(\lambda,u',u'')u'v'\big)\,\mathrm{d}x, \tag{2.10}$$

which is well defined by this formula and is C^1 smooth under the assumptions (2.1)–(2.7) (see [11, Corollary 18] for details). Hence, our variational inequality (1.1) reads

$$\begin{gathered}\lambda\in\mathbb{R},\ h\ge 0,\ u\in U\cap K_h: \\ \int_{-1}^{1} a(u')u''(\varphi-u)''-f(\lambda,u',u'')u'(\varphi-u)'\,\mathrm{d}x\ge 0\ \text{ for all }\varphi\in K_h.\end{gathered} \tag{2.11}$$

For a particular choice of J, a, and f (see [11]), we obtain a model of an elastic beam which is simply fixed at its ends, compressed by a force proportional to λ, and supported by a fixed unilateral obstacle from above at the height h (cf. also [8], where (2.11) is studied for the particular choice of a and f with more general unilateral conditions, and [5, Example 4.3] for the corresponding semilinear differential operators).

In addition to (2.11), we will consider its "linearization"

$$\begin{gathered}\lambda\in\mathbb{R},\ h\ge 0,\ u\in K_h: \\ \int_{-1}^{1}\big(a(0)u''(\varphi-u)''-\lambda u'(\varphi-u)'\big)\,\mathrm{d}x\ge 0\ \text{ for all }\varphi\in K_h.\end{gathered} \tag{2.12}$$

In fact we will deal with solutions to (2.11) and (2.12) for which the contact set $A_h(u):=\{x\in[-1,1]:\ u(x)=h\}$ (for $h>0$ and $u\in K_h$) is an interval.

The proof of the following assertion is standard.

Proposition 2.1. *A triplet* (λ,h,u) *with* $A_h(u)=[\alpha,\beta]$, $h>0$ *satisfies* (2.11) *if and only if*

$$u\in C^2([-1,1])\cap C^4([-1,1]\setminus(\alpha,\beta)), \tag{2.13}$$

$$\big(a(u')u''\big)'' + \big(f(\lambda, u', u'')u'\big)' = 0 \quad \textit{in } [-1,1] \setminus (\alpha, \beta), \tag{2.14}$$
$$u = h \quad \textit{in } [\alpha, \beta], \quad u < h \quad \textit{in } [-1,1] \setminus [\alpha, \beta], \tag{2.15}$$
$$u(\pm 1) = u''(\pm 1) = 0, \tag{2.16}$$
$$u'''(\alpha-) > 0, \; u'''(\beta+) < 0. \tag{2.17}$$

In particular, the conditions (2.13) *and* (2.15) *imply*

$$u \in W^{3,2}(-1,1), \tag{2.18}$$
$$u(\alpha) = u(\beta) = h, \quad u'(\alpha) = u'(\beta) = u''(\alpha) = u''(\beta) = 0. \tag{2.19}$$

Further, if (λ, h, u) *is an arbitrary solution to* (2.11) *and* Ω *is a connected component of the set* $\{x \in (-1,1) : u(x) < h\}$*, then* $u \in C^4(\overline{\Omega})$ *and* u *satisfies the equation from* (2.14) *in* Ω*.*

The assertions remain valid if we replace (2.11) *by* (2.12) *and* (2.14) *by*

$$a(0)u'''' + \lambda u'' = 0 \quad \textit{in } [-1,1] \setminus (\alpha, \beta). \tag{2.20}$$

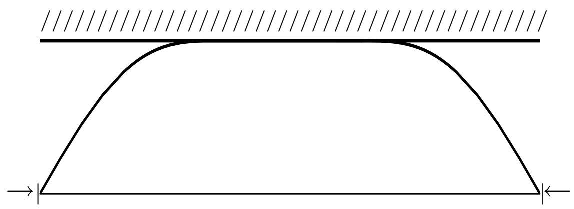

Fig. 1 — Graph of the function $\hat{u}(\lambda, h)$.

Let us introduce the following notation for $k, \ell, n \in \mathbb{N}$:

$$\Lambda_n := a(0)(\tfrac{n}{2}\pi)^2, \; \alpha_k(\lambda) := -1 + \sqrt{\tfrac{a(0)}{\lambda}}k\pi, \; \beta_\ell(\lambda) := 1 - \sqrt{\tfrac{a(0)}{\lambda}}\ell\pi.$$

The numbers Λ_n are eigenvalues of a certain associated eigenvalue problem—see Observation 3.3. It is easy to verify that

$$\alpha_k(\lambda) < \beta_\ell(\lambda) \text{ if and only if } \lambda > \Lambda_{k+\ell}.$$

Theorem 2.2. *Let the assumptions* (2.1)–(2.7) *be fulfilled. Then for any* $k, \ell \in \mathbb{N}$ *there exist a relatively open subset* $\mathcal{U}_{k\ell}$ *of the closed halfplane* $\{(\lambda, h) \in \mathbb{R}^2; \, h \geq 0\}$ *and* C^1 *mappings* $\overline{u}_{k\ell} : \mathcal{U}_{k\ell} \to H$*,* $\overline{\alpha}_{k\ell}, \overline{\beta}_{k\ell} : \mathcal{U}_{k\ell} \to \mathbb{R}$ *such that*

$$\{(\lambda, h) \in \mathcal{U}_{k\ell} : h = 0\} = \left\{(\lambda, 0) \in \mathbb{R}^2 : \lambda > \Lambda_{k+\ell}\right\},$$

$\overline{u}_{k\ell}(\lambda,0) = 0$, $\overline{\alpha}_{k\ell}(\lambda,0) = \alpha_k(\lambda)$, $\overline{\beta}_{k\ell}(\lambda,0) = \beta_\ell(\lambda)$ for all $\lambda > \Lambda_{k+\ell}$, and $(\lambda,h,\overline{u}_{k\ell}(\lambda,h))$ satisfies (2.11) with $A_h(\overline{u}_{k\ell}(\lambda,h)) = [\overline{\alpha}_{k\ell}(\lambda,h),\overline{\beta}_{k\ell}(\lambda,h)]$, $-1 < \overline{\alpha}_{k\ell}(\lambda,h) < \overline{\beta}_{k\ell}(\lambda,h) < 1$ for all $(\lambda,h) \in \mathcal{U}_{k\ell}$, $h > 0$.

Further, there exists a relatively open subset $\mathcal{Z}$ of $\{(\lambda,h,u) \in \mathbb{R}^2 \times H : h \geq 0\}$ such that

$$\{(\lambda,h,u) \in \mathcal{Z} : h = 0, u = 0\} = \left\{(\lambda,0,0) \in \mathbb{R}^2 \times H : \lambda > a(0)\pi^2\right\},$$

and that for any $(\lambda,h,u) \in \mathcal{Z}$ satisfying (2.11) with $A_h(u) = [\alpha,\beta]$, $-1 < \alpha < \beta < 1$, there exists (k,ℓ) such that $(\lambda,h) \in \mathcal{U}_{k\ell}$, $u = \overline{u}_{k\ell}(\lambda,h)$, $\alpha = \overline{\alpha}_{k\ell}(\lambda,h)$, and $\beta = \overline{\beta}_{k\ell}(\lambda,h)$.

Remark 2.3. Theorem 2.2 implies that for any $\lambda_0 > a(0)\pi^2 = \Lambda_2$, the couple $(\lambda,h) = (\lambda_0,0)$ is a bifurcation point of (2.11) at which smooth families of nontrivial solutions bifurcate from the trivial solution $u = 0$. If $\lambda_0 \in (\Lambda_n,\Lambda_{n+1})$, $n \geq 2$, then the number of such families is equal to $\frac{n(n-1)}{2}$ (the number of different pairs $(k,\ell) \in \mathbb{N}^2$ with $k+\ell \leq n$). For $\lambda_0 \in (\Lambda_2,\Lambda_3)$ there is one branch ($u = \overline{u}_{11}(\lambda,h)$ in the notation of Theorem 2.2), for $\lambda_0 \in (\Lambda_3,\Lambda_4)$ there are three branches ($u = \overline{u}_{11}(\lambda,h)$, $u = \overline{u}_{12}(\lambda,h)$, and $u = \overline{u}_{21}(\lambda,h)$), for $\lambda_0 \in (\Lambda_4,\Lambda_5)$ there are six branches, and so on.

Remark 2.4. If $-\xi \in J$ and $a(-\xi) = a(\xi)$ for all $\xi \in J$, $f(\lambda,-\xi,\eta) = f(\lambda,\xi,\eta)$ for all $\lambda,\eta \in \mathbb{R}$, $\xi \in J$, and $k = \ell$ (i.e., $\alpha_k(\lambda) = -\beta_\ell(\lambda)$), then the functions $\overline{u}_{kk}(\lambda,h)$ from Theorem 2.2 are symmetric in x, and therefore $\overline{\alpha}_{kk}(\lambda,h) = -\overline{\beta}_{kk}(\lambda,h)$. This follows immediately from the uniqueness assertion of Theorem 2.2.

Proof of Theorem 2.2 will be given in Section 4 on the basis of the following "local" Theorem 2.5.

Theorem 2.5. *Let the assumptions (2.1)–(2.7) be fulfilled, let k and ℓ be positive integers, and suppose $\lambda_0 > \Lambda_{k+\ell}$. Then there exist $\varepsilon > 0$ and neighbourhoods $V \subset \mathbb{R}$, $W \subset H$, and $Z \subset \mathbb{R}^2$ of λ_0, 0, and $(\alpha_k(\lambda_0),\beta_\ell(\lambda_0))$, respectively, and C^1 mappings $\hat{u} : V \times [0,\varepsilon) \to H$ and $\hat{\alpha},\hat{\beta} : V \times [0,\varepsilon) \to \mathbb{R}$ such that $\hat{u}(\lambda,0) = 0$, $\hat{\alpha}(\lambda,0) = \alpha_k(\lambda)$, and $\hat{\beta}(\lambda,0) = \beta_\ell(\lambda)$ for any $\lambda \in V$, and that $(\lambda,h,u,\alpha,\beta) \in V \times (0,\varepsilon) \times W \times Z$ satisfies (2.11) and $A_h(u) = [\alpha,\beta]$ if and only if $u = \hat{u}(\lambda,h)$, $\alpha = \hat{\alpha}(\lambda,h)$, and $\beta = \hat{\beta}(\lambda,h)$.*

Remark 2.6. For a given λ_0, the neighbourhoods V, W, Z, and ε can be chosen independently of k and ℓ, because only finitely many k and ℓ satisfy $\lambda_0 > \Lambda_{k+\ell}$. However, these neighborhoods depend on the choice of λ_0 and the maps $\hat{u}$, $\hat{\alpha}$, and $\hat{\beta}$ depend on the choice of k, ℓ, and λ_0. If we need to emphasize the dependence on k and ℓ, then we will write $\hat{u}_{k\ell}$, $\hat{\alpha}_{k\ell}$, and

$\hat{\beta}_{k\ell}$ instead of $\hat{u}$, $\hat{\alpha}$, and $\hat{\beta}$. We omitted the indices in Theorem 2.5 because keeping them systematically would formally complicate the proof.

Main idea of the proof of Theorem 2.5. This is to show that in a neighbourhood of $(\lambda_0, 0, 0)$, our variational inequality (2.11) together with the conditions $A_h(u) = [\alpha, \beta]$, α, β in a neighbourhood of α_0, β_0, $-1 < \alpha_0 < \beta_0 < 1$, are equivalent to an operator equation, and to apply a scaling technique and the implicit function theorem to this equation. In order to explain this equivalence, let us denote

$$D := \{(\alpha, \beta) \in \mathbb{R}^2 : -1 < \alpha < \beta < 1\} \tag{2.21}$$

and fix $(\alpha_0, \beta_0) \in D$. Further, we will need the following transformations $\varphi_{\alpha,\beta}$ of the interval $[-1, 1]$ and linear maps $\Phi_{\alpha,\beta} : H \to H$ defined for any $(\alpha, \beta) \in D$ as follows:

$\varphi_{\alpha,\beta}(x) := \frac{1+\alpha_0}{1+\alpha}(1 + x) - 1 \quad \text{for } x \in [-1, \alpha],$
$\varphi_{\alpha,\beta}(x) := -\frac{1-\beta_0}{1-\beta}(1 - x) + 1 \quad \text{for } x \in [\beta, 1],$
$\varphi_{\alpha,\beta}(x) \in (\alpha_0, \beta_0)$ for $x \in (\alpha, \beta)$,
$(\alpha, \beta, x) \in D \times [-1, 1] \mapsto \varphi_{\alpha,\beta}(x) \in \mathbb{R}$ is C^3 smooth,
$\varphi_{\alpha,\beta}$ is a diffeomorphism of $[-1, 1]$ onto itself for any fixed $(\alpha, \beta) \in D$,
$\varphi_{\alpha_0,\beta_0}(x) = x$ for all $x \in [-1, 1]$,
$(\Phi_{\alpha,\beta}u)(x) := u(\varphi_{\alpha,\beta}(x))$ for $x \in [-1, 1]$.

Let us remark that $\Phi_{\alpha,\beta}$ is invertible on H (for α, β fixed), $\Phi^{-1}_{\alpha,\beta}u = u \circ \varphi^{-1}_{\alpha,\beta}$. Moreover, $\Phi_{\alpha,\beta}$ is bounded because of the C^2 smoothness of $\varphi_{\alpha,\beta}$, and the mapping $(\alpha, \beta, u) \in D \times H \mapsto \Phi_{\alpha,\beta}u \in H$ is continuous (see the Appendix in [11]). Let us introduce a closed subspace H_0 in H by

$$H_0 := \{u \in H : u = 0 \text{ in } [\alpha_0, \beta_0]\}$$

and choose fixed functions $v_0, w_0 \in H$ satisfying the conditions

$$v_0(\alpha_0) = v_0(\beta_0) = v_0'(\beta_0) = w_0(\alpha_0) = w_0(\beta_0) = w_0'(\alpha_0) = 0,\ v_0'(\alpha_0) = w_0'(\beta_0) = 1.$$

The local equivalence of the variational inequality (2.11) and an operator equation mentioned above is described by the following theorem.

Theorem 2.7. *Let the assumptions* (2.1)–(2.7) *be fulfilled, and let* (λ_0, u_0) *be a solution to* (2.12) *with* $h = 1$ *satisfying* $A_1(u_0) = [\alpha_0, \beta_0]$ *with some* $(\alpha_0, \beta_0) \in D$*. Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that the following holds:*

(i) *For any solution* $(\lambda, h, u) \in \mathbb{R}^2 \times H$ *to* (2.11) *with* $h > 0$*,* $A_h(u) = [\alpha, \beta]$*,* $(\alpha, \beta) \in D$*, and* $|\lambda - \lambda_0| + h + \|u\| + |\alpha - \alpha_0| + |\beta - \beta_0| < \delta$

there exists $v \in H_0$ *with* $\|v\| < \varepsilon$, *such that* $(\lambda, h, \alpha, \beta, v)$ *satisfies*

$$(\lambda, h, \alpha, \beta) \in \mathbb{R}^2 \times D,\ h > 0,\ v \in H_0 :\ h\Phi_{\alpha,\beta}(u_0 + v) \in U, \tag{2.22}$$

$$\langle F(\lambda, h\Phi_{\alpha,\beta}(u_0 + v)), \Phi_{\alpha,\beta}\varphi\rangle = 0 \quad \text{for all } \varphi \in H_0 \oplus \operatorname{span}\{v_0, w_0\},$$

$$u = h\Phi_{\alpha,\beta}(u_0 + v). \tag{2.23}$$

(ii) *For any solution* $(\lambda, h, \alpha, \beta, v)$ *to* (2.22) *with* $|\lambda - \lambda_0| + h + |\alpha - \alpha_0| + |\beta - \beta_0| + \|v\| < \delta$, *the triplet* (λ, h, u) *with* u *from* (2.23) *satisfies* (2.11), $\|u\| + \left\|\frac{u}{h} - u_0\right\| < \varepsilon$ *and* $A_h(u) = [\alpha, \beta]$.

We will see in Corollary 3.4 that under the assumptions of Theorem 2.7 there are $k, \ell \in \mathbb{N}$ such that $\alpha_0 = \alpha_k(\lambda_0)$, $\beta_0 = \beta_\ell(\lambda_0)$, $\lambda_0 > \Lambda_{k+\ell}$, and $u_0 = u_{k\ell}(\lambda_0)$ can be expressed in a closed form.

3. Auxiliary assertions

In this section, the conditions (2.1)–(2.7) will be always automatically assumed.

Lemma 3.1. *Let* $\gamma < \delta$. *Let* $(\lambda_n, \xi_n, v_n) \in \mathbb{R}^2 \times C^4([\gamma, \delta])$, $n \in \mathbb{N}$, *satisfy* $\xi_n v_n'(x) \in J$ *for all* $x \in [\gamma, \delta]$ *and the equation*

$$\xi^2 \left(a(\xi v')v''\right)'' + \left(f(\lambda, \xi v', \xi^2 v'')v'\right)' = 0 \quad \text{in } (\gamma, \delta). \tag{3.1}$$

Suppose $|\lambda_n - \lambda| + |\xi_n - \xi| + \|v_n - v\|_{W^{2,2}(\gamma,\delta)} \to 0$ *with a certain* $(\lambda, \xi, v) \in \mathbb{R}^2 \times W^{2,2}(\gamma, \delta)$. *Then* $v \in C^4([\gamma, \delta])$, (λ, ξ, v) *satisfies* (3.1), *and* $\|v_n - v\|_{C^4([\gamma,\delta])} \to 0$.

If, moreover, $v = 0$, *then* $\frac{v_n}{\|v_n\|_{C^4([\gamma,\delta])}} \to w$ *in* $C^4([\gamma, \delta])$ *and* (λ, ξ, w) *satisfies*

$$\xi^2 a(0) w'''' + \lambda w'' = 0 \quad \text{in } (\gamma, \delta). \tag{3.2}$$

Remark 3.2. Let $\gamma < \delta$. If $w_n \in C^4([\gamma, \delta])$, $w_n \to w$ in $C^3([\gamma, \delta])$, $w_n'''' \to z$ in $C([\gamma, \delta])$, then $z = w''''$. Indeed, for any smooth φ with a compact support in (γ, δ) we have $\int_\gamma^\delta w_n''''\varphi \,\mathrm{d}x = -\int_\gamma^\delta w_n'''\varphi' \,\mathrm{d}x$, the limiting process gives $\int_\gamma^\delta z''''\varphi \,\mathrm{d}x = -\int_\gamma^\delta w'''\varphi' \,\mathrm{d}x$, and our assertion follows.

Proof of Lemma 3.1. The first assertion coincides with [11, Lemma 5]. Let us prove the other. Let us realize that the embedding $C^4([\gamma, \delta])$ into $C^3([\gamma, \delta])$ is compact and assume $w_n := \frac{v_n}{\|v_n\|_{C^4([\gamma,\delta])}} \to w$ in $C^3([\gamma, \delta])$. Dividing (3.1) by $\|v_n\|_{C^4([\gamma,\delta])}$, we can write the resulting equation in the form

$$\begin{aligned} w_n'''' = -\left(a(\xi_n v_n')\right)^{-1} \Big[& a'(\xi_n v_n')\,\xi_n v_n'' w_n''' + \left(a'(\xi_n v_n')\,\xi_n v_n'' w_n''\right)' \\ & + \xi_n^{-2}\left(f\left(\lambda_n, \xi_n v_n', \xi_n^2 v_n''\right) w_n'\right)' \Big] \quad \text{in } (\gamma, \delta). \end{aligned} \tag{3.3}$$

We already know from the first assertion that $\|v_n\|_{C^4([\gamma,\delta])} \to 0$. It follows from (3.3) with help of (2.2) and Remark 3.2 that $w_n \to w$ in $C^4([\gamma,\delta])$. In fact, our procedure can be done for any subsequence of $\{w_n\}$ converging in $C^3([\gamma,\delta])$, and it follows that really the whole sequence $\{w_n\}$ converges in $C^4([\gamma,\delta])$. The limiting process in (3.3) realizing (2.3) implies that (λ,ξ,w) satisfies (3.2). □

Observation 3.3. *Let $\gamma < \delta$. Then the problem*

$$a(0)u'''' + \lambda u'' = 0 \quad \text{in } (\gamma,\delta), \tag{3.4}$$

$$u(\gamma) = u''(\gamma) = u(\delta) = u''(\delta) = 0 \tag{3.5}$$

has a nontrivial solution if and only if $\lambda = a(0)(\frac{k\pi}{\delta-\gamma})^2$ for some $k = 1,2,\ldots$. The same holds for the problem (3.4),

$$u(\gamma) = u''(\gamma) = u'(\delta) = u''(\delta) = 0. \tag{3.6}$$

The eigenfunctions of (3.4), (3.5) and of (3.4), (3.6) have the form

$$u(x) = C\sin\frac{k\pi}{\delta-\gamma}(x-\gamma), \qquad x \in [\gamma,\delta], \tag{3.7}$$

$$u(x) = C\Big((-1)^{k+1}\sin\frac{k\pi}{\delta-\gamma}(x-\gamma) + \frac{k\pi}{\delta-\gamma}(x-\gamma)\Big), \quad x \in [\gamma,\delta], \tag{3.8}$$

respectively, with some $C \in \mathbb{R}$. If, moreover, the function from (3.8) satisfies

$$u(\delta) = 1, \tag{3.9}$$

then

$$u(x) = \frac{(-1)^{k+1}}{k\pi}\sin\frac{k\pi}{\delta-\gamma}(x-\gamma) + \frac{x-\gamma}{\delta-\gamma}, \qquad x \in [\gamma,\delta]. \tag{3.10}$$

As a consequence of Proposition 2.1 and Observation 3.3 we obtain

Corollary 3.4. *The problem (2.12) with $h = 1$ has a solution (λ,u) with $A_1(u) = [\alpha,\beta]$, $-1 < \alpha \le \beta < 1$, if and only if $\alpha = \alpha_k(\lambda)$, $\beta = \beta_\ell(\lambda)$, $\lambda \ge \Lambda_{k+\ell}$, and $u = u_{k\ell}(\lambda)$ for some positive integers k and ℓ, where*

$$\begin{aligned}(u_{k\ell}(\lambda))(x) &= \frac{(-1)^{k+1}}{k\pi}\sin\sqrt{\frac{\lambda}{a(0)}}(1+x) + \frac{1+x}{1+\alpha_k(\lambda)}, \qquad x \in [-1,\alpha_k(\lambda)],\\ &= 1, \quad x \in [\alpha_k(\lambda),\beta_\ell(\lambda)], \qquad (3.11)\\ &= \frac{(-1)^{\ell+1}}{\ell\pi}\sin\sqrt{\frac{\lambda}{a(0)}}(1-x) + \frac{1-x}{1-\beta_\ell(\lambda)}, \quad x \in [\beta_\ell(\lambda),1].\end{aligned}$$

Remark 3.5. The problem (2.12) with $h > 0$ is positively homogeneous in (h,u); i.e., (λ,h,u) satisfies (2.12) if and only if $(\lambda,1,\frac{u}{h})$ satisfies (2.12).

Lemma 3.6. *Let $\gamma < \delta$. For any compact set $K \subset (0,+\infty)$ and any $R > 0$ there exist $\varepsilon, c > 0$ such that for arbitrary $(\lambda, h, v, \xi) \in (0,R) \times (0,+\infty) \times C^4([\gamma,\delta]) \times K$ satisfying* (3.1),

$$v(\gamma) = v''(\gamma) = v'(\delta) = v''(\delta) = 0, \tag{3.12}$$

$$v(\delta) = h \tag{3.13}$$

with $\|v\| + h < \varepsilon$, we have $\|v\|_{C^4([\gamma,\delta])} \le ch$.

Proof. Let us assume for the sake of contradiction that there exist $(\lambda_n, h_n, v_n, \xi_n) \in (0,R)\times(0,+\infty)\times C^4([\gamma,\delta])\times K$ satisfying (3.1), (3.12), $v_n(\delta) = h_n$, $\lambda_n \to \lambda$, $h_n \to 0$, $\|v_n\| \to 0$, $\|v_n\|_{C^4([\gamma,\delta])} \ge nh_n$, and $\xi_n \to \xi \in K$. Similarly as in the proof of Lemma 3.1, dividing (3.1) by $\|v_n\|_{C^4([\gamma,\delta])}$ and setting $w_n := \frac{v_n}{\|v_n\|_{C^4([\gamma,\delta])}}$, we can write the resulting equation in the form (3.3). Since w_n is bounded in $C^4([\gamma,\delta])$, we can assume that $w_n \to w$ in $C^3([\gamma,\delta])$. It follows from (3.3) with help of (2.2), (2.3), and Remark 3.2 that $w_n \to w$ in $C^4([\gamma,\delta])$ and (λ,ξ,w) satisfies (3.2). Therefore w satisfies (3.4) and (3.6) with λ replaced by $\xi^{-2}\lambda$, and Observation 3.3 gives that $\lambda = \xi^2\left(\frac{k\pi}{\delta-\gamma}\right)^2 a(0)$ with a positive integer k and w has the form (3.8). Moreover, since $v_n(\delta) = h_n$, we have

$$w_n(\delta) = \frac{v_n(\delta)}{\|v_n\|_{C^4([\gamma,\delta])}} = \frac{h_n}{\|v_n\|_{C^4([\gamma,\delta])}} \le \frac{1}{n} \to 0;$$

i.e., $w(\delta) = 0$ holds. Hence, it follows from (3.8) that $w = 0$ in $[\gamma,\delta]$. This is a contradiction, and the assertion of Lemma 3.6 follows. □

The following assertion is similar to Lemma 3.1, but now we will consider the sequence $w_n := \frac{v_n}{h_n}$ instead of $w_n := \frac{v_n}{\|v_n\|_{C^4([\gamma,\delta])}}$.

Lemma 3.7. *Let $\gamma < \delta$. Let $(\lambda_n, h_n, \xi_n, v_n) \in \mathbb{R}^3 \times C^4([\gamma,\delta])$, $n \in \mathbb{N}$, satisfy $h_n > 0$, $\xi_n v_n'(x) \in J$ for all $x \in [\gamma,\delta]$, the equation* (3.1) *with* (3.12), *and $v_n(\delta) = h_n$. Suppose $|\lambda_n - \lambda| + |h_n| + |\xi_n - \xi| + \|v_n\|_{W^{2,2}(\gamma,\delta)} \to 0$ with a certain $(\lambda,\xi) \in \mathbb{R}^2$, $\xi > 0$. Then $\frac{v_n}{h_n} \to w$ in $C^4([\gamma,\delta])$, and (λ,ξ,w) satisfies* (3.2), (3.12), *and* (3.13) *with h replaced by* 1.

Proof. Lemma 3.6 gives that $\|\frac{v_n}{h_n}\|_{C^4([\gamma,\delta])}$ are bounded, and the proof now continues similarly to that of Lemma 3.1. □

Remark 3.8. We have $\varphi'_{\alpha,\beta}(x) = \frac{1+\alpha_0}{1+\alpha}$ for $x \in [-1,\alpha]$ and $\varphi'_{\alpha,\beta}(x) = \frac{1-\beta_0}{1-\beta}$ for $x \in [\beta,1]$. In particular, $\varphi^{(m)}_{\alpha,\beta}(x) = 0$ for $x \in [-1,\alpha] \cup [\beta,1]$ and $m > 1$. Hence, for $w \in C^4([-1,\alpha]\cup[\beta,1])$ and $m = 1,2,3,4$, we have

$$\left[(\Phi_{\alpha,\beta} w)\right]^{(m)}(x) = w^{(m)}(\varphi_{\alpha,\beta}(x))\left(\varphi'_{\alpha,\beta}(x)\right)^m \text{ for } x \in [-1,\alpha]\cup[\beta,1]. \tag{3.14}$$

For a given $(\alpha_0, \beta_0) \in D$ we will denote $E_0 := (-1, \alpha_0) \cup (\beta_0, 1)$.

Lemma 3.9. *Let* $(\lambda_n, h_n, u_n) \in \mathbb{R}^2 \times H$*,* $n \in \mathbb{N}$*, satisfy* (2.11) *and* $A_{h_n}(u_n) = [\alpha_n, \beta_n]$ *with* $(\alpha_n, \beta_n) \in D$*. Let* $|\lambda_n - \lambda_0| + h_n + \|u_n\| + |\alpha_n - \alpha_0| + |\beta_n - \beta_0| \to 0$ *for* $n \to +\infty$ *with certain* λ_0*,* α_0*,* $\beta_0 \in \mathbb{R}$*. Then there exist positive integers* k *and* ℓ *such that* $-1 < \alpha_0 \le \beta_0 < 1$*,* $\alpha_0 = \alpha_k(\lambda_0)$*,* $\beta_0 = \beta_\ell(\lambda_0)$*,* $\lambda_0 \ge \Lambda_{k+\ell}$*, and* $\frac{u_n}{h_n} \to u_{k\ell}(\lambda_0)$ *in* H *for* $n \to +\infty$ *with* $u_{k\ell}(\lambda_0)$ *from* (3.11).

Proof. First, let us show $\alpha_0 > -1$ (the case $\beta_0 < 1$ can be treated in a similar way). Due to Proposition 2.1 the functions u_n satisfy (3.1) with $\xi = 1$, $\gamma = -1$, and $\delta = \alpha_n$. Multiplying it by u_n and integrating by parts we get

$$\int_{-1}^{\alpha_n} a(u_n')\left(u_n''\right)^2 \mathrm{d}x + a(0)u_n'''(\alpha_n-)h_n = \int_{-1}^{\alpha_n} f(\lambda_n, u_n', u_n'')\left(u_n'\right)^2 \mathrm{d}x. \tag{3.15}$$

We have $u_n \to 0$ in H, consequently in $C^1([-1,1])$, and therefore $u_n'(x)$ belong to a compact subset of J for all $x \in [-1,1]$ and $n \in \mathbb{N}$. Using the assumption (2.2) and realizing that

$$|u_n'(x)|^2 = \left| -\int_x^{\alpha_n} u_n''\,\mathrm{d}y\right|^2 \le (1+\alpha_n)\int_{-1}^{\alpha_n} \left|u_n''\right|^2 \mathrm{d}y,$$

we obtain the existence of $a_0 > 0$ such that

$$\int_{-1}^{\alpha_n} a(u_n')\left(u_n''\right)^2 \mathrm{d}x \ge \frac{a_0}{1+\alpha_n} \sup_{-1 \le x \le \alpha_n} |u_n'(x)|^2.$$

On the other hand, it follows from the assumption (2.4) that there exists $c > 0$ such that

$$\left|\int_{-1}^{\alpha_n} f(\lambda_n, u_n', u_n'')(u_n')^2\,\mathrm{d}x\right| \le c \sup_{-1 \le x \le \alpha_n} |u_n'(x)|^2.$$

Clearly $\sup_{-1 \le x \le \alpha_n} |u_n'(x)|^2 > 0$. Hence, $\alpha_n \to -1$ would mean $\frac{a_0}{1+\alpha_n} \to +\infty$ and (3.15) would lead to a contradiction because $a(0)u_n'''(\alpha_n-)h_n > 0$ by Proposition 2.1.

Let $v_n := \Phi_{\alpha_n,\beta_n}^{-1} u_n$. Then $A_{h_n}(v_n) = [\alpha_0, \beta_0]$, and it follows from Proposition 2.1 by using Remark 3.8 that the v_n satisfy (3.1) in $(-1, \alpha_0)$ and $(\beta_0, 1)$ with $\xi = \frac{1+\alpha_0}{1+\alpha_n}$ and $\xi = \frac{1-\beta_0}{1-\beta_n}$, respectively. Clearly, $\|v_n\| \to 0$ and Lemma 3.1 gives $\|v_n\|_{C^4(\overline{E_0})} \to 0$. If we denote $z_n := \frac{v_n}{h_n}$, then $z_n \in K_1$, $z_n = 1$ in $[\alpha_0, \beta_0]$, and Lemma 3.6 implies that $\|z_n\|_{C^4(\overline{E_0})}$ are bounded. Moreover, the z_n satisfy

$$z_n'''' = -\left(a\left(\xi_n v_n'\right)\right)^{-1}\left[a'\left(\xi_n v_n'\right)\xi_n v_n'' z_n''' + \left(a'\left(\xi_n v_n'\right)\xi_n v_n'' z_n''\right)'\right. \tag{3.16}$$

$$+\xi_n^{-2}\left(f\left(\lambda_n,\xi_n v_n',\xi_n^2 v_n''\right)z_n'\right)'\Big] \quad \text{in } E_0.$$

Let us realize that the embedding $C^4(\overline{E_0})$ into $C^3(\overline{E_0})$ is compact and assume that $z_n \to z$ in $C^3(\overline{E_0})$. The equation (3.16) implies $z_n'''' \to \tilde{z}$ in $C(\overline{E_0})$ with some $\tilde{z}$, and Remark 3.2 ensures $z_n \to z$ in $C^4(\overline{E_0})$. In fact, this procedure can be done for any subsequence of $\{z_n\}$ converging in $C^3(\overline{E_0})$, and it follows that really the whole sequence $\{z_n\}$ converges in $C^4(\overline{E_0})$. The limiting process gives that (λ_0, z) satisfies (3.2) with $\xi = 1$ in E_0. Taking the information about u_n (i.e., also about v_n) from Proposition 2.1, we obtain by the limiting process

$$z(\pm 1) = z''(\pm 1) = 0, \tag{3.17}$$

$$z(\alpha_0) = z(\beta_0) = 1, \quad z'(\alpha_0) = z'(\beta_0) = z''(\alpha_0) = z''(\beta_0) = 0, \tag{3.18}$$

$z'''(\alpha_0-) \geq 0 \geq z'''(\beta_0+)$. However, $z'''(\alpha_0-) = 0$ or $z'''(\beta_0+) = 0$ would imply $z \equiv 1$ in $[-1,\alpha_0]$ or $[\beta_0, 1]$, and therefore $z'''(\alpha_0-) > 0 > z'''(\beta_0+)$ must hold. In particular, we get $z_n \to z$ in $C^2([-1,1])$ and therefore in H. It follows from the form of the solution to (3.2) (see Corollary 3.4) that $z < 1$ in E_0. Hence, $(\lambda_0, 1, z)$ satisfies (2.12) by Proposition 2.1. Moreover, Corollary 3.4 gives $z = u_{k\ell}(\lambda_0)$, $\alpha_0 = \alpha_k(\lambda_0)$, $\beta_0 = \beta_\ell(\lambda_0)$, and $\lambda_0 \geq \Lambda_{k+\ell}$ for some positive integers k and ℓ.

It is easy to see that $\frac{u_n}{h_n} = \Phi_{\alpha_n,\beta_n} z_n$. The continuity of the mapping $(\alpha,\beta,z) \in D \times H \mapsto \Phi_{\alpha,\beta} z \in H$ (see [11, Lemma 13]) implies $\frac{u_n}{h_n} \to z$ in H, and the assertion is proved. □

Lemma 3.10. ([11, Lemma 8]) *Let $\lambda \in \mathbb{R}$, $h > 0$, and $u \in U \cap K_h$ with $A_h(u) = [\alpha,\beta]$, $-1 < \alpha < \beta < 1$. Then (λ, h, u) satisfies* (2.11) *if and only if*

$$\int_{-1}^{1} \left(a(u')u''\varphi'' - f(\lambda, u', u'')u'\varphi'\right) \mathrm{d}x = 0 \tag{3.19}$$

for any $\varphi \in \{\psi \in H : \psi = 0$ in $[\alpha,\beta]\} \oplus \operatorname{span}\{\Phi_{\alpha,\beta} v_0, \Phi_{\alpha,\beta} w_0\}$.

Remark 3.11. The functions $\Phi_{\alpha,\beta} v_0$ and $\Phi_{\alpha,\beta} w_0$ have analogous properties at α, β as the functions v_0, w_0 do at α_0, β_0:

$$\Phi_{\alpha,\beta} v_0(\alpha) = \Phi_{\alpha,\beta} v_0(\beta) = \left(\Phi_{\alpha,\beta} v_0\right)'(\beta) = 0, \quad \left(\Phi_{\alpha,\beta} v_0\right)'(\alpha) = \frac{1+\alpha_0}{1+\alpha} \neq 0,$$

$$\Phi_{\alpha,\beta} w_0(\alpha) = \Phi_{\alpha,\beta} w_0(\beta) = \left(\Phi_{\alpha,\beta} w_0\right)'(\alpha) = 0, \quad \left(\Phi_{\alpha,\beta} w_0\right)'(\beta) = \frac{1-\beta_0}{1-\beta} \neq 0.$$

In particular, the test functions in (3.19) have free derivatives at α, β. Thus, if (3.19) is true and $A_h(u) = [\alpha,\beta]$, then standard considerations imply that (2.13), (2.14), (2.16), and, in particular, (2.18) and (2.19) hold.

4. Proof of main results

Again, for a given $(\alpha_0, \beta_0) \in D$ we will denote $E_0 := (-1, \alpha_0) \cup (\beta_0, 1)$.

Proof of Theorem 2.7. Because of Proposition 2.1 and Corollary 3.4 we have $u_0 = u_{k\ell}(\lambda_0)$, $\alpha_0 = \alpha_k(\lambda_0)$, $\beta_0 = \beta_\ell(\lambda_0)$, and $\lambda_0 > \Lambda_{k+\ell}$ for some positive integers k and ℓ. Hence, u_0 satisfies (2.13), (2.20), (2.16), and

$$u_0(\alpha_0) = u_0(\beta_0) = 1,\ u_0'(\alpha_0) = u_0''(\alpha_0) = 0,\ u_0'(\beta_0) = u_0''(\beta_0) = 0, \tag{4.1}$$

$$u_0'''(\alpha_0+) > 0 > u_0'''(\beta_0-). \tag{4.2}$$

Let us prove (i). Let $\varepsilon > 0$ be given. Since we assume $A_h(u) = [\alpha, \beta]$, Lemma 3.10 implies that (3.19) is valid. The mapping $\Phi_{\alpha,\beta}$ is invertible, and therefore there is $v \in H$ such that (2.23) holds and $A_1(u_0 + v) = [\alpha_0, \beta_0]$. It follows that $v \equiv 0$ on $[\alpha_0, \beta_0]$; i.e., $v \in H_0$. Lemma 3.9 implies that for any $\delta_1 > 0$ there is $\delta > 0$ such that $\left\|\frac{u}{h} - u_0\right\| < \delta_1$ if $|\lambda - \lambda_0| + h + |\alpha - \alpha_0| + |\beta - \beta_0| + \|u\| < \delta$. Now, with the help of the continuity of the mapping $(\alpha, \beta, z) \in D \times H \mapsto \Phi^{-1}_{\alpha,\beta} z \in H$ (see [11, Lemma 13]) we get $\|v\| = \|\frac{1}{h}\Phi^{-1}_{\alpha,\beta} u - u_0\| = \|\Phi^{-1}_{\alpha,\beta}\frac{u}{h} - \Phi^{-1}_{\alpha_0,\beta_0} u_0\| < \varepsilon$ if δ_1 and δ are sufficiently small. We have $\varphi \in H_0 \oplus \operatorname{span}\{v_0, w_0\}$ if and only if $\Phi_{\alpha,\beta}\varphi \in \{\psi \in H : \psi = 0 \text{ in } [\alpha, \beta]\} \oplus \operatorname{span}\{\Phi_{\alpha,\beta} v_0, \Phi_{\alpha,\beta} w_0\}$. Hence, (2.22) follows from (3.19) in Lemma 3.10, and the assertion (i) is proved.

Let us prove (ii). Let $\varepsilon > 0$ be given. Let $(\lambda, h, \alpha, \beta, v) \in \mathbb{R}^4 \times H_0$ satisfy (2.22), $h > 0$, and $|\lambda - \lambda_0| + h + |\alpha - \alpha_0| + |\beta - \beta_0| + \|v\| < \delta$ with $\delta > 0$ small. Let us set $u := h\Phi_{\alpha,\beta}(u_0 + v)$. Clearly $A_h(u) = [\alpha, \beta]$. Moreover, it is easy to see that (λ, h, u) satisfies (3.19). In particular, u satisfies (2.13) and (2.19) by Remark 3.11. Since u_0 also satisfies (2.13) and (2.19) (but with $\alpha = \alpha_0$ and $\beta = \beta_0$), we obtain by using (3.14) that $v \in C^4(\overline{E_0})$,

$$v(\alpha_0) = v'(\alpha_0) = v''(\alpha_0) = 0, \quad v(\beta_0) = v'(\beta_0) = v''(\beta_0) = 0. \tag{4.3}$$

Let us assume for a moment that

$$\|v'''\|_{C(\overline{E_0})} < \|u_0'''\|_{C(\overline{E_0})} \tag{4.4}$$

if δ is small enough. Then the condition (4.2) implies $u \in K_h$ for δ small. Lemma 3.10 ensures that (λ, h, u) satisfies (2.11). Clearly, $\|u\| + \left\|\frac{u}{h} - u_0\right\| < \varepsilon$ if δ is small enough. Hence, (ii) is valid.

It remains to prove that (4.4) holds for all v under consideration if δ is small enough. Assume the contrary. Then there are $(\lambda_n, h_n, \alpha_n, \beta_n, v_n) \in \mathbb{R}^4 \times H_0$ satisfying (2.22), $(\lambda_n, h_n, \alpha_n, \beta_n) \to (\lambda_0, h_0, \alpha_0, \beta_0)$, $\|v_n\| \to 0$, and

$$\|v_n'''\|_{C(\overline{E_0})} \geq \|u_0'''\|_{C(\overline{E_0})}. \tag{4.5}$$

It follows from (2.10), (2.22), and Remark 3.8 that the functions $h_n(u_0+v_n)$ are classical solutions to (3.1) with $\xi_n := \frac{1+\alpha_0}{1+\alpha_n}$ in $(-1,\alpha_0)$ and with $\xi_n := \frac{1-\beta_0}{1-\beta_n}$ in $(\beta_0,1)$. Moreover, since u_0 satisfies (2.13), (2.16), and (4.1) and the v_n satisfy (4.3), we obtain that $h_n(u_0+v_n)$ satisfy (2.13), (2.15), and (2.16) (in particular, (2.19)) with (α,β,h) replaced by (α_0,β_0,h_n). It follows from Lemma 3.7 that $u_0+v_n \to z$ in $C^4(\overline{E_0})$, where z satisfies (3.2) with $\lambda=\lambda_0$ and $\xi=1$, and (2.16) and (2.19) with (α,β,h) replaced by $(\alpha_0,\beta_0,1)$. Therefore z satisfies (3.4) and (3.6) with (λ,γ,δ) replaced by $(\lambda_0,-1,\alpha_0)$ and $(\lambda_0,\beta_0,1)$, respectively. Furthermore, $z=1$ in $[\alpha_0,\beta_0]$. Hence, $z=u_0$ by Proposition 2.1 and Corollary 3.4. Therefore

$$\|v_n\|_{C^4(\overline{E_0})} = \|u_0+v_n-u_0\|_{C^4(\overline{E_0})} = \|u_0+v_n-z\|_{C^4(\overline{E_0})} \to 0.$$

This is a contradiction to (4.2) and (4.5), and (4.4) is proved. □

Proof of Theorem 2.5. Let k, ℓ, and λ_0 be given. Set $\alpha_0=\alpha_k(\lambda_0)$ and $\beta_0=\beta_\ell(\lambda_0)$. Let $\widetilde{H}_0 := H_0 \oplus \operatorname{span}\{v_0,w_0\}$. It follows from Corollary 3.4 that there is u_0 satisfying the assumptions of Theorem 2.7. First, we will describe all solutions $(\lambda,h,\alpha,\beta,v)$ close to $(\lambda_0,0,\alpha_0,\beta_0,0)$ with $h>0$ of the problem (2.22). According to Theorem 2.7, in this way we obtain simultaneously all solutions of our variational inequality (2.11) which are close to $(\lambda_0,0,0)$ and satisfy $h>0$ and $A_h(u)=[\alpha,\beta]$ with some (α,β) close to (α_0,β_0).

For any fixed $(\lambda,u)\in\mathbb{R}\times H$, let us define a linear mapping $L(\lambda,u): H\to H$ by

$$L(\lambda,u) := \int_0^1 \frac{\partial F}{\partial u}(\lambda,su)\,\mathrm{d}s.$$

Then $F(\lambda,u)=L(\lambda,u)u$.

Let us introduce a mapping G of a neighbourhood of $(\lambda_0,0,\alpha_0,\beta_0,0)$ in $\mathbb{R}^4\times H_0$ into $\widetilde{H}_0$ by

$$\langle G(\lambda,h,\alpha,\beta,v),\varphi\rangle := \langle L(\lambda,h\Phi_{\alpha,\beta}(u_0+v))\Phi_{\alpha,\beta}(u_0+v), \Phi_{\alpha,\beta}\varphi\rangle \text{ for all } \varphi\in\widetilde{H}_0.$$

Now, the problem (2.22) is equivalent to

$$(\lambda,h,\alpha,\beta)\in\mathbb{R}^2\times D,\ h>0,\ v\in H_0: h\Phi_{\alpha,\beta}(u_0+v)\in U,\ G(\lambda,h,\alpha,\beta,v)=0. \tag{4.6}$$

Since $u_0=1$ and $v=0$ in $[\alpha_0,\beta_0]$ for $v\in H_0$, we have $(\Phi_{\alpha,\beta}(u_0+v))'=0$ in $[\alpha,\beta]$. It follows by using (2.10), Remark 3.8, and the substitution $y:=\varphi_{\alpha,\beta}(x)$ that for any $\varphi\in\widetilde{H}_0$ we have

$$\langle G(\lambda,h,\alpha,\beta,v),\varphi\rangle$$
$$= \int_{-1}^{\alpha_0} a\Big(\frac{1+\alpha_0}{1+\alpha}h(u_0+v)'\Big)(u_0+v)''\Big(\frac{1+\alpha_0}{1+\alpha}\Big)^3\varphi''$$

$$- f\left(\lambda, \frac{1+\alpha_0}{1+\alpha} h\,(u_0+v)', \left(\frac{1+\alpha_0}{1+\alpha}\right)^2 h\,(u_0+v)''\right)(u_0+v)'\frac{1+\alpha_0}{1+\alpha}\varphi'\,\mathrm{d}y$$
$$+ \int_{\beta_0}^{1} a\left(\frac{1-\beta_0}{1-\beta}h(u_0+v)'\right)(u_0+v)''\left(\frac{1-\beta_0}{1-\beta}\right)^3\varphi''$$
$$- f\left(\lambda, \frac{1-\beta_0}{1-\beta} h\,(u_0+v)', \left(\frac{1-\beta_0}{1-\beta}\right)^2 h\,(u_0+v)''\right)(u_0+v)'\frac{1-\beta_0}{1-\beta}\varphi'\,\mathrm{d}y.$$

It follows from the assumptions (2.1)–(2.7) that G is C^1 smooth. (See [11, Corollary 16], where we put $w := h(u_0+v)$ and $z := u_0+v$ to obtain $G(\lambda,h,\alpha,\beta,v) = \widetilde{G}(\lambda,\alpha,\beta,h(u_0+v),u_0+v)$—the smoothness of G now follows from the smoothness of $\widetilde{G}$.)

Let us denote by $M : \mathbb{R}^2\times H_0 \to \widetilde{H}_0$ the partial derivative of G with respect to (α,β,v) at the point $(\lambda_0,0,\alpha_0,\beta_0,0)$. Then for all $(\xi,\eta,w)\in\mathbb{R}^2\times H_0$ and $\varphi\in\widetilde{H}_0$ we obtain (see also (2.3))

$$\langle M(\xi,\eta,w),\varphi\rangle = \int_{E_0} a(0)w''\varphi'' - \lambda_0 w'\varphi'\,\mathrm{d}y$$
$$- \frac{\xi}{1+\alpha_0}\int_{-1}^{\alpha_0} 3a(0)u_0''\varphi'' - \lambda_0 u_0'\varphi'\,\mathrm{d}y + \frac{\eta}{1-\beta_0}\int_{\beta_0}^{1} 3a(0)u_0''\varphi'' - \lambda_0 u_0'\varphi'\,\mathrm{d}y.$$

Below we will prove that M is an isomorphism from $\mathbb{R}^2\times H_0$ onto $\widetilde{H}_0$. Let us assume that this is true. Then it follows from the implicit function theorem that there are $\varepsilon>0$, neighbourhoods $V\subset\mathbb{R}$, $W\subset H_0$, and $Z\subset\mathbb{R}^2$ of λ_0, 0, and (α_0,β_0), respectively, and C^1 mappings $\hat{v} : V\times(-\varepsilon,\varepsilon)\to H_0$ and $\hat{\alpha},\hat{\beta} : V\times(-\varepsilon,\varepsilon)\to\mathbb{R}$ such that $\hat{v}(\lambda_0,0)=0$, $\hat{\alpha}(\lambda_0,0)=\alpha_0$, and $\hat{\beta}(\lambda_0,0)=\beta_0$, and that $G(\lambda,h,\alpha,\beta,v)=0$ for $(\lambda,h)\in V\times(-\varepsilon,\varepsilon)$, $(\alpha,\beta,v)\in Z\times W$ if and only if $v=\hat{v}(\lambda,h)$, $\alpha=\hat{\alpha}(\lambda,h)$, and $\beta=\hat{\beta}(\lambda,h)$. Let us define $\hat{u}(\lambda,h) := h\Phi_{\hat{\alpha}(\lambda,h),\hat{\beta}(\lambda,h)}(u_0+\hat{v}(\lambda,h))$ for $(\lambda,h)\in V\times[0,\varepsilon)$. Then it follows by using Theorem 2.7 that $(\lambda,h,u,\alpha,\beta)\in V\times(0,\varepsilon)\times W\times Z$ satisfies (2.11) and $A_h(u)=[\alpha,\beta]$ if and only if $u=\hat{u}(\lambda,h)$, $\alpha=\hat{\alpha}(\lambda,h)$, and $\beta=\hat{\beta}(\lambda,h)$. We have $u_0,\hat{u}(\lambda,h)\in W^{3,2}(-1,1)$ by Proposition 2.1; therefore also $\hat{v}(\lambda,h)\in W^{3,2}(-1,1)$. The mapping $(\alpha,\beta)\mapsto\Phi_{\alpha,\beta}w$ of D into H is C^1 smooth for any $w\in W^{3,2}(-1,1)$ (see [11, Lemma 17]), and elementary considerations (using the uniform boundedness principle) give that the mapping $\hat{u}$ is C^1 smooth from V into H.

If $\lambda_1\in V$ is arbitrarily given, then we can repeat our considerations with λ_0 replaced by λ_1 and obtain functions $\widetilde{u}(\lambda,h)$, $\widetilde{\alpha}(\lambda,h)$, and $\widetilde{\beta}(\lambda,h)$ with properties analogous to $\hat{u}(\lambda,h)$, $\hat{\alpha}(\lambda,h)$, and $\hat{\beta}(\lambda,h)$ but with some neighbourhoods $\widetilde{V}$, $\widetilde{W}$, and $\widetilde{Z}$ of λ_1, 0, and $(\alpha_k(\lambda_1), \beta_\ell(\lambda_1))$, respectively, and with some $\widetilde{\varepsilon}$. In particular, $\widetilde{u}(\lambda_1, 0) = 0$, $\widetilde{\alpha}(\lambda_1, 0) = \alpha_k(\lambda_1)$, and $\widetilde{\beta}(\lambda_1, 0) = \beta_\ell(\lambda_1)$. It follows from the uniqueness assertion obtained from the implicit function theorem that $\hat{u}(\lambda, h) = \widetilde{u}(\lambda, h)$, $\hat{\alpha}(\lambda, h) = \widetilde{\alpha}(\lambda, h)$, and $\hat{\beta}(\lambda, h) = \widetilde{\beta}(\lambda, h)$ for all $(\lambda, h) \in (\widetilde{V} \times [0, \widetilde{\varepsilon})) \cap (V \times [0, \varepsilon))$. Since $\lambda_1 \in V$ was arbitrary we can conclude that $\hat{u}(\lambda, 0) = 0$, $\hat{\alpha}(\lambda, 0) = \alpha_k(\lambda)$, and $\hat{\beta}(\lambda, 0) = \beta_\ell(\lambda)$ for all $\lambda \in V$.

Hence, $\hat{u}(\lambda, h)$, $\hat{\alpha}(\lambda, h)$, and $\hat{\beta}(\lambda, h)$ have all properties announced in Theorem 2.5.

It remains to prove that M is an isomorphism from $\mathbb{R}^2 \times H_0$ onto $\widetilde{H}_0$. Let us write $M = a(0)A + B$, where $A, B : \mathbb{R}^2 \times H_0 \to \widetilde{H}_0$ are bounded linear operators defined by

$$\langle A(\xi, \eta, w), \varphi\rangle := \int_{E_0} w''\varphi''\,\mathrm{d}y - \frac{3\xi}{1+\alpha_0}\int_{-1}^{\alpha_0} u_0''\varphi''\,\mathrm{d}y + \frac{3\eta}{1-\beta_0}\int_{\beta_0}^{1} u_0''\varphi''\,\mathrm{d}y$$

for any $(\xi, \eta, w) \in R^2 \times H_0$ and $\varphi \in \widetilde{H}_0$, $B := M - a(0)A$. Let us show that A is invertible. First, let $A(\xi, \eta, w) = 0$ for some $(\xi, \eta, w) \in \mathbb{R}^2 \times H_0$. Then

$$\begin{aligned}
&w'''' - \frac{3\xi}{1+\alpha_0}u_0'''' = 0 \quad \text{in } (-1, \alpha_0),\\
&w(-1) = w''(-1) = w(\alpha_0) = w'(\alpha_0) = w''(\alpha_0) = 0,\\
&w'''' + \frac{3\eta}{1-\beta_0}u_0'''' = 0 \quad \text{in } (\beta_0, 1),\\
&w(1) = w''(1) = w(\beta_0) = w'(\beta_0) = w''(\beta_0) = 0.
\end{aligned}$$

If $\xi = \eta = 0$, then clearly also $w = 0$. Let $\xi \neq 0$. It follows that

$$w'' - \frac{3\xi}{1+\alpha_0}u_0'' + c_1(x - \alpha_0) + c_2 = 0 \quad \text{in } (-1, \alpha_0)$$

with some $c_1, c_2 \in \mathbb{R}$. Let us recall that $u_0(\pm 1) = u_0''(\pm 1) = u_0'(\alpha_0) = u_0'(\beta_0) = u_0''(\alpha_0) = u_0''(\beta_0) = 0$ and $u_0(\alpha_0) = u_0(\beta_0) = 1$. The choice $x = \alpha_0$ (and the boundary conditions) implies $c_2 = 0$; the choice $x = -1$ gives $c_1 = 0$. Furthermore, it follows by using the assumption (2.2) that

$$\begin{aligned}
&w' - \frac{3\xi}{1+\alpha_0}u_0' + c_3 = 0 \quad \text{in } (-1, \alpha_0),\\
&w - \frac{3\xi}{1+\alpha_0}u_0 + c_3(x - \alpha_0) + c_4 = 0 \quad \text{in } (-1, \alpha_0).
\end{aligned}$$

The choice $x = \alpha_0$ and $x = -1$ gives $c_4 = \frac{3\xi}{1+\alpha_0}$ and $c_3 = \frac{3\xi}{(1+\alpha_0)^2}$. Substituting into the former from the last two equations we obtain $w'(\alpha_0) =$

$-\frac{3\xi}{(1+\alpha_0)^2} \neq 0$, which is a contradiction. The assumption $\eta \neq 0$ leads to a contradiction in the same way. Hence, A is injective.

Now, let $z \in \widetilde{H}_0$ be arbitrary. Analogously as in the proof of Theorem 2 in [11] (see [11, Remark 12] for details) it is not hard to show that there exist $w_1, w_2, w_3 \in \{\varphi \in W^{2,2}(E_0) : \varphi(\pm 1) = \varphi(\alpha_0) = \varphi(\beta_0) = 0\}$ such that

$$\int_{E_0} w_1'' \varphi'' \,\mathrm{d}y = \langle z, \varphi \rangle \quad \text{for any } \varphi \in \widetilde{H}_0, \tag{4.7}$$

$$\int_{-1}^{\alpha_0} w_2'' \varphi'' \,\mathrm{d}y = \frac{3}{1+\alpha_0} \int_{-1}^{\alpha_0} u_0'' \varphi'' \,\mathrm{d}y \quad \text{for any } \varphi \in \widetilde{H}_0, \tag{4.8}$$

$$\int_{\beta_0}^{1} w_3'' \varphi'' \,\mathrm{d}y = -\frac{3}{1-\beta_0} \int_{\beta_0}^{1} u_0'' \varphi'' \,\mathrm{d}y \quad \text{for any } \varphi \in \widetilde{H}_0, \tag{4.9}$$

and such that $w_2 = 0$ in $[\beta_0, 1]$ and $w_3 = 0$ in $[-1, \alpha_0]$ (in particular, $w_2'(\beta_0) = w_3'(\alpha_0) = 0$). It follows from the previous step that $w_2'(\alpha_0) \neq 0$ and $w_3'(\beta_0) \neq 0$. Hence, there exist $\xi, \eta \in \mathbb{R}$ such that $w := w_1 + \xi w_2 + \eta w_3$ satisfies $w'(\alpha_0) = w'(\beta_0) = 0$; therefore, w can be extended onto a function from H_0. Then $Aw = z$. Hence, A maps $\mathbb{R}^2 \times H_0$ onto $\widetilde{H}_0$.

The mapping B is compact due to the compact embedding of $W^{2,2}$ into C^1. Hence, the mapping M is Fredholm, and therefore for proving its bijectivity it is sufficient to show its injectivity.

Let $M(\xi, \eta, w) = 0$ for some $\xi, \eta \in \mathbb{R}$, $w \in H_0$. Then the following equations are fulfilled in the weak sense and consequently also in the classical sense (in particular, $w \in C^4(\overline{E_0})$):

$$a(0)w'''' + \lambda_0 w'' = \frac{\xi}{1+\alpha_0}\left(3a(0)u_0'''' + \lambda_0 u_0''\right) \quad \text{in } (-1, \alpha_0), \tag{4.10}$$

$$w(-1) = w''(-1) = 0,\ w(\alpha_0) = w'(\alpha_0) = w''(\alpha_0) = 0, \tag{4.11}$$

$$a(0)w'''' + \lambda_0 w'' = -\frac{\eta}{1-\beta_0}\left(3a(0)u_0'''' + \lambda_0 u_0''\right) \quad \text{in } (\beta_0, 1), \tag{4.12}$$

$$w(1) = w''(1) = 0,\ w(\beta_0) = w'(\beta_0) = w''(\beta_0) = 0. \tag{4.13}$$

If $\xi = 0$ in (4.10), then w is a solution of (3.4) and (3.5) with $(\lambda, \gamma, \delta)$ replaced by $(\lambda_0, -1, \alpha_0)$, and it must have the form (3.7) with $(\gamma, \delta) = (-1, \alpha_0)$ (cf. Observation 3.3). The fifth boundary condition $w'(\alpha_0) = 0$ implies $w = 0$.

If $\xi \neq 0$, then the equation (4.10) together with (4.11) is solvable only if the right-hand side of (4.10) is perpendicular in $L^2(-1, \alpha_0)$ to any solution $v_1 \in C^4([-1, \alpha_0])$ of $a(0)w'''' + \lambda_0 w'' = 0$ with $w(-1) = w''(-1) = 0$, $w(\alpha_0) = w''(\alpha_0) = 0$. It follows from Observation 3.3 that such a v_1 has the form (3.7) with (γ, δ) replaced by $(-1, \alpha_0)$ and $\frac{k\pi}{1+\alpha} = \sqrt{\frac{\lambda_0}{a(0)}}$. Since u_0 has the form (3.11) we obtain for a nontrivial v_1 (i.e., $C \neq 0$ in the expression (3.7)) that

$$\int_{-1}^{\alpha_0} v_1[3a(0)u_0'''' + \lambda_0 u_0''] \,\mathrm{d}x = \frac{2C\lambda_0^2(-1)^{k+1}}{a(0)k\pi} \int_{-1}^{\alpha_0} \sin^2 \sqrt{\frac{\lambda_0}{a(0)}}(1+x)\,\mathrm{d}x \neq 0.$$

Hence, (4.10), (4.11) is not solvable. By the same considerations we can deal with η, (4.12), (4.13). Hence, M is injective, and the proof is completed. □

Proof of Theorem 2.2. First, let $\lambda_0 > a(0)\pi^2$ be fixed. We will prove that there exists $\delta > 0$ such that

$$\begin{aligned} &\text{for any } (\lambda, h, u) \text{ satisfying (2.11)},\ h > 0,\ A_h(u) = [\alpha, \beta] \text{ with } (\alpha, \beta) \in D,\\ &|\lambda - \lambda_0| + h + \|u\| < \delta,\ \text{there are } k, \ell \in \mathbb{N} \text{ such that} \\ &\Lambda_{k+\ell} < \lambda_0,\ u = \hat{u}_{k\ell}(\lambda, h),\ \alpha = \hat{\alpha}_{k\ell}(\lambda, h),\ \beta = \hat{\beta}_{k\ell}(\lambda, h), \end{aligned} \tag{4.14}$$

where $\hat{u}_{k\ell}(\lambda, h)$, $\hat{\alpha}_{k\ell}(\lambda, h)$, and $\hat{\beta}_{k\ell}(\lambda, h)$ are from Theorem 2.5 in the notation of Remark 2.6. According to the uniqueness assertion of Theorem 2.5 it is sufficient to show that for any $\xi > 0$ there is $\delta > 0$ such that if (λ, h, u) satisfies (2.11), $h > 0$, $A_h(u) = [\alpha, \beta]$ with $(\alpha, \beta) \in D$, and $|\lambda - \lambda_0| + h + \|u\| < \delta$, then there are positive integers k and ℓ with $\Lambda_{k+\ell} < \lambda_0$ such that $|\alpha - \alpha_k(\lambda_0)| + |\beta - \beta_\ell(\lambda_0)| < \xi$. Let us assume the contrary. Then there are $(\lambda_n, h_n, u_n, \alpha_n, \beta_n)$ satisfying (2.11), $h_n > 0$, $A_{h_n}(u_n) = [\alpha_n, \beta_n]$, $(\alpha_n, \beta_n) \in D$, $|\alpha_n - \alpha_k(\lambda_0)| + |\beta_n - \beta_\ell(\lambda_0)| \geq \xi$ for all k and ℓ with $\Lambda_{k+\ell} < \lambda_n$, and $|\lambda_n - \lambda_0| + h_n + \|u_n\| + |\alpha_n - \alpha_0| + |\beta_n - \beta_0| \to 0$ with some $-1 \leq \alpha_0 \leq \beta_0 \leq 1$. Lemma 3.9 implies that $\frac{u_n}{h_n} \to u_{k\ell}(\lambda_0)$ in H for some positive integers k and ℓ, $-1 < \alpha_0 \leq \beta_0 < 1$, $\alpha_0 = \alpha_k(\lambda_0)$, and $\beta_0 = \beta_\ell(\lambda_0)$. This is a contradiction, and (4.14) is proved.

Consider an arbitrary fixed $\eta > 1$. The number δ in (4.14) can be chosen independent of $\lambda_0 \in \left[a(0)\pi^2 + \eta^{-1}, a(0)\pi^2 + \eta\right]$. Let us denote it by $\delta(\eta)$. For any $\lambda_0 \in \left[a(0)\pi^2 + \eta^{-1}, a(0)\pi^2 + \eta\right]$, the neighbourhood V of λ_0 and ε in Theorem 2.5 can be chosen independently of k and ℓ satisfying $\Lambda_{k+\ell} < \lambda_0$, and such that $\|\hat{u}(\lambda, h)\| < \delta(\eta)$ for all $(\lambda, h) \in V \times [0, \varepsilon)$. We will have always in mind such neighbourhoods.

Further, let (k, ℓ) be a fixed couple of arbitrary positive integers. There exists a finite number p of points $\lambda_j \in \left[\Lambda_{k+\ell} + \eta^{-1}, \Lambda_{k+\ell} + \eta\right]$, $j = 1, \ldots, p$, such that the corresponding neigbourhoods V of $\lambda_0 = \lambda_j$, $j = 1, \ldots, p$, from Theorem 2.5 form a finite covering of $\left[\Lambda_{k+\ell} + \eta^{-1}, \Lambda_{k+\ell} + \eta\right]$. Denote the particular neighbourhoods V and W, the number ε, and the mappings $\hat{u}$, $\hat{\alpha}$, and $\hat{\beta}$ corresponding to $\lambda_0 = \lambda_j$ from Theorem 2.5 by V_j, W_j, ε_j and $\hat{u}_{k\ell j}$, $\hat{\alpha}_{k\ell j}$, and $\hat{\beta}_{k\ell j}$, respectively. We will show below that there is

$\varepsilon(\eta) \le \min_{j=1,\dots,p} \varepsilon_j$ such that

$$\text{if } (\lambda,h) \in (V_i \cap V_j) \times [0,\varepsilon(\eta)),\ \hat{u}_{k\ell j}(\lambda,h) = \hat{u}_{rsi}(\lambda,h) \quad \text{with some } i,j = 1,\dots,p,\ \Lambda_{r+s} < \lambda_i,\ \text{then } (r,s) = (k,\ell). \tag{4.15}$$

Let us assume that (4.15) is true, and let us prove that

$$\hat{u}_{k\ell j}(\lambda,h) = \hat{u}_{k\ell i}(\lambda,h),\ \hat{\alpha}_{k\ell j}(\lambda,h) = \hat{\alpha}_{k\ell i}(\lambda,h),\ \hat{\beta}_{k\ell j}(\lambda,h) = \hat{\beta}_{k\ell i}(\lambda,h) \tag{4.16}$$

for $(\lambda,h) \in (V_j \cap V_i) \times [0,\varepsilon(\eta))$. Clearly, it is sufficient to prove the first equality, the others follow. If $(\lambda,h) \in (V_j \cap V_i) \times [0,\varepsilon(\eta))$, then the condition (4.14) (for $\lambda_0 = \lambda_i$ and $u = \hat{u}_{k\ell j}(\lambda,h)$) ensures that $\hat{u}_{k\ell j}(\lambda,h) = \hat{u}_{rsi}(\lambda,h)$ for some r and s with $\Lambda_{r+s} < \lambda_i$. It follows from (4.15) that $(r,s) = (k,\ell)$, and (4.16) is proved. Set $V(\eta) = \cup_{j=1}^{p} V_j$. It follows from (4.16) that we can define mappings $\hat{u}^{\eta}_{k\ell} : V(\eta) \times [0,\varepsilon(\eta)) \to H$, $\hat{\alpha}^{\eta}_{k\ell}$, $\hat{\beta}^{\eta}_{k\ell} : V(\eta) \times [0,\varepsilon(\eta)) \to \mathbb{R}$ by

$$\hat{u}^{\eta}_{k\ell}(\lambda,h) = \hat{u}_{k\ell j}(\lambda,h),\ \hat{\alpha}^{\eta}_{k\ell}(\lambda,h) = \hat{\alpha}_{k\ell j}(\lambda,h),\ \hat{\beta}^{\eta}_{k\ell}(\lambda,h) = \hat{\beta}_{k\ell j}(\lambda,h)$$

for all $(\lambda,h) \in V_j \times [0,\varepsilon(\eta))$, $j = 1,\dots,p$.

Now, let us set $\mathcal{U}_{k\ell} = \cup_{\eta>1} V(\eta) \times [0,\varepsilon(\eta))$. Clearly, the mappings introduced above for different η_1 and η_2 coincide on $[V(\eta_1) \times [0,\varepsilon(\eta_1))] \cap [V(\eta_2) \times [0,\varepsilon(\eta_2))]$, and therefore we can define the mappings $\overline{u}_{k\ell} : \mathcal{U}_{k\ell} \to H$ and $\overline{\alpha}_{k\ell}, \overline{\beta}_{k\ell} : \mathcal{U}_{k\ell} \to \mathbb{R}$ by

$$\overline{u}_{k\ell}(\lambda,h) = \hat{u}^{\eta}_{k\ell}(\lambda,h),\ \overline{\alpha}_{k\ell}(\lambda,h) = \hat{\alpha}^{\eta}_{k\ell}(\lambda,h),\ \overline{\beta}_{k\ell}(\lambda,h) = \hat{\beta}^{\eta}_{k\ell}(\lambda,h)$$

for $(\lambda,h) \in V(\eta) \times [0,\varepsilon(\eta))$, $\eta > 0$ arbitrary. All properties of the set $\mathcal{U}_{kl}$ and the functions $\overline{u}_{k\ell}$, $\overline{\alpha}_{k\ell}$, and $\overline{\beta}_{k\ell}$ from the first part of Theorem 2.2 follow from their construction and Theorem 2.5.

Finally, set $\mathcal{Z}(\eta) = \{(\lambda,h,u) \in \mathbb{R}^2 \times H;\ \lambda \in (a(0)\pi^2 + \eta^{-1}, a(0)\pi^2 + \eta),\ h \in [0,\delta(\eta)),\ \|u\| < \delta(\eta)\}$. It follows by using (4.14) and noting the construction of the functions $\overline{u}_{k\ell}$, $\overline{\alpha}_{k\ell}$, and $\overline{\beta}_{k\ell}$, that the set $\mathcal{Z} := \cup_{\eta>1} \mathcal{Z}(\eta)$ has the properties from the second part of Theorem 2.2.

It remains to prove (4.15). Assume for the sake of contradiction that there are $(\lambda_n,h_n) \in (V_j \cap V_i) \times [0,\varepsilon(\eta))$ and $(r,s) \ne (k,\ell)$ such that $\hat{u}_{k\ell j}(\lambda_n,h_n) = \hat{u}_{rsi}(\lambda_n,h_n)$, $\lambda_n \to \lambda$, $h_n \to 0$. Then it follows by using the properties of $\hat{\alpha}_{k\ell j}$ and $\hat{\alpha}_{rsi}$ obtained from the assertion of Theorem 2.5 for $\lambda_0 = \lambda_j$ and $\lambda_0 = \lambda_i$, respectively, that

$$\alpha_k(\lambda) = \lim_{n\to\infty} \hat{\alpha}_{k\ell j}(\lambda_n,h_n) = \lim_{n\to\infty} \hat{\alpha}_{rsi}(\lambda_n,h_n) = \alpha_r(\lambda)$$

and similarly $\beta_\ell(\lambda) = \beta_s(\lambda)$. That means $k = r$ and $\ell = s$, which is a contradiction. □

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