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ALMOST PERIODICITY OF SOLUTIONS FOR ALMOST PERIODIC EVOLUTION EQUATIONS

Zuosheng Hu and Angelo B. Mingarelli
School of Mathematics and Statistics, Carleton University
1125 Colonel By Drive, Ottawa, Ontario, Canada K1S 5B6

(Submitted by: Reza Aftabizadeh)

Abstract. In this paper, we discuss almost periodicity of solutions for evolution equations in Banach space. We introduce the concept of *difference-variation equation.* For a special Hilbert space, we establish some results on almost periodicity of all solutions for evolution equations. In particular, we extend Haraux's result on R^2 to R^n.

1. Introduction and preliminaries

In this paper, we consider evolution equations in Banach spaces

$$u'(t) + A(t)u(t) = f(t) \tag{1.1}$$

where $A(t) : X \to X$ is an operator for any $t \in R$, $f : R \to X$ is a continuous function and X is a Banach space.

It is well-known that if $X = R^n$, $f(t)$ is periodic, and $A(t)$ is linear, does not depend on t, or depends on t in a periodic manner, then all bounded solutions of (1.1) are Bohr almost periodic, but these results cannot be extended to more general cases; for example, the case that $A(t)$ is almost periodic, or the case that X is infinite dimensional (see [17]). Even though all solutions are bounded, there still may not exist any nontrivial solution which is Bohr almost periodic (see [11]), or Stepanov almost periodic (see [12]).

The main problem here is then: *Under what conditions is any bounded solution of* (1.1) *almost periodic?* This is a very complex question and one which has aroused the interest of many mathematicians over the years. Many authors have tried to make some progress on this question since the question was raised and some results were established (see [1, 5, 6, 7, 8, 9, 13, 15, 17] and the references cited there)

In [7], A. Haraux dealt with the case that $X = R^2$ and $A(t)$ is independent of t, nonlinear, and monotone. The main result is that, under the above assumptions, all bounded solutions of (1.1) are Bohr almost periodic.

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Encouraged by the idea of this paper, we deal with the case $X = H$, a separable Hilbert space, and in which $A(t)$ can depend on t in some manner. We introduce the *difference-variation operator* of $A(t)$ and *difference-variation equations* of (1.1). Using some properties of difference-variation equations, we can infer some properties of the solutions of (1.1). Our results can be used to deduce several previous results (see [6, 17, etc.]). In particular, we generalize the result of [7] to the case that H is R^n.

Let H be a real separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ its orthonormal basis, $||\cdot||$ the norm of H. For any $t \in R$, let $A(t) : D(A(t)) \to H$ be a densely defined operator (linear or nonlinear), $f(t) : R \to H$ a function. Throughout this paper, we always assume that

(a) $A(t)$ and $f(t)$ are almost periodic on R;

(b) for any $t \in R, A(t)$ is a monotone operator (defined below);

(c) for all $t \in R, A(t)$ is Fréchet differentiable on $D(A(t))$ and the derivative operator of $A(t)$, say $A'(t,u)$, is continuous on $D(A(t))$, and for any $u \in D(A(t))$,

$$\lim_{||h||\to 0} \left|\left| \frac{A(t)(u+h) - A(t)u}{h} - A'(t,u) \right|\right| = 0, \qquad \text{uniformly on } t \in R \ ;$$

(d) for any $(t_0, u_0) \in R \times D(A(t_0))$, the equation (1.1) has a unique solution through (t_0, u_0).

From the assumption (c) and the properties of almost periodic functions, it is easy to see that $A'(t,u)$ is almost periodic in t on R. And if $u(t)$ is an almost periodic solution of (1.1), then $A'(t, u(t))$ is also almost periodic. If $A'(t,u)$ is continuous on $D(A(t))$, then for any $u_0 \in D(A(t)), h \in H$ and $u_0 + sh \in D(A(t))$ for $s \in [0,1]$, we have

$$A(t)(u_0 + h) - A(t)u_0 = \int_0^1 A'(t, u_0 + sh)h ds \tag{1.2}$$

(cf. [16]).

Let $u_0(t)$ be an almost-periodic solution of (1.1). The equation

$$y' + \int_0^1 A'(t, u_0(t) + sy)y ds = 0 \tag{1.3}$$

is called the *difference-variation equation* of (1.1) along the solution $u_0(t)$. Obviously, if $A(t)$ is linear, then the difference-variation equation of (1.1) is simply the homogeneous equation

$$u'(t) + A(t)u(t) = 0 \tag{1.4}$$

associated with (1.1).

Let $U(t,s)$ be the *process* generated by the equation (1.1). Then $U(t,s)$ is an almost-periodic process for any $t \in R$, $s \in R^+$. We denote the trajectory of $U(t,s)$ (or solution of (1.1)) through (s_0, u_0) by $u(t, s_0, u_0)$. We use $U_{(s_0,u_0)}(t,s)$ to denote the process generated by the equation (1.3) with $u_0(t) = u(t, s_0, u_0)$. We call this the *difference-variation process* of $U(t,s)$ along $u(t,s_0,u_0)$. Evidently, if $u(t,s_0,u_0)$ is almost periodic, then $U_{(s_0,u_0)}(t,s)$ is also almost periodic. At the same time, we have the following fact: If $u(t,s_0,u_0)$ is a trajectory of $U(t,s)$, then for any other trajectory $u(t,s_1,u_1)$ of $U(t,s)$, $u(t,s_1,u_1)$ - $u(t,s_0,u_0)$ is a trajectory of $U_{(s_0,u_0)}(t,s)$.

We say that $U(t.s)$ is *contractive* (or non-expansive) if for any $x, y \in H$, $t \in R$, $s \in R^+$, we have

$$||U(t,s)x - U(t,s)y|| \leq ||x-y||.$$

A trajectory $u(t)$ of $U(t,s)$ is said to be *relatively compact* in H (or with relatively compact range in H) if the set $\{u(t)|t \in R\}$ is relatively compact in H.

Let $A : H \to H$ be a nonlinear operator. We call A *monotone* if for any $u, v \in D(A)$ (the domain of A), we have that

$$(Av - Au, v-u) \geq 0. \tag{1.5}$$

We use symbols $\mathcal{H}(A)$, $\mathcal{H}(f)$ to denote the uniform hull of $A(t)$ and $f(t)$, respectively. We also use the following preliminary results for completeness.

Lemma 1.1. (S. Bochner [2]) *A function $u \in C(R,H)$ is almost periodic if and only if for any sequences $\{a_n\}$, $\{b_n\}$ contained in R, there exist common subsequences $\{a'_n\} \subset \{a_n\}$, $\{b'_n\} \subset \{b_n\}$ such that $\{u(t+a'_n)\}$ converges to some function $v(t)$ pointwise in R, and $\{v(t+b'_n)\}$ and $\{u(t+a'_n+b'_n)\}$ converge to the same limit function pointwise in R.*

Lemma 1.2. (H. Bohr [3], or M. Fink [6]) *Let $F(t)$ be a real continuous function. If $f(t) = \exp(iF(t))$ is almost periodic, then there exist a real number c and a real almost-periodic function $G(t)$ such that $F(t) = ct + G(t)$.*

Lemma 1.3. (C. M. Dafermos [5]) *Let U be an almost-periodic process on H which is contractive and has two (distinct) trajectories with relatively compact range in H. Then each $V \in \mathcal{H}(U)$ has two trajectories $v_1(t)$ and $v_2(t)$, with relatively compact range in H, such that $||v_1(t) - v_2(t)|| =$ constant for all $t \in R$.*

Lemma 1.4. (H. Ishii [13], or A. Haraux [7]) *Let U be an almost-periodic process. If $u_1(t)$ and $u_2(t)$ are two trajectories of U with relatively compact ranges in H, then the norm $||u_1(t) - u_2(t)||$ is non–increasing in t.*

Lemma 1.5. (H. Ishii [13]) *Let U be an almost-periodic process on H which is contractive. Assume that for any $V \in \mathcal{H}(U)$, there exists a trajectory of V which is continuous on R and has relatively compact range in H. Then, for any $V \in \mathcal{H}(U)$, there exists a trajectory of V which is almost periodic on R.*

In order to obtain more general results, we introduce the following concepts.

Let H be a separable Hilbert space, S a subset of H. We call S *synchronous* if for any $\epsilon > 0$, there exists an integer $N > 0$ such that for any $x \in S, x = \sum_{i=1}^{\infty} a_i(x)e_i$, we have

$$||x - \sum_{i=1}^{N} a_i(x)e_i|| < \epsilon, \tag{1.6}$$

where $\{e_i\}_{i=1}^{\infty}$ is the basis of H.

We call a separable Hilbert space H *compact synchronous* if any compact subset of H is synchronous.

Obviously, if H is finite dimensional, then it is compact synchronous.

Lemma 1.6. *Let H be compact synchronous and $f : R \to H$ a function with $f(t) = \sum_{i=1}^{\infty} f_i(t)e_i$. If $f(t)$ has relatively compact range in H and for each i, $f_i(t)$ is almost periodic, then $f(t)$ is almost periodic.*

Proof. Since $f(t)$ has relatively compact range in H, there exists a compact subset K of H such that $f(t) = \sum_{i=1}^{\infty} f_i(t)e_i \in K$ for all $t \in R$. Since H is compact synchronous, for any $\epsilon > 0$, we can choose an $N > 0$ such that

$$||f(t) - \sum_{i=1}^{N} f_i(t)e_i|| < \epsilon, \tag{1.7}$$

Let $F_N(t) = \sum_{i=1}^{N} f_i(t)e_i$. Then $F_N(t)$ converges to $f(t)$ uniformly for all $t \in R$. Since, for each i, $f_i(t)$ is almost periodic, so is $F_N(t)$, and thus $f(t)$ is almost periodic.

2. GENERAL RESULTS

In this section, we establish some general results which are extensions of previous results. First, we have the following obvious result.

Lemma 2.1. *If $A(t)$ is monotone for any $t \in R$, then $U(t, s)$ is contractive.*

According to the definition of $U(t, s)$, it is easy to show this lemma (see [7]). We omit it. From this lemma and Lemma 1.5, we have the following result.

Theorem 2.2. *Assume that for any $(t_0, u_0) \in R \times H$, the trajectory $u(t, t_0, u_0)$ of $U(t, s)$ is continuous on R and there exists a trajectory of $U(t, s)$ which is relatively compact in H. Then there exists a trajectory of $U(t, s)$ which is almost periodic on R.*

We always assume that the hypotheses of Theorem 2.2 hold and denote the almost-periodic trajectory of $U(t, s)$ by $u_0(t)$. We also denote the difference-variation process of $U(t, s)$ along $u_0(t)$ simply by $U_0(t, s)$. Now we establish the following hypotheses (to be used at various places in the sequel).

Condition 1. *For any almost-periodic trajectory $u(t, t_0, u_0)$ of $U(t, s)$, $U_{(t_0,u_0)}(t, s)$ is such that if $y(t)$ is a relatively compact trajectory of $U_{(t_0,u_0)}(t, s)$ with $||y(t)|| = constant$ for all $t \in R$, then $y(t)$ is almost periodic on R.*

Condition 2. *For any $V(t, s) \in \mathcal{H}(U(t, s))$ and any almost-periodic trajectory $v(t, t_0, u_0)$ of $V(t, s)$, $V_{(t_0,u_0)}(t, s)$ is such that if $y_1(t)$, $y_2(t)$ are two relatively compact trajectories of $V_{(t_0,u_0)}(t, s)$ with $||y_1(t)|| = r_1 > 0$, $||y_2(t)|| = r_2 > 0$, $||y_1(t) - y_2(t)|| = r_0 > 0$ for all $t \in R$ where $r_i (i = 0, 1, 2)$ are constants and $0 < r_0 < r_1 + r_2$, then $y_1(t)$, $y_2(t)$ are almost periodic on R.*

Theorem 2.3. *Suppose that Condition 1 holds. If $u(t)$ is a bounded uniformly continuous trajectory of $U(t, s)$ which is relatively compact in H, then it is almost periodic on R.*

Proof. Let $u_0(t)$ be the almost-periodic trajectory of $U(t, s)$ and $u(t)$ be a bounded uniformly continuous trajectory of $U(t, s)$ which is relatively compact in H. Now, since $U(t, s)$ and $u_0(t)$ are almost periodic, we can choose a sequence $\{t_n\} \subset R$ such that as $n \to +\infty$, $t_n \to +\infty$, $U(t, s - t_n) \to U(t, s)$ and $u_0(t - t_n) \to u_0(t)$ uniformly on R. At the same time, since $u(t)$ is relatively compact in H, we can choose t_n also satisfying $u(t - t_n) \to v(t)$ uniformly on any compact subset of R as $n \to +\infty$, where $v(t) : R \to H$ is a function. Obviously, $v(t)$ is a trajectory of $U(t, s)$ with relatively compact range in H. Let $y(t) = v(t) - u_0(t)$, then $y(t)$ is a trajectory of $U_0(t, s)$ and $||y(t)|| = constant$ for all $t \in R$ from Lemma 1.4. By Condition 1, $y(t)$ is almost periodic on R, and so is $v(t)$.

Choose a subsequence $\{t'_n\} \subset \{t_n\}$ such that $v(t + t'_n) \to u^*(t)$ uniformly on R. Since $lim_{n\to\infty} ||u(-t'_n) - u^*(-t'_n)|| = ||v(0) - v(0)|| = 0$, we have that $u(t) = u^*(t)$ for all $t \in R$ by Lemma 1.4. Obviously, $u^*(t)$ is almost periodic, and so is $u(t)$. This completes the proof of this theorem.

Theorem 2.4. *Suppose that Condition 2 holds. If there exist three uniformly continuous trajectories $u_i(t)$ $(i = 0, 1, 2)$ of $U(t, s)$ such that $u_i(t)$ has* *relatively compact range in* H $(i = 0, 1, 2)$, $u_0(t)$ *is almost periodic, and*

$$||u_0(t) - u_i(t)|| = r_i > 0, t \in R, i = 1, 2, \tag{2.1}$$

$$||u_1(t) - u_2(t)|| = r_0 > 0, t \in R, \tag{2.2}$$

where

$$r_0 < r_1 + r_2, \tag{2.3}$$

then for any $V(t, s) \in \mathcal{H}(U(t, s))$, *any bounded uniformly continuous trajectory of* $V(t, s)$ *with relatively compact range in* H *is almost periodic on* R.

Proof. Since $u_0(t) - u_i(t)$ $(i = 1, 2)$ is a uniformly continuous trajectory of $U_0(t, s)$ with relatively compact range in H, $u_0(t) - u_i(t)$ $(i = 1, 2)$ is almost periodic on R by Condition 2. And so $u_i(t)$ $(i = 1, 2)$ is almost periodic.

Now for any $V(t, s) \in \mathcal{H}(U(t, s))$, let $v_0(t) \in \mathcal{H}(u_0(t))$ and $v_0(t)$ be a trajectory of $V(t, s)$. By Lemma 1.3, there exist two other trajectories of $V(t, s)$, say $v_1(t)$, $v_2(t)$, such that

$$||v_0(t) - v_i(t)|| = r_i > 0 \quad (i = 1, 2) \tag{2.4}$$

for all $t \in R$,

$$||v_1(t) - v_2(t)|| = r_0 > 0 \tag{2.5}$$

for all $t \in R$, and $v_i(t)(i = 1, 2)$ are all almost periodic on R.

Now let $v(t)$ be any trajectory of $V(t, s)$ with relatively compact range in H. Since $V(t, s), v_i(t), i = 0, 1, 2$ are almost periodic, we can choose a sequence $\{t_n\} \subset R$ with $t_n \to +\infty$ as $n \to \infty$ satisfying $V(t - t_n, s)x \to V(t, s)x$ uniformly in $t \in R$ and pointwise in $(s, x) \in R^+ \times H$, $v_i(t - t_n) \to v_i(t)$ uniformly in $t \in R$ for $i \in \{0, 1, 2\}$, and $v(t - t_n) \to w(t)$ uniformly on any bounded interval of R as $n \to \infty$, where $w(t)$ is a trajectory of $V(t, s)$ with relatively compact range in H. Clearly, $||w(t) - v_i(t)||$ is constant on R for $i \in \{0, 1, 2\}$ and from (2.4) and (2.5), we can choose k and l such that $w(t) - v_k(t), w(t) - v_l(t)$, and $v_k(t) - v_l(t)$ satisfy Condition 2. And thus $w(t)$ is almost periodic on R.

By Lemma 1.5, we can choose a subsequence $\{t_k\}$ of $\{t_n\}$ and a periodic trajectory $\tilde{v}(t)$ of V such that $w(t+t_k)$ converges uniformly to $\tilde{v}(t)$ as $k \to \infty$. Then, using Lemma 1.3 we have that $||v(t) - \tilde{v}(t)||$ is nonincreasing on R and $\lim_{k\to\infty} ||\tilde{v}(-t_k) - v(-t_k)|| = 0$ because $\tilde{v}(-t_k)$ and $v(-t_k)$ both tend to $w(0)$ as $k \to \infty$. Hence $v(t) = \tilde{v}(t)$ for all $t \in R$. This completes the proof of the theorem.

Lemma 2.5. *Assume Condition* 2. *Let* $v(t)$ *be a uniformly continuous trajectory of* $U(t, s)$ *with relatively compact range in* H, $u_0(t)$ *as before. If*

$w(t) = v(t) - u_0(t)$ is not almost periodic and $||w(t)|| = constant > 0$ for all $t \in R$, then $u_0(t) - w(t)$ is a trajectory of $U(t,s)$.

In [7], Haraux gave a proof of this lemma for the case $H = R^2$. For the general case, the proof is similar. We omit it here.

Lemma 2.6. *Let H be compact synchronous and suppose that Condition 2 holds. Let $u(t)$ be a bounded uniformly continuous trajectory of $U(t,s)$ with relatively compact range in H. If $||u(t) - u_0(t)|| = r$ (constant) > 0 for all $t \in R$, then $u(t)$ is almost periodic.*

Proof. On the contrary, we assume that $u(t)$ is not almost periodic. This means that $w(t) = u(t) - u_0(t)$ is not almost periodic. By Lemma 2.5, $u_0(t) - w(t)$ is a trajectory of $U(t,s)$. Now, for any $V(t,s) \in \mathcal{H}(U(t,s))$, $v_0(t) \in \mathcal{H}\ (u_0(t))$, there exists a function $\gamma(t)$ with $||\gamma(t)|| = r$ such that $v_0(t) + \gamma(t)$ and $v_0(t) - \gamma(t)$ are two trajectories of $V(t,s)$. We assert that if $y(t) \in C(R,H)$ with $||y(t)|| = r$ for all $t \in R$ and $v_0(t) + y(t)$ is a trajectory of $V(t,s)$, then $y(t) = \gamma(t)$, or $y(t) = -\gamma(t)$.

In fact, $v_0(t) + \gamma(t)$ and $v_0(t) - \gamma(t)$ are two trajectories of $V(t,s)$. By Lemma 1.4, $||y(t) - \gamma(t)||$ and $||y(t) + \gamma(t)||$ are both nonincreasing, so they are constants. If $y(t) = \gamma(t)$ and $y(t) = -\gamma(t)$ are not both true, then $y(t)$, $\gamma(t)$ are two trajectories of $V_0(t,s)$ such that $||y(t)|| = ||\gamma(t)|| = r$ and $||y(t) - \gamma(t)|| = constant < 2r$. By Condition 2, $\gamma(t)$ is almost periodic and so is $w(t)$. This gives a contradiction. So, $y(t) = \gamma(t)$ or $y(t) = -\gamma(t)$.

Since $u_0(t)$ is almost periodic, $u_0(t)$ has relatively compact range in H by [14, p.2, Property 1]. Again since $u(t)$ has relatively compact range in H, we can choose a compact subset K of H such that $w(t) \in K$ for all $t \in R$. We write $w(t)$ as $w(t) = \sum_{j=1}^{\infty} w_j(t)e_j$, where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of H.

Now, we show that for any $j \in N$, $w_j(t)$ is almost periodic. For any $j \in N$, we define

$$c_j(w(t)) = w_j(t) + i\,\mathbf{sign}\,w_j(t)||w^j(t)|| \tag{2.6}$$

where $w^j(t) = \sum_{k \neq j} w_k(t)e_k$. Obviously, if $c_j(w(t))$ is almost periodic, then $w_j(t)$ is also almost periodic. And we have that $c_j(-w(t)) = -c_j(w(t))$ and $|c_j(w(t))|^2 = ||w(t)||^2$ for all $t \in R$.

Now we show that for any $j \in N$, $c_j^2(w(t))$ is almost periodic. For any sequences $\{\alpha_n\}$, $\{\beta_n\}$ of R, since $U(t,s)$, $u_0(t)$ are almost periodic, by Lemma 1.1 there exists common subsequences $\{\alpha_n'\} \subset \{\alpha_n\}$, $\{\beta_n'\} \subset \{\beta_n\}$ such that $U(t, s+\alpha_n')$ converges to some process $U_1(t,s)$, $u_0(t+\alpha_n')$ converges to some function $u^*(t)$, and $U(t, s+\alpha_n'+\beta_n')$ and $U_1(t, s+\beta_n')$ have the same limit $V(t,s)$, and $u_0(t+\alpha_n')$ and $u^*(t+\beta_n')$ have the same limit $v_0(t)$. At the same time, we can take $\{\alpha'_n\}$, $\{\beta'_n\}$ such that $u(t+\alpha'_n+\beta'_n)$ converges to some function $v_1(t)$, $u(t+\alpha'_n)$ converges to some function $v_2(t)$, and $v_2(t+\beta'_n)$ converges to some function $v_3(t)$ uniformly on any compact subset of R. Obviously, $v_1(t)$ and $v_3(t)$ are all trajectories of $V(t,s)$. Now, let $w_1(t) = v_1(t) - v_0(t)$, $w_3(t) = v_3(t) - v_0(t)$. Then $||w_1(t)|| = r$, $||w_3(t)|| = r$ for all $t \in R$. By the first part of this lemma, $w_3(t) = w_1(t)$ or $w_3(t) = -w_1(t)$.

Now, $c_j^2(w(t+\alpha'_n+\beta'_n))$ converges to $c_j^2(w_1(t))$, $c_j^2(w(t+\alpha'_n))$ converges to $c_j^2(v_2(t)-u^*(t))$, and $c_j^2(v_2(t+\beta'_n)-u^*(t+\beta'_n))$ converges to $c_j^2(v_3(t)-v_0(t)) = c_j^2(w_3(t)) = c_j^2(w_1(t))$ since $w_3(t) = w_1(t)$ or $w_3(t) = -w_1(t)$. From Lemma 1.1, $c_j^2(w(t))$ is almost periodic, and obviously, $|c_j^2(w(t))| = r^2$ for all $t \in R$.

Using Theorem 1.2, there exists a real number a and a real almost-periodic function $\theta(t) : R \to R$ such that

$$c_j^2(w(t)) = r^2 e^{i(at+\theta(t))} \tag{2.7}$$

for all $t \in R$. Now, let

$$y_j(t) = c_j(w(t)) e^{-\frac{1}{2}i(at+\theta(t))} \tag{2.8}$$

for all $t \in R$. Then, $y_j(t)$ is continuous on R and $y_j^2(t) = c_j^2(w(t))e^{-i(at+\theta(t))}$ $= r^2$ for all $t \in R$. This implies that $c_j(w(t)) = y_j(t)e^{\frac{1}{2}i(at+\theta(t))}$ is almost periodic on R.

But, we have shown that for any $j \in N$, $c_j(w(t))$ is almost periodic. So, $w_j(t)$ is almost periodic for all $j \in N$. From Lemma 1 $w(t)$ is almost periodic. This contradiction completes the proof of this lemma.

Theorem 2.7. *Let H be compact synchronous and suppose that Condition 2 holds. If $u(t)$ is a bounded, uniformly continuous trajectory of $U(t,s)$ with relatively compact range in H, then it is almost periodic on R.*

Proof. Let $u_0(t)$ be the almost-periodic trajectory of $U(t,s)$ and let $u(t)$ be a bounded, uniformly continuous trajectory of $U(t,s)$ which is relatively compact in H. Choose $t_n \to +\infty$ as $n \to +\infty$ such that $U(t, s-t_n) \to U(t,s)$, $u_0(t-t_n) \to u_0(t)$ uniformly on R and $u(t-t_n) \to v(t)$ uniformly on any compact subset of R, where $v(t) : R \to H$ is a function. Obviously, $v(t)$ is a trajectory of $U(t,s)$ with relatively compact range in H. Let $y(t) = v(t) - u_0(t)$, then $y(t)$ is a trajectory of $U_0(t,s)$ and $||y(t)|| = constant$ for all $t \in R$ from Lemma 1.4. By Lemma 2.6, $y(t)$ is almost periodic on R; thus, so is $v(t)$.

Choose a subsequence $\{t'_n\} \subset \{t_n\}$ such that $v(t+t'_n) \to u^*(t)$ uniformly on R. Since $lim_{n\to\infty}||u(-t'_n) - u^*(-t'_n)|| = ||v(0) - v(0)|| = 0$, we have that $u(t) = u^*(t)$ for all $t \in R$ by Lemma 1.4. Obviously, $u^*(t)$ is almost periodic, and so is $u(t)$. This completes the proof of this theorem.

3. Applications to special cases

In this section, we apply our main results (Theorem 2.3, 2.7) to some special cases. First, we have the following result.

Theorem 3.1. *Let $A(t)$ be linear for any $t \in R$. Suppose that any bounded, uniformly continuous solution $y(t)$ of the homogeneous equation of* (1.1)

$$y' + A(t)y = 0 \tag{3.1}$$

has the property that if $y(t)$ has relatively compact range in H and $||y(t)|| =$ constant for any $t \in R$, then $y(t)$ is almost periodic. Then any bounded, uniformly continuous solution of (1.1) *with relatively compact range in H is almost periodic.*

Corollary 3.2. *Let $H = R^n$, $A(t)$ be linear for any $t \in R$. If $A(t)$ is not dependent on t or depends on t in a periodic manner, then any bounded solution of* (1.1) *is almost periodic* (*cf.* [6, *Theorem* 5.8, 6.4]).

Proof. Under the assumptions of this corollary, any bounded solution of (3.1) is almost periodic. So, if $||y(t)|| = constant$ for all $t \in R$, then $y(t)$ is bounded, and thus is almost periodic from Theorem 3.1.

Corollary 3.3. *Let $A(t) = A$ be linear and independent of t. Suppose that $\sigma(A) \cap iR$ is countable. Then any bounded, uniformly continuous solution of* (1.1) *with relatively compact range is almost periodic* (*cf.* [14, P_{94}, *Theorem* 5]).

Corollary 3.4. *Let $A(t)$ be linear and $A(t+1) = A(t)$ for all $t \in R$. Let V be the monodromy operator of the semi-group generated by $A(t)$. Assume that $\sigma(V) \cap \{\lambda \in C : |\lambda| = 1\}$ is countable. Then any bounded, uniformly continuous solution of* (1.1) *with relatively compact range is almost periodic* (*cf.* [17, *Theorem* 4.2]).

Remark. Compared with the results we cited, these corollaries require a stronger condition, namely, the monotonicity of A. This is because they are implied by a more general result, Theorem 3.1, which can be applied to more general almost-periodic cases (see the example in the next section). But the previous results can only be applied to special cases.

Lemma 3.5. *Let $H = R^n$, $A(t) = A$ be independent of t. Suppose that u_1, u_2, $u_3 \in D(A)$ and u_1, u_2, u_3 are affinely independent such that*

$$(Au_i - Au_j, u_i - u_j) = 0, \quad i, j = 1, 2, 3. \tag{3.2}$$

where $(\cdot, \cdot)$ is the inner product of R^n. Then there exist a vector $a \in R^n$ and a skew-symmetric linear operator $L \in \mathcal{L}(R^n)$ such that for any $u \in$

$Int\{u|u = \sum_{i=1}^3 \alpha_i u_i,\ 0 \le \alpha_i \le 1,\ \sum_{i=1}^3 \alpha_i = 1\}$,

$$Au = Lu + a. \tag{3.3}$$

In [7], A. Haraux proved this lemma for the case $n = 2$, and he also mentioned that it is valid in a more general framework. See Lemma 1.3 in [7] for details.

Theorem 3.6. *Let* $H = R^n$, $A(t) = A$ *be independent of* t *and compact. Then any bounded, uniformly continuous solution of* (1.1) *is almost periodic on* R.

Proof. It is obvious that R^n is compact synchronous. By Theorem 2.4, it is sufficient to show that Condition 2 holds under such an assumption. Let $u_0(t)$ be the almost-periodic solution of (1.1), and $U_0(t,s)$ be the difference-variation process of $U(t,s)$ along $u_0(t)$. Since $A(t)$ is independent of t, we only need to show that Condition 2 holds for $U_0(t,s)$. Let $\xi_i(t)$ $(i = 1,2)$ be any two distinct trajectories of $U_0(t,s)$ such that $\xi_i(t)$ is bounded, $||\xi_i(t)|| \equiv r_i > 0$, $||\xi_1(t) - \xi_2(t)|| \equiv r_0 > 0$ for all $t \in R$, and $0 < r_0 < r_1 + r_2$. We will show that $\xi_i(t)$ are almost periodic $(i = 1,2)$. Let $\eta_i = u_0(t) - \xi_i(t)$ $(i = 1,2)$. According to the definition of $U_0(t,s)$, $\eta_i(t)$ is a trajectory of $U(t,s)$ for $i = 1,2$, and we have $||u_0(t) - \eta_i(t)|| \equiv r_i > 0$, $||\eta_1(t) - \eta_2(t)|| \equiv r_0 > 0$, for all $t \in R$. Since $0 < r_0 < r_1 + r_2$, $u_0(t), \eta_1(t), \eta_2(t)$ are affinely independent and

$$(A\eta_i(t) - A\eta_j(t), \eta_i(t) - \eta_j(t)) \equiv 0, \quad i,j = 0,1,2 \tag{3.4}$$

for all $t \in R$, where $\eta_0(t) = u_0(t)$. Let $\Omega(t) = \{u : u = \sum_{j=0}^2 \alpha_j \eta_j(t), 0 \le \alpha_j \le 1, \sum_{j=0}^2 \alpha_j = 1\}$. From Lemma 3.5, there exists a linear operator $L(t)$ and an element $a(t) \in H$ such that for any $u(t) \in Int\Omega(t)$, we have

$$Au = L(t)u + a(t). \tag{3.5}$$

Since $\Omega(t) \cap \Omega(\tau) \ne \phi$ for τ close to t, a and L, in fact, are independent of t. Now let $u(t) = \sum_{j=0}^2 \alpha_j \eta_j$, $0 < \alpha_j < 1$, $\sum_{j=0}^2 \alpha_j = 1$. Then, $u(t)$ is a bounded, uniformly continuous solution of the equation

$$u' + Lu = f(t) - a \tag{3.6}$$

and $u(t)$ is bounded. So $u(t)$ is almost periodic. Letting $\alpha_2 \to 0$, we get that $\eta_1(t)$ is almost periodic and we have the same result for $\eta_2(t)$. Finally, since $u_0(t)$ is almost periodic, so are $\xi_1(t)$ and $\xi_2(t)$. This completes the proof of the theorem.

Remark. When $n = 2$, this theorem is Theorem 2.1 in [7].

4. Examples

Example 1. Consider the linear system

$$x' + A(t)x = f(t), \tag{4.1}$$

where $x \in R^3$, $f : R \to R^3$ is almost periodic, and

$$A(t) = \begin{pmatrix} 0 & -a(t) & 0 \\ a(t) & 0 & 0 \\ 0 & 0 & b(t) \end{pmatrix}$$

such that $a(t)$ and $b(t)$ are real almost-periodic functions.

Suppose that $a(t) = a_0 + a^*(t)$ where a_0 is a constant, $a^*(t)$ is almost periodic, $\int_0^t a^*(s)ds$ is bounded on R, and $b(t) \geq 0$ for all $t \in R$. Then, any bounded solution of (4.1) is almost periodic. In fact, for any $t \in R$, $A(t)$ is monotone. Since the homogeneous equation of (4.1)

$$y' + A(t)y = 0 \tag{4.2}$$

has the three linearly independent solutions

$$y_1(t) = \begin{pmatrix} cos \int_0^t a(s)ds \\ -sin \int_0^t a(s)ds \\ 0 \end{pmatrix}, \quad y_2(t) = \begin{pmatrix} sin \int_0^t a(s)ds \\ cos \int_0^t a(s)ds \\ 0 \end{pmatrix},$$

$$y_3(t) = \begin{pmatrix} 0 \\ 0 \\ e^{-\int_0^t b(s)ds} \end{pmatrix},$$

it follows that for any solution $y(t)$ of (4.2), with $||y(t)|| \equiv \mathit{constant}$ for all $t \in R$, we have $y(t) = c_1y_1(t)+c_2y_2(t)$, unless $b(t) \equiv 0$. From the assumption on $a(t)$ and Lemma 1.3, $y(t)$ is almost periodic. So, by Theorem 2.4, any bounded solution of (4.1) is almost periodic.

Example 2. Let $l > 0$ and $F : R \to R$ be a monotone function and $f : R \to R$ be almost periodic. Consider the third-order differential equation

$$u''' + lu'' + u' - u + F(u'' + u) = f(t). \tag{4.3}$$

Equation (4.3) is equivalent to the three-dimensional system

$$\begin{cases} x' - y = 0 \\ y' + x - lz = 0 \\ z' + ly + \frac{1}{l}F(lz) = \frac{1}{l}f(t) \end{cases}$$

or

$$X'(t) + AX(t) = G(t), \tag{4.4}$$

where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3, AX = \begin{pmatrix} -y \\ x - ly \\ ly + \frac{1}{l}F(lz) \end{pmatrix}, \text{and} \quad G(t) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{l}f(t) \end{pmatrix}$$

Since F is monotone, it is easy to verify that A is monotone. By Theorem 3.6, any bounded solution of (4.4) is almost periodic if it exists. From this, we obtain that if a solution $x(t)$ of (4.3) satisfies that $x(t)$, $x'(t)$, $x''(t)$ are all bounded on R, then $x(t)$ is almost periodic.

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